

MDPC-McEliece: New McEliece Variants from Moderate Density Parity-Check Codes

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Abstract. In this work, we propose two McEliece variants. One from Moderate Density Parity-Check (MDPC) codes and other from quasi-cyclic MDPC codes. MDPC codes are LDPC codes of higher density than what is usually adopted for telecommunication applications, providing an worse error-correction capability. However, in cryptography, we are not necessarily interested in correcting many errors, but only a number which ensures an adequate security level, a condition satisfied by MDPC codes. The benefits of their employment are many. Under a reasonable assumption, MDPC codes reduce the key-distinguishing McEliece problem to the problem of decoding a linear code. This leads the security of our variant to rely only on a single, well studied coding-theory problem. Furthermore, in the quasi-cyclic case, our proposal provides extremely compact-keys (for 80-bits of security, public-keys have only 4800 bits).

Keywords: post-quantum cryptography, code-based cryptography, coding-theory, LDPC codes.

1 Introduction

Code-Based Cryptography. All cryptosystems based on the hardness of factoring or discrete logarithm can be attacked [37] in polynomial time with a quantum computer (see [10] for an extensive report). This threatens most if not all public-key cryptosystems deployed in practice, such as RSA [34] or DSA [23]. Code-based cryptography is believed to be quantum resistant and is therefore considered as a viable replacement for those schemes in future applications. Yet, independently of their so-called “post-quantum” nature, code-based cryptosystems offer other benefits even for present-day applications due to their excellent algorithmic efficiency, which is up to several orders of complexity better than traditional schemes.

The McEliece cryptosystem [26] is the first code-based cryptosystem, originally proposed using Goppa codes. Its security is based on two assumptions, the indistinguishability of the code family and the hardness of decoding a generic linear code. It is namely proved in [13] that if an adversary is not able to distinguish a Goppa code from a random code, then he is challenged to decode a generic linear code, a problem proved to be NP-complete [8]. However in [15] a distinguisher for Goppa codes of high rate (like those originally suggested for CFS

signature [13] and for some realistic secure parameters of McEliece cryptosystems) is presented. Although this fact does not represent an effective attack, it would be more satisfactory to use other code families which have less remarkable algebraic structure.

Although efficient, this cryptosystem suffers from an extremely large key size. There is a way to reduce considerably the key size which consists in choosing codes with a large automorphism group, such as quasi-cyclic codes [18]. It has been followed by several other proposals such as [27, 7]. However, an structural algebraic attack [16] succeeds in breaking many of them (except the binary case of [27]). The underlying codes of these proposals, subfamilies of alternant codes, come with an algebraic structure which allows a cryptanalysis consisting in setting up an algebraic system of equations. This system is based on the alternant description of the code and can be solved with Gröbner bases techniques. Several particular features of the algebraic system make this attack feasible: the system is bihomogeneous and bilinear and most importantly the quasi-cyclic or the quasi-dyadic structure of these schemes allows a drastic reduction of the number of unknowns in the system. This kind of attack is exponential in nature and can be easily prevented by choosing more conservative parameters. However codes without any algebraic code structure would thwart completely this approach.

Related work. Low-Density Parity Check (LDPC) codes [19] are good candidates for this purpose. These are codes with no algebraic structure which meet a very simple combinatorial property: they admit a sparse parity-check matrix. This sparsity is used for decoding. They have been repeatedly suggested for the McEliece scheme [28, 4, 5, 3, 2]. The very first proposal [28] analyzes the use of a simple LDPC code in the original setup of McEliece: the private-key is the sparse parity-check matrix H of constant row weight w of a code \mathcal{C} and the public-key is a dense generator matrix $G' = S \cdot G \cdot P$ of a code \mathcal{C}' , where S is a scrambling matrix, G is a generator matrix for \mathcal{C} and P is a permutation matrix. However, finding low-weight codewords in the dual of \mathcal{C}' is feasible and enough to reconstruct a sparse parity-check matrix which efficiently decodes. In [3], a proposal to fix this problem is suggested. It consists in replacing the permutation matrix P by an invertible matrix Q of some small constant row weight m and choosing S sparse. Properly choosing w and m , finding codewords of weight wm in \mathcal{C}' is unfeasible. Nevertheless, the unfortunate choices for the structure of these matrices allowed to successfully cryptanalyze the scheme [30]. In [2], proposing a dense matrix S and a more general construction for Q , the variant seems to be immune against the attack suggested in [30]. The authors also suggest a quasi-cyclic structure achieving compact keys. For 80-bits of security, the authors suggest public-keys composed by 3 rows of 4 circulant blocks of size 4032×4032 . This implies in a public-key of $3 \times 4 \times 4032 = 48384$ bits³.

