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MAXIMUM MODULUS CONVEXITY AND THE LOCATION OF ZEROS OF AN ENTIRE FUNCTION

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ABSTRACT. Let f be an entire function with non-negative Maclaurin coefficients and let $b(r) = r(rf'(r)/f(r))'$. It is shown that if all the zeros of f lie in the angle $|\arg z| \leq \delta$, where $0 < \delta \leq \pi$, then $\limsup_{r\to\infty} b(r) \geq \frac{1}{4}\operatorname{cosec}^2 \frac{1}{2}\delta$. In particular, we always have $\limsup_{r\to\infty} b(r) > \frac{1}{4}$ for such functions.

1. INTRODUCTION

Let f be an entire function and let $M(r) = M(r, f) = \sup_{|z|=r} |f(z)|$ be its maximum modulus. It is known that the function

$$b(r) = d^2 \log M(r)/d(\log r)^2 \tag{1.1}$$

exists and is continuous except at isolated points and $b(r) \geq 0$ by Hadamard's convexity theorem. Some time ago Hayman [2] showed that in certain situations a little more can be said about $b(r)$. Specifically Hayman showed that if f is transcendental, then $\limsup_{r\to\infty} b(r) \geq A_0$ where $A_0 > .18$. Hayman conjectured that $\frac{1}{4}$ is the best possible value of A_0 but this was disproved by Kjellberg [3]. At about the same time Boichuck and Gol′dberg [1] proved that the best possible value of A_0 is indeed $\frac{1}{4}$ if discussion is restricted to entire functions with positive coefficients. They also showed that more information about $b(r)$ may be obtained if the class of functions under consideration is further restricted. In fact they proved that if $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ is entire and $A = \limsup_{k\to\infty}(n_{k+1} - n_k)$ then $A_0 \geq \frac{1}{4}A^2$ when $a_k > 0$ for all k. Thus the presence of gaps in the Maclaurin series of f tends to increase the size or growth of $b(r)$.

In this note we consider the connection between the size of $b(r)$ and the location of the zeros of f. It turns out that there is a simple and direct relationship between the size of $b(r)$ as measured by $\limsup b(r)$ and the location of the zeros relative to the negative x-axis. The smallest value of $\limsup b(r)$ occurs when all but a finite number of the zeros lie on or in the direction of the

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negative x-axis, and it increases as we swing the zeros away from the negative x-axis. Our main result may be stated as follows.

Theorem 1. *Let f be an entire function with non-negative Maclaurin coefficients and suppose that the equation $f(z) = w$ has infinitely many roots in the angle $|\arg z| \leq \delta$, where $0 < \delta \leq \pi$. Then*

$$\limsup_{r \to \infty} b(r) \geq \tfrac{1}{4} \operatorname{cosec}^2 \frac{1}{2}\delta . \tag{1.2}$$

In particular if f is any transcendental entire function with non-negative coefficients then

$$\limsup_{r \to \infty} b(r) \geq \tfrac{1}{4} . \tag{1.3}$$

Note that (1.3) is Hayman's $\frac{1}{4}$-conjecture for functions with positive coefficients.

A result slightly more general than Theorem 1 may also be obtained by our method.

Theorem 2. *Let f be an entire function with non-negative Maclaurin coefficients and g an entire function satisfying $M(r, g) = o(f(r))$ as $r \to \infty$. Assume that*

$$\frac{1}{4\alpha} = \limsup_{r \to \infty} b(r) < +\infty . \tag{1.4}$$

Then for every $\delta \in (0, 2\sin^{-1}\sqrt{\alpha})$ the equation $f(z) = g(z)$ has at most a finite number of roots in $|\arg z| \leq \delta$.

In the original paper [2], Hayman showed that when f is a polynomial then $\limsup_{r\to\infty} b(r) = 0$; but $\sup_{r>0} b(r) \geq A_0 > .18$ if f is not of the form cz^m. Hayman asked whether there exist entire functions other than cz^m which satisfy $b(r+0) = b(r-0) = 0$ for a value of $r > 0$. An example of such a function was constructed by London [4]. Our next result shows that when the coefficients are non-negative, the vanishing of $b(r)$ at one point $r > 0$ implies $f(z) = cz^m$.

