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Maximum mass and radius of strange stars in the linear approximation of the EOS

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Abstract. The Chandrasekhar limit for strange stars described by a linear equation of state (describing quark matter with density-dependent quark masses) is evaluated. The maximum mass and radius of the star depend on the fundamental constants and on the energy density of the quark matter at zero pressure. By comparing the expression for the mass of the star with the limiting mass formula for a relativistic degenerate stellar configuration one can obtain an estimate of the mass of the strange quark.

Key words. dense matter – equation of state – stars: fundamental parameters

1. Introduction

One of the most important characteristics of compact relativistic astrophysical objects is their maximum allowed mass. The maximum mass is crucial for distinguishing between neutron stars and black holes in compact binaries and in determining the outcome of many astrophysical processes, including supernova collapse and the merger of binary neutron stars. The theoretical value of the maximum mass for white dwarfs and neutron stars was found by Chandrasekhar and Landau and is given by $M_{\rm max} \sim \left(\frac{\hbar c}{G} m_{\rm B}^{-4/3}\right)^{3/2}$ (Shapiro & Teukolsky 1983), where $m_{\rm B}$ is the mass of the baryons (in the case of white dwarfs, even pressure comes from electrons; most of the mass is in baryons). Thus, with the exception of composition-dependent numerical factors, the maximum mass of a degenerate star depends only on fundamental physical constants. The radius $R_{\rm max}$ of the degenerate star obeys the condition $R_{\rm max} \leq \frac{\hbar}{mc}\left(\frac{\hbar c}{G m_{\rm B}^2}\right)^{1/2}$, with m being the mass of either electron (white dwarfs) or neutron (neutron stars) (Shapiro & Teukolsky 1983). White dwarfs are supported against of gravitational collapse by the degeneracy pressure of electrons whereas for neutron stars this pressure comes mainly from the nuclear force between nucleons (Shapiro & Teukolsky 1983). For non-rotating neutron stars with the central pressure at their center tending to the limiting value $\rho_{\rm c}c^2$, an upper bound of around 3 $M_{\odot}$ has been found (Rhoades & Ruffini 1974).

The quark structure of the nucleons, suggested by quantum cromodynamics, opens the possibility of a hadron-quark phase transition at high densities and/or temperatures, as suggested by Witten (1984). In the theories of strong interaction, quark bag models suppose that breaking of physical vacuum takes place inside hadrons. As a result, vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure on the bag wall equilibrates the pressure of quarks, thus stabilizing the system. If the hypothesis of the quark matter is true, then some neutron stars could actually be strange stars, built entirely of strange matter (Alcock et al. 1986; Haensel et al. 1986). For a review of strange star properties, see Cheng et al. (1998).

Most of the investigations of quark star properties have been done within the framework of the so-called MIT bag model. Assuming that interactions of quarks and gluons are sufficiently small, neglecting quark masses and supposing that quarks are confined to the bag volume (in the case of a bare strange star, the boundary of the bag coincides with the stellar surface), the energy density ρc^2 and pressure p of a quark-gluon plasma at temperature T and chemical potential $\mu_{\rm f}$ (the subscript f denotes the various quark flavors u, d, s etc.) are related, in the MIT bag model, by the equation of state (EOS) (Cheng et al. 1998)

$$p = \frac{(\rho - 4B)c^2}{3}, \tag{1}$$

where B is the difference between the energy density of the perturbative and non-perturbative QCD vacuum (the bag constant). Equation (1) is essentially the equation of state of a gas of massless particles with corrections due to the QCD trace anomaly and perturbative interactions. These corrections are always negative, reducing the energy density at given temperature by about a factor of two (Farhi & Jaffe 1984). For quark stars obeying the bag model equation of state (1) the Chandrasekhar limit has been evaluated, from simple energy balance relations, in

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(Bannerjee et al. 2000). In addition to fundamental constants, the maximum mass also depends on the bag constant.

