Progress In Electromagnetics Research B, Vol. 8, 115–127, 2008

MAXWELL'S EQUATIONS, SYMPLECTIC MATRIX, AND GRID

W. Sha

Department of Electrical and Electronic Engineering
the University of Hong Kong
Pokfulam Road, Hong Kong

X. L. Wu and Z. X. Huang

Key Laboratory of Intelligent Computing & Signal Processing
Anhui University
Feixi Road 3, Hefei 230039, China

M. S. Chen

Department of Physics and Electronic Engineering
Hefei Teachers College
Jinzhai Road 327, Hefei 230061, China

Abstract—The connections between Maxwell's equations and symplectic matrix are studied. First, we analyze the continuous-time Maxwell's differential equations in free space and verify its time evolution matrix (TEMA) is symplectic-unitary matrix for complex space or symplectic-orthogonal matrix for real space. Second, the spatial differential operators are discretized by pseudo-spectral (PS) approach with collocated grid and by finite-difference (FD) method with staggered grid. For the PS approach, the TEMA conserves the symplectic-unitary property. For the FD method, the TEMA conserves the symplectic-orthogonal property. Finally, symplectic integration scheme is used in the time direction. In particular, we find the symplectiness of the TEMA also can be conserved. The mathematical proofs presented are helpful for further numerical study of symplectic schemes.

1. INTRODUCTION

Most non-dissipative physical or chemical phenomenons can be modeled by Hamiltonian differential equations whose time evolution is symplectic transform and flow conserves the symplectic structure [1]. The symplectic schemes include a variety of different temporal discretization strategies designed to preserve the global symplectic structure of the phase space for a Hamiltonian system. Compared with other non-symplectic methods, the symplectic schemes have demonstrated their advantages in numerical computation for the Hamiltonian system [2], especially under long-term simulation. Since Maxwell's equations can be written as an infinite-dimensional Hamiltonian system, a stable and accurate solution can be obtained by using the symplectic schemes, which preserve the energy of the Hamiltonian system constant.

Recently, many scientists and engineers from computational electromagnetics society have focused on the symplectic scheme for solving Maxwell's equations. Symplectic finite-difference time-domain (FDTD) method [3–5], symplectic pseudo-spectral time-domain (PSTD) approach [6], and multi-symplectic scheme [7] are proposed and advanced. Although some numerical results on electromagnetic propagation, penetration, and scattering have been reported, rigorous mathematical background on the issue is seldom studied.

What are the connections between Maxwell's equations [8, 9] and symplectic matrix? Can the symplectiness of Maxwell's equations be persevered if we discretize the continuous-time differential equations both in spatial domain and in time domain? For answering the questions, we present the convincing mathematical proofs in detail.

2. PRELIMINARY KNOWLEDGE

Definition 1.1. For $p_{2n}^0, q_{2n}^0 \in R_{2n}$, real-symplectic inner product [1] can be defined as

$$\varpi\left(p^0, q^0\right) = p^0 J \left(q^0\right)^T \tag{1}$$

where T denotes transpose and $J = \begin{bmatrix} \{0\}_{n\times n} & I_{n\times n} \\ -I_{n\times n} & \{0\}_{n\times n} \end{bmatrix}_{2n\times 2n}$ which satisfies skew-symmetric and orthogonal properties, i.e., $J = -J^T$, $J^{-1} = J^T = -J$.

The real-symplectic inner product has the following properties:

(1) Bilinear property:

$$\varpi\left(p^0+q^0,r^0+s^0\right)=\varpi\left(p^0,r^0\right)+\varpi\left(p^0,s^0\right)+\varpi\left(q^0,r^0\right)+\varpi\left(q^0,s^0\right),$$
$$\varpi\left(\lambda p^0,\eta q^0\right)=\lambda\eta\varpi\left(p^0,q^0\right),r^0,s^0\in R_{2n},\text{ and }\lambda,\eta\in R;$$

(2) Skew-symmetric property: $\varpi(p^0,q^0)=-\varpi(q^0,p^0)$;

(3) Non-degenerate property: $\forall q^0\neq 0$, $\exists p^0$, $\varpi(p^0,q^0)=0\Rightarrow p^0=0$.

