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MAXIMUM DEGREE AND FRACTIONAL MATCHINGS IN UNIFORM HYPERGRAPHS

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Let $\mathscr{H}$ be a family of r-subsets of a finite set X. Set $D(\mathscr{H})=\max_{x\in X}|\{E\colon x\in E\in\mathscr{H}\}|$, (maximum degree). We say that $\mathscr{H}$ is intersecting if for any $H, H'\in\mathscr{H}$ we have $H\cap H'\neq\emptyset$. In this case, obviously, $D(\mathscr{H})\geqq|\mathscr{H}|/r$. According to a well-known conjecture $D(\mathscr{H})\geqq|\mathscr{H}|/(r-1+1/r)$. We prove a slightly stronger result. Let $\mathscr{H}$ be an r-uniform, intersecting hypergraph. Then either it is a projective plane of order $r-1$, consequently $D(\mathscr{H})=|\mathscr{H}|/(r-1+1/r)$, or $D(\mathscr{H})\geqq|\mathscr{H}|/(r-1)$. This is a corollary to a more general theorem on not necessarily intersecting hypergraphs.

1. Introduction, definitions

1.1 Some well-known definitions

We list the basic definitions and notation to be used throughout:
hypergraph $\mathscr{H}$ — a finite collection of non-empty finite sets (edges);
vertex set of $\mathscr{H}$ — $V(\mathscr{H})=\cup\,\{E\colon E\in\mathscr{H}\}$;
rank of $\mathscr{H}$ — $r(\mathscr{H})=\max\,\{|E|\colon E\in\mathscr{H}\}$,
$\mathscr{H}$ is *r-uniform* if the cardinality of every $E\in\mathscr{H}$ is r;
degree of a vertex x (in $\mathscr{H}$) — $d_{\mathscr{H}}(x)=|\{E\colon x\in E\in\mathscr{H}\}|$;
$D(\mathscr{H})=\max\,\{d_{\mathscr{H}}(x)\colon x\in V(\mathscr{H})\}$;
$\mathscr{H}$ is *D-regular* if the degree of every vertex x is D;
partial hypergraph — $\mathscr{H}'\subset\mathscr{H}$;
matching — partial hypergraph of $\mathscr{H}$ whose edges are pairwise disjoint;
$\nu(\mathscr{H})$ — *matching number* — maximum number of edges in a matching;
intersecting hypergraph — $\nu(\mathscr{H})=1$;
transversal (or cover) — a set $T\subset V(\mathscr{H})$ which meets all the edges;
$\tau(\mathscr{H})$ — *transversal number* — minimum cardinality of a transversal.

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1.2 Fractional transversals and matchings

A survey with applications on fractional hypergraph theory can be found e.g. in Berge [1] or Lovász [12], [16]. Here we shall define only the most important concepts of this theory which appeared in first papers published on this topic. (Berge and Simonovits [2], Lovász [11].)

A *fractional transversal* of a hypergraph $\mathcal{H}$ is a weight function $t\colon V(\mathcal{H})\to\mathbf{R}$ satisfying

$$t(x)\geqq 0 \quad \text{for every} \quad x\in V(\mathcal{H}),$$

and

$$\sum_{x\in E} t(x)\geqq 1 \quad \text{for every edge } E\in\mathcal{H}.$$

The *value* of a fractional transversal t is

$$|t| = \sum_{x\in V(\mathcal{H})} t(x).$$

The minimum of $|t|$ when t ranges over all fractional transversals is called the *fractional transversal number* and is denoted by

$$\tau^*(\mathcal{H})=\min\{|t|\colon\ t \text{ is a fractional transversal of } \mathcal{H}\}.$$

Similarly the *fractional matching number* is the maximum value of the fractional matchings of $\mathcal{H}$, i.e.

$$\nu^*(\mathcal{H}) = \max\Big\{\sum_{E\in\mathcal{H}} w(E)\,\Big|\, w\colon \mathcal{H}\to\mathbf{R},\ w(E)\geqq 0,\ \forall x\in V(\mathcal{H}) \text{ we have } \sum_{E\ni x} w(E)\Big\}\leqq 1.$$

