Maxwell and Lamé Eigenvalues on Polyhedra

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Abstract. *In a* convex *polyhedron, a part of the Lamé eigenvalues with hard simple support boundary conditions does not depend on the Lamé coefficients and coincide with the Maxwell eigenvalues. The other eigenvalues depend linearly on a parameter* s *linked to the Lamé coefficients and the associated eigenmodes are the gradients of the Laplace-Dirichlet eigenfunctions. In a* non-convex *polyhedron, such a splitting of the spectrum disappears partly or completely, in relation with the non-* H^2 *singularities of the Laplace-Dirichlet eigenfunctions. From the Maxwell equations point of view, this means that in a non-convex polyhedron, the spectrum cannot be approximated by finite element methods using* H^1 *elements. Similar properties hold in polygons. We give numerical results for two L-shaped domains.*

Introduction

The time-harmonic Maxwell equations have some properties that set them apart from standard elliptic systems, even if one reformulates them as vector Helmholtz equations. One example is the weak regularity on non-smooth domains. The standard "electric" and "magnetic" boundary value problems generically have solutions that are one order less regular than solutions of the corresponding Dirichlet problem. Thus on non-convex polyhedra, one does not have H^1 regularity. See COSTABEL & DAUGE [9] for a thorough discussion of singularities of solutions of Maxwell boundary value problems.

These solutions can be approximated by special finite element methods using a direct variational formulation coercive in $H(\mathbf{curl})$ and discontinuous (edge) elements, see NEDELEC [15]. This formulation presents problems for the computation of eigenvalues, due to the infinite-dimensional null-space of the curl operator. To overcome these problems, mixed variational formulations have been constructed, see LEVILLAIN [14] and BOFFI, FERNANDES, GASTALDI & PERUGIA [7, 6], BOFFI, BREZZI & GASTALDI [5].

Another standard way to avoid these problems is to use a variational formulation ("regularized formulation", see HAZARD & LENOIR [13]) containing the divergence explicitly and coercive in $H(\mathbf{curl})\cap H(\mathrm{div})$. This corresponds to writing the Maxwell equations as vector Helmholtz or Lamé equations. While the resolvent then becomes compact and the spectrum therefore more suited for numerical approximation (see the discussion and the numerical results in ADAM, ARBENZ & GEUS [1]) the fact remains that the solutions, in general, are not in H^1 .

On the other hand, it has been known for some time [8] that the same boundary value problems are solvable in H^1. If one uses conforming finite element methods, it is this second, non-Maxwell, solution that is approximated. This solution can be interpreted as the solution of a boundary value problem for the Lamé equations where the "electric" boundary conditions correspond to a hard simply supported boundary and the "magnetic" conditions to a sliding boundary.

In order to understand the importance of this discrepancy between the Maxwell non- H^1 solutions and the H^1 non-Maxwell solutions, we analyze here the corresponding eigenvalue problems. In particular, we study the dependence of the eigenvalues on a parameter s which can be considered as a penalization parameter for the vanishing divergence condition in the Maxwell equations. For the H^1 (Lamé) problem, s is equal to $\lambda + 2$ if (λ, μ) are the Lamé constants and $\mu = 1$. We present theoretical and numerical evidence showing that this dependence is completely different for the cases of convex or non-convex polyhedra.

In section 1, we introduce the Maxwell eigenproblems and prove the simple dependence of the eigenvalues on the parameter s: some are constant and the others are linear. In section 2, we give the key of the comparison of Maxwell and Lamé problems: the coincidence of different bilinear forms (based on the curl, the divergence or the strain tensor) for H^1 fields in a polyhedra. In section 3, we investigate Lamé eigenproblems and prove the convergence of eigenvalues to the Stokes eigenvalues as $s \to \infty$. In section 4, we give peculiarities of two-dimensional problems and numerical results showing the different behaviors of Maxwell and Lamé eigenvalues in a non-convex polygon.

We assume everywhere that Ω is a *simply connected Lipschitz domain*. We call *polyhedron* a three-dimensional Lipschitz domain with piecewise plane boundary and *curved polyhedron* a Lipschitz domain with piecewise smooth boundary such that in any point of $\partial\Omega$, Ω is locally $\mathscr{C}^\infty$-diffeomorphic to a neighborhood of a boundary point of a polyhedron. Similarly, a *polygon* is a two-dimensional Lipschitz domain with piecewise straight boundary and a *curvilinear polygon* is defined like curved polyhedra.

In general, we will denote by bold letters the functional spaces for the vector fields. Thus $H^s(\Omega)$ denotes the usual Sobolev space on Ω and $\boldsymbol{H}^s(\Omega)$ denotes $H^s(\Omega)^3$. The space of $H^1(\Omega)$ functions with zero traces is denoted by $\overset{\circ}{H}{}^1(\Omega)$. Finally, as usual for Maxwell equations, we need spaces of fields with square integrable curls:

$$\boldsymbol{H}(\mathbf{curl}\,;\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \mid \quad \mathbf{curl}\,\boldsymbol{u} \in L^2(\Omega)^3\}, \tag{0.1}$$

and with square integrable divergences

$$\boldsymbol{H}(\mathrm{div}\,;\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \mid \quad \mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)\}. \tag{0.2}$$

We will use $\langle\cdot,\cdot\rangle$ for the duality bracket and $(\cdot,\cdot)$ for the L^2 product.

1 Maxwell eigenproblems

The eigenfrequency problem in a domain Ω consist in finding non-zero L^2 electromagnetic eigenfields $(\boldsymbol{E}, \boldsymbol{H})$ and non-zero eigenfrequency ω such that

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = 0 \quad \text{in} \quad \Omega. \tag{1.1}$$

Here $\boldsymbol{E}$ is the electric part and $\boldsymbol{H}$ the magnetic part of the electromagnetic field. The exterior boundary conditions on $\partial\Omega$ are those of the perfect conductor ($\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$):

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here we assume ε and μ positive constants, and without restriction we normalize the product $\varepsilon\mu$ to 1 .

Natural spaces for a variational formulation of problem (1.1)-(1.2) are $\boldsymbol{X}_N(\Omega)$ for the electric field and $\boldsymbol{X}_T(\Omega)$ for the magnetic field according to

$$\boldsymbol{X}_N(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}\,;\Omega) \cap \boldsymbol{H}(\mathrm{div}\,;\Omega) \mid \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\},$$

$\boldsymbol{X}_T(\Omega)$ being the similar space with the zero normal component boundary condition. Then, problem (1.1)-(1.2) can equivalently be written: find non-zero $\boldsymbol{E}$ and non-zero ω such that

$$\boldsymbol{E} \in \boldsymbol{X}_N(\Omega), \quad \forall \widetilde{\boldsymbol{E}} \in \boldsymbol{X}_N(\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{E}} = \omega^2 \int_\Omega \boldsymbol{E} \cdot \widetilde{\boldsymbol{E}}. \tag{1.3}$$

