The number of terms required to calculate $E'(N, k, p)$ is $m + 1$. This is also the minimum number of subdivisions into which an element must be divided by a laboratory technician to be at least 99 per cent confident that he will know the history of the group before exhausting any element. Values of $m + 1$ corresponding to many $p$'s will be found in Table I.

**5. An error expression.** Define $\delta$ to be $E(N, k, p) - E'(N, k, p)$. In other words, $\delta = (N/k)\sum_{i=m+1}^{k} \{\Pr_k(i)E_k(i)\}$.

Since $E_k(k) = 2k - 1$, it follows that $\delta \leqq (2k - 1)\ (N/k)\sum_{i=m+1}^{k} \Pr_k(i)$.

Arbitrarily, $m$ is chosen large enough to make $\sum_{i=0}^{m} \Pr_k(i)$ greater than 0.99. Therefore, $\sum_{i=m+1}^{k} \Pr_k(i)$ is less than 0.01. Consequently,

$$\delta < (2k - 1)(N/k)(0.01) = [2 - (1/k)]/100 \cdot N.$$

That is, $\delta$ is less than $2 - (1/k)$ for each 100 items of the universe. This is a generous error since it was assumed that every pool containing more than $m$ defectives contains $k$ defective elements.

**6. Conclusions.** Using $E'(N, k, p)$, the optimum $k$ and their corresponding economies are determined for many prevalence rates in the range $0.001 \leqq p \leqq 0.38$. Values of $E'(N, k, p)$ are calculated for $k = 4, 8, 12, \cdots$ and at the intermediate integral values necessary to insure that the minimum value is found. Results of this work and comparison with Dorfman's efficiencies are found in Table I.

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# MAXIMUM LIKELIHOOD ESTIMATES IN A SIMPLE QUEUE

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**0. Summary.** The problem of obtaining maximum likelihood estimates for the parameters involved in a stationary single-channel, Markovian queuing process is considered. A method of taking observations is presented which simplifies this problem to that of determining a root of a certain quadratic equation. A useful and even simpler rational approximation is also studied.

**1. Introduction.** By a simple queue is meant a queue having a Poisson input and a negative exponential service time (type $M/M/1$ in the notation of Kendall

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[1]). That is, the arrival of individuals at the tail of the queue is assumed to be a Poisson process with parameter $\lambda$, ($\lambda$ = the mean number of arrivals per unit time), while the time required for an individual at the head of the queue to pass through the service mechanism is assumed to have a negative-exponential distribution with frequency function of the form $\mu e^{-\mu t}$, ($1/\mu$ = the mean length of an individual service time), the individual service times being independent of each other and of the arrival times. An equivalent description of the service mechanism is that individual departures form a Poisson process with parameter $\mu$, independent of the arrivals, provided the queue is nonempty; when the queue is empty, no departures can occur.

The quantity

$$\rho = \lambda/\mu$$

is known as the *traffic intensity* of the system. It is well known, [2], that, if $\rho < 1$, then the distribution of the number, $n(t)$, of individuals in the queue at time $t$ approaches a limiting distribution as $t \to \infty$, independent of the initial queue length. This limiting distribution is geometric with common ratio $\rho$. If $\rho \geqq 1$, then no such limiting distribution exists and the mean queue length becomes infinite. This paper is concerned with the statistical estimation of $\rho$, as well as the individual parameters $\lambda$ and $\mu$.

The most obvious method of estimating $\lambda$ and $\mu$ would be to observe the operation of the queue for a fixed time $s$, note the number of arrivals $n$, the number of departures $m$, and the busy time $\tau$ (that is, the total time during which $n(t) > 0$). Then $\lambda$ would be estimated by $n/s$, and $\mu$ by $m/\tau$. In the non-stationary case, $\rho \geqq 1$, this may be the best one can do. However in the stationary case, $\rho < 1$, the initial value of $n(t)$ is available, which, under the assumption that the process has attained its stationary state, constitutes extra information from which one should be able to make more accurate estimates. However, in order to obtain maximum likelihood estimates, it is necessary to study the distribution of the random variables involved under the condition that the total observation time is fixed, and this turns out to be extremely complicated. In the following it is noted that if the process is observed for a constant *busy time* $\tau$, rather than total time, a considerable simplification is achieved and the problem then admits of an elementary solution.