Our contribution. Our first observation is that it is not necessary to replace the permutation matrix by a matrix of some small constant row weight to use LDPC codes into the McEliece scheme. Simply by increasing enough the length

³ The authors did not consider a CCA-2 secure conversion, which would allow public-keys in systematic form, reducing the key size to 12096 bits.

and the row weight of the secret sparse parity-check matrix allows to avoid all known message (using standard decoding algorithms) and key recovery attacks (aiming at finding low weight codewords in the dual of the public code). For instance, for a code rate $\frac{1}{2}$ and 80 bits of security, we chose the secret parity-check matrix to be of size 4800×9600 and rows of weight 90 (whereas the LDPC codes used in practice for error correcting purposes have much lower row weights, typically less than 10). We call them MDPC codes⁴ (which stands for Moderate Parity Check Codes) to insist on the fact that they admit a parity-check which is only moderately sparse. However the error correction performance of MDPC codes degrades significantly when compared to standard LDPC codes. For instance, we correct for the aforementioned example only 84 errors, whereas any decent LDPC code of same size would correct about 700-800 errors. Despite this fact, this number of errors is still large enough so that standard decoding algorithms for correcting errors in a generic linear code are thwarted by such parameters. Furthermore, our proposal is scalable for any security level and code rate so that standard attacks completely fail.

We also give a quite satisfactory security reduction towards a well studied coding-theory problem, namely decoding random linear codes. To achieve this goal, we make a single, natural assumption: distinguishing an MDPC code from a random linear code amounts to being able to ascertain the existence of low weight codewords in its dual code. This provides a strong argument in favor of the security of our scheme. Besides, adding a quasi-cyclic structure, our proposal provides extremely compact keys. The aforementioned example, for 80-bits of security, has a public-key of only 4800 bits. Note that the state of the art indicates that a quasi-cyclic structure, by itself, does not imply a significant improvement for an adversary. All previous attacks on McEliece schemes are based on the *combination* of a quasi-cyclic/dyadic structure with some algebraic code information.

2 Preliminaries

We gather here a few basic definitions which are used in this paper.

Definition 1 (Hamming distance and weight). *The Hamming weight (or simply weight) of a vector $x \in \mathbb{F}_2^n$ is the number $\text{wt}(x)$ of its nonzero components.*

Definition 2 (Linear codes). *A binary (n, r) -linear code \mathcal{C} of length n , dimension $n - r$ and codimension r , is a $(n - r)$ -dimensional vector subspace of \mathbb{F}_2^n . It is spanned by the rows of a matrix $G \in \mathbb{F}_2^{(n-r) \times n}$, called a generator matrix of \mathcal{C} . Equivalently, it is the kernel of a matrix $H \in \mathbb{F}_2^{r \times n}$, called a parity-check matrix*

⁴ This terminology has already been proposed in the communications theory literature before for the very same concept [31]. The authors showed that certain quasi-cyclic MDPC codes may perform well at moderate lengths for correcting a rather large number of errors by using a variation of the standard belief propagation taking advantage of the quasi-cyclic structure.

of \mathcal{C} . The codeword $c \in \mathcal{C}$ of a vector $m \in \mathbb{F}_2^{(n-r)}$ is $c = mG$. The syndrome $s \in \mathbb{F}_2^r$ of a vector $e \in \mathbb{F}_2^n$ is $s = He^T$. The dual \mathcal{C}^\perp of \mathcal{C} is the linear code spanned by the rows of any parity-check matrix of \mathcal{C} .

Definition 3 (Quasi-cyclic code). An (n, r) -linear code is quasi-cyclic (QC) if there is some integer n_0 such that every cyclic shift of a codeword by n_0 places is again a codeword.

When $n = n_0p$, for some integer p , it is possible and convenient to have both generator and parity check matrices composed by $p \times p$ circulant blocks. A circulant block is completely described by its first row (or column) and the algebra of $p \times p$ binary circulant matrices is isomorphic to the algebra of polynomials modulo $x^p - 1$ over \mathbb{F}_2 .

Definition 4 (MDPC codes). An (n, r, w) -MDPC code is a linear code of length n , codimension r admitting a parity check matrix with constant row weight w .

When they are also quasi-cyclic, we call them (n, r, w) -QC-MDPC codes. LDPC codes have typically small constants row weights (usually, less than 10). For MDPC codes, we assume row weights which scale in $O(\sqrt{n \log n})$.

3 Moderate Density Parity-Check McEliece variants

In this section, we present the construction of our codes and the description of our variant.

(n, r, w) -MDPC code construction. A random (n, r, w) -MDPC code is easily generated by picking a random $r \times n$ matrix with rows of weight w :

1. Generate r vectors $(h_i \in \mathbb{F}_2^n)_{0 \leq i < r}$, of weight w uniformly at random.
2. The (n, r, w) -MDPC code is defined by a parity-check matrix $H \in \mathbb{F}_2^n$ of i -th row h_i .

With overwhelming probability this matrix is of full rank and the rightmost $r \times r$ block is always invertible after possibly swapping a few columns.

(n, r, w) -QC-MDPC code construction. We are specially interested in (n, r, w) -QC-MDPC codes, where $n = n_0p$ and $r = p$. This means that the parity-check matrix has the form $H = [H_0|H_1|\dots|H_{n_0-1}]$. Basically, we pick one random word of length $n = n_0p$ and weight w . The other $r - 1$ rows are obtained from $r - 1$ quasi-cyclic shifts.

