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# MAXIMUM OF PARTIAL SUMS AND AN INVARIANCE PRINCIPLE FOR A CLASS OF WEAK DEPENDENT RANDOM VARIABLES

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Abstract. The aim of this paper is to investigate the properties of the maximum of partial sums for a class of weakly dependent random variables which includes the instantaneous filters of a Gaussian sequence having a positive continuous spectral density. The results are used to obtain an invariance principle and the convergence of the moments in the central limit theorem.

## 1. Introduction

Let $\mathcal{A},\mathcal{B}$ be two $\sigma$-algebras. Define the strong mixing coefficient by

$$\alpha(\mathcal{A},\mathcal{B})=\sup_{A\in\mathcal{A},B\in\mathcal{B}}|P(AB)-P(A)P(B)|$$

and the maximal coefficient of correlation

$$\rho(\mathcal{A},\mathcal{B})=\sup_{f\in L_2(\mathcal{A}),g\in L_2(\mathcal{B})}|\operatorname{corr}(f,g)|.$$

Obviously $\alpha(\mathcal{A},\mathcal{B})\le\rho(\mathcal{A},\mathcal{B})$. Let $\{X_n\}_{n\in Z}$ be a sequence of random variables and for $T\subset Z$ denote $\mathcal{F}_T=\sigma(X_i,i\in T)$. For two sets $T\subset Z$ and $S\subset Z$ let

$$\alpha(T,S)=\alpha(\mathcal{F}_T,\mathcal{F}_S),$$

$$\rho(T,S)=\rho(\mathcal{F}_T,\mathcal{F}_S).$$

**Definition 1.1.** We call the strictly stationary sequence $\{X_n\}_{n\in Z}$ *strongly mixing* if $\alpha_n\to 0$ as $n\to\infty$, where $\alpha_n$ is defined by

$$\alpha_n=\alpha(\{\dots,-2,-1,0\},\{n,n+1,\dots\}).$$

We say that the sequence is *$\rho$-mixing* if $\rho_n\to 0$ as $n\to\infty$, where

$$\rho_n=\rho(\{\cdots,-2,-1,0\},\{n,n+1,\dots\}).$$

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It should be noted that the functional CLT for stationary centered strongly mixing sequences requires a combination of a polynomial mixing rate and the existence of moments strictly higher than two, while for the $\rho$-mixing sequences having only second moments a logarithmic mixing rate should be imposed. (See Peligrad [12] for a survey.) For a sequence of random variables $\{X_n\}_{n\in Z^d}$, not necessarily stationary, we define

$$\alpha_n^* = \sup \alpha(S,T)$$

and

$$\rho_n^* = \sup \rho(S,T),$$

where the suprema are taken over all pairs of nonempty disjoint subsets $S, T$ of $Z^d, d \geq 1$, such that $dist(S,T) \geq n$. According to Bradley [3], for every $n \geq 1$, $\alpha_n^* \leq \rho_n^* \leq 2\pi\alpha_n^*$. Therefore $\alpha_n^* \to 0$ and $\rho_n^* \to 0$ are equivalent. Let us note that by Rosenblatt ([15], p. 73, Theorem 7) or Bradley ([2], Theorem 1) a Gaussian sequence having a continous positive spectral density satisfies the condition $\rho_n^* \to 0$. Therefore, instantaneous functions $\{f(X_n)\}_{n\in Z}$ of such a sequence provides a class of examples for $\rho^*$-mixing sequences. Other examples are discussed in Bradley [4]. Bryc and Smolensky [6] pointed out that $\rho_n^* \to 0$ follows from a hypercontractivity condition:

(H) There exist $q(k) \to \infty$ (as $k \to \infty$) such that if $S, T \subset N$ satisfy $dist(S,T) \geq k$, then the norm of conditional expectation $E\{\cdot|\mathcal{F}_S\}$ as a linear operator from $L_2(\mathcal{F}_T)$ to $L_{q(k)}(\mathcal{F}_S)$ is 1.