**2. Sampling method.** One observes that the time axis may be decomposed into two random sequences of intervals: the *busy intervals* consisting of all times when the queue size $n(t)$ is greater than zero, and the *free intervals* consisting of all times when $n(t) = 0$. The assignment of the endpoints of these intervals is arbitrary and immaterial. By the *busy time between* $t_1$ *and* $t_2$ is meant the sum of the lengths of all the busy intervals, or parts of intervals, between time $t_1$ and time $t_2 > t_1$. During any busy interval arrivals and departures proceed as independent Poisson processes with parameters $\lambda$ and $\mu$.

Let the process now be observed until the busy time reaches some preassigned fixed value $\tau$, and the values of the following random variables noted:

$\nu = n(0)$, the initial queue size;

$m =$ the total number of departures during this period;

$T =$ the time of the $m$th departure;

$n =$ the total number of arrivals up to time $T$.

**3. Construction of a likelihood function.** It is assumed that the system is proceeding in equilibrium, i.e., $\rho < 1$ and $\nu$ is taken to have a geometric distribution with common ratio $\rho$.

Let us further define the random variables $x_i =$ the $i$th arrival time, $y_i =$ the $i$th departure time, and $z_i =$ busy time up to the $i$th departure, $i = 1, 2, \cdots$, ($x_i = y_i = z_i = 0$ for $i \leqq 0$).

Note that

$$y_i = \max [y_{i-1}, x_{i-\nu}] + z_i - z_{i-1}.$$

Thus $y_1, \cdots, y_k$ are determined recursively by $z_1, \cdots, z_k$ and $x_1, \cdots, x_{k-\nu}$, and consequently the entire queuing process may be described by specifying $\nu$, the sequence $x_1, x_2, \cdots$, and the sequence $z_1, z_2, \cdots$. The sequences $x_1, x_2, \cdots$ and $z_1, z_2, \cdots$ represent the transition times of independent Poisson processes having parameters $\lambda$ and $\mu$, both processes being independent of $\nu$. Since $z_1, z_2, \cdots$ refers only to busy time, when arrivals and departures proceed independently, the $x_1, x_2, \cdots$ process will be independent of the $z_1, z_2, \cdots$ process.

The likelihood function $L$ may now be constructed stepwise as follows:

(a) $\nu$ has a geometric distribution with frequency function

$$\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}, \qquad \nu = 0, 1, 2, \cdots.$$

(b) $m$ is a function of the $z_i$ only—namely, the maximum index for which $z_m \leqq \tau$. Thus $m$ is independent of $\nu$ and has a Poisson distribution with frequency function

$$e^{-\mu\tau}\frac{(\mu\tau)^m}{m!}, \qquad m = 0, 1, 2, \cdots.$$

(c) The conditional distribution of $z_1, \cdots, z_m$, given $m$, is that of a random subdivision of a fixed interval of length $\tau$ into a fixed number, $m + 1$, of parts, and is thus independent of $\mu$, (and, of course, of $\lambda$).

(d) When $\nu$ and $m$ are given, $x_1, \cdots, x_{m-\nu}$ will be independent of $z_1, \cdots, z_m$, and will have joint frequency function

$$\lambda^{m-\nu}e^{-\lambda x_{m-\nu}}, \qquad 0 \leqq x_1 \leqq \cdots \leqq x_{m-\nu} < \infty.$$

(e) When $\nu$, $m$, $z_1, \cdots, z_m$, and $x_1, \cdots, x_{m-\nu}$ are given, $T = y_m$ is determined, and the number of arrivals from time $x_{m-\nu}$ to time $T$, namely

$n - m + \nu$, will have a Poisson distribution with frequency function

$$e^{-\lambda(T - x_{m-\nu})} \frac{[\lambda(T - x_{m-\nu})]^{n-m+\nu}}{(n - m + \nu)!}, \qquad n - m + \nu = 0, 1, 2, \cdots ; n + \nu > 0.$$

Multiplying all these conditional frequency functions together, one obtains for the likelihood function

$$L = \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu\tau - \lambda T} \cdot \mu^{m-\nu} \cdot \lambda^{n+\nu} \cdot K,$$

where $K$ depends on $\tau$, $m$, $n$, $z_1, \cdots, z_m$, and $x_{m-\nu}$, but not on $\lambda$ or $\mu$. This formula was derived under the assumption that $m - \nu > 0$, but it is easily seen that the same formula holds when $m - \nu \leqq 0$.