1. Generate a vector $h \in \mathbb{F}_2^n$ of weight w uniformly at random.
2. The (n, r, w) -QC-MDPC code is defined by a quasi-cyclic parity-check matrix $H \in \mathbb{F}_2^n$ of first row h .

- The other $r - 1$ rows of H are obtained from the $r - 1$ quasi-cyclic shifts of h .

Each block H_i has row weight w_i , such that $w = \sum_{i=0}^{n_0-1} w_i$. In general, a smooth distribution is expected for the sequence of w_i 's. A generator matrix G in row reduced echelon form can be easily derived from the H_i 's blocks: assuming H_{n_0-1} is non-singular (this particularly implies w_{n_0-1} odd, otherwise the rows of H_{n_0-1} would sum up to 0), we compute:

$$G = \left[\begin{array}{c|c} & \begin{matrix} (H_{n_0-1}^{-1} \cdot H_0)^T \\ (H_{n_0-1}^{-1} \cdot H_1)^T \\ \vdots \\ (H_{n_0-1}^{-1} \cdot H_{n_0-2})^T \end{matrix} \\ \hline \mathbf{I} & \end{array} \right]$$

This generation algorithm supports the security reduction presented in Section 5.1, providing a strong argument in favor of the security of our scheme. However, it also leads to a worse error-correction capability due to the existence of short cycles in the Tanner graph associated to the code⁵.

Key-Generation.

- Generate a parity-check matrix $H \in \mathbb{F}_2^{r \times n}$ of a t -error-correcting (n, r, w) -MDPC or (n, r, w) -QC-MDPC code, as described above.
- Generate its corresponding generator matrix $G \in \mathbb{F}_2^{(n-r) \times n}$ in row reduced echelon form.

The public key is G and the private key is H .

Encryption. To encrypt a plaintext $m \in \mathbb{F}_2^{(n-r)}$ into $x \in \mathbb{F}_2^n$:

- Generate $e \in \mathbb{F}_2^n$ of $wt(e) \leq t$ at random.
- Compute $x \leftarrow mG + e$.

Decryption. Let Ψ_H be an LDPC decoding algorithm equipped with the knowledge of the sparse parity-check matrix H . To decrypt $x \in \mathbb{F}_2^n$ into $m \in \mathbb{F}_2^{(n-r)}$,

- Compute $mG \leftarrow \Psi_H(mG + e)$.
- Extract the plaintext m from the first $(n - r)$ positions of mG .

Note that this description gets rid⁶ of the usual scrambling matrix S and permutation matrix P . Note also that the use of a CCA2-secure conversion, e.g.

⁵ In [4], a construction based on random difference families avoids such a problem at the price of adding an algebraic relation on how the weight is distributed.

⁶ A folklore reasoning assigns security functions to those matrices. However it is enough that the public-key does not reveal any useful information for decoding, a condition satisfied by the dense public matrix.

[22], allows for G in systematic-form, without bringing any security-flaw. Thus the QC-MDPC variant has public-key of size $(n - r)$ and the MDPC variant of size $r(n - r)$. In practice, the MDPC variant obtains huge keys whilst the QC-MDPC allows for extremely compact keys. Regarding the QC case, note that the state of the art indicates that a quasi-cyclic structure, by itself, does not imply a significant improvement for an adversary. All previous attacks on McEliece schemes are based on the *combination* of a quasi-cyclic/dyadic structure with some algebraic code information.

4 Decoding MDPC codes

Our approach is to decode MDPC codes using the same decoding framework available for LDPC codes. These decoding algorithms are iterative and provide an error correction capability which increases linearly with the code-length and which decreases when the row weight of the parity-check matrix increases. Thus a degradation in the error correction capability is expected when using such algorithms for MDPC decoding.

4.1 Decoding algorithm

Basically, we can divide LDPC decoding algorithms in two groups. The first one gathers simple and fast algorithms, e.g. the bit-flipping algorithm [19]. These algorithms do not achieve optimal error correction capability. The second group gathers more involved algorithms with better error-correction capability, e.g. the Sum-Product algorithm [20]. For MDPC codes, the first kind of algorithm seems to be much more appropriate: the gain in decoding complexity more than outweighs the slight improvement in error correction capability of the second group.

Next, we propose a variant of Gallager's bit-flipping algorithm suitable for MDPC decoding. At first, we describe the idea on how bit-flipping works in general. At each iteration, the syndrome of the message is computed and the number of unsatisfied parity-check equations associated to each bit of the message is computed. Each bit associated to more than b unsatisfied equations is flipped. This process is repeated until either the syndrome becomes a zero-vector or after a maximum number of iteration. The difference in our variant regards how the threshold value b is chosen. Below three approaches are described.

- I. Gallager precomputes a value for b at each iteration (see Inequality 4.16, page 46 of [19]);
- II. In [21], b is taken as the maximum number Max_{upc} of unsatisfied parity-check equations;
- III. Our approach: $b = \text{Max}_{\text{upc}} - \delta$, for a small positive integer δ .