Theorem 3. *Let f be an entire function with non-negative Maclaurin coefficients. Then*

(a) $\limsup_{r\to\infty} b(r) = 0$ *if and only if f is a polynomial;*
(b) *there exists $R > 0$ such that $b(R) = 0$ if and only if $f(z) = cz^m$;*
(c) *if f is a polynomial other than cz^m then* $\sup_{r>0} b(r) \geq \frac{1}{4}$

and equality holds if and only if $f(z) = cz^n(1 + z/a)$ where $c > 0$, $a > 0$ and n is a non-negative integer.

2. An auxiliary lemma

An entire function f with non-negative coefficients cannot vanish on the positive x-axis unless $a_n = 0$ for all n where a_n is its nth Maclaurin coefficient. Thus if one of its coefficients is positive, then $f(r) > 0$ for all $r > 0$ and,

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by continuity, we may find an open set containing the positive x-axis where f never vanishes. Is there a simple way of describing some such set? The following Lemma which is fundamental to our proofs arose as an answer to the above question.

Lemma 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function satisfying* $a_n \geq 0$ *for* $n = 0, 1, 2, \cdots$ *and let* $b(r)$ *be defined by* (1.1). *If* $r \geq 0$ *and* $z = re^{i\theta}$ *then we have*

$$f^2(r) - |f(z)|^2 \leq (4\sin^2 \tfrac{1}{2}\theta) f^2(r) b(r). \tag{2.1}$$

Proof. We have

$$\begin{aligned} f^2(r) - |f(z)|^2 &= \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} a_{n-k} a_k (1 - e^{i(n-2k)\theta}) \\ &= \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} a_{n-k} a_k \cdot 2\sin^2(n-2k)\tfrac{1}{2}\theta \\ &\leq (2\sin^2 \tfrac{1}{2}\theta) \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} a_{n-k} a_k (n-2k)^2 \\ &= (4\sin^2 \tfrac{1}{2}\theta) f^2(r) b(r). \end{aligned}$$

The second equality above is due to the fact that the sum is real; while the inequality in the third step is a result of $|\sin(mt)| \leq |m|\,|\sin t|$ which is valid for all integers m and all real t. The last equality is obtained as follows: Since $a_n \geq 0$ we have $M(r) = f(r)$ for all $r \geq 0$ and so $b(r)$ can be expressed explicitly in terms of $f(r)$ and its first two derivatives. Indeed using the definition (1.1) of $b(r)$ we have

$$f^2(r) b(r) = r^2 f(r) f''(r) + r f(r) f'(r) - (r f'(r))^2. \tag{2.2}$$

If the terms appearing on the righthand side of (2.2) are expressed as series we obtain

$$\begin{aligned} f^2(r) b(r) &= \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} a_{n-k} a_k \{k(k-1) + k - k(n-k)\} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{2} r^n \sum_{k=0}^{n} a_{n-k} a_k \{k^2 - k(n-k) + (n-k)^2 - (n-k)k\} \\ &= \tfrac{1}{2} \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} a_{n-k} a_k (n-2k)^2, \qquad (r \geq 0). \end{aligned} \tag{2.3}$$

This finishes the proof of Lemma 1.

We now turn to the proofs of our results. We shall prove Theorem 3 first since we are going to use one of its assertions in the proof of Theorem 1.

Proof of Theorem 3. (a). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function with non-negative coefficients and suppose that $\limsup_{r \to \infty} b(r) = 0$ where $b(r)$ is

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defined in (1.1). Let ϵ be a positive number less than 1. Then there exists $R > 0$ such that $b(r) < \frac{1}{4}(1-\epsilon)$ for all $r > R$. If this, together with $\sin^2 \frac{1}{2}\theta \leq 1$, is used in (2.1) we obtain

$$|f(z)| \geq \sqrt{\epsilon} f(r) \qquad (|z| = r > R). \tag{2.4}$$

Now if $a_n = 0$ for all $n \geq 1$, there is nothing to prove. Otherwise $a_n > 0$ for at least one value of $n \geq 1$ in which case $f(r) \to \infty$ as $r \to \infty$ and then, (2.4) implies that $|f(z)| \to \infty$ as $|z| \to \infty$. But then f must be a polynomial being an entire function with a pole at ∞. Conversely, if f is a polynomial then by [2, p. 212] $\limsup_{r\to\infty} b(r) = 0$ and this does not require the hypothesis of positivity of coefficients.