Bradley [2] proved that the condition $\rho_n^* \to 0$ gives enough information to assure the CLT for stationary random fields without assuming mixing rates or moments higher than 2. Miller [10] proved a CLT for block sums from sequences of strictly stationary random fields satisfying a Lindeberg condition and uniformly satisfying $\lim_{n\to\infty} \rho_n^* = 0$. Bryc and Smolenski [6] found bounds for the moments of partial sums for a sequence of random variables satisfying

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.1}$$

It should be said that, according to the proof of Theorem 2 in Bradley [2] and Remark 3 in Bryc and Smolenski [6], if $\{X_n\}_{n\in Z}$ is a strictly stationary Gaussian sequence which has a bounded positive spectral density $f(t)$, i.e. $0 < m < f(t) < M$ for every $t$, then the sequence defined by

$$Z_i = f(X_i, X_{i+1}, \ldots, X_{i+u})$$

for some $u \geq 1$ (where $f : R^u \to R$ is a measurable function) has the property that $\lim_{n\to\infty} \rho_n^* \leq 1 - \dfrac{m}{M}$. For $u = 0$ the sequence $Y_i = f(X_i)$ is such that $\rho_1^* < 1$.

In the nonstationary context Peligrad [14] studied the importance of condition (1.1) in the CLT for strongly mixing sequences of random variables. In this paper we shall investigate the maximum of partial sums for a sequence satisfying (1.1). We shall first prove some properties of uniform integrability for the maximum of partial sums which will allow us to obtain invariance principles for random elements associated to sums of strongly mixing sequences of random variables. What is notable is that only the second moment is assumed and no mixing rates are imposed. In the following text we shall denote by $[x]$ the integer part of $x$; $\Longrightarrow$ denotes weak convergence. The symbol $\ll$ is the Vinogradov symbol, and $\|\cdot\|_p$ denotes the norm in $L_p$.

## 2. Results

Denote $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \operatorname{var} S_n$. For $0 \le t \le 1$ let $W_n(t) = \frac{S_{[nt]}}{\sigma_n}$, and denote by $W(t)$ the standard Brownian motion on $[0,1]$. We shall establish

**Theorem 2.1.** *Suppose $\{X_n\}_{n\in Z}$ is a strictly stationary strongly mixing sequence of random variables which is centered and has finite second moments. Assume*

$$\lim_{n\to\infty} \sigma_n^2 = \infty \quad \text{and} \quad \lim_{n\to\infty} \rho_n^* < 1. \tag{2.1}$$

*Then*

$$0 < \liminf_n \sigma_n^2/n \le \limsup_n \sigma_n^2/n < \infty \tag{2.2}$$

*and*

(2.3)

$$W_n(t) \Longrightarrow W(t) \quad \text{in the space} \quad D[0,1] \quad \text{endowed with the Skorohod topology.}$$

*In addition, for every $p \le 2$*

$$E\left[\frac{\max_{1\le i\le n}|S_i|}{\sigma_n}\right]^p \to E\sup_t |W(t)|^p \quad \text{as} \quad n\to\infty. \tag{2.4}$$

*Remark* 2.1. Bradley [2] proved that the condition

$$|\operatorname{corr}(S_Q, S_{Q^*})| \le r < 1, \tag{2.5}$$

for every $Q \subset \{1,2,\dots,n\}$, $Q^* = \{1,2,\dots,n\} - Q$ and every $n$, implies

$$\frac{1-r}{1+r}\sum_{i=1}^n EX_i^2 \le \sigma_n^2 \le \frac{1+r}{1-r}\sum_{i=1}^n EX_i^2 \quad \text{for every} \quad n \ge 1.$$

As a consequence condition (2.5) implies $\sigma_n^2 \to \infty$.

Because $\rho_n^* \to 0$ as $n \to \infty$ implies $\alpha_n \to 0$ as $n \to \infty$, we can formulate a functional form of a CLT contained in Theorem 4 of Bradley [2]. The next corollary follows directly from Theorem 2.1 and the comment following Theorem 4 in Bradley [2].