More sophisticated investigations of quark-gluon interactions have shown that Eq. (1) represents a limiting case of more general equations of state. For example, MIT bag models with massive strange quarks and lowest- order QCD interactions lead to some corrections terms in the equation of state of quark matter. Models incorporating restoration of chiral quark masses at high densities and giving absolutely stable strange matter can no longer be accurately described by using Eq. (1). On the other hand, in models in which quark interaction is described by an interquark potential originating from gluon exchange and by a density-dependent scalar potential which restores the chiral symmetry at high densities (Dey et al. 1998), the equation of state $P = P(\rho)$ can be well approximated by a linear function in the energy density ρ (Gondek-Rosinska et al. 2000). It is interesting to note that Frieman & Olinto (1989) and Haensel & Zdunik (1989) have already mentioned the approximation of the EOS by a linear function (see also Prakash et al. 1990; Lattimer et al. 1990). Recently Zdunik (2000) has studied the linear approximation of the equation of state, obtaining all parameters of the EOS as polynomial functions of strange quark mass, the QCD coupling constant and bag constant. The scaling relations have been applied to the determination of the maximum frequency of a particle in a stable circular orbit around strange stars.

It is the purpose of this paper to obtain, by using a simple phenomenological approach (which is thermodynamical in its essence), the maximum mass and radius (the Chandrasekhar limits) for strange stars obeying a linear equation of state. Of course the maximum mass of compact astrophysical objects is a consequence of General Relativity and not of the character of motion of matter constituents. However, the formulae for maximum mass and radius, due to their simple analytical form, give a better insight into the underlying physics of quark stars, also allowing us to obtain some results which cannot be obtained by numerical methods. For example, from the obtained relations one can find the scaling relations for the maximum mass and radius of strange stars in a natural way.

The present paper is organized as follows. The maximum mass and radius of quark stars with a general linear equation of state is obtained in Sect. 2. In Sect. 3 we discuss our results and conclude the paper.

2. Maximum mass and radius for strange stars in the linear approximation of the EOS

We assume that the strange star obeys an equation of state that can be obtained by interpolation with a linear function of density in the form:

$$p = a(\rho - \rho_0)c^2, \tag{2}$$

where a and ρ_0 are non-negative constants. ρ_0 is the energy density at zero pressure.

Such an equation of state has been proposed mainly to describe the strange matter built of u, d and s quarks (Gondek-Rosinska et al. 2000; Zdunik 2000). The physical consistency of the model requires $\rho_0 > 0$.

The particle number density and the chemical potential corresponding to EOS (2) are given respectively by (Zdunik 2000)

$$n(p) = n_0 \left(1 + \frac{a+1}{a}\frac{p}{\rho_0 c^2}\right)^{1/(a+1)}, \tag{3}$$

$$\mu(p) = \mu_0 \left(1 + \frac{a+1}{a}\frac{p}{\rho_0 c^2}\right)^{a/(a+1)}, \tag{4}$$

where n_0 is the particle number density at zero pressure and $\mu_0 = \rho_0 c^2/n_0$.

The parameters a and ρ_0 can be calculated, for realistic equations of state, by using a least squares fit method (Gondek-Rosinska et al. 2000; Zdunik 2000). For the equations of state incorporating restoration of chiral quark masses at high densities proposed in Dey et al. (1998) one obtains the values $a = 0.463$, $\rho_0 = 1.15 \times 10^{15}$ g/cm^3 and $a = 0.455$, $\rho_0 = 1.33 \times 10^{15}$ g/cm^3, respectively (Gondek-Rosinska et al. 2000). The standard bag model corresponds to $a = 0.333$ and $\rho_0 = 4 \times 10^{14}$ g/cm^3 (Cheng et al. 1998).

