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# MBJ-neutrosophic ideals of $BCK/BCI$-algebras

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**Abstract:** The notion of MBJ-neutrosophic ideal is introduced, and its properties are investigated. Conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal are provided. In a $BCK/BCI$-algebra, a condition for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal is given. In a $BCK$-algebra, a condition for an MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal is given. In a $BCI$-algebra, conditions for an MBJ-neutrosophic ideal to be an MBJ-neutrosophic subalgebra are considered. In an $(S)$-$BCK$-algebra, we show that every MBJ-neutrosophic ideal is an MBJ-neutrosophic $\circ$-subalgebra, and a characterization of an MBJ-neutrosophic ideal is established.

**Keywords:** MBJ-neutrosophic set, MBJ-neutrosophic subalgebra, MBJ-neutrosophic ideal, MBJ-neutro-sophic $\circ$-subalgebra

**MSC:** 06F35, 03G25, 03E72

## 1 Introduction

Different types of uncertainties are encountered in many complex systems and/or in many practical situations like behavioral, biologial and chemical etc. In order to handle uncertainties in many real applications, the fuzzy set was introduced by L.A. Zadeh [1] in 1965. The intuitionistic fuzzy set on a universe X was introduced by K. Atanassov in 1983 as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is developed by Smarandache [2–4]. Neutrosophic algebraic structures in $BCK/BCI$-algebras are discussed in the papers [5–14] and [15]. In [16], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, it is applied to $BCK/BCI$-algebras. Mohseni et al. [16] introduced the concept of MBJ-neutrosophic subalgebras in $BCK/BCI$-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a $BCI$-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra.

In this paper, we apply the notion of MBJ-neutrosophic sets to ideals of $BCK/BI$-algebras. We introduce the concept of MBJ-neutrosophic ideals in $BCK/BCI$-algebras, and investigate several properties. We provide a condition for an MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal in a $BCK$-algebra. We provide conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal in a $BCK/BCI$-algebra.

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We discuss relations between MBJ-neutrosophic subalgebras, MBJ-neutrosophic ∘-subalgebras and MBJ-neutrosophic ideals. In a $BCI$-algebra, we provide conditions for an MBJ-neutrosophic ideal to be an MBJ-neutrosophic subalgebra. In an $(S)$-$BCK$-algebra, we consider a characterization of an MBJ-neutrosophic ideal.

## 2 Preliminaries

By a *$BCI$-algebra*, we mean a set $X$ with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y$

for all $x, y, z \in X$. If a $BCI$-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X)\,(0 * x = 0)$,

then $X$ is called a *$BCK$-algebra*.

By a *weakly $BCK$-algebra* (see [17]), we mean a $BCI$-algebra $X$ satisfying $0 * x \le x$ for all $x \in X$.

Every $BCK/BCI$-algebra $X$ satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,\big((x * y) * z = (x * z) * y\big)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,\big((x * z) * (y * z) \le x * y\big)\,, \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. Any $BCI$-algebra $X$ satisfies the following conditions (see [17]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{2.5}$$

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{2.6}$$

A $BCI$-algebra $X$ is said to be *$p$-semisimple* (see [17]) if

$$(\forall x \in X)(0 * (0 * x) = x). \tag{2.7}$$

In a $p$-semisimple $BCI$-algebra $X$, the following holds:

$$(\forall x, y \in X)(0 * (x * y) = y * x,\ \ x * (x * y) = y). \tag{2.8}$$

A $BCI$-algebra $X$ is said to be *associative* (see [17]) if

$$(\forall x, y, z \in X)((x * y) * z = x * (y * z)). \tag{2.9}$$

By an *$(S)$-$BCK$-algebra*, we mean a $BCK$-algebra $X$ such that, for any $x, y \in X$, the set

$$\{z \in X \mid z * x \le y\}$$

has the greatest element, written by $x \circ y$ (see [18]).