**Corollary 2.1.** *Assume $\{X_n\}_{n\in Z}$ is a strictly stationary sequence of centered random variables having second order moments such that $\rho_n^* \to 0$ as $n \to \infty$. Assume either $\sigma_n^2 \to \infty$ or (2.5) holds. Then the conclusions (2.3) and (2.4) of Theorem 2.1 hold. In addition, $\lim_{n\to\infty} \frac{\sigma_n^2}{n} = \sigma^2 > 0$.*

As an intermediate step in proving these results we also prove the following proposition which has significance in itself.

**Proposition 2.1.** *Assume $\{X_n\}_{n\in Z}$ is a strictly stationary sequence of random variables such that $\lim_{n\to\infty} \rho_n^* < 1, EX_1 = 0, EX_1^2 < \infty$. Then*

$$\text{the family} \quad \left\{\frac{\max_{1\le i\le n} S_i^2}{n}\right\}_{n\ge 1} \quad \text{is uniformly integrable.} \tag{2.6}$$

## 3. Proofs

The first proposition relates higher moments of the maximum of partial sums to its first moment.

**Proposition 3.1.** *Assume $\{X_n\}_{n\in Z}$ is a sequence of centered random variables, and $E|X_k|^q<\infty$ for some $q\geq 1$ and every $k\in Z$. Assume $\rho_1^*<1$. Then we can find a constant $K_1=K_1(q)$ such that for every $n\geq 1$*

$$\left\|\max_{1\leq i\leq n}|S_i|\right\|_q\leq\left\|\max_{1\leq i\leq n}|S_i|\right\|_1+K_1\left\|\left(\sum_{j=1}^n X_j^2\right)^{1/2}\right\|_q. \tag{3.1}$$

*Moreover, when $2\leq q\leq 4$, then for $K_2=K_2(q)$ and for every $n\geq 1$ we have*

$$E\max_{1\leq i\leq n}|S_i|^q\leq K_2\left(\left(E\max_{1\leq i\leq n}|S_i|\right)^q+\left(\sum_{j=1}^n EX_j^2\right)^{q/2}+\sum_{j=1}^n E|X_j|^q\right). \tag{3.2}$$

*Proof.* Let $Q\subset\{1,2,\dots,n\}$ and $Q^*=\{1,2,\dots,n\}-Q$. Denote $S_{Q,i}=\sum_{\substack{j=1\\ j\in Q}}^{i}X_j$, $M_{Q,n}=\max_{1\leq i\leq n}|S_{Q,i}|$ and $M_n=\max_{1\leq i\leq n}|S_i|$. Because $S_n=S_{Q,n}+S_{Q^*,n}$ we have

$$2M_{Q,n}-\max_{1\leq i\leq n}|S_{Q,i}-S_{Q^*,i}|\leq M_n.$$

By adding to this relation the similar one with $Q$ replaced by $Q^*$ we get

$$M_{Q,n}+M_{Q^*,n}\leq M_n+\max_{1\leq i\leq n}|S_{Q,i}-S_{Q^*,i}|. \tag{3.3}$$

We also have

$$\|M_n\|_q\leq\|M_{Q,n}\|_q+\|M_{Q^*,n}\|_q. \tag{3.4}$$

By a well known consequence of the Hahn-Banach theorem applied to each term of the right hand side of (3.4) separately we get

$$\|M_{Q,n}\|_q+\|M_{Q^*,n}\|_q=EYM_{Q,n}+EY^*M_{Q^*,n},$$

where $Y$ (respectively $Y^*$) is $\mathcal{F}_Q$-measurable (respectively $\mathcal{F}_{Q^*}$-measurable) and $E|Y^*|^p=E|Y|^p=1$, where $1/p+1/q=1$. As a consequence, by (3.4) we get

$$\begin{aligned}\|M_n\|_q\leq{}&E(Y-EY)M_{Q,n}+E(Y^*-EY^*)M_{Q^*,n}\\&+EY\,EM_{Q,n}+EY^*EM_{Q^*,n}.\end{aligned} \tag{3.5}$$