Definition 1.2. If V is a vector space defined on R_{2n} and the mapping $\varpi : V\times V\rightarrow R$ is real-symplectic, (V,ϖ) is called real-symplectic space and ϖ is called real-symplectic structure.

Definition 1.3. A linear transform $T : V\rightarrow V$ is called real-symplectic transform, if it meets $\varpi(Tp^0,Tq^0)=\varpi(p^0,q^0)$, $\forall p^0,q^0\in R_{2n}$.

Definition 1.4. The matrix T is called real-symplectic matrix if $T^TJT=J$ and $\varpi(Tp^0,Tq^0)=\varpi(p^0,q^0)$. The group including all the real-symplectic matrices is called real-symplectic group. We sign it as $Sp(2n,R)$.

Definition 1.5. B is an infinitesimally real-symplectic matrix if $B^TJ+JB=0$. The infinitesimally real-symplectic matrices can be composed of Lie algebra via anti-commutable Lie Poisson bracket $[A,B]=AB-BA$.

Theory 1. B is an infinitesimally real-symplectic matrix $\Rightarrow$ $\exp(B)\in Sp(2n,R)$.

Above mentioned definitions and theory can be extended to complex space [10].

Definition 2.1. For $p^0_{2n},q^0_{2n}\in C_{2n}$, complex-symplectic inner product can be defined as

$$\varpi\left(p^0,q^0\right)=p^0J\left(q^0\right)^H \tag{2}$$

where H denotes complex conjugate transpose or adjoint.

The complex-symplectic inner product has the following properties:

(1) Conjugate bilinear property:

$$\varpi\left(p^0+q^0,r^0+s^0\right)=\varpi\left(p^0,r^0\right)+\varpi\left(p^0,s^0\right)+\varpi\left(q^0,r^0\right)+\varpi\left(q^0,s^0\right),$$

$\varpi(\lambda p^0,\eta q^0)=\lambda\,\overline{\eta}\,\varpi(p^0,q^0)$, $r^0,s^0\in C_{2n}$, $\lambda,\eta\in C$, and $\overline{\eta}$ is the conjugate complex of η;

(2) Skew-Hermitian property: $\varpi(p^0,q^0)=-\overline{\varpi(q^0,p^0)}$;

(3) Non-degenerate property: $\forall q^0\neq 0$, $\exists p^0$, $\varpi(p^0,q^0)=0\Rightarrow p^0=0$.

Definition 2.2. If V is a vector space defined on C_{2n} and the mapping $\varpi : V \times V \rightarrow C$ is complex-symplectic, (V, ϖ) is called complex-symplectic space and ϖ is called complex-symplectic structure.

Definition 2.3. A linear transform $T : V \rightarrow V$ is called complex-symplectic transform, if it meets $\varpi(Tp^0, Tq^0) = \varpi(p^0, q^0)$, $\forall p^0, q^0 \in C_{2n}$.

Definition 2.4. The matrix T is called complex-symplectic matrix if $T^H JT = J$ and $\varpi(Tp^0, Tq^0) = \varpi(p^0, q^0)$. The group including all the complex-symplectic matrices is called complex-symplectic group. We sign it as $Sp(2n, C)$.

Definition 2.5. B is an infinitesimally complex-symplectic matrix if $B^H J + JB = 0$. The infinitesimally complex-symplectic matrices can be composed of Lie algebra via anti-commutable Lie Poisson bracket $[A, B] = AB - BA$.

Theory 2. B is an infinitesimally complex-symplectic matrix $\Rightarrow \exp(B) \in Sp(2n, C)$.

Definition 3. If $p^0 = (p_1, p_2, \ldots p_n)$, $q^0 = (q_1, q_2, \ldots q_n)$, $(p^0, q^0) \in \Omega \subseteq R_{2n}$, and $t_0 \in I$, the Hamiltonian canonical equations [2] can be written as

$$\frac{dp_i}{dt_0} = -\frac{\partial H}{\partial q_i},\ \frac{dq_i}{dt_0} = +\frac{\partial H}{\partial p_i},\ i = 1, 2, \ldots n \tag{3}$$

where $H(p^0, q^0, t_0)$ is the Hamiltonian function, Ω is the phase space, and $\Omega \times I$ is the extended phase space.