The above bilinear form $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)$ is not coercive on $\boldsymbol{X}_N(\Omega)$. Moreover the gradient $\mathbf{grad}\,\Phi$ of any function $\Phi \in \mathring{H}^1(\Omega)$ yields a solution of this problem associated with the value $\omega = 0$. To avoid these drawbacks, a standard procedure is the penalization of the above equation by the $(\mathrm{div}\,\cdot, \mathrm{div}\,\cdot)$ form: for any $s > 0$, we introduce the new problem: find non-zero $\boldsymbol{u}$ such that

$$\begin{aligned} &\boldsymbol{u} \in \boldsymbol{X}_N(\Omega), \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N(\Omega), \\ &\qquad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + s\,\mathrm{div}\,\boldsymbol{u}\;\mathrm{div}\,\boldsymbol{v} = \omega^2 \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}. \end{aligned} \tag{1.4}$$

Thus (1.4) is nothing but the spectral problem for the operator M_s defined from $\boldsymbol{X}_N(\Omega)$ into its dual by

$$\forall \boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{X}_N(\Omega), \quad -\langle M_s\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + s\,\mathrm{div}\,\boldsymbol{u}\;\mathrm{div}\,\boldsymbol{v}. \tag{1.5}$$

Since the bilinear form defining $-M_s$ is coercive on $\boldsymbol{X}_N(\Omega)$, and since $\boldsymbol{X}_N(\Omega)$ is compactly embedded in $L^2(\Omega)^3$, see WEBER [16], for any $s > 0$ the spectrum of $-M_s$ is discrete and formed by a sequence of eigenvalues $\sigma_k[M_s]$ (with repetition according to the multiplicity) which tend to infinity. Let $\boldsymbol{u}_k[M_s]$ be an associated eigenvector, *i.e.* satisfying

$$-M_s\boldsymbol{u}_k = \sigma_k\boldsymbol{u}_k.$$

For the comparison of eigenproblems (1.1)-(1.2) and (1.4), we need the Laplace-Dirichlet operator Δ^{Dir} defined from $\mathring{H}^1(\Omega)$ into its dual $H^{-1}(\Omega)$ by

$$\forall\Phi,\ \Psi\in\mathring{H}^1(\Omega),\quad -\langle\Delta^{\mathrm{Dir}}\Phi,\Psi\rangle = \int_\Omega \mathbf{grad}\,\Phi\ \mathbf{grad}\,\Psi\ . \tag{1.6}$$

Theorem 1.1 *The eigenpairs* $(\sigma_k[M_s],\boldsymbol{u}_k[M_s])$ *span the two following disjoint families:*

(i) *the pairs* $(\omega^2,\boldsymbol{E})$ *with* $\omega\neq 0$, $\boldsymbol{E}$ *and* $\boldsymbol{H}=-\frac{i}{\omega\mu}\,\mathbf{curl}\,\boldsymbol{E}$ *the solutions of problem* (1.1)-(1.2),

(ii) *the pairs* $(s\tau,\mathbf{grad}\,\Phi)$ *with* (τ,Φ) *the eigenpairs of* $-\Delta^{\mathrm{Dir}}$.

PROOF. *a)* If $\omega\neq 0$, $\boldsymbol{E}$ and $\boldsymbol{H}=-\frac{i}{\omega\mu}\,\mathbf{curl}\,\boldsymbol{E}$ solve the problem (1.1)-(1.2), then $\operatorname{div}\boldsymbol{E}=0$ and $\boldsymbol{E}$ belongs to $\boldsymbol{X}_N(\Omega)$. Thus $(\omega^2,\boldsymbol{E})$ is an eigenpair of $-M_s$.
If (τ,Φ) is an eigenpair of $-\Delta^{\mathrm{Dir}}$, then $\mathbf{grad}\,\Phi$ belongs to $\boldsymbol{X}_N(\Omega)$ and $(s\tau,\mathbf{grad}\,\Phi)$ is an eigenpair of $-M_s$.
b) Conversely, let $(\sigma,\boldsymbol{u})$ be an eigenpair of $-M_s$. Let $p=\operatorname{div}\boldsymbol{u}$.
If p is zero then $\omega=\sqrt{\sigma}$, $\boldsymbol{E}=\boldsymbol{u}$ and $\boldsymbol{H}=-\frac{i}{\omega\mu}\,\mathbf{curl}\,\boldsymbol{u}$ is solution of problem (1.1)-(1.2).
If p is non-zero, taking as test functions the gradients of all $\Psi\in\mathring{H}^1(\Omega,\Delta)$, *i.e.* the functions in $\mathring{H}^1(\Omega)$ such that $\Delta\Psi\in L^2(\Omega)$, we obtain

$$\forall\Psi\in\mathring{H}^1(\Omega,\Delta),\quad \int_\Omega sp\,\Delta^{\mathrm{Dir}}\Psi = \sigma\int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,\Psi.$$

Since for any $\Psi\in\mathring{H}^1(\Omega)$ and $\boldsymbol{u}\in\boldsymbol{X}_N(\Omega)$, there holds $(\boldsymbol{u},\mathbf{grad}\,\Psi)=-(\operatorname{div}\boldsymbol{u},\Psi)$, we have

$$\forall\Psi\in\mathring{H}^1(\Omega,\Delta),\quad \int_\Omega p\,(s\Delta^{\mathrm{Dir}}\Psi+\sigma\Psi)=0.$$

As p is non-zero, σ/s belongs to the spectrum of $-\Delta^{\mathrm{Dir}}$. We deduce from the Fredholm alternative that p belongs to the associated eigenspace. Thus p belongs to $\mathring{H}^1(\Omega,\Delta)$ and the field $\boldsymbol{w}$ defined as

$$\boldsymbol{w} := \boldsymbol{u}+\frac{s}{\sigma}\,\mathbf{grad}\,p\in\boldsymbol{X}_N(\Omega)$$

satisfies

$$\operatorname{div}\boldsymbol{w}=p+\frac{s}{\sigma}\Delta p=0 \quad\text{and}\quad \mathbf{curl}\,\boldsymbol{w}=\mathbf{curl}\,\boldsymbol{u}.$$

Thus for any $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega)$ there holds

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{v} &= \int_\Omega \sigma\boldsymbol{u}\cdot\boldsymbol{v} - sp\,\operatorname{div}\boldsymbol{v}\\ &= \int_\Omega (\sigma\boldsymbol{u} + s\,\mathbf{grad}\,p)\cdot\boldsymbol{v} \;=\; \int_\Omega \sigma\boldsymbol{w}\cdot\boldsymbol{v}.\end{aligned}$$

Whence, either $\boldsymbol{w} = 0$ and $\boldsymbol{u} = -\frac{s}{\sigma}\,\mathbf{grad}\,p$ belongs to the second family, or the pair $(\sigma, \boldsymbol{w})$ belongs to the first family and σ is a multiple eigenvalue of $-M_s$. In the latter case, we have obtained that the corresponding eigenspace splits according to the two families *(i)* and *(ii)*. ■

Remark 1.2 We will see in §4 that in plane domains, the eigenpairs of the first family are the pairs $(\tau, \mathbf{curl}\,\Phi)$ with (τ, Φ) the eigenpairs of $-\Delta^{\mathrm{Neu}}$, with Δ^{Neu} the Laplace operator with Neumann boundary conditions. ■

2 Maxwell and Lamé bilinear forms

We have just seen that the bilinear forms $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot) + s(\operatorname{div}\cdot, \operatorname{div}\cdot)$ can be associated for any $s \geq 0$ to the Maxwell equations. As well known, the bilinear form $2\mu(\varepsilon\cdot, \varepsilon\cdot) + \lambda(\operatorname{div}\cdot, \operatorname{div}\cdot)$ is associated to the elasticity system for a material of Lamé coefficients λ and μ. These two families of forms are strongly linked to each other as we are going to show now.