Now we apply Lemma 1 of Bryc and Smolenski [6] to the $\sigma$-fields $\mathcal{F}_Q$ and $\mathcal{F}_{Q^*}$ and to the random variables $Y-EY$ and $-Y^*+EY^*$. Because $\rho_1^*<1$, according to that lemma we can construct a random variable $Z$ such that

$$E(Z|\mathcal{F}_Q)=Y-EY,\quad E(Z|\mathcal{F}_Q^*)=-Y^*+EY^* \tag{3.6}$$

and, for a certain constant $C_p$,

$$\|Z\|_p\leq C_p(\|Y-EY\|_p+\|Y^*-EY^*\|_p)\leq 4C_p. \tag{3.7}$$

By substituting (3.6), and (3.3) in (3.5) we get

$$\|M_n\|_q \le EZ(M_{Q,n} - M_{Q^*,n}) + EM_n + E \max_{1\le i\le n} |S_{Q,i} - S_{Q^*,i}|.$$

Hence by the Cauchy-Schwarz inequality and (3.7) we have

$$\|M_n\|_q \le \|M_n\|_1 + (4C_p + 1)\| \max_{1\le i\le n} |S_{Q,i} - S_{Q^*,i}|\|_q.$$

We can rewrite this inequality as

$$(\|M_n\|_q - \|M_n\|_1)^q \le (4C_p + 1)^q E \max_{1\le i\le n} |S_{Q,i} - S_{Q^*,i}|^q.$$

We sum this inequality over all the subsets $Q \subset \{1, 2, \dots, n\}$ and get for $K_p = 4C_p + 1$

$$(\|M_n\|_q - \|M_n\|_1)^q \le K_p^q \frac{1}{2^n} \sum_Q E \max_{1\le i\le n} |S_{Q,i} - S_{Q^*,i}|^q \tag{3.8}$$

Now let $\{\varepsilon_n\}$ be a Rademacher sequence independent of $\{X_n\}$, i.e., $\{\varepsilon_n\}$ are i.i.d. with $P(\varepsilon_1 = \pm 1) = 1/2$.

We can easily see that the inequality (3.8) can be rewritten as

$$(\|M_n\|_q - \|M_n\|_1)^q \le K_p^q E \max_{1\le i\le n} \left| \sum_{j=1}^{i} \varepsilon_j \cdot X_j \right|^q.$$

By conditioning on $\{X_n\}$ and then integrating in Levy's inequality for the maximum of a symmetric sequence we get

$$(\|M_n\|_q - \|M_n\|_1)^q \le 2K_p^q E \left| \sum_{j=1}^{n} \varepsilon_j \cdot X_j \right|^q,$$

and by Khintchine's inequality we get (3.1).

The relation (3.2) is obtained from (3.1), and its proof can be completed by the same kind of arguments used by Bryc and Smolenski in the proof of their Lemma 3. □

**Proposition 3.2.** *Assume* $\{X_n\}_{n\in Z}$ *is a sequence of centered random variables such that* $\rho_1^* < 1$ *and*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n EX_i^2 I(|X_i| > \sigma_n)}{\sigma_n^2} = 0. \tag{3.9}$$

*Then the family*

$$\left\{ \frac{\max_{1\le i\le n} S_i^2}{E \max_{1\le i\le n} S_i^2} \right\}_{n\ge 1}$$

*is uniformly integrable.*

*Proof.* We shall truncate the variables at the level $\sigma_n$. Define

$$X_n' = X_n I(|X_n| \le \sigma_n) - EX_n I(|X_n| \le \sigma_n)$$

and

$$X_n'' = X_n I(|X_n| > \sigma_n) - EX_n I(|X_n| > \sigma_n).$$

Denote $M_n' = \max_{1\le i\le n}|S_i'|$ and $M_n'' = \max_{1\le i\le n}|S_i''|$. By Proposition 3.1 and relation (3.1) we find a constant $K$ such that for every $n \ge 1$

$$\|M_n''\|_2 \le \|M_n''\|_1 + K\left(\sum_{i=1}^n E(X_i'')^2\right)^{1/2}. \tag{3.10}$$