Clearly, to determine the fractional transversal number and the fractional matching number is a problem of linear programming. This is a dual pair so by the duality principle of linear programming we have $\tau^*(\mathcal{H})=\nu^*(\mathcal{H})$ for every hypergraph $\mathcal{H}$. Thus

$$1\leqq \nu\leqq \nu^* = \tau^*\leqq \tau\leqq r\nu.$$

In view of the fact that $w(E)\equiv 1/D$ and $t(x)\equiv 1/\min|E|$ are a fractional matching resp. fractional transversal we have

$$\frac{|\mathcal{H}|}{D(\mathcal{H})}\leqq \tau^*(\mathcal{H})\leqq \frac{|V(\mathcal{H})|}{\min\{|E|\colon E\in\mathcal{H}\}}. \tag{1}$$

1.3 An important example

If $\mathcal{H}$ is D-regular and r-uniform then (1) yields

$$|\mathcal{H}|/D = |V(\mathcal{H})|/r = \tau^*(\mathcal{H}). \tag{2}$$

For $r\geqq 3$ write $\mathcal{P}_r$ for the hypergraph consisting of the lines of the r-uniform finite projective plane (if there exists) further let $\mathcal{P}_2$ consist of the 2-tuples of a 3-element set (i.e. $\mathcal{P}_2$ is a triangle) and let $\mathcal{P}_1$ be the hypergraph having only 1 point. It is well-known that $\mathcal{P}_r$ exists provided $r=P+1$, where P is a prime power.

It is evident that every line of the projective plane $\mathcal{P}_r$ is a minimal transversal of $\mathcal{P}_r$. For $r=1, 2, 3$ there is no other minimal transversal. For the projective plane $\mathscr{P}_r$ with $r>3$, J. Pelikán [18] proved that the only transversals of cardinality r are the edges, and

(3) *all other transversal sets have size* $\geqq r+2$.

Summing up: $|\mathscr{P}_r|=|V(\mathscr{P}_r)|=r^2-r+1$, $\mathscr{P}_r$ is r-uniform and r-regular, $\nu(\mathscr{P}_r)=1$, $\tau(\mathscr{P}_r)=r$, $\tau^*(\mathscr{P}_r)=r-1+1/r$.

2. Results

Considering all the r-tuples of an underlying set with $r\nu+r-1$ elements it can be seen that the inequality $\tau\leqq r\nu$ cannot be improved in general. Nevertheless, as L. Lovász observed, the inequality $\tau^*\leqq r\nu$ is not sharp. He showed (see [14], [15]), that, for any hypergraph $\mathscr{H}$, $\tau^*(\mathscr{H})<r(\mathscr{H})\nu(\mathscr{H})$, furthermore

$$\tau^*(r,\nu)=\sup\{\tau^*(\mathscr{H}):\ r(\mathscr{H})\leqq r,\ \nu(\mathscr{H})\leqq \nu\}<r\nu.$$

For $\nu=1$ he proved that $\tau^*(r,1)\leqq r-1+2/(r+1)$ and he conjectured that $\tau^*(r,1)\leqq r-1+1/r$. In this paper we shall prove a bit more.

Theorem. *Let $\mathscr{H}$ be a hypergraph of rank $r\geqq 3$, $\nu(\mathscr{H})=\nu$. Suppose further that $\mathscr{H}$ does not contain a partial hypergraph which consists of $p+1$ copies of pairwise disjoint r-uniform projective planes. Then*

$$\tau^*(\mathscr{H})\leqq (r-1)\nu+p/r.$$

(The proof of the Theorem is in § 5.) We mention that the inequality of the Theorem is sharp. To see this consider the hypergraph $\mathscr{P}'_r$ which we get from $\mathscr{P}_r$ by omitting a line. $(\tau^*(\mathscr{P}'_r)=r-1.)$

The case $r=1$ is of no importance. For $r=2$ the Theorem does not hold true, because for the odd circuits C_{2n+1} one has $\nu(C_{2n+1})=n$, $p=0$ but $\tau^*(C_{2n+1})=n+1/2$. L. Lovász [13] proved that for an ordinary graph G

$$\tau^*(G)\leqq \frac{1}{2}(\tau+\nu)\leqq \frac{3}{2}\nu.$$

The following corollaries are true even if $r<3$.