The results of this section are more specific about the closure of $\mathscr{C}^\infty$ fields in the spaces X_N and X_T than [8] where the closedness of $X_N \cap \boldsymbol{H}^1$ in X_N and of $X_T \cap \boldsymbol{H}^1$ in X_T was observed. A related result is the closedness of $H^2 \cap \mathring{H}^1$ in $\{u \in \mathring{H}^1 \mid \Delta u \in L^2\}$ on a polyhedron, which has been known for a long time, see HANNA & SMITH [12].

Let us recall that the strain tensor $\varepsilon_{jk}(\boldsymbol{u})$ is $\frac{1}{2}(\partial_j u_k + \partial_k u_j)$ and $\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})$ denotes the product $\sum_{jk} \varepsilon_{jk}(\boldsymbol{u})\varepsilon_{jk}(\boldsymbol{v})$. Similarly $\mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v}$ denotes $\sum_{jk} \partial_j u_k \partial_j v_k$ and we write $\mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top$ for $\sum_{jk} \partial_j u_k \partial_k v_j$. Then it is easy to obtain the two following formulas for any vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$:

$$\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} \;=\; \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v} \;-\; \mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top, \tag{2.1}$$

$$2\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \;=\; \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v} \;+\; \mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top, \tag{2.2}$$

while the comparison of $(\mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top)$ with $(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v})$ requires two integrations by parts and the introduction of a few geometrical objects.

We assume that Ω is a curved polyhedron. Each of its faces Γ^i is part of a smooth surface S^i. Let $\boldsymbol{n}^i$ be the unique extension of the (exterior to Ω) normal vector field to S^i as a smooth vector field of unit length. We recall that $\partial_{n^i}\boldsymbol{n}^i \equiv 0$. On the boundary $\Gamma = \cup_i \Gamma^i$ of Ω we can define for any vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ its normal and tangential components on the boundary

$$u_n = \boldsymbol{u}\cdot\boldsymbol{n} \quad \text{and} \quad \boldsymbol{u}_\top = \boldsymbol{n}\times(\boldsymbol{u}\times\boldsymbol{n}) = \boldsymbol{u} - u_n\boldsymbol{n},$$

and for $\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$ the bilinear form

$$b(\boldsymbol{u},\boldsymbol{v}) = \int_\Gamma \Big(\boldsymbol{u}_\top\cdot\mathscr{B}(\boldsymbol{v}_\top) + (\operatorname{tr}\mathscr{B})\,u_n v_n\Big)\,\mathrm{d}\sigma \tag{2.3}$$

where on each Γ^i, $\mathscr{B} = \mathbf{grad}\,\boldsymbol{n}^i$ is the second fundamental form of S^i. Thus, on Γ^i, $\operatorname{tr}\mathscr{B} = \operatorname{div}\boldsymbol{n}^i$ and $\boldsymbol{u}_\top\cdot\mathscr{B}(\boldsymbol{v}_\top) = \boldsymbol{u}\cdot\mathscr{B}(\boldsymbol{v}) = \sum_{jk} u_j v_k \partial_j n^i_k$.

Finally, for $\boldsymbol{u}\in\boldsymbol{H}^2(\Omega)$ and $\varphi\in H^2(\Omega)$ we can define on each Γ_i the tangential divergence $\operatorname{div}_\top\boldsymbol{u}_\top$ and the tangential gradient $\mathbf{grad}_\top\varphi$ and for $\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{H}^2(\Omega)$, the following form makes sense

$$d(\boldsymbol{u},\boldsymbol{v}) = \sum_i \int_{\Gamma^i}\Big(\mathbf{grad}_\top u_n\cdot\boldsymbol{v}_\top - \operatorname{div}_\top\boldsymbol{u}_\top\;v_n\Big)\,\mathrm{d}\sigma^i. \tag{2.4}$$

The following partial integration formulas are similar to those of GRISVARD [11, Thm 3.1.1.2].

Lemma 2.1 *Let Ω be a curved polyhedron. For any $\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{H}^2(\Omega)$ there holds*

$$\int_\Omega \mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top\,\mathrm{d}x = \int_\Omega \operatorname{div}\boldsymbol{u}\;\operatorname{div}\boldsymbol{v}\,\mathrm{d}x - b(\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{u},\boldsymbol{v}). \tag{2.5}$$

PROOF. Two integrations by parts yield

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\boldsymbol{u} : (\mathbf{grad}\,\boldsymbol{v})^\top\,\mathrm{d}x &= -\int_\Omega \mathbf{grad}(\operatorname{div}\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}x + \sum_{j,k}\int_\Gamma(\partial_j u_k\,n_k v_j)\,\mathrm{d}\sigma\\ &= \int_\Omega \operatorname{div}\boldsymbol{u}\;\operatorname{div}\boldsymbol{v}\,\mathrm{d}x + \sum_{j,k}\int_\Gamma(\partial_j u_k\,n_k v_j - \operatorname{div}\boldsymbol{u}\,v_n)\,\mathrm{d}\sigma.\end{aligned}$$

The integrand over Γ is equal to

$$\partial_n\boldsymbol{u}\cdot\boldsymbol{v} - (\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n})\cdot\boldsymbol{v} - \operatorname{div}\boldsymbol{u}\,v_n.$$

Using the relations

$$\begin{aligned}\operatorname{div}\boldsymbol{u} &= \operatorname{div}_\top\boldsymbol{u}_\top + \partial_n u_n + u_n\operatorname{div}\boldsymbol{n}\\ \partial_n\boldsymbol{u} &= \mathbf{grad}\,u_n + \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} - (\boldsymbol{u}\cdot\mathbf{grad})\boldsymbol{n}\\ (\mathbf{grad}\,u_n)\cdot\boldsymbol{v} &= (\mathbf{grad}_\top u_n)\cdot\boldsymbol{v}_\top + (\partial_n u_n)\,v_n,\end{aligned}$$

we obtain that the integrand over Γ is equal to $d(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{u},\boldsymbol{v})$. ■

As a consequence of (2.1)-(2.5) we obtain

Lemma 2.2 *Let Ω be a curved polyhedron. For any $\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{H}^2(\Omega)$ there holds*

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v}\,\mathrm{d}x \tag{2.6}$$

$$
\begin{aligned}
&\qquad\qquad + b(\boldsymbol{u},\boldsymbol{v}) - d(\boldsymbol{u},\boldsymbol{v}) \\
= &\int_\Omega 2\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) - \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}x \\
&\qquad\qquad + 2\Big(b(\boldsymbol{u},\boldsymbol{v}) - d(\boldsymbol{u},\boldsymbol{v})\Big).
\end{aligned}
\tag{2.7}
$$

If moreover the faces of Ω *are plane, the above formulas hold with* $b(\boldsymbol{u},\boldsymbol{v}) \equiv 0$ *.*