A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $I$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies:

$$0 \in I, \tag{2.10}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.11}$$

A subset $I$ of a $BCI$-algebra $X$ is called a *closed ideal* of $X$ (see [17]) if it is an ideal of $X$ which satisfies:

$$(\forall x \in X)(x \in I \Rightarrow 0 * x \in I). \tag{2.12}$$

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $I$, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := [a_1^-, a_1^+]$ and $\tilde{a}_2 := [a_2^-, a_2^+]$ . Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}\right], \\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}\right], \\
&\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow a_1^- \ge a_2^-,\ a_1^+ \ge a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let $X$ be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in $X$. Let $[I]^X$ stand for the set of all IVF sets in $X$. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element $x$ to $A$, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in $X$ which are called a *lower fuzzy set* and an *upper fuzzy set* in $X$, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let $X$ be a non-empty set. A *neutrosophic set* (NS) in $X$ (see [3]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

We refer the reader to the books [17, 18] for further information regarding $BCK/BCI$-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let $X$ be a non-empty set. By an *MBJ-neutrosophic set* in $X$ (see [16]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\},$$

where $M_A$ and $J_A$ are fuzzy sets in $X$, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in $X$ which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let $X$ be a $BCK/BCI$-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in $X$ is called an *MBJ-neutrosophic subalgebra* of $X$ (see [16]) if it satisfies:

$$(\forall x, y \in X) \left( \begin{array}{l} M_A(x * y) \ge \min\{M_A(x), M_A(y)\}, \\ \tilde{B}_A(x * y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ J_A(x * y) \le \max\{J_A(x), J_A(y)\}. \end{array} \right) \tag{2.13}$$

## 3 MBJ-neutrosophic ideals of $BCK/BCI$-algebras

**Definition 3.1.** Let $X$ be a $BCK/BCI$-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in $X$ is called an *MBJ-neutrosophic ideal* of $X$ if it satisfies:

$$(\forall x \in X)\begin{pmatrix} M_A(0) \geq M_A(x) \\ \tilde{B}_A(0) \succeq \tilde{B}_A(x) \\ J_A(0) \leq J_A(x) \end{pmatrix} \tag{3.1}$$

and

$$(\forall x, y \in X)\begin{pmatrix} M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \\ \tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} \\ J_A(x) \leq \max\{J_A(x * y), J_A(y)\} \end{pmatrix}. \tag{3.2}$$

An MBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of a $BCI$-algebra $X$ is said to be *closed* if

$$(\forall x \in X)\begin{pmatrix} M_A(0 * x) \geq M_A(x) \\ \tilde{B}_A(0 * x) \succeq \tilde{B}_A(x) \\ J_A(0 * x) \leq J_A(x) \end{pmatrix}. \tag{3.3}$$

**Example 3.2.** Consider a set $X = \{0, 1, 2, a\}$ with the binary operation $*$ which is given in Table 1. Then $(X; *, 0)$ is a $BCI$-algebra (see [17]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 2.

**Table 1:** Cayley table for the binary operation "$*$".

| $*$ | 0 | 1 | 2 | $a$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $a$ |
| 1 | 1 | 0 | 0 | $a$ |
| 2 | 2 | 2 | 0 | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 |

**Table 2:** MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

| $X$ | $M_A(x)$ | $\tilde{B}_A(x)$ | $J_A(x)$ |
|---|---|---|---|
| 0 | 0.7 | $[1.0, 1.0]$ | 0.2 |
| 1 | 0.5 | $[0.2, 0.6]$ | 0.2 |
| 2 | 0.4 | $[0.2, 0.6]$ | 0.7 |
| $a$ | 0.3 | $[0.2, 0.6]$ | 0.7 |

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$.

**Proposition 3.3.** *Let $X$ be a $BCK/BCI$-algebra. Then every MBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of $X$ satisfies the following assertion.*

$$x * y \leq z \Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ \tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases} \tag{3.4}$$

*for all $x, y, z \in X$.*

*Proof.* Let $x, y, z \in X$ be such that $x * y \le z$. Then

$$M_A(x * y) \ge \min\{M_A((x * y) * z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$\tilde{B}_A(x * y) \succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\} = \mathrm{rmin}\{\tilde{B}_A(0), \tilde{B}_A(z)\} = \tilde{B}_A(z),$$

and

$$J_A(x * y) \le \max\{J_A((x * y) * z), J_A(z)\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \ge \min\{M_A(x * y), M_A(y)\} = \min\{M_A(y), M_A(z)\},$$

$$\tilde{B}_A(x) \succeq \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} = \mathrm{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\},$$

and

$$J_A(x) \le \max\{J_A(x * y), J_A(y)\} = \max\{J_A(y), J_A(z)\}.$$

This completes the proof. □

**Theorem 3.4.** *Every MBJ-neutrosophic set in a BCK/BCI-algebra $X$ satisfying* (3.1) *and* (3.4) *is an MBJ-neutrosophic ideal of $X$.*