Corollary 1. *If $\mathscr{H}$ is the union of ν pairwise disjoint copies of $\mathscr{P}_r$, then $\tau^*(\mathscr{H})=(r-1+1/r)\nu$ otherwise $\tau^*(\mathscr{H})\leqq(r-1+1/r)\nu-1/r$.*

Proof. The inequality $\tau^*(\mathscr{H})>(r-1+1/r)\nu-1/r$ implies that $\mathscr{H}$ has a partial hypergraph $\mathscr{H}_\nu$ which is the disjoint union of ν copies of $\mathscr{P}_r$. That is $\mathscr{H}_\nu\subset\mathscr{H}$. Then it follows from (3) that $\mathscr{H}_\nu\equiv\mathscr{H}$. ∎
(The case $r=2$ is left to the reader).

Corollary 2. *Let r be a positive integer for which $\mathscr{P}_r$ does exist. Then $\tau^*(r,\nu)=(r-1+1/r)\nu$. If $\mathscr{P}_r$ does not exist then $\tau^*(r,\nu)\leqq(r-1)\nu$.* ∎

I think that for the time being the determination of the exact value of $\tau^*(r,\nu)$ for other r's is hopelessly difficult, because to solve this problem one probably has to decide whether or not the projective plane $\mathscr{P}_r$ does exist for a given r.

3. Applications

3.1 The maximum degree of hypergraphs

Let us consider an intersecting, r-uniform hypergraph $\mathscr{H}$. Obviously $D(\mathscr{H}) \geqq |\mathscr{H}|/r$. It is a well-known conjecture [6, 7] that $D \geqq |\mathscr{H}|/(r-1+1/r)$. From the Theorem and from (1) a slightly stronger result follows.

Corollary 3. *Let $\mathscr{H}$ be an r-uniform, intersecting hypergraph. Then either it is a projective plane of order $r-1$ and consequently $D(\mathscr{H})=|\mathscr{H}|/(r-1+1/r)$ or $D(\mathscr{H}) \geqq |\mathscr{H}|/(r-1)$.* ∎

In general $D(\mathscr{H}) \geqq |\mathscr{H}|/(r-1+1/r)\nu$. This was proved for $r=2$ by B. Bollobás [4] in a slightly different form. This Corollary is sharp because $D(\mathscr{P}_r')=|\mathscr{P}_r'|/(r-1)$.

3.2 The number of vertices of regular hypergraphs

Using his result $(\tau^*(r, 1) \leqq r-1+2(r+1))$ mentioned above L. Lovász [14], [15] proved the following conjecture of P. Erdős [6] and B. Bollobás [4]:

If $\mathscr{H}$ is an intersecting, r-uniform and regular hypergraph, then $|V(\mathscr{H})| \leqq r^2-r+1$.

By the Theorem and (2) we generalize this result as follows. (For $r=2$ see Bollobás—Eldridge [5]).

Corollary 4. *If $\mathscr{H}$ is r-uniform and regular then $|V(\mathscr{H})| \leqq (r^2-r+1)\nu$. Moreover equality holds if and only if $\mathscr{H}$ is the disjoint union of projective planes or order $r-1$. Furthermore if there is no such r-uniform plane then $|V(\mathscr{H})| \leqq (r^2-r)\nu$.* ∎

By omitting from $\mathscr{P}_r$ a point together with all the lines containing it we get the hypergraph $\mathscr{P}_r''$ which is $(r-1)$-regular, r-uniform and intersecting. It has r^2-r points. This example shows that Corollary 4 is sharp, too.

3.3 Fractional transversal number of r-partite hypergraphs

The hypergraph $\mathscr{H}$ is said to be *r-partite* if $V(\mathscr{H})$ is the disjoint union of $X_1, \ldots, X_r$, and for each $E \in \mathscr{H}$: $|E \cap X_i|=1$ holds $(i=1, 2, \ldots, r)$.
A well-known conjecture of H. J. Ryser states that for an r-partite hypergraph $\tau \leqq (r-1)\nu$. (In particular for $r=2$ this is simply König's Theorem (see [12]). For some small values of r and ν this conjecture has recently been proved by Zs. Tuza [19]). A. Gyárfás [10] proved an easier version of Ryser's conjecture. His result follows from our Theorem because $\mathscr{P}_r$ is not r-partite $(r \geqq 2)$.