Approach II leads to better error-correcting capability when compared to I, at the price of an increased number of iterations. Approach III combines the

benefits of I and II. It reduces the overall number of iterations obtained by Approach II since much more bits are flipped at each iteration. If the algorithm fails to decode, the value of δ is decreased by 1 and the process is restarted. Obviously when $\delta = 0$, we are back to Approach II, ensuring at least its error-correcting capability. The actual, optimal value of δ is determined empirically. For the parameters suggested in Section 6, choosing $\delta \approx 5$ is usually fine, reducing the number of iterations from ~ 65 to $\lesssim 10$. Algorithm 1 describes our decoding approach. “U.p.c.” is the abbreviation for “unsatisfied parity-check” and $\mathbf{0}$ stands to a zero-vector of length r .

Algorithm 1 A variant of the bit-flipping algorithm suitable for MDPC codes.

Input: $\text{Max}_{\text{it}} \in \mathbb{N}^*$,
 $\delta \in \mathbb{N}$,
 $y \in \mathbb{F}_2^n$,
 $H \in \mathbb{F}_2^{r \times w}$. // H stores the non-zero positions of a sparse matrix in $\mathbb{F}_2^{r \times n}$.

Output: $c \in \mathbb{F}_2^n$, such that $Hc^T = \mathbf{0}$, or FAIL.

- 1: **while** $\delta \geq 0$ **do**
- 2: $c \leftarrow y$; Iteration $\leftarrow 0$;
- 3: **while** Iteration $<$ Max_{it} **do**
- 4: $\text{Max}_{\text{upc}} \leftarrow 0$
- 5: $(\text{counter}_i \leftarrow 0)_{1 \leq i \leq n}$ // “counter” stores the number of u.p.c. for each bit.
- 6: $s \leftarrow Hc^T$
- 7: **for** $i = 1$ **to** (r) **do**
- 8: **if** $s[i] = 1$ **then**
- 9: **for** $j = 1$ **to** (w) **do**
- 10: $\text{counter}_{H[i][j]} = \text{counter}_{H[i][j]} + 1$ // Counting the u.p.c. for each bit.
- 11: **end for**
- 12: **end if**
- 13: **end for**
- 14: **for** $i = 1$ **to** (n) **do**
- 15: **if** $\text{Max}_{\text{upc}} >$ counter_i **then**
- 16: $\text{Max}_{\text{upc}} \leftarrow \text{counter}_i$ // Defining the maximal number of u.p.c.
- 17: **end if**
- 18: **end for**
- 19: **for** $i = 1$ **to** (n) **do**
- 20: **if** $\text{counter}_i \geq (\text{Max}_{\text{upc}} - \delta)$ **then**
- 21: Flip c_i // Flipping the appropriate bits.
- 22: **end if**
- 23: **end for**
- 24: **if** $Hc^T = \mathbf{0}$ **then**
- 25: **return** c
- 26: **end if**
- 27: **end while**
- 28: $\delta \leftarrow \delta - 1$ // In case of decoding failure, δ is decreased.
- 29: **end while**
- 30: **return** FAIL

It is easy to see that Algorithm 1 has complexity $O(nwI)$, where I stands for the average number of iterations. Note that Max_{upc} tends to decrease at each iteration. Another variant might use this information to avoid the computation of the maximal number of unsatisfied parity-check equations. This would eliminate the first and second for-loops at the price of an increased number of iterations.

4.2 Error-correction capability estimation

The error correction capability estimation for LDPC (or MDPC) codes is a hard task. In general, two steps are needed. The first one provides what is known as the waterfall threshold, an asymptotic estimation of the maximal number of errors from which reliable decoding can be expected (i.e. correct decoding is achieved when the code-length goes to infinity). In Appendix A, a way to compute this initial threshold value, inspired in [19], is described. A second step is based on exhaustive decoding simulation, providing a decoding failure rate.

Thus a valid approach for choosing code parameters is to evaluate the failure decoding rate for a code trying to correct the initial number of errors computed by the waterfall threshold. If the decoding failure rate is not satisfactory, then the number of errors is decreased until a negligible decoding failure rate is reached. Using our bit-flipping variant, the parameters suggested in Section 6 reach a decoding failure rate of at most 10^{-7} , verified through exhaustive simulation.

4.3 Dealing with decoding failures

In cryptography, a non-zero probability of decoding failure requires some special treatment. We present three approaches to deal with this problem.

- A. The first approach consists in conservatively choosing the number of errors compared to the code-length so that the decoding failure rate is negligible. An approach adopted by error-correcting applications is in scaling this decoding failure rate to be smaller than the failure rate of the machine where the system is deployed.
- B. A second approach deals with these unlikely events on the fly. In the case of a decoding failure, more sophisticated decoding algorithms with better error correction capability are applied, e.g. the Sum-Product algorithm [20].
- C. A third approach consists in using a CCA-2 secure conversion [22]. In short, a CCA2-secure conversion uses hash functions and random sequences to ensure the indistinguishability of the encrypted messages. Thus after a decoding failure, when the application allows it, a new encryption is requested. Since the encrypted messages behave like random sequences, the adversary would not be able to extract any information from this redundancy.