An important particular case is the situation where either the tangential or the normal components of the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ are zero on the boundary of Ω . Moreover, in such a situation, the equalities deduced from (2.6)-(2.7) extend to vector fields which have only the H^1 regularity. Let us define

$$
\boldsymbol{H}_N(\Omega) = \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N(\Omega) \quad \text{and} \quad \boldsymbol{H}_T(\Omega) = \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T(\Omega).
$$

Theorem 2.3 *Let* Ω *be a curved polyhedron. For any* $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{H}_N(\Omega)$ *and for any* $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{H}_T(\Omega)$ *there holds*

$$
\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{v}) \tag{2.8}
$$

$$
= \int_\Omega 2\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) - \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}x + 2b(\boldsymbol{u},\boldsymbol{v}). \tag{2.9}
$$

If moreover the faces of Ω *are plane, the above formulas hold with* $b(\boldsymbol{u},\boldsymbol{v}) \equiv 0$ *: for any* $\boldsymbol{u}$ *in* $\boldsymbol{H}_N(\Omega)$ *or* $\boldsymbol{H}_T(\Omega)$ *there holds*

$$
\int_\Omega |\,\mathbf{curl}\,\boldsymbol{u}|^2 + |\operatorname{div}\boldsymbol{u}|^2 = \int_\Omega |\,\mathbf{grad}\,\boldsymbol{u}|^2 = \int_\Omega 2|\varepsilon(\boldsymbol{u})|^2 - |\operatorname{div}\boldsymbol{u}|^2. \tag{2.10}
$$

PROOF. For any $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$ satisfying either $\boldsymbol{u}_\top = \boldsymbol{v}_\top = 0$ or $u_n = v_n = 0$ on Γ , the form $d(\boldsymbol{u},\boldsymbol{v})$ is zero and formulas (2.8)-(2.9) are obvious. In order to extend them to vector fields in $\boldsymbol{H}_N(\Omega)$ or $\boldsymbol{H}_T(\Omega)$, we need a density result of $\boldsymbol{H}^2 \cap \boldsymbol{H}_N(\Omega)$ in $\boldsymbol{H}_N(\Omega)$ (and not only the well-known density of H^2 in H^1).

This result requires a rather technical proof (see [10] for a complete version) and can be roughly described in the following way. A vector field $\boldsymbol{u} \in \boldsymbol{H}_N(\Omega)$ being given, the first step consists in approximating it by fields of the form $\varphi_k\boldsymbol{u}$ where φ_k are smooth cut-off functions which are zero in a neighborhood of the corners and edges of Ω . This can be achieved by functions of the type $1 - \chi(\frac{\rho}{\varepsilon})$ at each corner (with ρ the distance to this corner), but has to be combined by the multiplication by r^α with $\alpha \to 0$ at each edge (with r the distance to this edge). The second step is a standard regularizing procedure, combined with the lifting of the traces which should be zero inside each face. ■

Remark 2.4 In Lemma 2.2, all terms except $d(\boldsymbol{u},\boldsymbol{v})$ extend directly by continuity to $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. It follows that $d(\boldsymbol{u},\boldsymbol{v})$ admits such an extension, too. With this extended definition of $d(\boldsymbol{u},\boldsymbol{v})$, the formulas (2.6)-(2.7) remain valid for $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, *cf* [8]. ■

Corollary 2.5 *Let Ω be a curved polyhedron. Then $\boldsymbol{H}_N(\Omega)$ is a closed subspace of $\boldsymbol{X}_N(\Omega)$, and $\boldsymbol{H}_T(\Omega)$ is a closed subspace of $\boldsymbol{X}_T(\Omega)$.*

PROOF. It suffices to note that the bilinear form $b(\boldsymbol{u},\boldsymbol{v})$ in (2.8) is compact with respect to the H^1 norm. ■

3 Lamé eigenproblems

Everywhere in this section, Ω is supposed to be a polyhedron (i.e. with plane faces). As a consequence of polarization of formula (2.10) the Lamé bilinear form associated with Lamé constants $\mu = 1$ and $\lambda = s-2$ coincides *in* $\boldsymbol{H}_N(\Omega)$ with the Maxwell bilinear form

$$\forall \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{H}_N(\Omega),\ \int_\Omega 2\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v}) + (s-2)\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v} = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}.$$

Let us introduce the operator L_s defined from $\boldsymbol{H}_N(\Omega)$ into its dual by

$$\forall \boldsymbol{u},\ \boldsymbol{v}\in \boldsymbol{H}_N(\Omega),\quad -\langle L_s\boldsymbol{u},\boldsymbol{v}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\boldsymbol{u}\ \operatorname{div}\boldsymbol{v}. \tag{3.1}$$

Thus L_s is defined by the *same* bilinear form as M_s in (1.5), but on a different space in general. The natural boundary condition is $\operatorname{div}\boldsymbol{u} = 0$ on Γ. Let us recall, see BIRMAN & SOLOMYAK [3, 4]:

Theorem 3.1 *Let Ω be a curved polyhedron. Then*

$$\boldsymbol{X}_N(\Omega) = \boldsymbol{H}_N(\Omega) + \Big\{\mathbf{grad}\,\Phi \mid \Delta^{\mathrm{Dir}}\Phi \in L^2(\Omega)\Big\}.$$

In particular for a polyhedron, $\boldsymbol{H}_N(\Omega) = \boldsymbol{X}_N(\Omega)$ if and only if Ω is convex.

Thus M_s and L_s coincide when Ω is convex and it is interesting to study the spectrum of L_s in the other case. From the self-adjointness of L_s and general results we have

Lemma 3.2 *Let Ω be a curved polyhedron. The spectrum of L_s is discrete for any $s > 0$. The eigenvalues $\sigma_k[L_s]$ of L_s ordered in a non-decreasing sequence, are continuous non-decreasing functions of s. After a possible renumbering, these eigenvalues depend analytically on s.*

From Theorem 1.1, we see that the eigenvectors of M_s do not depend on s. In the case of L_s there holds

Theorem 3.3 *Let Ω be a polyhedron. The eigenvectors of L_s which do not depend on s, are the eigenvectors of M_s which belong to $\boldsymbol{H}^1(\Omega)$.*

PROOF. It is obvious that the $\boldsymbol{H}^1$ eigenvectors of M_s are eigenvectors of L_s. Conversely, let $\boldsymbol{u}$ be an eigenvector of L_s for any $s > 0$ associated with the eigenvalue $\sigma(s)$. Then, derivating with respect to s the equality $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - s\,\mathbf{grad}\,\mathrm{div}\,\boldsymbol{u} = \sigma(s)\boldsymbol{u}$ yields

$$\forall s > 0, \quad -\mathbf{grad}\,\mathrm{div}\,\boldsymbol{u} = \sigma'(s)\boldsymbol{u}.$$

Thus σ' is constant. If $\sigma' \neq 0$, then $\mathrm{div}\,\boldsymbol{u} \in H^2 \cap \mathring{H}^1(\Omega)$ and $-\Delta^{\mathrm{Dir}}(\mathrm{div}\,\boldsymbol{u}) = \sigma'\,\mathrm{div}\,\boldsymbol{u}$, moreover $\boldsymbol{u} = \frac{1}{\sigma'}\mathbf{grad}\,\mathrm{div}\,\boldsymbol{u}$: we are in family *(ii)* of Theorem 1.1. If $\sigma' = 0$, then $\mathrm{div}\,\boldsymbol{u} = 0$ and $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} = \sigma\boldsymbol{u}$: we are in family *(i)* of Theorem 1.1. ■

Let $\sigma_k[S]$, $k \geq 1$, be the non-decreasing sequence of the eigenvalues of the Stokes operator S:

$$\boldsymbol{u} \in \boldsymbol{V}_N(\Omega), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_N(\Omega), \qquad \int_\Omega \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v} = \sigma \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}, \tag{3.2}$$

where

$$\boldsymbol{V}_N(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}_N(\Omega) \mid \mathrm{div}\,\boldsymbol{u} = 0\}.$$

And we recall that $\sigma_k[L_s]$, $k \geq 1$, are the eigenvalues of L_s ordered in a non-decreasing sequence.