Corollary 5. *If the hypergraph $\mathscr{H}$ is r-partite then $\tau^*(\mathscr{H}) \leqq (r-1)\nu(\mathscr{H})$.* ∎

This Corollary is sharp for $\mathscr{P}_r'$ is r-partite and $\tau^*(\mathscr{P}_r')=r-1$.

3.4 Some further applications of this Theorem to extremal graphs and to extremal set-systems can be found in J. Pach—L. Surányi [17], Z. Füredi [9] and P. Frankl—Z. Füredi [8], respectively.

4. The reduction lemma

We denote by $(\mathbf{a}_i, b_i, \mathbf{c}, I)$ the following linear program with minimum value M.

$$(4) \qquad \begin{cases} \mathbf{x} \geqq \mathbf{0} \\ \mathbf{a}_i\mathbf{x} \geqq b_i \quad \text{for all } i \in I. \\ \min \mathbf{cx} = M(\mathbf{a}_i, b_i, \mathbf{c}, I) \end{cases}$$

Here $\mathbf{a}_i$, $\mathbf{x}$ and $\mathbf{c}$ are n-dimensional vectors, b_i's are real numbers and $|I|$ is the number of conditions of the program $(|I|<\infty)$.
The following Proposition is well known in the theory of linear programming. For the sake of completness we give its short proof in the Appendix.

Proposition. *If the linear program* $(\mathbf{a}_i, b_i, \mathbf{c}, I)$ *with n variables has a finite optimum, then there exists a* $J \subset I$ *such that* $M(\mathbf{a}_i, b_i, \mathbf{c}, I) = M(\mathbf{a}_i, b_i, \mathbf{c}, J)$ *and* $|J| \leqq n$.

In other words this Proposition states that the number of conditions of a linear program can be reduced to n without changing the optimum value.

The proof of the Theorem is based on the following Lemma which may help to determine τ^* in some other cases as well.

Lemma. *For any hypergraph* $\mathscr{H}$ *there exists a partial hypergraph* $\mathscr{H}' \subset \mathscr{H}$ *such that* $\tau^*(\mathscr{H}') = \tau^*(\mathscr{H})$ *and* $|\mathscr{H}'| \leqq |V(\mathscr{H}')|$.

Proof. To determine τ^* one has to solve a linear program of dimension $|V(\mathscr{H})|$, with index set I, where $|I| = |\mathscr{H}|$. Of course, this program always has a finite optimum. So by applying the Proposition (possibly several times) one can find a suitable $\mathscr{H}' \subset \mathscr{H}$. ∎

5. Proof of the Theorem

Let $\mathscr{H}$ be an r-uniform hypergraph which does not contain $p+1$ disjoint copies of the projective plane $\mathscr{P}_r$ and $v(\mathscr{H}) = v$. Suppose $r \geqq 3$. (Our proof can be applied for $r=2$, too, but the details are left to the reader.)

It is sufficient to give a suitable fractional transversal t of $\mathscr{H}$. We shall give it by induction on v, while r is fixed. The proof in the case $v=1$ is similar to that one for $v>1$ and that is why we do not separate them, but sometimes we mention the differences.

For $\mathscr{H} = \emptyset$ put $\tau^*(\mathscr{H}) = 0$. By the Lemma we may suppose that $|\mathscr{H}| \leqq |V(\mathscr{H})|$. Consequently,

$$(5) \qquad \min_{x \in V(\mathscr{H})} d(x) \leqq \frac{\sum \{d(x) \colon x \in V(\mathscr{H})\}}{|V(\mathscr{H})|} = \frac{r|\mathscr{H}|}{|V(\mathscr{H})|} \leqq r.$$

Case 1. There exists $x_0 \in V(\mathscr{H})$ *with* $d_{\mathscr{H}}(x_0) = k < r$. Put $\mathscr{H}_0 = \{E \in \mathscr{H} \colon x_0 \in E\} = \{E_1, \ldots, E_k\}$, and $\mathscr{H}_i = \{E \in \mathscr{H} \colon E \cap E_i = \emptyset\}$ for $1 \leqq i \leqq k$.