5 Security Assessment

The security assessment of our proposal is divided in two parts: its security reduction and the practical security assessment.

5.1 Security reduction

By security reduction, we mean a proof that an adversary able to attack the scheme is able to solve some (presumably hard) computational problem with a similar effort.

We start by giving the generic security reduction presented in [35] for the Niederreiter cryptosystem [29]. This scheme is equivalent in terms of security to the McEliece cryptosystem [24]. It is easy to see that this security reduction also holds for the McEliece scheme, at the price of more involved probability space and statements. After the generic security reduction, we provide the discussion regarding our proposal.

Notation:

- $\mathcal{F}_{n,r,w}$: a t -error correcting code family which can be either (n, r, w) -MDPC or (n, r, w) -QC-MDPC. We assume the public key is a parity check matrix of some code in $\mathcal{F}_{n,r,w}$.
- $\mathcal{K}_{n,r,w}$: the key space of $\mathcal{F}_{n,r,w}$.
- $\mathcal{H}_{n,r} \supset \mathcal{K}_{n,r,w}$: the apparent key space of $\mathcal{F}_{n,r,w}$:
 - MDPC case: $\mathcal{H}_{n,r}$ is the set of all full rank matrices in $\mathbb{F}_2^{r \times n}$.
 - QC-MDPC case: $\mathcal{H}_{n,r}$ is the set of all full rank matrices in $\mathbb{F}_2^{r \times n}$, restricted to block circulant matrices.

Note that all the statements in this section are valid in both (MDPC and QC-MDPC) cases.

Generic Reduction. Let $\mathcal{S}_n(0, t)$ denote the sphere centered in zero of radius t in the Hamming space \mathbb{F}_2^n and let Ω denote the probability space consisting of the sample space $\mathcal{H}_{n,r} \times \mathcal{S}_n(0, t)$ equipped with a uniform distribution. We define:

Distinguisher. A program $\mathcal{D} : \mathcal{H}_{n,r} \rightarrow \{0, 1\}$ is a (T, ϵ) -distinguisher for $\mathcal{K}_{n,r,w}$ (vs. $\mathcal{H}_{n,r}$) if it runs in time at most T and the *advantage of \mathcal{D} for $\mathcal{K}_{n,r,w}$*

$$Adv(\mathcal{D}, \mathcal{K}_{n,r,w}) = |\Pr_{\Omega}[\mathcal{D}(H) = 1 \mid H \in \mathcal{K}_{n,r,w}] - \Pr_{\Omega}[\mathcal{D}(H) = 1]|$$

is greater than ϵ .

Decoder. A program $\phi : \mathcal{H}_{n,r} \times \mathbb{F}_2^r \rightarrow \mathcal{S}_n(0, t)$ is a (T, ϵ) -decoder for $(\mathcal{H}_{n,r}, t)$ if it runs in time at most T and its *success probability*

$$Succ(\phi) = \Pr_{\Omega}[\phi(H, eH^T) = e]$$

is greater than ϵ .

Adversary. A program $\mathcal{A} : \mathcal{H}_{n,r} \times \mathbb{F}_2^n \rightarrow \mathcal{S}_n(0, t)$ is a (T, ϵ) -adversary against $\mathcal{K}_{n,r,w}$ -Niederreiter if it runs in time at most T its *success probability*

$$\text{Succ}(\mathcal{A}, \mathcal{K}_{n,r,w}) = \Pr_{\Omega}[\mathcal{A}(H, eH^T) = e \mid H \in \mathcal{K}_{n,r,w}]$$

is greater than ϵ .

An adversary against $\mathcal{K}_{n,r,w}$ -McEliece could be defined as a program $\mathcal{H}_{n,r} \times \mathbb{F}_2^n \rightarrow \mathbb{F}_2^{(n-r)} \times \mathcal{S}_n(0, t)$ of probability space Ω and sample set $\mathcal{H}_{n,r} \times \mathbb{F}_2^k \times \mathcal{S}_n(0, t)$. As stated before, this setup would only make all the statements and proofs more cumbersome. Next, the proposition from [35] which supports the security reduction.

Proposition 1 ([35]). *Given the security parameters (n, r, w) and t , if there exists a (T, ϵ) -adversary against $\mathcal{K}_{n,r,w}$ -Niederreiter, then there exists either a $(T, \epsilon/2)$ -decoder for $(\mathcal{H}_{n,r}, t)$ or a $(T + O(n^2), \epsilon/2)$ -distinguisher for $\mathcal{K}_{n,r,w}$ vs. $\mathcal{H}_{n,r}$.*

Proof. Let $\mathcal{A} : \mathcal{H}_{n,r} \times \mathbb{F}_2^r \rightarrow \mathcal{S}_n(0, t)$ be a (T, ϵ) -adversary against $\mathcal{K}_{n,r,w}$ -Niederreiter. We define the following distinguisher:

\mathcal{D} : input $H \in \mathcal{H}_{n,r}$.
 $e \leftarrow \mathcal{S}_n(0, t)$ //pick randomly and uniformly
if $(\mathcal{A}(H, eH^T) = e)$ **then return 1 else return 0.**

which implies:

$$\begin{aligned} \Pr_{\Omega}[\mathcal{D}(H) = 1] &= \Pr_{\Omega}[\mathcal{A}(H, eH^T) = e] \\ &= \text{Succ}(\mathcal{A}) \\ \Pr_{\Omega}[\mathcal{D}(H) = 1 \mid H \in \mathcal{K}_{n,r,w}] &= \Pr_{\Omega}[\mathcal{A}(H, eH^T) = e \mid H \in \mathcal{K}_{n,r,w}] \\ &= \text{Succ}(\mathcal{A}, \mathcal{K}_{n,r,w}) \end{aligned}$$

thus $\text{Adv}(\mathcal{D}, \mathcal{K}_{n,r,w}) = |\text{Succ}(\mathcal{A}, \mathcal{K}_{n,r,w}) - \text{Succ}(\mathcal{A})|$ and particularly:

$$\text{Adv}(\mathcal{D}, \mathcal{K}_{n,r,w}) + \text{Succ}(\mathcal{A}, \mathcal{K}_{n,r,w}) \geq \text{Succ}(\mathcal{A})$$

Since $\text{Succ}(\mathcal{A}, \mathcal{K}_{n,r,w}) \geq \epsilon$, we either have $\text{Adv}(\mathcal{D}, \mathcal{K}_{n,r,w})$ or $\text{Succ}(\mathcal{A})$ greater or equal to $\epsilon/2$ (recall that both are positive). The running time of \mathcal{D} is equal to the running time of \mathcal{A} increased by the cost for picking e and computing the product eH^T , which cannot exceed $O(n^2)$. So either \mathcal{A} is a (T, ϵ) -decoder for $(\mathcal{H}_{n,r}, t)$ or \mathcal{D} is a $(T + O(n^2), \epsilon/2)$ -distinguisher for $\mathcal{K}_{n,r,w}$. \square

A distinguisher for $\mathcal{K}_{n,r,w}$ vs. $\mathcal{H}_{n,r}$ and a decoder for $(\mathcal{H}_{n,r}, t)$ provide a solution respectively to the two following problems:

Problem 1 (Code distinguishing problem).

Parameters: $\mathcal{K}_{n,r,w}, \mathcal{H}_{n,r}$.

Instance: a matrix $H \in \mathcal{H}_{n,r}$.

Question: is $H \in \mathcal{K}_{n,r,w}$?

Problem 2 (Computational syndrome decoding problem).

Parameters: $\mathcal{H}_{n,r}$, an integer $t > 0$.

Instance: a matrix $H \in \mathcal{H}_{n,r}$ and a vector $s \in \mathbb{F}_2^r$.

Problem: find a vector $e \in \mathcal{S}_n(0,t)$ such that $eH^T = s$.

Thus, from Proposition 1, it is enough to assume that none of those problems can be solved efficiently to ensure that no efficient adversary against the scheme exists.

The MDPC and the QC-MDPC cases. We introduce an additional problem which consists in deciding the existence of words of given weight in a given linear code. Note that the code we consider below has a *generator matrix* $H \in \mathcal{H}_{n,r}$, it is thus the dual of a code in $\mathcal{F}_{n,r,w}$.

Problem 3 (Codeword existence problem).

Parameters: $\mathcal{H}_{n,r}$, an integer $w > 0$.

Instance: a matrix $H \in \mathcal{H}_{n,r}$.

Question: is there a codeword of weight at most w in the code of generator matrix H ?

Ideally, we would like to replace Problem 1 by Problem 3 in Proposition 1. Unfortunately, one would need to replace the distinguisher advantage by the quantity:

$$\text{Adv}(\mathcal{E}, \mathcal{K}_{n,r,w}) = |\Pr_{\Omega}[\mathcal{E}(H) = 1 \mid H \in \mathcal{K}_{n,r,w}] - \Pr_{\Omega}[\mathcal{E}(H) = 1]|$$

where \mathcal{E} denotes a program deciding the existence of a word of weight w in a given code. However this quantity is not directly related to the hardness of Problem 3 and therefore cannot be considered. Nevertheless we reach our purpose if we assume the following assumption.

Assumption 1 *Solving Problem 1 for parameters $(\mathcal{H}_{n,r}, \mathcal{K}_{n,r,w})$ is not easier than solving Problem 3 for the parameters $(\mathcal{H}_{n,r}, w)$.*

Within this assumption we could modify the reduction to a claim that the $\mathcal{K}_{n,r,w}$ -McEliece scheme is at least as hard as either Problem 2 and Problem 3. However we can do much better. Consider the computational problem associated to Problem 3 and the two following lemmas:

Problem 4 (Codeword finding problem).

Parameters: $\mathcal{H}_{n,r}$, an integer $w > 0$.

Instance: a matrix $H \in \mathcal{H}_{n,r}$.

Problem: find a codeword of weight at most w in the code of generator matrix H .