Theorem 3.4 *Let Ω be a polyhedron. For any $k \geq 1$, $\sigma_k[L_s]$ tends to $\sigma_k[S]$ as $s \to \infty$ and the corresponding Lamé eigenspaces tend to the Stokes eigenspaces.*

PROOF. In order to keep the proof as simple as possible, we assume that the Stokes eigenvalues are simple and that each Lamé eigenvalue $\sigma_k[L_s]$ is simple for s large enough.

a) Let $(\sigma^0, \boldsymbol{u}^0)$ be a Stokes eigenpair. Let us assume that it is simple. Thus there exists $p^0 \in L^2(\Omega)$, $p^0|_\Gamma = 0$ such that

$$-\Delta\boldsymbol{u}^0 - \mathbf{grad}\,p^0 = \sigma^0\boldsymbol{u}^0 \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}^0 = 0 \quad \text{in } \Omega\,.$$

We can construct a sequence $(\sigma^\ell, \boldsymbol{u}^\ell, p^\ell)$, $\ell \geq 1$, with $\sigma^\ell \in \mathbb{R}$, $\boldsymbol{u}^\ell \in \boldsymbol{H}_N(\Omega)$ and $p^\ell \in L^2(\Omega)$, $p^\ell|_\Gamma = 0$ such that

$$-\Delta\boldsymbol{u}^\ell - \mathbf{grad}\,p^\ell = \sigma^0\boldsymbol{u}^\ell + \sigma^1\boldsymbol{u}^{\ell-1} + \ldots + \sigma^\ell\boldsymbol{u}^0 \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}^\ell = p^{\ell-1} \quad \text{in } \Omega\,.$$

Indeed, after subtraction of a solution $\boldsymbol{u}_* \in \boldsymbol{H}_N(\Omega)$ of the equation $\mathrm{div}\,\boldsymbol{u}_* = p^{\ell-1}$, the above equation has the form $(S - \sigma_0)\tilde{\boldsymbol{u}}^\ell = \boldsymbol{f}^\ell + \sigma^\ell\boldsymbol{u}^0$, which is solvable if its right hand side is orthogonal to $\boldsymbol{u}^0$: this can be achieved by the choice of σ^ℓ:

$$\sigma^\ell = -\frac{(\boldsymbol{f}^\ell, \boldsymbol{u}_0)}{(\boldsymbol{u}_0, \boldsymbol{u}_0)}.$$

With $\varepsilon = 1/(s-1)$, we see that the power series in ε $(\sum_{\ell\geq 0}\varepsilon^\ell\sigma^\ell, \sum_{\ell\geq 0}\varepsilon^\ell\boldsymbol{u}^\ell)$ is a formal eigenpair of L_s since

$$-L_s\boldsymbol{u} = -\Delta\boldsymbol{u} - (s-1)\,\mathbf{grad}\,\mathrm{div}\,\boldsymbol{u}.$$

Let us prove now that this also holds in the sense of asymptotic expansions: Setting for any $m \geq 1$,

$$\underline{\sigma}^m = \sum_{\ell=0}^{m} \varepsilon^\ell \sigma^\ell \quad \text{and} \quad \underline{\boldsymbol{u}}^m = \sum_{\ell=0}^{m} \varepsilon^\ell \boldsymbol{u}^\ell$$

we immediately check that

$$L_s(\underline{\boldsymbol{u}}^m) + \underline{\sigma}^m \underline{\boldsymbol{u}}^m = \varepsilon^m \Delta \boldsymbol{u}^m + \sum_{\ell=m}^{2m} \varepsilon^\ell \sum_{k=\ell-m}^{m} \sigma^k \boldsymbol{u}^{\ell-k}.$$

Whence the uniform estimate for ε small enough

$$\|L_s(\underline{\boldsymbol{u}}^m) + \underline{\sigma}^m \underline{\boldsymbol{u}}^m\|_{\boldsymbol{H}_N(\Omega)'} \leq C\varepsilon^m \|\underline{\boldsymbol{u}}^m\|_{\boldsymbol{H}_N(\Omega)}.$$

But, with $\boldsymbol{u}_k$ the normalized (in $L^2(\Omega)$) eigenvector associated to the eigenvalue $\sigma_k = \sigma_k[L_s]$, for any $\sigma \in \mathbb{R}$ and $\boldsymbol{u} \in \boldsymbol{H}_N(\Omega)$ there holds

$$\begin{aligned} \|L_s(\boldsymbol{u}) + \sigma \boldsymbol{u}\|^2_{\boldsymbol{H}_N(\Omega)'} &= \sum_{k\geq 1} (\sigma - \sigma_k)^2 (\boldsymbol{u}, \boldsymbol{u}_k)^2 \|\boldsymbol{u}_k\|^2_{\boldsymbol{H}_N(\Omega)'} \\ &\geq C \sum_{k\geq 1} \frac{(\sigma - \sigma_k)^2}{\sigma_k} (\boldsymbol{u}, \boldsymbol{u}_k)^2. \end{aligned}$$

Therefore

$$\sum_{k\geq 1} \frac{(\underline{\sigma}^m - \sigma_k)^2}{\sigma_k} (\underline{\boldsymbol{u}}^m, \boldsymbol{u}_k)^2 \leq C\varepsilon^{2m}.$$

Hence there exists k_0 such that $\lim_{s\to\infty} \sigma_{k_0}(s)$ is equal to $\lim_{\varepsilon\to 0} \underline{\sigma}^m = \sigma^0$, and

$$\exists \delta > 0, \quad \forall k \neq k_0, \quad \frac{(\underline{\sigma}^m - \sigma_k)^2}{\sigma_k} \geq \delta,$$

thus

$$\sum_{k\neq k_0} (\underline{\boldsymbol{u}}^m, \boldsymbol{u}_k)^2 \leq C\varepsilon^{2m}.$$

On the other hand, for $k = k_0$, we must have $|\underline{\sigma}^m - \sigma_{k_0}| \leq c\varepsilon^m$. This proves that $\sum_{\ell\geq 0} \varepsilon^\ell \sigma^\ell$ is the asymptotic development of σ_{k_0} as $s \to \infty$.