From Eqs. (3)–(4) it follows that the particle number and chemical potential are related by the equation

$$\mu = \frac{\rho_0 c^2}{n_0^{a+1}} n^a. \tag{5}$$

From the numerical studies of strange star models we know that the density profile of this type of astrophysical object is quite uniform (Glendenning 1996). Therefore we can approximate $n \approx N/V$, which leads to

$$\frac{\mu}{\mu_0} = \left(\frac{4\pi}{3}\right)^{-a} \left(\frac{N}{n_0}\right)^{a} R^{-3a}, \tag{6}$$

where N is the total number of particles in a star of radius R and volume V.

With the use of Eqs. (2)–(6) one obtains the energy density of the star in the form

$$\rho = \frac{\rho_0}{a+1}\left(\frac{4\pi}{3}\right)^{-a-1}\left(\frac{N}{n_0}\right)^{a+1} R^{-3(a+1)} + \frac{a}{a+1}\rho_0. \tag{7}$$

The total mass M of the star is defined according to $M = 4\pi \int_0^R \rho r^2 \mathrm{d}r \approx \frac{4\pi}{3}\rho R^3$ and is given by

$$M = \frac{\rho_0}{a+1}\left(\frac{4\pi}{3}\right)^{-a}\left(\frac{N}{n_0}\right)^{a+1} R^{-3a} + \frac{4\pi}{3}\frac{a}{a+1}\rho_0 R^3, \tag{8}$$

where we assumed that the energy density is approximately constant inside the star.

Extremizing the mass with respect to the radius R by means of $\partial M/\partial R = 0$ gives the relation

$$\frac{\rho_0}{a+1}\left(\frac{4\pi}{3}\right)^{-a}\left(\frac{N}{n_0}\right)^{a+1} R^{-3a} = \frac{4\pi}{3}\frac{1}{a+1}\rho_0 R^3. \tag{9}$$

Substituting Eq. (9) into Eq. (8) we obtain the maximum mass of the strange star in the linear approximation of the EOS:

$$M=\frac{4\pi}{3}\rho_0R^3. \tag{10}$$

This expression is very similar to the expression for the maximum mass of the quark star obtained assuming that the star is composed of three-flavour masslesss quarks, confined in a large bag (Bannerjee et al. 2000; Cheng & Harko 2000). From a physical point of view, Eq. (10) describes a uniform density zero pressure stellar type configuration.

The maximum equilibrium radius corresponds to a minimum total energy of the star (including the gravitational one), for any radius. For ordinary compact stars, the mass is entirely due to baryons, and the corresponding (Newtonian) gravitational potential energy is of the order $E_{\rm G}\sim -\alpha GM^2/R$ ($\alpha=-3/5$ for constant density Newtonian stars). For quark stars, assumed to be formed of massless quarks, the total mass can be calculated from the total (thermodynamic as well as confinment) energy in the star. One possibility for the estimation of the gravitational energy per fermion is to define an effective quark mass incorporating all the energy contributions (Bannerjee et al. 2000).

The gravitational energy per particle (the strange star is assumed to be formed from fermions) is

$$E_{\rm G}=-\frac{GMm_{\rm eff}}{R}, \tag{11}$$

where $m_{\rm eff}$ is the effective mass of the particles inside the star, incorporating also effects such as quark confinment. For a star with N particles one can write $M=Nm_{\rm eff}=\rho_0V$, or $m_{\rm eff}=\rho_0/n$. On the other hand one can assume $\mu=\rho_0/2n$ (Bannerjee et al. 2000), leading to $m_{\rm eff}=2\mu/c^2$. Hence, with the use of Eqs. (6), (9) and (10) we can express the gravitational energy per particle as

$$E_{\rm G}=-2\left(\frac{4\pi}{3}\right)^2G\frac{\rho_0^2}{N}R^5. \tag{12}$$

The energy density per particle of the fermions follows from Eq. (7) and is given by:

$$E_{\rm F}=\frac{4\pi}{3}\frac{1}{a+1}\frac{\rho_0}{N}R^3. \tag{13}$$

The total energy E per particle is

$$E=\frac{4\pi}{3}\frac{1}{a+1}\frac{\rho_0}{N}R^3-2\left(\frac{4\pi}{3}\right)^2G\frac{\rho_0^2}{N}R^5. \tag{14}$$