Lemma 1. *Problem 3 is polynomially equivalent to Problem 4.*

Proof. Let $\mathcal{G}_{n,k}$ denote a subset of $\mathbb{F}_2^{k \times n}$ composed by full rank matrices. A matrix $G \in \mathcal{G}_{n,k}$ is the generator matrix of some binary linear code \mathcal{C} of length n and dimension k . For any $1 \leq i \leq n$, we denote \mathcal{C}_i the code shortened at i , that is

$$\mathcal{C}_i = \{c = (c_1, \dots, c_n) \in \mathcal{C} \mid c_i = 0\}.$$

We will denote by G_i a generator matrix of \mathcal{C}_i . We assume we have a solution to Problem 3, that is a program $\mathcal{E} : \mathcal{G}_{n,k} \rightarrow \{0, 1\}$ such that $\mathcal{E}(G) = 1$ if and only if there exists a word of weight w in the code spanned by G . The following program called on input G such that $\mathcal{E}(G) = 1$

```

A: input  $G \in \mathcal{G}_{n,k}$ 
  for  $i$  from 1 to  $n$  while  $G$  has a rank  $> 1$ 
    if  $\mathcal{E}(G_i) = 1$  then  $G \leftarrow G_i$  // false at most  $w$  times
  return the first row of  $G$  of weight at most  $w$ 

```

will return a word of weight at most w in the code spanned by G . It calls the program \mathcal{E} at most n times. Conversely a solution to Problem 4 obviously provides a solution to Problem 3. □

Lemma 2. *Problem 4 is polynomially equivalent to Problem 2.*

Proof. For a matter of simplicity, we rewrite Problem 4 (codeword finding) to receive as input the parity-check matrix of the code, instead of its generator-matrix. Obviously, both descriptions are polynomially equivalent since one matrix can be obtained from the other in polynomial time. Let $\mathcal{H}_{n,r}$ denote a subset of $\mathbb{F}_2^{r \times n}$ composed by full rank matrices. A matrix $H \in \mathcal{H}_{n,r}$ is the parity check matrix of some binary linear code \mathcal{C} of length n and dimension $k = n - r$.

Problem 4 (Codeword finding problem).

Parameters: $\mathcal{H}_{n,r}$, an integer $w > 0$.

Instance: a matrix $H \in \mathcal{H}_{n,r}$.

Problem: find a codeword of weight w in the code of parity check matrix H .

1. Let us assume that we have a program \mathcal{B} which solves the Problem 4 for parameters $(\mathcal{H}_{n+1,r}, w + 1)$, we define the following program

```

A: input  $H \in \mathcal{H}_{n,r}, s \in \mathbb{F}_2^r$ 
   $H' \leftarrow (H \mid s^T)$  //  $s$  serves as  $(n+1)$ -th column of  $H'$ 
   $e \leftarrow \mathcal{B}(H')$  //  $e = (e_1, \dots, e_n, e_{n+1})$ 
  if  $e_{n+1} = 1$  then return  $(e_1, \dots, e_n)$  else FAIL

```

If $w + 1$ is smaller than the minimum distance of the code of parity check matrix H , the call $\mathcal{A}(H)$ will never fail. This provides a solution to Problem 2 with parameters $(\mathcal{H}_{n,r}, w)$.

2. Conversely, let us assume that we have a program \mathcal{A} which solves the Problem 2 for parameters $(\mathcal{H}_{n,r+1}, w)$

```

 $\mathcal{B}$ : input  $H \in \mathcal{H}_{n,r}$ 
       $(g_1, \dots, g_k) \leftarrow$  a basis of  $\mathcal{C}$  // where  $\mathcal{C}$  is the code of parity check matrix  $H$ 
      for  $j$  from 1 to  $n$ 
         $H' \leftarrow$  parity check matrix of  $\bigoplus_{i \neq j} \langle g_i \rangle$  // subcode of  $\mathcal{C}$  without  $g_j$ 
        if  $\mathcal{A}(H', g_j H'^T) \neq \text{FAIL}$  then
           $z \leftarrow \mathcal{A}(H', g_j H'^T)$ 
          return  $z + g_j$ 
      FAIL //  $\mathcal{A}$  fails to decode for all  $j$ 

```

If there exists a codeword of weight w , the decoder \mathcal{A} will succeed for at least one value of j . The above program provide a solution to Problem 4 for parameters $(\mathcal{H}_{n,r}, w)$. \square

Within Assumption 1, Lemma 1 and Lemma 2, we are able to produce strong security statements.

Proposition 2. *Given Assumption 1:*

- *Breaking the MDPC variant of McEliece or Niederreiter is not easier than solving the syndrome decoding problem for a random code.*
- *Breaking the QC-MDPC variant of McEliece or Niederreiter is not easier than solving the syndrome decoding problem for a random quasi-cyclic linear code.*

Proof. This follows directly from Lemma 1 and Lemma 2. \square

5.2 Practical security

In this section, we analyze the practical attacks against the proposed scheme. Key attacks aim either at recovering the secret decoder or simply distinguish the public key from a random matrix (what invalidates the security reduction). Message attacks try to decode one particular message considered as a noisy codeword.