b) Conversely, let us fix $\sigma(s) = \sigma_k[L_s]$ the k-th eigenvalue of L_s. From part *a)* of the proof we deduce that $\sigma(s) \leq \sigma_k[S]$. Thus $\sigma(s)$ is bounded as $s \to \infty$ and has a limit σ^0. The corresponding normalized eigenvectors $\boldsymbol{u}(s) = \boldsymbol{u}_k[L_s]$ are thus bounded in the domain of any power of L_s, thus in $\boldsymbol{H}^{1+\delta}(\Omega)$ for $\delta > 0$ small enough. Thus $\boldsymbol{u}(s)$ has a limit $\boldsymbol{u}^0$ in $\boldsymbol{H}_N(\Omega)$. Going back to the equations satisfied by $\boldsymbol{u}(s)$ we find that

$$\operatorname{div} \boldsymbol{u}^0 = 0 \quad \text{and} \quad (s-1)\, \mathbf{grad} \operatorname{div} \boldsymbol{u}(s) \longrightarrow -\Delta \boldsymbol{u}^0 - \sigma^0 \boldsymbol{u}^0 \quad \text{in } \boldsymbol{H}_N(\Omega)'.$$

Setting $p(s) = (s-1) \operatorname{div} \boldsymbol{u}(s)$, we obtain that it converges in $L^2(\Omega)$ to a limit p^0. Thus $(\sigma^0, \boldsymbol{u}^0)$ is a Stokes eigenpair. ■

Combining the result of Theorem 3.4 with the mini-max formulas for the eigenvalues, we obtain the following inequalities:

Corollary 3.5 *Let* Ω *be a polyhedron. We recall that* $\sigma_k[M_s]$ *,* $\sigma_k[L_s]$ *,* $\sigma_k[S]$ *denote the* k *-th eigenvalue with repetition according to the multiplicity of the self-adjoint operators* M_s *,* L_s *and* S *respectively. For any* $k \geq 1$ *and any* $s > 0$ *there holds*

$$\sigma_k[M_s] \leq \sigma_k[L_s] \leq \sigma_k[S]. \tag{3.3}$$

4 Examples in dimension 2

Let Ω be a plane polygon. Thus, when defined with the bilinear form

$$\boldsymbol{u}, \boldsymbol{v} \longrightarrow \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} + s \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v}$$

with $\operatorname{curl} \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$ the scalar curl, M_s and L_s have the same properties as in polyhedra. Moreover there holds:

Theorem 4.1 *Let* Ω *be a curvilinear polygon. Then the solutions of the Maxwell problem* (1.1)-(1.2) *are the pairs* $(\tau, \mathbf{curl}\,\Psi)$ *with* (τ, Ψ) *the eigenpairs of* $-\Delta^{\mathrm{Neu}}$.

PROOF. Let $(\omega^2, \boldsymbol{u})$ be a solution of (1.1)-(1.2). Then, setting $\Psi = \operatorname{curl} \boldsymbol{u}$ we have

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{and} \quad -\mathbf{curl}\,\Psi = \omega^2 \boldsymbol{u}.$$

Therefore $\mathbf{curl}\,\Psi \in L^2$ thus Ψ belongs to $H^1(\Omega)$ (note that we are in dimension 2). Moreover, for all $\Phi \in H^1(\Omega) = H(\mathbf{curl}; \Omega)$ there holds

$$\begin{aligned} \int_\Omega \mathbf{grad}\,\Psi \cdot \mathbf{grad}\,\Phi &= \int_\Omega \mathbf{curl}\,\Psi \cdot \mathbf{curl}\,\Phi = \int_\Omega \omega^2 \boldsymbol{u} \cdot \mathbf{curl}\,\Phi \\ &= \omega^2 \int_\Omega \operatorname{curl} \boldsymbol{u}\ \Phi = \int_\Omega \omega^2 \Psi\, \Phi. \end{aligned}$$

Hence ω^2 belongs to the spectrum of Δ^{Neu} and Ψ is an associated eigenvector. Finally $\boldsymbol{u} = \frac{1}{\omega^2}\,\mathbf{curl}\,\Psi$. ■

Remark 4.2 In dimension 2 , the spectrum of M_s only derives from the Dirichlet and Neumann problems for the Laplace operator. A consequence of this is the continuity of the spectrum of M_s with respect to the domain. For example if we consider a sequence of regular polygons Ω_m which tend to the unit disk Ω , each eigenvalue of M_s on Ω_m converges to the corresponding eigenvalue of M_s on Ω . On each Ω_m , according to Theorem 3.1 the spectrum of L_s is equal to the the spectrum of M_s . But at the limit, the bilinear form of L_s on Ω involves the curvature:

$$(\boldsymbol{u}, \boldsymbol{v}) \longmapsto \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} + s \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \, \mathrm{d}x - 2 \int_\Gamma u_n v_n \, \mathrm{d}\sigma.$$

Its spectrum is necessarily different from the limit as $m \to \infty$. This behavior can be compared with the Babuška polygon-circle paradox [2]. ■

We show results of computations of Maxwell and Lamé eigenvalues (the $\sigma[M_s]$ and $\sigma[L_s]$) versus s in two different L-shaped domains: each has one non-convex corner, thus $\boldsymbol{X}_N(\Omega)$ and $\boldsymbol{H}_N(\Omega)$ differ by a one-dimensional space

$$\boldsymbol{X}_N(\Omega) = \boldsymbol{H}_N(\Omega) \oplus \mathrm{span}\Big\{\mathbf{grad}(\chi(r)\, r^{2/3} \sin \tfrac{2\theta}{3})\Big\} \tag{4.1}$$

with (r, θ) polar coordinates at the non-convex corner and χ a smooth cut-off function $\equiv 1$ in a neighborhood of this corner.

The computations were made with the program Stress Check* using a p-version finite element method. Since this program is meant for computations in elasticity theory, it insists on having a positive bulk modulus λ which means $s > 2$.

*Stress Check is a trademark of Engineering Software Research and Development, Inc., St. Louis, Missouri, USA.

Example 1. $\Omega = \Sigma_0 \setminus \Sigma_1$ where Σ_0 is the square $[0,1] \times [0,1]$ and Σ_1 the square $[\frac{3}{4},1] \times [\frac{3}{4},1]$. The important feature of this domain is its *symmetry* with respect to the first diagonal.