Extremizing the total energy with respect to the radius (with the total particle number kept constant), $\left(\frac{\partial E}{\partial R}\right)_{N={\rm const.}}=0$, it follows that the maximum radius of the equilibrium configuration in the linear approximation of the EOS is given by:

$$R_{\max}=R_0\frac{c}{\sqrt{\pi\left(a+1\right)G\rho_0}}. \tag{15}$$

The maximum mass of the star can be calculated from Eq. (10) and is:

$$M_{\max}=\frac{4}{3}\frac{R_0^3}{\left(a+1\right)^{3/2}}\frac{c^3}{G}\frac{1}{\sqrt{\pi G\rho_0}}. \tag{16}$$

In Eqs. (15) and (16) R_0 is a numerical factor of the order $R_0\approx 0.474$.

The maximum radius of the quark star given by Eq. (15) is the radius corresponding to the maximum mass. On the other hand for the existing models of strange stars, the configuration with maximum mass has a radius which is lower than the maximum radius. For example, for strange stars described by the bag model equation of state, the maximum radius is 11.40 km, while the radius corresponding to the maximum mass is 10.93 km, which is 4% lower than the maximum radius. This difference is neglected in Eq. (15).

The maximum mass and radius of the star are strongly dependent on the numerical value of the coefficient R_0 and estimations based on other physical models could lead to different numerical estimates of the limiting values of the basic parameters of the static strange stars.

3. Discussions and final remarks

In the present paper we have shown that there is a maximum mass and radius (the Chandrasekhar limits) for quark stars whose equation of state can be approximated by a linear function of the density. We have also obtained the explicit expressions for $M_{\max}$ and $R_{\max}$.

With respect to the scaling of the parameter ρ_0 of the form $\rho_0\rightarrow k\rho_0$, the maximum mass and radius have the following scaling behaviors:

$$R_{\max}\rightarrow k^{-1/2}R_{\max}, M_{\max}\rightarrow k^{-1/2}M_{\max}. \tag{17}$$

For the maximum mass of the strange stars this scaling relation has also been found from the numerical study of the structure equations in the framework of the bag model (Witten 1984; Haensel et al. 1986).

A rescaling of the parameter a of the form $a+1\rightarrow K\left(a+1\right)$, with ρ_0 unscaled, leads to a transformation of the radius and mass of the form

$$R_{\max}\rightarrow K^{-1/2}R_{\max}, M_{\max}\rightarrow K^{-3/2}M_{\max}. \tag{18}$$

A simultaneous rescaling of both a and ρ_0, with $a+1\rightarrow K\left(a+1\right), \rho_0\rightarrow k\rho_0$ gives

$$\begin{aligned} R_{\max} &\rightarrow k^{-1/2}K^{-1/2}R_{\max},\\ M_{\max} &\rightarrow k^{-1/2}K^{-3/2}M_{\max}. \end{aligned} \tag{19}$$

The maximum mass and radius of strange stars with linear EOS is strongly dependent on the numerical value of ρ_0, the mass decreasing with increasing ρ_0. For $\rho_0=4B$, with the bag constant $B=10^{14}$ g/cm^3 (56 MeVfm^{-3}) we obtain $M_{\max}=1.83\ M_{\odot}$, a value that must be compared to the value $M_{\max}=2\ M_{\odot}$ obtained by numerical methods (Witten 1984; Haensel et al. 1986). The difference between the numerical and theoretical predictions is around 10%. For $\rho_0 = 1.33 \times 10^{15}$ gcm^{-3} the maximum mass of the star is about 1 $M_\odot$.