(b). Suppose that $b(R) = 0$ for some $R > 0$. Then (2.1) gives $|f(Re^{i\theta})| \geq f(R)$ for all θ; and since the opposite inequality is always true under our assumptions (non-negativity of coefficients) we must have

$$|f(z)| = f(R) \qquad \text{for all } |z| = R. \tag{2.5}$$

We may assume that $f(R) \neq 0$ since otherwise $f(z) \equiv 0$. Let $\{z_1, z_2, \ldots, z_p\}$ be the possible zeros of f in $0 < |z| < R$, necessarily finite in number since otherwise $f(z) \equiv 0$. Let $m \geq 0$ be the multiplicity of the possible zero of f at the origin. Put $\phi_\alpha(z) = R(z-\alpha)/(R^2 - \overline{\alpha}z)$ and note that $|\phi_\alpha(z)| = 1$ when $|z| = R$. Now put $g(z) = f(z)/z^m \prod_{k=1}^{p} \phi_{z_k}(z)$. Then g is analytic and never vanishes in $|z| \leq R$, and $|g(z)| = |z^{-m} f(z)| = R^{-m} f(R)$ on $|z| = R$. By the maximum principle applied to g and to $1/g$, we conclude that g must be a constant in $|z| \leq R$. That is $g(z) = c$ where $|c| = R^{-m} f(R)$. It follows that

$$f(z) = cz^m \prod_{k=1}^{p} \phi_{z_k}(z) \qquad \text{for all } |z| \leq R, \tag{2.6}$$

and so for all $z \neq z_k$. But f has no poles and hence we must have $f(z) = cz^m$. Since the converse is trivial the proof of part (b) is complete.

(c). Suppose that $f(z)$ is a polynomial other than cz^m with positive coefficients. Then f has zeros $\{z_1, \overline{z}_1, \ldots, z_N, \overline{z}_N\}$ away from the origin. If these zeros are ordered so that $0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_N \leq \pi$ where $\theta_k = \arg z_k$ then by using z_1 in (2.1) we obtain $1 \leq (4\sin^2 \frac{1}{2}\theta_1) b(r_1) \leq 4b(r_1)$ where $r_1 = |z_1|$. Hence $\sup_{r>0} b(r) \geq \frac{1}{4}\csc^2 \frac{1}{2}\theta_1 \geq \frac{1}{4}$.

For the remaining parts of (c) note first that if $f(z) = cz^n(1 + z/a)$ with positive c and a then $b(r) = ra/(r+a)^2$ from which it follows immediately that $\sup_{r>0} b(r) = \frac{1}{4}$. Suppose next that f is an entire function with positive coefficients and that $\sup_{r>0} b(r) = \frac{1}{4}$. If $z = re^{i\theta}$ is a zero of f away from the origin then (2.1) gives $1 \leq (4\sin^2 \frac{1}{2}\theta) b(r) \leq \sin^2 \frac{1}{2}\theta$ and this implies that $\theta = \pi$. Thus all the zeros of f must lie on the negative x-axis. Since $\log M(r, f) = O(\log r)^2$ we may write $f(z) = cz^n \prod_{k=1}^{N}(1 + z/z_k)$ where $z_k > 0$ and $1 \leq N \leq \infty$. But then we will have $b(r) = \sum_{k=1}^{N}(rz_k/(r+z_k)^2)$ and $b(z_1)$ will be

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greater than $\frac{1}{4}$ unless $N = 1$. Hence $N = 1$ and f has the desired form. This finishes the proof of (c) and the proof of Theorem 3 is complete.