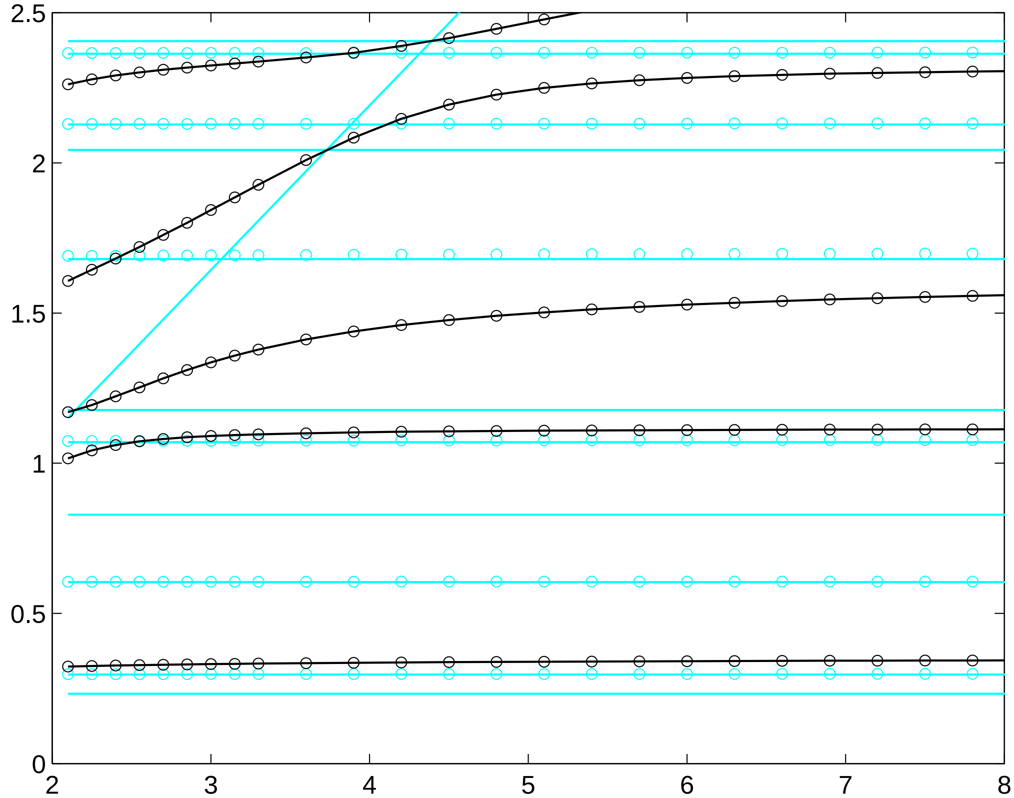

Figure 1: Lowest eigenvalues (symmetric domain)

In Fig. 1, the grey lines are the Maxwell eigenvalues *divided by* $4\pi^2$, deduced from the computation of the Dirichlet and Neumann eigenvalues of the Laplace operator. We can compare the Neumann eigenvalues with those on the square Σ_0, which are, once divided by $4\pi^2$: double at $\frac{1}{4}$, simple at $\frac{1}{2}$, double at 1, etc...

The circles show the results of the computation of the Lamé eigenvalues, still divided by $4\pi^2$. We can see sets of eigenvalues (grey) which are aligned and coincide with Maxwell eigenvalues. We join the other sets by black lines.

We observe in Fig. 1 that the first Lamé eigenvalue coincides with the *second* Maxwell eigenvalue: this is due to the presence of the singularity in the first Maxwell eigenvector, and to its absence in the second one. And more generally, we see that one out of two Maxwell eigenvalues (of family *(i)*, independent of s) coincide with Lamé eigenvalues. Since Ω is symmetric, the eigenvectors of Δ^{Neu} are alternatively odd and even with respect to the first diagonal. Those which are even cannot contain the strong non-H^2 singularity which is odd ($r^{2/3} \cos \frac{2\theta}{3}$).

We also see the first Maxwell eigenvalue of family *(ii)*, generated by the first Dirichlet eigenvalue: the first Dirichlet eigenvector has always the strongest singularities because it has a constant sign. This Maxwell eigenvalue does not correspond to any Lamé eigenvalue.

In the second figure the eigenvalues between 2.5 and 7.5 are represented.

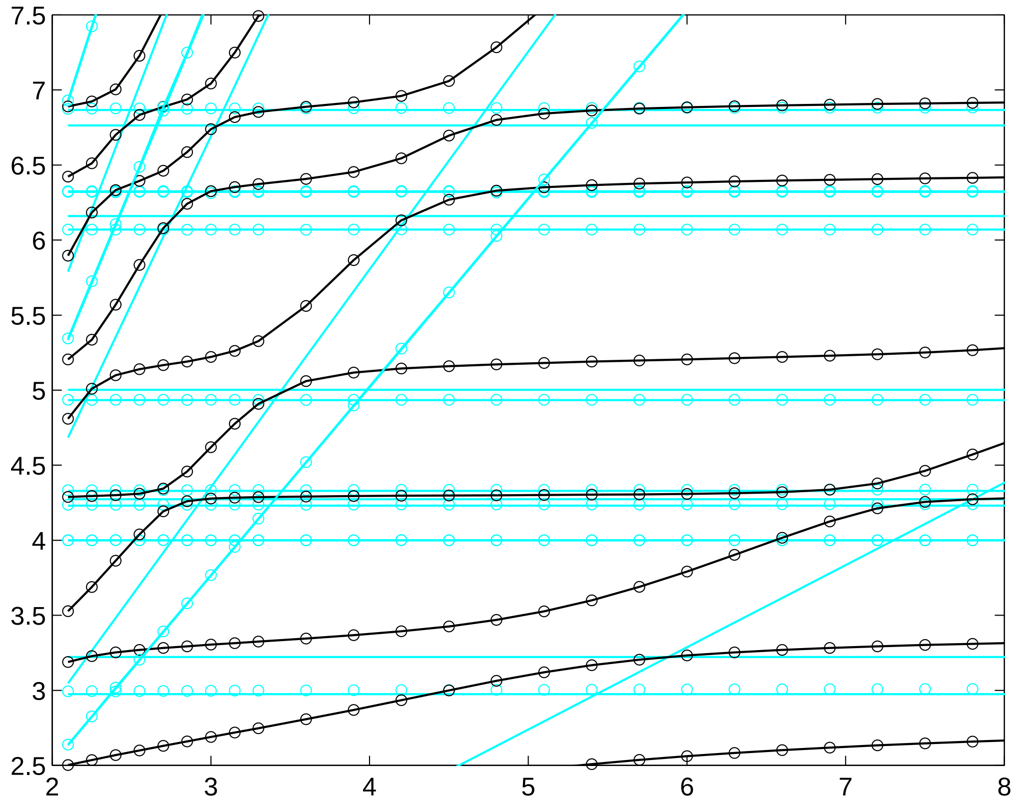

Figure 2: Further eigenvalues (symmetric domain)

In Fig. 2, the grey lines are still the Maxwell eigenvalues: now the two families appear. Concerning family *(ii)*, we see the continuation of the first one (corresponding to the Dirichlet eigenvalue $\frac{1}{2}$ on Σ_0) and the second one (corresponding to the Dirichlet eigenvalue $\frac{5}{4}$ on Σ_0).

We still join by solid lines the sets of computed Lamé eigenvalues which are not aligned on Maxwell eigenvalues of family *(i)*. Now appear curved lines like before (in black), but also sets of eigenvalues (in grey) exactly aligned on Maxwell eigenvalues of family *(ii)*. Whereas the Maxwell eigenvalues issued from the first Dirichlet eigenvalue coincide with no Lamé eigenvalues, the set of Maxwell eigenvalues issued from the second Dirichlet eigenvalue is fully populated by Lamé eigenvalues. This is due to the H^2 regularity of the second Dirichlet eigenvalue, which is odd with respect to the diagonal: concerning family *(i)*, the situation is exactly converse to that of family *(i)* (odd Dirichlet eigenvectors are H^2).

On both Fig. 1 and 2, we can see the effects of inequality (3.3).