**4. Maximum likelihood estimates and approximations.** Standard methods of obtaining maximum likelihood estimates can now be applied. It has been assumed that the process is stationary, $\rho = (\lambda/\mu) < 1$. Consequently $\lambda$ and $\mu$ must be confined to the region $0 < \lambda < \mu$. It is easily seen that at least one interior maximum exists. On differentiating $L$ with respect to $\lambda$ and $\mu$ and setting the derivatives equal to zero, after some simplification, the following equations for the maximum likelihood estimates $\hat{\lambda}$ and $\hat{\mu}$ are obtained:

$$\hat{\lambda} = (\hat{\mu} - \hat{\lambda})(n + \nu - \hat{\lambda}T),$$
$$\hat{\lambda} = (\hat{\lambda} - \hat{\mu})(m - \nu - \hat{\mu}T).$$

Substituting $\hat{\lambda} = \hat{\mu}\hat{\rho}$, and eliminating $\hat{\mu}$, one obtains the following quadratic equation for the maximum likelihood estimate $\hat{\rho}$ of the traffic intensity $\rho$:

$$f(\hat{\rho}) = (m - \nu - 1)T\hat{\rho}^2 - [(m - \nu)T + (n + \nu + 1)\tau]\hat{\rho} + (n + \nu)\tau = 0.$$

Since $f(0) = (n + \nu)\tau > 0, f(1) = -\tau - T < 0$, exactly one solution of this equation lies in the interval $0 < \hat{\rho} < 1$. This unique solution will be the required estimate.

In order to obtain a simple rational approximation to $\hat{\rho}$, one notes that if the terms $m - \nu - 1$ and $n + \nu + 1$ are replaced by $m - \nu$ and $n + \nu$ respectively in the above quadratic equation, the resulting equation will have unity as one root and the other root will be

$$\rho_1 = \frac{(n + \nu)\tau}{(m - \nu)T}.$$

Presumably, under certain conditions, $\rho_1$ will be an approximation to $\hat{\rho}$. More precisely, by a straightforward computation one can show that, provided $0 < \rho_1 < 1$, then

$$\hat{\rho} < \rho_1$$

and

$$0 < \rho_1 - \hat{\rho} < \frac{2\rho_1}{(1 - \rho_1)(m - \nu)}.$$

Consequently, $\rho_1$ will be a good approximation to $\hat{\rho}$ whenever $\rho_1$ is bounded away from unity and $m - \nu$ is large.

On substituting back, one obtains for the maximum likelihood estimates of $\lambda$ and $\mu$

$$\hat{\lambda} = \frac{(n + m)\hat{\rho}}{\hat{\rho}T + \tau},$$

$$\hat{\mu} = \frac{n + m}{\hat{\rho}T + \tau}.$$

Whenever the approximation of $\rho_1$ for $\hat{\rho}$ is valid, the following simple approximations for $\hat{\lambda}$ and $\hat{\mu}$ result:

$$\hat{\lambda} \approx \frac{n + \nu}{T},$$

$$\hat{\mu} \approx \frac{m - \nu}{\tau}.$$

Note the difference between these formulas and the formulas $n/T$ and $m/\tau$ which would result if the initial distribution was neglected as mentioned in Sec. 1.

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# MOST POWERFUL RANK-TYPE TESTS

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For some non-parametric problems the use of the invariance principle reduces the class of suitable tests to those based on ranks of ordered observations. To obtain among these the test that is most powerful from some specified alternative distribution, it is necessary to have the marginal probability distribution of the rank statistic under the alternative. Hoeffding [1] gives a method that expresses the probabilities of such a distribution in terms of an expectation taken with respect to the hypothesis distribution. Applications have been made to the problem of location (Hoeffding [1]) and to the problem of randomness (Lehmann [2] and Terry [3]). We extend Hoeffding's method and, for the problem of location with symmetry, derive a locally most powerful rank-type test against normal alternatives.

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