For the hypergraph $\mathscr{H}_i$ the induction hypothesis can be applied, because $v(\mathscr{H}_i) \leqq v-1$ and of course $\mathscr{H}_i$ does not contain more than p disjoint $\mathscr{P}_r$ as partial hypergraphs. Hence there exists a fractional transversal $t_i\colon V(\mathscr{H}_i)\to\mathbf{R}$ of $\mathscr{H}_i$, such that $|t_i|\leqq(r-1)(v-1)+p/r$. (If here $v=1$ then $\mathscr{H}_i=\emptyset$, $t_i\equiv 0$.)

Put

$$t(x)=\begin{cases}0 & \text{if}\quad x=x_0\\ \dfrac{1}{k}\left(d_0(x)+\sum_{i=1}^{k}t_i(x)\right) & \text{if}\quad x\in V(\mathscr{H})-\{x_0\},\end{cases}$$

where $d_0(x)$ is the degree of x in the hypergraph $\mathscr{H}_0$. We claim that this is a fractional transversal of $\mathscr{H}$. Indeed, $t(x)\geqq 0$, and for any $E\in\mathscr{H}_0$ we have

$$\sum_{x\in E}t(x)\geqq\frac{1}{k}\Big(\sum_{x\in E-\{x_0\}}d_0(x)\Big)\geqq\frac{1}{k}(r-1)\geqq 1.$$

If $E\in\mathscr{H}-\mathscr{H}_0$ then

$$\sum_{x\in E}t(x)=\frac{1}{k}\left(\sum_{x\in E}d_0(x)+\sum_{i=1}^{k}\Big[\sum_{x\in E}t_i(x)\Big]\right)$$

$$\geqq\frac{1}{k}\left(\sum_{\substack{i=1\\ E\cap E_i\neq\emptyset}}^{k}1+\sum_{\substack{i=1\\ E\cap E_i=\emptyset}}^{k}1\right)=1.$$

Finally

$$\tau^*(\mathscr{H})\leqq|t|=\frac{1}{k}\left(\sum_{x\in V(\mathscr{H})-\{x_0\}}d_0(x)+\sum_{i=1}^{k}\Big[\sum_{x\in V(\mathscr{H})}t_i(x)\Big]\right)$$

$$=\frac{1}{k}\left((r-1)k+\sum_{i=1}^{k}|t_i|\right)\leqq(r-1)v+p/r.$$

(For $v=1$ we get $\tau^*(\mathscr{H})\leqq r-1$.)

Case 2. $\min_{x\in V(\mathscr{H})}d_{\mathscr{H}}(x)\geqq r$.

Then, by (5), $\mathscr{H}$ is r-regular, so $|\mathscr{H}|=|V(\mathscr{H})|$. We shall show that $|\mathscr{H}|\leqq(r^2-r)v+p$ from here, by (2), the Theorem follows.

Suppose on the contrary that $|\mathscr{H}|\geqq(r^2-r)v+p+1$. Let E_1 be an arbitrary edge of $\mathscr{H}$ and put $\mathscr{H}_1=\{E\in\mathscr{H}\colon E\cap E_1=\emptyset\}$. Applying the induction hypothesis to $\mathscr{H}_1$ and using (1) we get that

$$|\mathscr{H}|=|\{E\colon E\cap E_1\neq\emptyset\}|+|\mathscr{H}_1|\leqq 1+r(r-1)+\tau^*(\mathscr{H}_1)D(\mathscr{H}_1)\leqq(r^2-r)v+p+1.$$

Here the right side is at most $(r^2-r)v+p$ if there is an edge E with $|E\cap E_1|\geqq 2$. Consequently it is enough to consider the following case.

(6) $$|\mathscr{H}|=|V(\mathscr{H})|=v(r^2-r)+p+1$$

$$|\mathscr{H}-\mathscr{H}_1|=r^2-r+1$$

(7) $$|E\cap E_1|=0\text{ or }1,\quad\text{for any edges}\quad E,E_1.$$

(If $v=1$ and $p=1$ then (6) yields a contradiction, because in this case $\mathscr{H}_1=\emptyset$. Similarly, for $v=1$, $p=0$ we have by (6) and (7) that $\mathscr{H}$ is an r-uniform, r-regular, intersecting system of sets on (r^2-r+1) points with the same number of edges.