Theory 3. If the solution of (3) at any time t_* is (p^*, q^*) and the (p^*, q^*) still satisfies the (3), the Jacobi matrix Θ is a symplectic matrix

$$\Theta^T J \Theta = J \tag{4}$$

where $\Theta = \frac{\partial(p^*, q^*)}{\partial(p^0, q^0)} = \begin{pmatrix} \partial p^*/\partial p^0 & \partial p^*/\partial q^0 \\ \partial q^*/\partial p^0 & \partial q^*/\partial q^0 \end{pmatrix}$.

Theory 4. If the time evolution operator of (3) from t_0 to t_* is $\Psi(t_*, t_0)$ and $(p^*, q^*) = \Psi(t_*, t_0)(p^0, q^0)$, the operator conserves the symplectic structure

$$\Psi(t_*, t_0)^* \varpi^* = \varpi^0 \tag{5}$$

where $\varpi^* = dp^* \wedge dq^*$, $\varpi^0 = dp^0 \wedge dq^0$, and $\Psi(t_*, t_0)^*$ is the conjugate operator of $\Psi(t_*, t_0)$. The time evolution operator is also called Hamiltonian flow or symplectic flow.

Theory 5. The matrix $L = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \Rightarrow \exp(L) = \begin{bmatrix} \cos(A) & \sin(A) \\ -\sin(A) & \cos(A) \end{bmatrix}$.

Theory 6. If the matrix $L = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}$ and $A = A^T$, we have: (1) L is skew-symmetric, i.e., $L = -L^T$; (2) $\exp(L)$ are both orthogonal and real-symplectic matrices. We call the $\exp(L)$ symplectic-orthogonal matrix.

Theory 7. If the matrix $L = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}$ and $A = A^H$, we have: (1) L is skew-Hermitian, i.e., $L = -L^H$; (2) $\exp(L)$ are both unitary and complex-symplectic matrices. We call the $\exp(L)$ symplectic-unitary matrix.

3. MAXWELL'S EQUATIONS AND SYMPLECTIC MATRIX

3.1. The Symplectiness of Continuous-time Continuous-space Maxwell's Equations

A helicity generating function [11] for Maxwell's differential equations in free space is introduced as

$$G(\mathbf{H}, \mathbf{E}) = \frac{1}{2}\left(\frac{1}{\varepsilon_0}\mathbf{H} \cdot \nabla \times \mathbf{H} + \frac{1}{\mu_0}\mathbf{E} \cdot \nabla \times \mathbf{E}\right) \tag{6}$$

where $\mathbf{E} = (E_x, E_y, E_z)^T$ is the electric field vector, $\mathbf{H} = (H_x, H_y, H_z)^T$ is the magnetic field vector, and ε_0 and μ_0 are the permittivity and permeability of free space.

The differential form of the Hamiltonian is

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{\partial G}{\partial \mathbf{E}}, \quad \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial G}{\partial \mathbf{H}} \tag{7}$$

According to the variational principle, we can derive Maxwell's equations of free space from (7)

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix} = L \begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix} \tag{8}$$

$$L = \begin{pmatrix} \{0\}_{3\times 3} & -\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} R_{3\times 3} \\ \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} R_{3\times 3} & \{0\}_{3\times 3} \end{pmatrix}, \hat{\mathbf{E}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{E} \tag{9}$$

$$R = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} = \nabla\times \tag{10}$$

where $\{0\}_{3\times3}$ is the 3×3 null matrix and R is the three-dimensional curl operator.

The time evolution of (8) from $t = 0$ to $t = \Delta_t$ can be written as

$$\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix}(\Delta_t) = \exp(\Delta_t L)\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix}(0) \tag{11}$$

where $\exp(\Delta_t L)$ is the time evolution matrix (TEMA) or symplectic flow of Maxwell's equations.

For infinite-dimensional real space, we define the inner product

$$< F(t,\mathbf{r}), G(t,\mathbf{r}) >= \int_{-\infty}^{\infty} F(t,\mathbf{r}) \cdot G(t,\mathbf{r}) d\mathbf{r} \tag{12}$$

where $\mathbf{r} = x\mathbf{e_x} + y\mathbf{e}_y + z\mathbf{e}_z$ is the position vector and t is the time variable.