Generally our formulae (15) and (16) underestimate the maximum values of the mass and radius because we have assumed that the density inside the star is uniform. It is obvious that near the surface the density is much lower than at the center of the compact object. Due to the approximations and simplifications used to derive the basic expressions, reflected mainly in the uncertainties in the exact value of the coefficient R_0, Eqs. (15) and (16) cannot provide high precison numerical values of the maximum mass and radius for linear EOS stars, which must be obtained by numerically integrating the gravitational field equations.

For the the linear EOS, $M_{\max}$ and $R_{\max}$ depend mainly on the fundamental constants c and G and on the zero pressure density ρ_0 (the bag constant). The Chandrasekhar expressions for the same physical parameters involve two more fundamental constants, $\hbar$ and the mass of the electron or neutron.

For quark stars, usually one assume they are composed of a three-flavour system of massless quarks, confined in a large bag. Hence the mass of the quark cannot play any role in the mass formula. But the linear EOS with arbitrary a can describe quark matter with non-zero quark masses (the mass of the strange quark $m_{\rm s} \approx 200$ MeV), forming a degenerate Fermi gas (Gondek-Rosinska et al. 2000; Zdunik 2000). Therefore this system should also be described by the same formulae as white dwarfs or neutron stars, not only by Eqs. (15)–(16). Generally ρ_0 is a function of the mass of the strange quark, so this mass implicitly appears in the expression of the maximum mass and radius. But on the other hand we can assume that the Chandrasekhar limit also applies to quark stars with the baryon mass substituted by an effective quark mass $m_{\rm qeff}$, representing the minimum mass of the quark bubbles composing the star. Hence we must have

$$\left(\frac{\hbar c}{G} m_{\rm qeff}^{-4/3}\right)^{3/2} \sim \frac{c^3}{G}\frac{1}{\sqrt{\pi G \rho_0}}. \tag{20}$$

Equation (20) leads to the following expression of the effective mass of the “elementary” quark bubble:

$$m_{\rm qeff} \sim \left(\frac{\hbar \rho_0^{1/3}}{c}\right)^{3/4}. \tag{21}$$

The effective quark mass is determined only by elementary particle physics constants and is independent of G. From its construction $m_{\rm qeff}$ should be relevant when the system is quantum mechanical and involves high velocities and energies. With respect to a scaling of the zero pressure density of the form $\rho_0 \to k\rho_0$, the effective quark mass has the scaling behavior $m_{\rm qeff} \to m_{\rm qeff} k^{1/4}$, similar to the scaling of the strange quark mass (Zdunik 2000).

For $\rho_0 = 4 \times 10^{14}$ gcm^{-3} we obtain $m_{\rm qeff} \sim 3.63 \times 10^{-25}$ g ≈ 204 MeV. For $\rho_0 = 1.33 \times 10^{15}$ gcm^{-3}, Eq. (21) gives $m_{\rm qeff} \sim 4.9 \times 10^{-25}$ g ≈ 275 MeV. The mass given by Eq. (21) can be considered as the minimum mass of the stable quark bubble. It is of the same order of magnitude as the mass $m_{\rm s}$ of the strange quark. Therefore the Chandrasekhar limit applies also for quark stars if we take $m_{\rm qeff}$ for the mass of the elementary constituent of the star.

In the present paper we have considered the maximum mass and radius of strange stars in the linear approximation of the equation of state and the dependence of these quantities on the parameter a has been found. We have also pointed out the existence of scaling relations for the maximum radius of strange stars, an aspect that has not been mentioned in previous investigations (Witten 1984; Haensel et al. 1986; Bannerjee et al. 2000; Zdunik 2000). Our formulae also lead to the transformation relations for the maximum mass and radius of strange stars with respect to separate and simultaneous scaling of the parameters a and ρ_0. On the other hand the possibility of estimation of the mass of the strange quark from general astrophysical considerations can perhaps give a better understanding of the deep connection between micro- and macro-physics.

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