*Proof.* Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ satisfying (3.1) and (3.4). Note that $x * (x * y) \le y$ for all $x, y \in X$. It follows from (3.4) that

$$M_A(x) \ge \min\{M_A(x * y), M_A(y)\},$$

$$\tilde{B}_A(x) \succeq \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\},$$

and

$$J_A(x) \le \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. □

**Theorem 3.5.** *Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a BCK/BCI-algebra $X$, if $(M_A, J_A)$ is an intuitionistic fuzzy ideal of $X$, and $B_A^-$ and $B_A^+$ are fuzzy ideals of $X$, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$.*

*Proof.* It is sufficient to show that $\tilde{B}_A$ satisfies the condition

$$(\forall x \in X)(\tilde{B}_A(0) \succeq \tilde{B}_A(x)) \tag{3.5}$$

and

$$(\forall x, y \in X)(\tilde{B}_A(x) \succeq \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}). \tag{3.6}$$

For any $x, y \in X$, we get

$$\tilde{B}_A(0) = [B_A^-(0), B_A^+(0)] \succeq [B_A^-(x), B_A^+(x)] = \tilde{B}_A(x)$$

and

$$\begin{aligned}\tilde{B}_A(x) &= [B_A^-(x), B_A^+(x)] \\ &\succeq [\min\{B_A^-(x * y), B_A^-(y)\}, \min\{B_A^+(x * y), B_A^+(y)\}] \\ &= \mathrm{rmin}\{[B_A^-(x * y), B_A^+(x * y)], [B_A^-(y), B_A^+(y)] \\ &= \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}.\end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. □

If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of a $BCK/BCI$-algebra $X$, then

$$\begin{aligned}[B_A^-(x), B_A^+(x)] = \tilde{B}_A(x) &\succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} \\ &= \operatorname{rmin}\{[B_A^-(x * y), B_A^+(x * y), [B_A^-(y), B_A^+(y)]\} \\ &= [\min\{B_A^-(x * y), B_A^-(y)\}, \min\{B_A^+(x * y), B_A^+(y)\}]\end{aligned}$$

for all $x, y \in X$. It follows that $B_A^-(x) \geq \min\{B_A^-(x * y), B_A^-(y)\}$ and $B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\}$. Thus $B_A^-$ and $B_A^+$ are fuzzy ideals of $X$. But $(M_A, J_A)$ is not an intuitionistic fuzzy ideal of $X$ as seen in Example 3.2. This shows that the converse of Theorem 3.5 is not true.

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a $BCK/BCI$-algebra $X$, we consider the following sets.

$$\begin{aligned}U(M_A; t) &:= \{x \in X \mid M_A(x) \geq t\}, \\ U(\tilde{B}_A; [\delta_1, \delta_2]) &:= \{x \in X \mid \tilde{B}_A(x) \succeq [\delta_1, \delta_2]\}, \\ L(J_A; s) &:= \{x \in X \mid J_A(x) \leq s\},\end{aligned}$$

where $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$.

**Theorem 3.6.** *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a $BCK/BCI$-algebra $X$ is an MBJ-neutrosophic ideal of $X$ if and only if the non-empty sets $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are ideals of $X$ for all $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$.*

*Proof.* Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. Let $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$ be such that $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are non-empty. Obviously, $0 \in U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$. For any $x, y, a, b, u, v \in X$, if $x * y \in U(M_A; t)$, $y \in U(M_A; t)$, $a * b \in U(\tilde{B}_A; [\delta_1, \delta_2])$, $b \in U(\tilde{B}_A; [\delta_1, \delta_2])$, $u * v \in L(J_A; s)$ and $v \in L(J_A; s)$, then

$$\begin{aligned}&M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \geq \min\{t, t\} = t, \\ &\tilde{B}_A(a) \succeq \operatorname{rmin}\{\tilde{B}_A(a * b), \tilde{B}_A(b)\} \succeq \operatorname{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2], \\ &J_A(u) \leq \max\{J_A(u * v), J_A(v)\} \leq \min\{s, s\} = s,\end{aligned}$$

and so $x \in U(M_A; t)$, $a \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $u \in L(J_A; s)$. Therefore $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are ideals of $X$.