Because

$$E|M_n''| \le \sum_{i=1}^n E|X_i''| \le \frac{2}{\sigma_n}\sum_{i=1}^n EX_i^2 I(|X_i| > \sigma_n)$$

and

$$\sum_{i=1}^n E(X_i'')^2 \le 2\sum_{i=1}^n EX_i^2 I(|X_i| > \sigma_n),$$

it follows from (3.9) and (3.10) that

$$\lim_{n\to\infty}\frac{\|M_n''\|_2}{\sigma_n} = 0$$

and therefore

$$\lim_{n\to\infty}\frac{E(M_n'')^2}{EM_n^2} = 0. \tag{3.11}$$

Because $|M_n' - M_n| \le M_n''$, (3.11) gives

$$\lim_{n\to\infty}\frac{E(M_n' - M_n)^2}{EM_n^2} = 0, \tag{3.12}$$

whence

$$\lim_{n\to\infty}\frac{E(M_n')^2}{EM_n^2} = 1. \tag{3.13}$$

Next we apply the relation (3.2) of Proposition 3.1 with $q = 4$ to the sequence $\{M_n'\}$, and we obtain

$$E(M_n')^4 \ll \|M_n'\|_2^2 + \left(\sum_{j=1}^n E(X_j')^2\right)^2 + \sum_{j=1}^n E(X_j')^4.$$

By (3.13), the definition of $X_j'$ and Remark 2.1 it follows that

$$E(M_n')^4 \ll (EM_n^2)^2 + \sigma_n^4 \ll (EM_n^2)^2. \tag{3.14}$$

Because

$$\begin{aligned}
&\frac{EM_n^2 I\left(M_n > \lambda\sqrt{EM_n^2}\right)}{EM_n^2}\\
&\quad\le 4\frac{E(M_n'')^2}{EM_n^2} + 4\frac{E(M_n')^2}{EM_n^2} I\left(M_n' > \frac{\lambda}{2}\sqrt{EM_n^2/2}\right)\\
&\quad\le 4\frac{E(M_n'')^2}{EM_n^2} + \frac{16}{\lambda^2}\frac{E(M_n')^4}{(EM_n^2)^2}
\end{aligned}$$

the result follows by (3.11) and (3.14) on letting first $n$ and then $\lambda$ go to infinity. $\square$

*Remark* 3.1. Proposition 3.2 can be formulated for triangular arrays of random variables. However, we are going to apply it for strictly stationary sequences. In this case we shall be able to find the estimate (3.15) from the following proposition.

**Proposition 3.3.** *Assume $\{X_n\}_{n\in Z}$ is a strictly stationary sequence of random variables which is centered, has finite second moments and $\lim_{n\to\infty}\rho_n^* < 1$. Then there is a constant $K$ such that*

$$E\max_{1\le i\le n} S_i^2 \le Kn \quad \textit{for every} \quad n\ge 1. \tag{3.15}$$

*Proof.* The proof of this proposition is indirect and requires three steps.

I. Let us note first that without restricting the generality we can assume $\rho_1^* < 1$. To see this let $p$ be a natural number such that $\rho_p^* < 1$, and notice that for each of the sequences $\{X_{jp+u}\}_{j\ge 0}, 1\le u\le p$, the first $\rho^*$-mixing coefficient is inferior to $\rho_p^*$ and therefore is strictly less than 1. In addition,

$$\max_{1\le i\le n}|S_i| \le \sum_{u=1}^{p}\max_{1\le i\le [\frac{n}{p}]}\left|\sum_{j=0}^{i} X_{jp+u}\right| + p\max_{1\le i\le n}|X_i|.$$

By stationarity it is enough to establish Propositions 3.3 and 2.1 for $\rho_1^* < 1$.

II. We now define a sequence of numbers

$$a_n = \sqrt{\max_{1\le k\le n} EM_k^2/k}$$

Notice that $\{a_n\}$ is increasing. Define the random element

$$V_n(t) = \frac{S_{[nt]}}{\sqrt{n}a_n}, \quad 0\le t\le 1.$$

Now we establish that $V_n(t)$ is tight in $D[0,1]$.