According to the identity both in generalized distribution space and in Hilbert space

$$< \frac{\partial}{\partial \delta}F, G >= - < F, \frac{\partial}{\partial \delta}G >, \quad \delta = x, y, z \tag{13}$$

we can know $\frac{\partial}{\partial \delta}$ is a skew-symmetric operator. Hence R is a symmetric operator, i.e., $R = R^T$.

Based on **Theory 6**, the TEMA of Maxwell's equations is a symplectic-orthogonal matrix in real space.

For infinite-dimensional complex space, we define the inner product

$$< F(t,\mathbf{r}), G(t,\mathbf{r}) >= \int_{-\infty}^{\infty} F(t,\mathbf{r}) \cdot \overline{G(t,\mathbf{r})} d\mathbf{r} \tag{14}$$

The forward and inverse Fourier transform for electromagnetic field components are respectively

$$\tilde{F}(t,\mathbf{k_0}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t,\mathbf{r}) \exp(j_0 \mathbf{k_0} \cdot \mathbf{r}) d\mathbf{r} \tag{15}$$

$$F(t,\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(t,\mathbf{k_0}) \exp(-j_0 \mathbf{k_0} \cdot \mathbf{r}) d\mathbf{k_0} \tag{16}$$

where j_0 is the imaginary unit and $\mathbf{k_0} = k_x\mathbf{e_x} + k_y\mathbf{e}_y + k_z\mathbf{e}_z$ is the wave vector. For simplicity, we can note (15) and (16) as $\tilde{F} = \phi F$ and $F = \phi^{-1}\tilde{F}$.

In the beginning, with the help of Parseval theorem

$$< \phi F, \tilde{G} >=< F, \phi^{-1}\tilde{G} > \tag{17}$$

we can know the Fourier operator ϕ is a unitary operator, i.e., $\phi^{-1} = \phi^H$.

Next, using the differential property of Fourier transform $\frac{\partial F}{\partial \delta} \leftrightarrow -j_0 k_\delta \tilde{F}, \delta = x, y, z$, we can obtain the spectral-domain form of Maxwell's equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\mathbf{H}} \\ \tilde{\hat{\mathbf{E}}} \end{pmatrix} = \begin{pmatrix} \{0\}_{3\times3} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} & \{0\}_{3\times3} \end{pmatrix}\begin{pmatrix} \tilde{\mathbf{H}} \\ \tilde{\hat{\mathbf{E}}} \end{pmatrix} \tag{18}$$

$$\tilde{R}_{3\times3} = \begin{pmatrix} 0 & j_0k_z & -j_0k_y \\ -j_0k_z & 0 & j_0k_x \\ j_0k_y & -j_0k_x & 0 \end{pmatrix} \tag{19}$$

where $\tilde{R}$ is a Hermitian matrix, i.e., $\tilde{R}^H = \tilde{R}$.

Finally, considering the unitary property of the Fourier operator, we can convert the spectral-domain form (18) into the spatial-domain form (20)

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix} &= \begin{pmatrix} \Phi_{3\times3}^{-1} & \{0\}_{3\times3} \\ \{0\}_{3\times3} & \Phi_{3\times3}^{-1} \end{pmatrix}\begin{pmatrix} \{0\}_{3\times3} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} & \{0\}_{3\times3} \end{pmatrix} \\
&\quad \begin{pmatrix} \Phi_{3\times3} & \{0\}_{3\times3} \\ \{0\}_{3\times3} & \Phi_{3\times3} \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix} = \begin{pmatrix} \Phi_{3\times3}^{H} & \{0\}_{3\times3} \\ \{0\}_{3\times3} & \Phi_{3\times3}^{H} \end{pmatrix} \\
&\quad \begin{pmatrix} \{0\}_{3\times3} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\tilde{R}_{3\times3} & \{0\}_{3\times3} \end{pmatrix}\begin{pmatrix} \Phi_{3\times3} & \{0\}_{3\times3} \\ \{0\}_{3\times3} & \Phi_{3\times3} \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix} \\
&= \begin{pmatrix} \{0\}_{3\times3} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\Phi_{3\times3}^{H}\tilde{R}_{3\times3}\Phi_{3\times3} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\Phi_{3\times3}^{H}\tilde{R}_{3\times3}\Phi_{3\times3} & \{0\}_{3\times3} \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ \hat{\mathbf{E}} \end{pmatrix}
\end{aligned} \tag{20}$$