Conversely, assume that the non-empty sets $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are ideals of $X$ for all $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$. Assume that $M_A(0) < M_A(a)$, $\tilde{B}_A(0) \prec \tilde{B}_A(a)$ and $J_A(0) > J_A(a)$for some $a \in X$. Then $0 \notin U(M_A; M_A(a)) \cap U(\tilde{B}_A; \tilde{B}_A(a)) \cap L(J_A; J_A(a)$, which is a contradiction. Hence $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. If

$$M_A(a_0) < \min\{M_A(a_0 * b_0), M_A(b_0)\}$$

for some $a_0, b_0 \in X$, then $a_0 * b_0 \in U(M_A; t_0)$ and $b_0 \in U(M_A; t_0)$ but $a_0 \notin U(M_A; t_0)$ for $t_0 := \min\{M_A(a_0 * b_0), M_A(b_0)\}$. This is a contradiction, and thus $M_A(a) \geq \min\{M_A(a * b), M_A(b)\}$ for all $a, b \in X$. Similarly, we can show that $J_A(a) \leq \max\{J_A(a * b), J_A(b)\}$ for all $a, b \in X$. Suppose that $\tilde{B}_A(a_0) \prec \operatorname{rmin}\{\tilde{B}_A(a_0 * b_0), \tilde{B}_A(b_0)\}$ for some $a_0, b_0 \in X$. Let $\tilde{B}_A(a_0 * b_0) = [\lambda_1, \lambda_2]$, $\tilde{B}_A(b_0) = [\lambda_3, \lambda_4]$ and $\tilde{B}_A(a_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] \prec \operatorname{rmin}\{[\lambda_1, \lambda_2], [\lambda_3, \lambda_4]\} = [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}],$$

and so $\delta_1 < \min\{\lambda_1, \lambda_3\}$ and $\delta_2 < \min\{\lambda_2, \lambda_4\}$. Taking

$$[\gamma_1, \gamma_2] := \tfrac{1}{2}\left(\tilde{B}_A(a_0) + \operatorname{rmin}\{\tilde{B}_A(a_0 * b_0), \tilde{B}_A(b_0)\}\right)$$

implies that

$$\begin{aligned}[\gamma_1, \gamma_2] &= \tfrac{1}{2}\left([\delta_1, \delta_2] + [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}]\right) \\ &= \left[\tfrac{1}{2}(\delta_1 + \min\{\lambda_1, \lambda_3\}), \tfrac{1}{2}(\delta_2 + \min\{\lambda_2, \lambda_4\})\right].\end{aligned}$$

It follows that

$$\min\{\lambda_1, \lambda_3\} > \gamma_1 = \tfrac{1}{2}(\delta_1 + \min\{\lambda_1, \lambda_3\}) > \delta_1$$

and

$$\min\{\lambda_2, \lambda_4\} > \gamma_2 = \tfrac{1}{2}(\delta_2 + \min\{\lambda_2, \lambda_4\}) > \delta_2.$$

Hence $[\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2] \succ [\delta_1, \delta_2] = \tilde{B}_A(a_0)$, and therefore $a_0 \notin U(\tilde{B}_A; [\gamma_1, \gamma_2])$. On the other hand,

$$\tilde{B}_A(a_0 * b_0) = [\lambda_1, \lambda_2] \succeq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2]$$

and

$$\tilde{B}_A(b_0) = [\lambda_3, \lambda_4] \succeq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2],$$

that is, $a_0 * b_0, b_0 \in U(\tilde{B}_A; [\gamma_1, \gamma_2])$. This is a contradiction, and therefore $\tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}$ for all $x, y \in X$. Consequently $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. □

**Theorem 3.7.** *Given an ideal $I$ of a $BCK/BCI$-algebra $X$, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by*

$$M_A(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A(x) = \begin{cases} [\gamma_1, \gamma_2] & \text{if } x \in I, \\ [0, 0] & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases} \tag{3.7}$$

*where $t \in (0, 1]$, $s \in [0, 1)$ and $\gamma_1, \gamma_2 \in (0, 1]$ with $\gamma_1 < \gamma_2$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ such that $U(M_A; t) = U(\tilde{B}_A; [\gamma_1, \gamma_2]) = L(J_A; s) = I$.*