By Theorems 15.5, 8.3, and the proof of Theorem 8.4 in Billingsley [1] and by stationarity we have only to prove that for each positives $\varepsilon$ and $\eta$ there exist $\delta, 0<\delta<1$, and an integer $n_0$ such that

$$\frac{1}{\delta}P(\max_{i\le n\delta}|S_i| > \varepsilon\sqrt{n}a_n) \le \eta \quad \text{for all} \quad n\ge n_0.$$

By Chebyshev's inequality and since $\{a_n\}$ is increasing, we have

(3.16)
$$\begin{aligned}\frac{1}{\delta}P\left(\max_{i\le n\delta}|S_i| > \varepsilon\sqrt{n}a_n\right) &\le \frac{1}{\delta}\frac{EM_{[n\delta]}^2}{na_n^2} I(M_{[n\delta]} > \varepsilon\sqrt{n}a_n)\\ &\le \frac{EM_{[n\delta]}^2}{[n\delta]a_{[n\delta]}^2} I\left(\frac{M_{[n\delta]}}{\sqrt{[n\delta]}a_{[n\delta]}} > \frac{\varepsilon}{\sqrt{\delta}}\right).\end{aligned}$$

Because $EM_n^2 \le na_n^2$, it follows by Proposition 3.2 that $\left\{\frac{M_n^2}{na_n^2}\right\}$ is a uniformly integrable family, and therefore, by (3.16),

$$\lim_{\delta\to 0}\sup_n \frac{1}{\delta}P(\max_{i\le n\delta}|S_i| > \varepsilon\sqrt{n}a_n) = 0,$$

which completes the proof of tightness.

III. In this step we prove that there is a constant $K$ such that (3.15) holds, i.e.

$$E\max_{1\le i\le n} S_i^2 \le Kn \quad \text{for every} \quad n\ge 1.$$

Let us assume for a contradiction that (3.15) does not hold, i.e. there is a subsequence $\{n_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \frac{EM_{n_k}^2}{n_k} = \infty$. This implies that $\max_{1\leq i\leq n_k} \frac{EM_i^2}{i} \to \infty$ as $k \to \infty$. Let $m_k, 1 \leq m_k \leq n_k$, be such that

$$\frac{EM_{m_k}^2}{m_k} = \max_{1\leq i\leq n_k} \frac{EM_i^2}{i} = \max_{1\leq i\leq m_k} \frac{EM_i^2}{i},$$

and observe that $\{m_k\}$ constructed this way has the property that $\frac{EM_{m_k}^2}{m_k} \to \infty$ as $k \to 0$, and in addition $a_{m_k}^2 = \max_{1\leq i\leq m_k} \frac{EM_i^2}{i} = \frac{EM_{m_k}^2}{m_k}$. The sequence of random elements $V_{m_k}(t) = \frac{S_{[m_k t]}}{\sqrt{EM_{m_k}^2}}$ is therefore a subsequence of $V_n(t) = \frac{S_{[nt]}}{\sqrt{n}a_n}$. By step II of this proof, $\{V_{m_k}(t)\}_k$ is tight in $D[0,1]$. Moreover, because $\frac{m_k}{EM_{m_k}^2} \to 0$ as $k \to \infty$, by Remark 2.1 it follows that for every $0 \leq t \leq 1$ we have $\frac{ES_{[m_k t]}^2}{EM_{m_k}^2} \to 0$ as $k \to \infty$. This implies that the finite dimensional distributions of $V_{m_k}(t)$ are convergent to those of 0 regarded as a continuous function. By Theorem 15.1 in Billingsley [1] it follows that $V_{m_k}(t) \overset{\mathcal{D}}{\Longrightarrow} 0$ as $k \to \infty$ in $D[0,1]$, and also that $\sup_{0\leq t\leq 1} |V_{m_k}(t)| \overset{\mathcal{D}}{\Longrightarrow} 0$ as $k \to \infty$; as a consequence

$$\frac{M_{m_k}}{\sqrt{EM_{m_k}^2}} \overset{\mathcal{D}}{\Longrightarrow} 0 \quad \text{as} \quad k \to \infty. \tag{3.17}$$

Because the family $\left\{\frac{M_{m_k}^2}{EM_{m_k}^2}\right\}$ is uniformly integrable, from the relation (3.17) and Theorem 5.4 in Billingsley [1] we deduce that

$$1 = \frac{EM_{m_k}^2}{EM_{m_k}^2} \to 0 \quad \text{as} \quad k \to \infty.$$

But this is obviously absurd. Therefore $\limsup_m \frac{EM_m^2}{m}$ cannot be $\infty$, and the claim of this step follows. □