where $\Phi_{3\times3} = diag(\phi, \phi, \phi)$, $\Phi^H_{3\times3} = diag(\phi^H, \phi^H, \phi^H)$, and $\Phi^{-1}_{3\times3} = diag(\phi^{-1}, \phi^{-1}, \phi^{-1})$. It is easy to show that $R = \Phi^H_{3\times3}\tilde{R}_{3\times3}\Phi_{3\times3}$ is a Hermitian matrix, i.e., $R = R^H$.

Based on **Theory 7**, the TEMA of Maxwell's equations is a symplectic-unitary matrix in complex space.

It is well known that the total energy of electromagnetic field in free space can be represented as

$$En=\frac{1}{2}\mu_0\Big(<\mathbf{H},\mathbf{H}>+<\hat{\mathbf{E}},\hat{\mathbf{E}}>\Big)=\iiint\limits_V\Big(\frac{1}{2}\mu_0|\mathbf{H}|^2+\frac{1}{2}\varepsilon_0|\mathbf{E}|^2\Big)dV \quad (21)$$

No matter in complex space or real space, the TEMA $\exp(\Delta_t L)$ accurately conserves the total energy of electromagnetic field components. In other words, the $\exp(\Delta_t L)$ only rotates the electromagnetic field (**Theory 5**). In addition, if an algorithm can accurately conserve the total energy of electromagnetic field, it is to be unconditionally stable.

3.2. The Symplectiness of Continuous-time Discrete-space Maxwell's Equations

For pseudo-spectral (PS) approximation, we discretize the infinite-dimensional electromagnetic field components with collocated grid, such as $\mathbf{E} \to \mathbf{E}^d(i,j,k)$ and $\mathbf{H} \to \mathbf{H}^d(i,j,k)$.

The three-dimensional discrete Fourier transform (DFT) and inverse DFT (IDFT) can be noted as

$$\tilde{F}^d = \phi_d F^d, \quad F^d = \phi_d^{-1}\tilde{F}^d \quad (22)$$

Similarly, ϕ_d is a $n \times n$ unitary matrix.

Using (22), the continuous-time discrete-space Maxwell's equations can be obtained.

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{H}^d\\ \hat{\mathbf{E}}^d\end{pmatrix}_{6n\times1} = L_d\begin{pmatrix}\mathbf{H}^d\\ \hat{\mathbf{E}}^d\end{pmatrix}_{6n\times1} \quad (23)$$

$$L_d = \begin{pmatrix}\{0\}_{3n\times3n} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\Phi_d^H\tilde{R}_d\Phi_d\\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\Phi_d^H\tilde{R}_d\Phi_d & \{0\}_{3n\times3n}\end{pmatrix} \quad (24)$$

where $\Phi_d = diag(\phi_d, \phi_d, \phi_d)_{3n\times3n}$, $\Phi_d^H = diag(\phi_d^H, \phi_d^H, \phi_d^H)_{3n\times3n}$, and $\tilde{R}_d$ is the discretized $3n \times 3n$ Hermitian matrix corresponding to $\tilde{R}$.

The $R_d = \Phi_d^H \tilde{R}_d \Phi_d$ is still a Hermitian matrix and therefore the TEMA $\exp(\Delta_t L_d)$ conserves the symplectic-unitary property.