*Proof.* Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$ and so

$$\begin{aligned} &M_A(x) = t = \min\{M_A(x * y), M_A(y)\}, \\ &\tilde{B}_A(x) = [\gamma_1, \gamma_2] = \operatorname{rmin}\{[\gamma_1, \gamma_2], [\gamma_1, \gamma_2]\} = \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}, \\ &J_A(x) = s = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If any one of $x * y$ and $y$ is contained in $I$, say $x * y \in K$, then $M_A(x * y) = t$, $\tilde{B}_A(x * y) = [\gamma_1, \gamma_2]$, $J_A(x * y) = s$, $M_A(y) = 0$, $\tilde{B}_A(y) = [0, 0]$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x) \geq 0 = \min\{t, 0\} = \min\{M_A(x * y), M_A(y)\}, \\ &\tilde{B}_A(x) \succeq [0, 0] = \operatorname{rmin}\{[\gamma_1, \gamma_2], [0, 0]\} = \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}, \\ &J_A(x) \leq 1 = \max\{s, 1\} = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If $x * y, y \notin K$, then $M_A(x * y) = 0 = M_A(y)$, $\tilde{B}_A(x * y) = [0, 0] = \tilde{B}_A(y)$ and $J_A(x * y) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x) \geq 0 = \min\{0, 0\} = \min\{M_A(x * y), M_A(y)\}, \\ &\tilde{B}_A(x) \succeq [0, 0] = \operatorname{rmin}\{[0, 0], [0, 0]\} = \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}, \\ &J_A(x) \leq 1 = \max\{1, 1\} = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

It is obvious that $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. Obviously, we have $U(M_A; t) = U(\tilde{B}_A; [\gamma_1, \gamma_2]) = L(J_A; s) = I$. □

**Theorem 3.8.** *For any non-empty subset $I$ of $X$, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ which is given in* (3.7). *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$, then $I$ is an ideal of $X$.*

*Proof.* Obviously, $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $M_A(x * y) = t = M_A(y)$, $\tilde{B}_A(x * y) = [\gamma_1, \gamma_2] = \tilde{B}_A(y)$ and $J_A(x * y) = s = J_A(y)$. Thus

$$\begin{aligned}
&M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = t,\\
&\tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} = [\gamma_1, \gamma_2],\\
&J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = s,
\end{aligned}$$

and hence $x \in I$. Therefore $I$ is an ideal of $X$. □

**Theorem 3.9.** *In a $BCK$-algebra, every MBJ-neutrosophic ideal is an MBJ-neutrosophic subalgebra.*

*Proof.* Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of a $BCK$-algebra $X$. Since $(x * y) * x \leq y$ for all $x, y \in X$, it follows from Proposition 3.3 that

$$\begin{aligned}
&M_A(x * y) \geq \min\{M_A(x), M_A(y)\},\\
&\tilde{B}_A(x * y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},\\
&J_A(x * y) \leq \max\{J_A(x), J_A(y)\}
\end{aligned}$$

for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of a $BCK$-algebra $X$. □

The converse of Theorem 3.9 may not be true as seen in the following example.

**Example 3.10.** Consider a $BCK$-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 4. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$, but it is not an MBJ-neutrosophic ideal of $X$ since

$$\tilde{B}_A(1) \not\succeq \operatorname{rmin}\{\tilde{B}_A(1 * 2), \tilde{B}_A(2)\}.$$

**Table 3:** Cayley table for the binary operation "$*$".

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

**Table 4:** MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

| $X$ | $M_A(x)$ | $\tilde{B}_A(x)$ | $J_A(x)$ |
|---|---|---|---|
| 0 | 0.7 | $[0.3, 0.8]$ | 0.2 |
| 1 | 0.4 | $[0.2, 0.6]$ | 0.3 |
| 2 | 0.4 | $[0.3, 0.8]$ | 0.4 |
| 3 | 0.6 | $[0.2, 0.6]$ | 0.5 |

We provide a condition for an MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal in a $BCK$-algebra.

**Theorem 3.11.** *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic subalgebra of a BCK-algebra X satisfying the condition* (3.4). *Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X.*

*Proof.* For any $x \in X$, we get

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\begin{aligned}\tilde{B}_A(0) = \tilde{B}_A(x * x) &\succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} \\ &= \operatorname{rmin}\{[B_A^-(x), B_A^+(x)], [B_A^-(x), B_A^+(x)]\} \\ &= [B_A^-(x), B_A^+(x)] = \tilde{B}_A(x),\end{aligned}$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$\begin{aligned}&M_A(x) \geq \min\{M_A(x * y), M_A(y)\}, \\ &\tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\}, \\ &J_A(x) \leq \max\{J_A(x * y), J_A(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. □

Theorem 3.9 is not true in a $BCI$-algebra as seen in the following example.