This in turn implies that $\mathscr{H}\equiv\mathscr{P}_r$ contradicting to $p=0$ From now on we suppose that $v\geqq 2$.)

Let $E_1, E_2, \ldots, E_v$ any fixed matching. We call an edge $E\in\mathscr{H}$ *crossing* if it intersects more than one E_i $(1\leqq i\leqq v)$. We count the number of edges of $\mathscr{H}-\{E_1\ldots E_v\}$ *with multiplicities* in the poins of $\bigcup_{i=1}^{v} E_i$. We get $vr(r-1)$. From the other hand $|\mathscr{H}-\{E_1, \ldots, E_v\}|=v(r^2-r)+p+1-v$, and so there are at most $v-1-p$ crossing edges. This means that there is an edge, say E_1, in the matching $\{E_1\ldots E_v\}$ which intersects at most one crossing edge.

Case 2a. E_1 is not intersected by a crossing edge.

In other words, the system $\{E_2, \ldots, E_v\}$ has no common point with edges intersecting E_1. Since $v(\mathscr{H})=v$ we get that $\mathscr{H}-\mathscr{H}_1$ is an intersecting family. Further $D(\mathscr{H}-\mathscr{H}_1)=r$, and this together with (6) and (7) implies that $\mathscr{H}-\mathscr{H}_1\equiv\mathscr{P}_r$. Moreover, the underlying sets $V(\mathscr{H}_1)$ and $V(\mathscr{H}-\mathscr{H}_1)$ are disjoint. Applying the induction hypothesis to $\mathscr{H}_1$ with parameters $v-1$ and $p-1$ we get by (1)

$$|\mathscr{H}-\mathscr{H}_1| \leqq \tau^*(\mathscr{H}-\mathscr{H}_1)r \leqq (r^2-r)(v-1)+(p-1),$$

contradicting to (6).

Case 2b. There is a unique crossing edge E' intersecting E_1.

It remains true that $\mathscr{H}-\mathscr{H}_1-\{E'\}$ is an intersecting family, $|\mathscr{H}-\mathscr{H}_1-\{E'\}|=$ $=r^2-r$, $D(\mathscr{H}-\mathscr{H}_1-\{E'\}|=r$. We claim that in this case $\mathscr{H}-\mathscr{H}_1-\{E'\}\equiv\mathscr{P}'_r$. Indeed, by (7), every edge of $\mathscr{H}-\mathscr{H}_1-\{E'\}$ contains a point of degree $r-1$. There are exactly r points of this type, they form a set T. It is easy to check that $(\mathscr{H}-\mathscr{H}_1-\{E'\})\cup\{T\}$ is a finite projective plane. $E'\neq T$ for E' is crossing. So there is an edge $E''\in\mathscr{H}-\mathscr{H}_1$ such that $E'\cap E''=\emptyset$, and

(8) *at least $r-1$ edges E of $\mathscr{H}-\mathscr{H}_1$ have the property that $E\cap E'\neq\emptyset$ and $E\cap E''\neq\emptyset$.*

Let $\mathscr{H}^1=\{E\in\mathscr{H}\colon E\cap E'=\emptyset$ and $E''\cap E=\emptyset\}$ and $\mathscr{H}^2=\mathscr{H}-\mathscr{H}^1$. Applying the induction hypothesis to $\mathscr{H}^1$ with parameters $v-2$ and p, and using that upper bound for $|\mathscr{H}^2|$ which follows from (8), we get

$$|\mathscr{H}| = |\mathscr{H}^1|+|\mathscr{H}^2| \leqq (v-2)(r^2-r)+p+2(r^2-r)+2-(r-1).$$

This again contradicts to (6), provided $r\geqq 3$. ∎

6. Appendix: Proof of the Proposition

Dropping some of the inequalities of (4) the minimal value of the program can only decrease. Hence we have to prove that there is a $J\subset I$, $|J|=n$ such that $M(\mathbf{a}_i, b_i, \mathbf{c}, J)\geqq M(\mathbf{a}_i, b_i, \mathbf{c}, I)=M$.