For finite-difference (FD) approximation, we discretize the infinite-dimensional electromagnetic field components with staggered grid, such as

$$E_x \to E_x^d\left(i+\frac{1}{2}, j, k\right), E_y \to E_y^d\left(i, j+\frac{1}{2}, k\right), E_z \to E_z^d\left(i, j, k+\frac{1}{2}\right)$$
$$H_x \to H_x^d\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right), H_y \to H_y^d\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right), H_z \to H_z^d\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right)$$

As a result, the continuous-time discrete-space Maxwell's equations are

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{H}^d \\ \hat{\mathbf{E}}^d \end{pmatrix}_{6n\times 1} = \begin{pmatrix} \{0\}_{3n\times 3n} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}} R_{d,E} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} R_{d,H} & \{0\}_{3n\times 3n} \end{pmatrix} \begin{pmatrix} \mathbf{H}^d \\ \hat{\mathbf{E}}^d \end{pmatrix}_{6n\times 1} \tag{25}$$

For (25), if the order of electric field components have not been rearranged we only have $R_{d,E}^T = R_{d,H}$ and $L_d^T = -L_d$ [12, 13]. Although it is the fact that $\exp(\Delta_t L_d)$ is an orthogonal matrix, the symplectiness of the TEMA seems not be hold.

Take a one-dimensional case for example. Figure 1 shows the distribution of electromagnetic field components.

Using the periodic boundary condition and the second-order centered difference, the (25) can be converted into (26) for the one-dimensional case.

$$\frac{\partial}{\partial t}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\kappa & 0 & 0 & 0 & \kappa \\ 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa \\ \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 \\ -\kappa & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \end{pmatrix} \tag{26}$$

where $\kappa = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\Delta_z}$. In addition, we can testify $R_{d,E}^T = R_{d,H}$.

Fortunately, both the matrix $R_{d,E}$ and the matrix $R_{d,H}$ are non-symmetric Toeplitz matrices. So we can change them into symmetric

Figure 1. The distribution of one-dimensional electromagnetic field components with staggered grid. (The positive z direction is directed from left to right.)

Hankel matrices by rearranging the electric field components.

$$\frac{\partial}{\partial t}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ \hat{E}_2 \\ \hat{E}_1 \\ \hat{E}_5 \\ \hat{E}_4 \\ \hat{E}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & -\kappa \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 \\ 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\kappa & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ \hat{E}_2 \\ \hat{E}_1 \\ \hat{E}_5 \\ \hat{E}_4 \\ \hat{E}_3 \end{pmatrix} \tag{27}$$

Here it is easy to see that $R_{d,E} = R_{d,H} = R_d$ and $R_d^T = R_d$. Based on **Theory 6**, the TEMA $\exp(\Delta_t L_d)$ can hold the symplectic-orthogonal property.

3.3. The Symplectiness of Discrete-time Discrete-space Maxwell's Equations

No matter in complex space and in real space, we can split the discretized L_d into U_d and V_d

$$L_d = U_d + V_d \tag{28}$$

$$U_d = \begin{pmatrix} \{0\}_{3n\times 3n} & -\dfrac{1}{\sqrt{\mu_0\varepsilon_0}} R_d \\ \{0\}_{3n\times 3n} & \{0\}_{3n\times 3n} \end{pmatrix}, V_d = \begin{pmatrix} \{0\}_{3n\times 3n} & \{0\}_{3n\times 3n} \\ \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} R_d & \{0\}_{3n\times 3n} \end{pmatrix} \tag{29}$$

The discretized TEMA can be approximated by the m-stage pth-order symplectic integration scheme [3, 14]

$$\exp\left(\Delta_t(U_d + V_d)\right) = \prod_{l=1}^{m} \exp\left(d_l \Delta_t V_d\right) \exp\left(c_l \Delta_t U_d\right) + O\left(\Delta_t^{p+1}\right) \tag{30}$$

where c_l and d_l are the symplectic integrators.

For real space, $R_d = R_d^T$ and therefore U_d and V_d are infinitesimally real-symplectic matrices. Likewise, for complex space, $R_d = R_d^H$ and therefore U_d and V_d are infinitesimally complex-symplectic matrices. In particular, we have: (1) U_d and V_d can be composed of Lie algebra. (2) $\exp(d_l \Delta_t V_d)$ and $\exp(c_l \Delta_t U_d)$ are the symplectic matrices.

Although the orthogonal properties can not be retained by the two matrices $\exp(d_l \Delta_t V_d)$ and $\exp(c_l \Delta_t U_d)$, the determinants of them are equal to 1 [15]. Thus the symplectic integration scheme is conditionally stable and does not have amplitude error.