**Example 3.12.** Let $(Y, *, 0)$ be a $BCI$-algebra and let $(\mathbb{Z}, -, 0)$ be an adjoint $BCI$-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then $X = Y \times \mathbb{Z}$ is a $BCI$-algebra and $I = Y \times \mathbb{N}$ is an ideal of $X$ where $\mathbb{N}$ is the set of all non-negative integers (see [17]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by

$$M_A(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A(x) = \begin{cases} [\gamma_1, \gamma_2] & \text{if } x \in I, \\ [0, 0] & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases} \tag{3.8}$$

where $t \in (0, 1]$, $s \in [0, 1)$ and $\gamma_1, \gamma_2 \in (0, 1]$ with $\gamma_1 < \gamma_2$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ by Theorem 3.7. But it is not an MBJ-neutrosophic subalgebra of $X$ since

$$M_A((0, 0) * (0, 1)) = M_A((0, -1)) = 0 < t = \min\{M_A((0, 0)), M_A(0, 1))\},$$

$$\tilde{B}_A((0, 0) * (0, 2)) = \tilde{B}_A((0, -2)) = [0, 0] \prec [\gamma_1, \gamma_2] = \operatorname{rmin}\{\tilde{B}_A((0, 0)), \tilde{B}_A(0, 2))\},$$

and/or

$$J_A((0, 0) * (0, 3)) = J_A((0, -3)) = 1 > s = \max\{J_A((0, 0)), J_A(0, 3))\}.$$

**Definition 3.13.** An MBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of a $BCI$-algebra $X$ is said to be *closed* if

$$(\forall x \in X)(M_A(0 * x) \geq M_A(x), \tilde{B}_A(0 * x) \succeq \tilde{B}_A(x), J_A(0 * x) \leq J_A(x)). \tag{3.9}$$

**Theorem 3.14.** *In a BCI-algebra, every closed MBJ-neutrosophic ideal is an MBJ-neutrosophic subalgebra.*

*Proof.* Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a closed MBJ-neutrosophic ideal of a $BCI$-algebra $X$. Using (3.2), (2.3), (III) and (3.3), we have

$$M_A(x * y) \geq \min\{M_A((x * y) * x), M_A(x)\} = \min\{M_A(0 * y), M_A(x)\} \geq \min\{M_A(y), M_A(x)\},$$

$$\tilde{B}_A(x * y) \succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * x), \tilde{B}_A(x)\} = \mathrm{rmin}\{\tilde{B}_A(0 * y), \tilde{B}_A(x)\} \succeq \mathrm{rmin}\{\tilde{B}_A(y), \tilde{B}_A(x)\},$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * x), J_A(x)\} = \max\{J_A(0 * y), J_A(x)\} \leq \max\{J_A(y), J_A(x)\}$$

for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$. □

**Theorem 3.15.** *In a weakly $BCK$-algebra, every MBJ-neutrosophic ideal is closed.*

*Proof.* Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of a weakly $BCK$-algebra $X$. For any $x \in X$, we obtain

$$M_A(0 * x) \geq \min\{M_A((0 * x) * x), M_A(x)\} = \min\{M_A(0), M_A(x)\} = M_A(x),$$

$$\tilde{B}_A(0 * x) \succeq \mathrm{rmin}\{\tilde{B}_A((0 * x) * x), \tilde{B}_A(x)\} = \mathrm{rmin}\{\tilde{B}_A(0), \tilde{B}_A(x)\} = \tilde{B}_A(x),$$

and

$$J_A(0 * x) \leq \max\{J_A((0 * x) * x), J_A(x)\} = \max\{J_A(0), J_A(x)\} = J_A(x).$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$. □

**Corollary 3.16.** *In a weakly $BCK$-algebra, every MBJ-neutrosophic ideal is an MBJ-neutrosophic subalgebra.*

The following example shows that any MBJ-neutrosophic subalgebra is not an MBJ-neutrosophic ideal in a $BCI$-algebra.