Proof of Theorem 1. Let $\delta \in (0, \pi]$ and assume that f has an infinite number of zeros in the angle $|\arg z| \leq \delta$. If we denote these zeros by $z_n = r_n e^{i\theta_n}$ where r_n increases to ∞, then $|\theta_n| \leq \delta$ and $\sin^2 \frac{1}{2}\theta_n \leq \sin^2 \frac{1}{2}\delta$. Using $z = z_n$ in (2.1) we obtain $1 \leq (4 \sin^2 \frac{1}{2}\delta) b(r)$ and this implies (1.2). To prove (1.3) we may assume that $\limsup_{r\to\infty} b(r) < +\infty$. Then $\log M(r, f) = O(\log r)^2$ and since f is now assumed to be transcendental it must have an infinity of zeros. Of course these zeros lie in the angle $|\arg z| \leq \pi$ and so taking $\delta = \pi$ we have $\csc^2 \frac{1}{2}\delta = 1$ and (1.3) follows from (1.2).

Proof of Theorem 2. Let f be an entire function with non-negative Maclaurin coefficients and suppose that (1.4) holds true. Then $\alpha \in (0, 1]$ by (1.3), and $\log M(r, f) = O(\log r)^2$. By part (a) of Theorem 3, (1.4) also implies that f is not a polynomial and so f is transcendental. Let g be an entire function satisfying $M(r, g) = o(f(r))$ and assume that the equation $f(z) = g(z)$ has an infinite number of roots in $|\arg z| \leq \delta$, where $\delta \in (0, 2\sin^{-1}\sqrt{\alpha})$. Denote these roots by $w_n = r_n e^{i\theta_n}$ so that $r_n = |w_n|$ increases to infinity and $|\theta_n| \leq \delta$. Then $\sin^2 \frac{1}{2}\theta_n \leq \sin^2 \frac{1}{2}\delta < \alpha$ and, by Lemma 1,

$$f^2(r_n) - |g(w_n)|^2 \leq (4\sin^2 \frac{1}{2}\theta_n) f^2(r_n) b(r_n) \leq (4 \sin^2 \tfrac{1}{2}\delta) f^2(r_n) b(r_n).$$

Since $f(r) \to \infty$ as $r \to \infty$ we have $f(r_n) \neq 0$ for all large n. Of course $|g(w_n)| = o(f(r_n))$ as $n \to \infty$. Dividing by $f(r_n)$ and passing to the limit as $n \to \infty$ we obtain $(4\sin^2 \frac{1}{2}\delta) \limsup_{r\to\infty} b(r) \geq 1$. This last inequality implies that $\limsup_{r\to\infty} b(r) > 1/4\alpha$ a contradiction to (1.4). Hence at most a finite number of the roots of the equation $f(z) = g(z)$ lie in the angle $|\arg z| \leq \delta$.

3. Examples

We define

$$g(z) = \prod_{k=1}^{\infty} (1 + ze^{-k^2}). \tag{3.1}$$

Then g has all its zeros on the negative x-axis, its Maclaurin coefficients are non-negative and by [2, p. 213] $\limsup_{r\to\infty} b(r, g) = \frac{1}{4}$. Let n be a positive integer and put $f(z) = f(z; b) = g(z^n)$. It is easy to verify that $b(r, f) = n^2 b(r^n, g)$ and so $\limsup_{r\to\infty} b(r, f) = n^2/4$. Thus for the function f, $\alpha = 1/n^2$ and the zeros nearest to the x-axis lie on the ray π/n. We have to compare π/n with $2\sin^{-1}\sqrt{\alpha}$. The inequality $2/n < 2\sin^{-1}(1/n) \leq \pi/n$ is easily verified for $n \geq 1$. Equality on the right side holds only for $n = 1$. This shows that the constant $2\sin^{-1}\sqrt{\alpha}$ in Theorem 2 is best possible when $\alpha = 1$. We conjecture that it is best possible for all $\alpha \in (0, 1)$.

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REFERENCES

1. V. S. Boichuk and A. A. Gol′dberg, *The three-lines theorem* (Russian), Mat. Zametki, **15** (1974), 45–53.
2. W. K. Hayman, *Note on Hadamard's convexity theorem*, Proc. of Pure Math., Vol. 11, *Entire functions and related parts of analysis*, Amer. Math. Soc., 210–213, 1968.
3. B. Kjellberg, *The convexity theorem of Hadamard-Hayman*, Proc. of the Sympos. in Math., Royal Institute of Technology, Stockholm (June 1973), 87–114.
4. R. R. London, *A note on Hadamard's three circles theorem*, Bull. London Math. Soc., **9** (1977), 182–185.

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