4. CONCLUSION

The mathematical proofs are presented for establishing the connections between Maxwell's equations and symplectic matrix. First, for continuous-time continuous-space Maxwell's equations, its TEMA which accurately conserves the electromagnetic energy is symplectic-orthogonal matrix for real space or symplectic-unitary matrix for complex space. Second, for continuous-time discrete-space Maxwell's equations, the TEMA is symplectic-unitary matrix for PS approximation with collocated grid or symplectic-orthogonal matrix for FD approximation with staggered grid. Third, for discrete-time discrete-space Maxwell's equations, the TEMA conserves the symplectiness and does not produce amplitude error with the symplectic integration scheme. The conclusions can be easily extended to homogeneous and lossless media and are helpful for further numerical study of symplectic schemes.

ACKNOWLEDGMENT

The first author expresses his sincere gratitude to Prof. Hans De Raedt, university of Groningen for his help. This work is supported by the Natural Science Foundation of the Anhui Higher Education Institution of China under Grant No. KJ2008A036 and No. KJ2008A100.

REFERENCES

1. Feng, K. and M. Z. Qin, *Symplectic Geometric Algorithm for Hamiltonian Systems*, Zhejiang Science & Technology Press, Hangzhou, 2003.
2. Sanz-Serna, J. M. and M. P. Calvo, *Numerical Hamiltonian Problems*, Chapman & Hall, London, U.K., 1994.

3. Hirono, T., W. Lui, S. Seki, and Y. Yoshikuni, "A three-dimensional fourth-order finite-difference time-domain scheme using a symplectic integrator propagator," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 49, 1640–1648, Sept. 2001.
4. Sha, W., X. L. Wu, and M. S. Chen, "A diagonal split-cell model for the high-order symplectic FDTD scheme," *PIERS Online*, Vol. 2, 715–719, Jun. 2006.
5. Sha, W., Z. X. Huang, M. S. Chen, and X. L. Wu, "Survey on symplectic finite-difference time-domain schemes for Maxwell's equations," *IEEE Transactions on Antennas and Propagation*, Vol. 56, 493–500, Feb. 2008.
6. Shi, Y. and C. H. Liang, "Multidomain pseudospectral time domain algorithm using a symplectic integrator," *IEEE Transactions on Antennas and Propagation*, Vol. 55, 433–439, Feb. 2007.
7. Reich, S., "Multi-symplectic Runge-Kutta collocation methods for Hamiltonian wave equations," *Journal of Computational Physics*, Vol. 157, 473–499, Jan. 2000.
8. Teixeira, F. L., "Geometric aspects of the simplicial discretization of Maxwell's equations," *Progress In Electromagnetics Research*, PIER 32, 171–188, 2001.
9. Zhou, X. L., "On independence completeness of Maxwell's equations and uniqueness theorems in electromagnetics," *Progress In Electromagnetics Research*, PIER 64, 117–134, 2006.
10. Everitt, W. N. and L. Markus, "Complex symplectic geometry with applications to ordinary differential operators," *Transactions of the American Mathematical Society*, Vol. 351, 4905–4945, Dec. 1999.
11. Anderson, N. and A. M. Arthurs, "Helicity and variational principles for Maxwell's equations," *International Journal of Electronics*, Vol. 54, 861–864, Jun. 1983.
12. Farago, I., R. Horvath, and W. H. A. Schilders, "Investigation of numerical time-integrations of Maxwell's equations using the staggered grid spatial discretization," *International Journal of Numerical Modelling-electronic Networks Devices and Fields*, Vol. 18, 149–169, Mar.–Apr. 2005.
13. Kole, J. S., M. T. Figge, and H. De Raedt, "Higher-order unconditionally stable algorithms to solve the time-dependent Maxwell equations," *Physical Review E*, Vol. 65, Jun. 2002.
14. Yoshida, H., "Construction of higher order symplectic integrators," *Physica D: Nonlinear Phenomena*, Vol. 46, 262–268,

Nov. 1990.

15. Dopico, F. M. and C. R. Johnson, "Complementary bases in symplectic matrices and a proof that their determinant is one," *Linear Algebra and Its Applications*, Vol. 419, 772–778, Dec. 2006.