**Example 3.17.** Consider a $BCI$-algebra $X = \{0, a, b, c, d, e\}$ with the $*$-operation in Table 5. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 6. It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$. But it is not an MBJ-neutrosophic ideal of $X$ since

$$M_A(d) < \min\{M_A(d * c), M_A(c)\},$$

$$\tilde{B}_A(d) \prec \mathrm{rmin}\{\tilde{B}_A(d * c), \tilde{B}_A(c)\},$$

and/or

$$J_A(d) > \max\{J_A(d * c), J_A(c)\}.$$

**Table 5:** Cayley table for the binary operation “$*$”.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $c$ | $b$ | $c$ | $c$ |
| $a$ | $a$ | 0 | $c$ | $b$ | $c$ | $c$ |
| $b$ | $b$ | $b$ | 0 | $c$ | 0 | 0 |
| $c$ | $c$ | $c$ | $b$ | 0 | $b$ | $b$ |
| $d$ | $d$ | $b$ | $a$ | $c$ | 0 | $a$ |
| $e$ | $e$ | $b$ | $a$ | $c$ | $a$ | 0 |

**Table 6:** MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

| $X$ | $M_A(x)$ | $\tilde{B}_A(x)$ | $J_A(x)$ |
|---|---|---|---|
| 0 | 0.7 | [0.4, 0.9] | 0.3 |
| $a$ | 0.4 | [0.04, 0.45] | 0.6 |
| $b$ | 0.7 | [0.4, 0.9] | 0.3 |
| $c$ | 0.7 | [0.4, 0.9] | 0.3 |
| $d$ | 0.4 | [0.04, 0.45] | 0.6 |
| $e$ | 0.4 | [0.04, 0.45] | 0.6 |

**Theorem 3.18.** *In a $p$-semisimple $BCI$-algebra $X$, the following are equivalent.*

(1) *$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$.*
(2) *$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$.*

*Proof.* (1) $\Rightarrow$ (2). See Theorem 3.14.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$. For any $x \in X$, we get

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\tilde{B}_A(0) = \tilde{B}_A(x * x) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} = \tilde{B}_A(x),$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Hence $M_A(0 * x) \geq \min\{M_A(0), M_A(x)\} = M_A(x)$, $\tilde{B}_A(0 * x) \succeq \mathrm{rmin}\{\tilde{B}_A(0), \tilde{B}_A(x)\} = \tilde{B}_A(x)$ and $J_A(0 * x) \leq \max\{J_A(0), J_A(x)\} = J_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} M_A(x) = M_A(y * (y * x)) &\geq \min\{M_A(y), M_A(y * x)\} \\ &= \min\{M_A(y), M_A(0 * (x * y))\} \\ &\geq \min\{M_A(x * y), M_A(y)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x) = \tilde{B}_A(y * (y * x)) &\succeq \mathrm{rmin}\{\tilde{B}_A(y), \tilde{B}_A(y * x)\} \\ &= \mathrm{rmin}\{\tilde{B}_A(y), \tilde{B}_A(0 * (x * y))\} \\ &\succeq \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} \end{aligned}$$

and

$$\begin{aligned} J_A(x) = J_A(y * (y * x)) &\leq \max\{J_A(y), J_A(y * x)\} \\ &= \max\{J_A(y), J_A(0 * (x * y))\} \\ &\leq \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$. □

Since every associative $BCI$-algebra is $p$-semisimple, we have the following corollary.

**Corollary 3.19.** *In an associative $BCI$-algebra $X$, the following are equivalent.*

(1) *$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$.*
(2) *$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $X$.*

**Corollary 3.20.** *In a $BCI$-algebra $X$, consider the following conditions*

(1) *every element $x$ in $X$ is minimal.*
(2) $X = \{0 * x \mid x \in X\}$.
(3) $(\forall x, y \in X)\,(x * (0 * y) = y * (0 * x))$.
(4) $(\forall x \in X)\,(0 * x = 0 \Rightarrow x = 0)$.
(5) $(\forall a, x \in X)\,(a * (a * x) = x)$.
(6) $(\forall a \in X)\, X = \{a * x \mid x \in X\}$.
(7) $(\forall x, y, a, b \in X)\,((x * y) * (a * b) = (x * a) * (y * b))$.
(8) $(\forall x, y \in X)\,(0 * (y * x) = x * y)$.