*Proof of Proposition 2.1.* According to the step 1 in the proof of Proposition 3.3 we can assume without losing the generality that $\rho_1^* < 1$. The uniform integrability in (2.6) follows as a combination of Propositions 3.2 and 3.3. □

*Proof of Theorem 2.1.* From the proof of Theorem 3 of Bradley [2] with $\lambda = 0$ one can easily deduce that condition (2.1) implies the left hand side of (2.2). The right hand side is an easy consequence of Remark 2.1. By Theorem 1.4 in Peligrad [12] we know that a strongly mixing strictly stationary sequence of random variables, which is centered, has second moments and $\sigma_n \to \infty$, satisfies the invariance principle if and only if $\left\{\frac{S_n^2}{\sigma_n^2}\right\}$ is uniformly integrable and for each positive $\varepsilon$ there is $\lambda > 1$ such that $P(\max_{1\leq i\leq n} |S_i| > \lambda\sigma_n) < \varepsilon/\lambda^2$. We remark that it is enough to prove that the family

$$\left\{\frac{\max_{1\leq i\leq n} S_i^2}{\sigma_n^2}\right\}$$

is an uniformly integrable family. This fact is a consequence of Proposition 2.1 and relation (2.2). The invariance principle is proved. The convergence of moments in (2.1) results from the above considerations and Theorem 5.4 in Billingsley [1]. □

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## References

1. P. Billingsley, *Convergence of Probability Measures*. New York: Wiley (1968). MR **38:**1718
2. R.C. Bradley, On the spectral density and asymptotic normality of weakly dependent random fields, J. Theor. Probab. **5** (1992), 355–373. MR **93e:**60094
3. R.C. Bradley, Equivalent mixing conditions for random fields, Ann. Probab. **21**, 4 (1993), 1921–1926. MR **94j:**60103
4. R.C. Bradley, On regularity conditions for random fields, Proc. Amer. Math. Soc. **121** (1994), 593–598. MR **94h:**60074
5. R.C. Bradley, Utev, S. On second order properties of mixing random sequences and random fields, Prob. Theory and Math. Stat., pp. 99–120, B. Grigelionis et al. (Eds.) VSP/TEV (1994).
6. W. Bryc, W. Smolenski, Moment conditions for almost sure convergence of weakly correlated random variables, Proc. A.M.S.**119**, 2 (1993), 629–635. MR **93k:**60071
7. P. Doukhan, Mixing Properties and Examples, Lecture Notes in Statistics **85**, Springer–Verlag (1994). MR **96b:**60090
8. I.A. Ibragimov, Y.A. Rozanov, Gaussian Random Processes, Berlin: Springer (1978). MR **80f:**60038
9. A.N. Kolmogorov, Y.A. Rozanov, On a strong mixing condition for stationary Gaussian processes, Theory Probab. Appl. **5** (1960), 204–208. MR **24:**A3009 (Russian original)
10. C. Miller, Three theorems on $\rho^*$–mixing random fields, J. of Theoretical Probability **7** , 4 (1994), 867–882. MR **95i:**60023
11. M. Peligrad, Invariance principles for mixing sequences of random variables, The Ann. of Probab.**10**, 4 (1982), 968–981. MR **84c:**60054
12. M. Peligrad, Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables, Progress in Prob. and Stat., Dependence in Prob. and Stat., **11**, pp. 193–223, E. Eberlein, M. Taqqu (eds.). Birkhauser (1986). MR **88j:**60053
13. M. Peligrad, Invariance principles under weak dependence J. of Multiv. Anal. **19**, 2 (1986), 299–310. MR **87m:**60077
14. M. Peligrad, On the asymptotic normality of sequences of weak dependent random variables, J. of Theoretical Probabilities **9** (1996), 703–715. MR **97e:**60046
15. M. Rosenblatt, *Stationary Sequences and Random Fields*. Boston: Birkhauser, (1985). MR **88c:**60077

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