Suppose, on the contrary, that for every $J\subset I$, $|J|=n$, we have $M(\mathbf{a}_i, b_i, \mathbf{c}, J)<M$. This means that any n of the halfspaces $\{\mathbf{y}\colon \mathbf{a}_i y\geqq b_i\}$ have a point in common with the open convex polytope $\{\mathbf{x}\in\mathbf{R}^n\colon \mathbf{cx}<M, \mathbf{x}\geqq 0\}$. The system (4) has a solution, hence any $n+1$ of the halfspaces $\{\mathbf{y}\colon \mathbf{a}_i\mathbf{y}\geqq b_i\}$ have a point

in common. Now Helly's Theorem in $\mathbf{R}^n$ implies that the intersection of the $|I|+1$ convex sets $\{\mathbf{y}: \mathbf{a}_i\mathbf{y} \geqq b_i\}$ and $\{\mathbf{x}: \mathbf{cx} < M, \mathbf{x} \geqq 0\}$ is not empty, i.e. it contains a point $\mathbf{x}_0$. This point $\mathbf{x}_0$ is feasible for the program (4) and $\mathbf{cx}_0 < M = M(\mathbf{a}_i, b_i, \mathbf{c}, I)$. This contradiction proves the existence of the appropriate J. ∎

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References

[1] C. Berge, Packing problems and hypergraph theory: a survey, *Annals of Discrete Math.* **4.** (1979), 3—37.

[2] C. Berge and M. Simonovits, The coloring number of the direct product of two hypergraphs, in [3], 21—33.

[3] C. Berge and D. K. Ray-Chaudhuri eds., *Hypergraph Seminar* 1972, *Lecture Notes* **411** (Springer-Verlag, Berlin, 1972).

[4] B. Bollobás, Disjoint triples in a 3-graph with given maximal degree, *Quart, J. Math. Oxford, Sec. Ser.* **28** (1977), 81—87.

[5] B. Bollobás and S. E. Eldridge, Maximal matchings in graphs with given minimal and maximal degrees, *Math. Proc. Cambridge Phil. Soc.* **79** (1976), 221—234.

[6] P. Erdős, *private communication.*

[7] P. Frankl, On intersecting families of finite sets, *J. Combinatorial Th. (A)* **24** (1978), 146—161.

[8] P. Frankl and Z. Füredi, Disjoint r-tuples in an r-graph with given maximal degree, submitted to *Quart, J. Math. Oxford.*

[9] Z. Füredi, Erdős—Ko—Rado type theorems with upper bounds on the maximal degree, in: *Algebraic Methods in Graph Theory,* (Proc. Conf. Szeged 1978, L. Lovász et al. eds.) *Coll. Math. Soc. J. Bolyai* **25,** North-Holland 1981, to appear.

[10] A. Gyárfás, *to appear.*

[11] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2** (1972), 253—267.

[12] L. Lovász, *Combinatorial problems and exercises,* Akadémiai Kiadó, Budapest—North-Holland, Amsterdam 1979.

[13] L. Lovász, Minimax theorems for hypergraphs, in [3], 111—126.

[14] L. Lovász, Doctoral thesis, Szeged 1977.

[15] L. Lovász, On minimax theorems of combinatorics (in Hungarian) *Matematikai Lapok* **26** (1975), 209—264.

[16] L. Lovász, Graph theory and integer programming, *Annals of Discrete Math.,* **2** (1978)

[17] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming, in *Algebraic Methods in Graph Theory,* (Proc. Conf. Szeged 1978, L. Lovász et al. eds.) *Coll. Math. Soc. J. Bolyai* **25,** North-Holland 1981, to appear.

[18] J. Pelikán, Properties of balanced incomplete block designs, in: *Combinatorial Theory and its Applications* (P. Erdős, A. Rényi and V. T. Sós, eds.) *Coll. Math. Soc. J. Bolyai* **4,** North-Holland 1971), 869—890.

[19] Zs. Tuza, Some special cases of Ryser's conjecture, to appear.