*If one of the conditions above is valid, then the following are equivalent.*

(1) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *is a closed MBJ-neutrosophic ideal of $X$.*
(2) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *is an MBJ-neutrosophic subalgebra of $X$.*

**Definition 3.21.** Let $X$ be an $(S)$-$BCK$-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in $X$ is called an *MBJ-neutrosophic $\circ$-subalgebra* of $X$ if the following assertions are valid.

$$\begin{aligned} &M_A(x \circ y) \geq \min\{M_A(x), M_A(y)\},\\ &\tilde{B}_A(x \circ y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},\\ &J_A(x \circ y) \leq \max\{J_A(x), J_A(y)\}\end{aligned} \tag{3.10}$$

for all $x, y \in X$.

**Lemma 3.22.** *Every MBJ-neutrosophic ideal of a $BCK/BCI$-algebra $X$ satisfies the following assertion.*

$$(\forall x, y \in X)\,\left(x \leq y \Rightarrow M_A(x) \geq M_A(y), \tilde{B}_A(x) \succeq \tilde{B}_A(y), J_A(x) \leq J_A(y)\right). \tag{3.11}$$

*Proof.* Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y),$$

$$\tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} = \operatorname{rmin}\{\tilde{B}_A(0), \tilde{B}_A(y)\} = \tilde{B}_A(y),$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y).$$

This completes the proof. □

**Theorem 3.23.** *In an $(S)$-$BCK$-algebra, every MBJ-neutrosophic ideal is an MBJ-neutrosophic $\circ$-subalgebra.*

*Proof.* Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of an $(S)$-$BCK$-algebra $X$. Note that $(x \circ y) * x \leq y$ for all $x, y \in X$. Using Lemma 3.22 and (3.2) implies that

$$M_A(x \circ y) \geq \min\{M_A((x \circ y) * x), M_A(x)\} \geq \min\{M_A(y), M_A(x)\},$$

$$\tilde{B}_A(x \circ y) \succeq \operatorname{rmin}\{\tilde{B}_A((x \circ y) * x), \tilde{B}_A(x)\} \succeq \operatorname{rmin}\{\tilde{B}_A(y), \tilde{B}_A(x)\},$$

and

$$J_A(x \circ y) \leq \max\{J_A((x \circ y) * x), J_A(x)\} \leq \max\{J_A(y), J_A(x)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic $\circ$-subalgebra of $X$. □

We provide a characterization of an MBJ-neutrosophic ideal in an $(S)$-$BCK$-algebra.

**Theorem 3.24.** *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in an $(S)$-$BCK$-algebra $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ if and only if the following assertions are valid.*

$$M_A(x) \geq \min\{M_A(y), M_A(z)\}, \tilde{B}_A(x) \succeq \operatorname{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\}, J_A(x) \leq \max\{J_A(y), J_A(z)\} \tag{3.12}$$

*for all $x, y, z \in X$ with $x \leq y \circ z$.*

*Proof.* Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ and let $x, y, z \in X$ be such that $x \leq y \circ z$. Using (3.1), (3.2) and Theorem 3.23, we have

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * (y \circ z)), M_A(y \circ z)\} \\ &= \min\{M_A(0), M_A(y \circ z)\} \\ &= M_A(y \circ z) \geq \min\{M_A(y), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x) &\succeq \operatorname{rmin}\{\tilde{B}_A(x * (y \circ z)), \tilde{B}_A(y \circ z)\} \\ &= \operatorname{rmin}\{\tilde{B}_A(0), \tilde{B}_A(y \circ z)\} \\ &= \tilde{B}_A(y \circ z) \succeq \operatorname{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x) &\leq \max\{J_A(x * (y \circ z)), J_A(y \circ z)\} \\ &= \max\{J_A(0), J_A(y \circ z)\} \\ &= J_A(y \circ z) \leq \max\{J_A(y), J_A(z)\}. \end{aligned}$$

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in an $(S)$-$BCK$-algebra $X$ satisfying the condition (3.12) for all $x, y, z \in X$ with $x \leq y \circ z$. Sine $0 \leq x \circ x$ for all $x \in X$, it follows from (3.12) that

$$M_A(0) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\tilde{B}_A(0) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} = \tilde{B}_A(x),$$

and

$$J_A(0) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Note that $x \leq (x * y) \circ y$ for all $x, y \in X$. Hence we have

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\}, \tilde{B}_A(x) \succeq \min\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} \text{ and } J_A(x) \leq \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. □

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