Example 2. We take another L-shaped domain: $\Omega = \Sigma_0' \setminus \Sigma_1'$ where Σ_0' is the rectangle $[0,1]\times[0,2^{-1/5}] \simeq [0,1]\times[0,0.87055]$ and Σ_1' the rectangle $[1-\frac{1}{\sqrt{13}},1]\times[e^{-1/2},2^{-1/5}] \simeq [0.7226,1]\times[0.60653,0.87055]$.

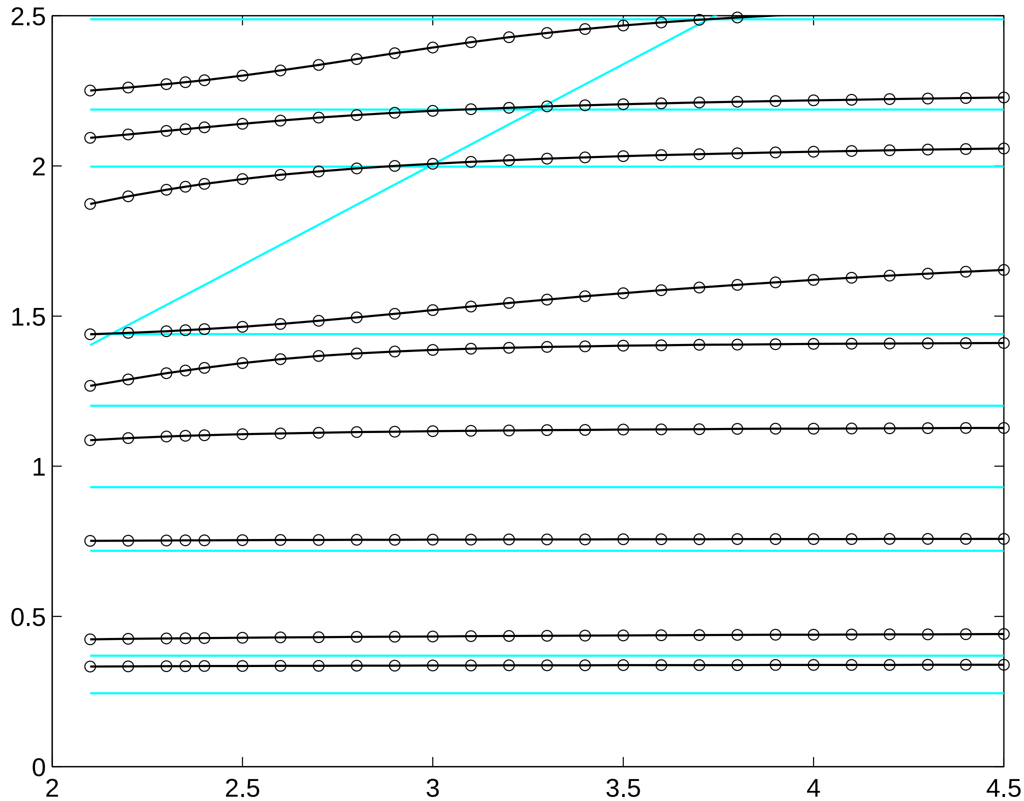

Figure 3: Lowest eigenvalues (non-symmetric domain)

In Fig. 3 and 4, the grey lines are still the Maxwell eigenvalues *divided by* $4\pi^2$, deduced from the computation of the Dirichlet and Neumann eigenvalues of the Laplace operator. The circles show the results of the computation of the Lamé eigenvalues, still divided by $4\pi^2$. In contrast to the previous case, the natural sets of Lamé eigenvalues follow the natural increasing order. We join them by black lines. We see very clearly the effects of inequality (3.3).

Another phenomenon appears: at each crossing of two Maxwell eigenvalues, one Lamé eigenvalue is present. The reason is the presence of only one non-convex corner and relation (4.1): thus in a Maxwell eigenspace of dimension 2, there is at least a one dimensional subspace contained in $\boldsymbol{H}_N(\Omega)$, thus a Lamé eigenspace.

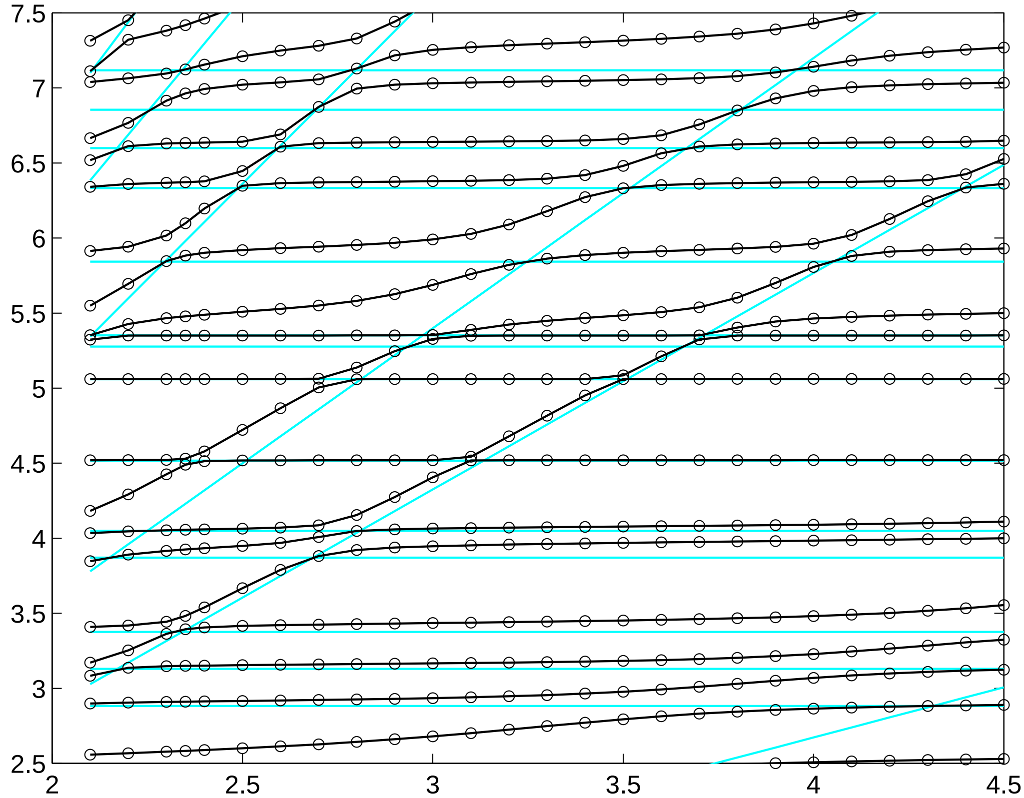

Figure 4: Further eigenvalues (non-symmetric domain)

We can also see that Lamé eigenvalues are more or less close to Maxwell eigenvalues. This is in proportion of the coefficient of the singularity in each Maxwell eigenvector. In the following table, we show these coefficients ("Generalized Stress Intensity Factors") as computed by Stress Check for some of the eigenvalues;

k	Maxwell Eigenvalue	Nearest Lamé Eigenv.	GSIF
14	3.8704	3.9998	-6.399
15	4.0496	4.1105	2.366
16	4.5168	4.5204	0.921
17	5.0586	5.0612	-1.015
18	5.2766		-12.927
19	5.3512	5.3510	-1.535
20	5.8428	5.9301	-7.417

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