Consider the system as an instantiation of the McEliece (or Niederreiter) scheme with an (n, r, w) -MDPC code, possibly quasy-cyclic, correcting t errors. We denote \mathcal{C} the hidden MDPC code defined by the public key (a generator matrix of \mathcal{C} for McEliece or a parity check matrix of \mathcal{C} for Niederreiter). We claim that the best attacks for each scenario are:

- *Key distinguishing attack:* exhibit one codeword of \mathcal{C}^\perp of weight w .
- *Key recovery attack:* exhibit r codewords of \mathcal{C}^\perp of weight w .
- *Decoding attack:* decode t errors in an $(n, n - r)$ -linear code.

For all those attacks we have to solve either the codeword finding problem or the computational syndrome decoding problem. For both those problems and for the considered parameters the best technique currently known is information

set decoding (ISD) [32]. In today’s state-of-the-art the best variants derive from Stern’s collision decoding algorithm [38]. There have been numerous contributions and improvements [14, 12, 11, 17, 9] until the recent asymptotic improvements [25, 6]. For selecting our parameters, we have analyzed all of them and an unpublished non-asymptotic analysis of [6] gives slightly lower workfactors (closed formulas⁷ in Appendix B). ISD workfactors are commonly used to estimate the practical security of code-based schemes. However there is a novelty related to the practical security of our proposal. The problem of finding a single low weight codeword in an MDPC code may admit *many* solutions.

We denote by $\text{WF}_{\text{isd}}(n, r, t)$ the cost for decoding t errors (or finding a codeword of weight t) in an (n, r) -binary linear code when there is a single solution of the problem. We start by giving a rough description of the ISD algorithms. These algorithms assume a pattern for the sought error vector and it proceeds analyzing a certain set of candidates until a solution is found. This set of candidates is usually stored in lists of a certain size \mathcal{L} and each candidate has a probability P to produce the solution. When the parameters algorithm are *optimal*, the workfactor $\text{WF}_{\text{isd}}(n, r, t)$ matches the ratio L/P , up to a small factor.

In [36], also mentioned by *Decoding One Out of Many* setting (DOOM), it is analyzed the gains when the decoding problem have multiple solutions and the adversary is satisfied with a single solution. In short, when the problem has N_s solutions, the probability of success P increases by a factor N_s (as long as $N_s P \ll 1$) and when N_i instances are treated simultaneously the list size L increases at most by a factor $\sqrt{N_i}$. Thus the DOOM technique [36] provides a gain⁸ of $N_s/\sqrt{N_i}$. This gain impacts on the practical security of our MDPC and QC-MDPC McEliece variants. Below we discuss the DOOM technique gain regarding each kind of attack against our scheme.

Key Distinguishing Attack. To distinguish a public key from a random matrix, it is enough to produce a word of weight w in the dual code \mathcal{C}^\perp . In this scenario, an adversary applying ISD to the all-zero syndrome will face a problem with r solutions (the r rows of the sparse parity check matrix). Then $N_s = r$ and $N_i = 1$ and the distinguishing attack cost drops by a factor of r :

$$\text{WF}_{\text{dist}}(n, r, w) = \frac{\text{WF}_{\text{isd}}(n, n - r, w)}{r}.$$

In the quasi-cyclic case, there is no obvious speedup and the distinguishing attack has the same cost as above.

Key Recovery Attack. To recover an equivalent private key, it is enough to recover all (or almost all) low weight parity check equations. All ISD variants are randomized and thus we can make r independent calls to a codeword finding algorithm. Each call costs on average $\frac{\text{WF}_{\text{isd}}(n, n - r, w)}{r}$ because there are r codewords

⁷ This is part of an unpublished work in progress.

⁸ In general, the real gain is in fact slightly smaller because these algorithms depend on optimal parameters which are not the same for multiple instances.

of weight w . Therefore on average, recovering all equations will cost:

$$\text{WF}_{\text{reco}}(n, r, w) = r \cdot \frac{\text{WF}_{\text{isd}}(n, n - r, w)}{r} = \text{WF}_{\text{isd}}(n, n - r, w).$$

In the quasi-cyclic case, any word of low weight will provide the sparse matrix (the sparse parity check matrix is the concatenation of several $r \times r$ circulant blocks) and thus the key recovery attack is no more expensive than the key distinguishing attack.

$$\text{WF}_{\text{reco}}^{\text{QC}}(n, r, w) = \text{WF}_{\text{dist}}^{\text{QC}}(n, r, w) = \frac{\text{WF}_{\text{isd}}(n, n - r, w)}{r}.$$

Decoding Attack. In the MDPC (*i.e.* non quasi-cyclic) case, the message security is related to the hardness of decoding t errors in a seemingly random binary linear code of length n and codimension r :

$$\text{WF}_{\text{dec}}(n, r, t) = \text{WF}_{\text{isd}}(n, r, t).$$

In the quasi-cyclic case, any cyclic shift of the target syndrome $s \in \mathbb{F}_2^r$ provides a new instance whose solution is equal to the one of the original syndrome, up to a block-wise cyclic shift. The number of instances and the number of solutions are thus $N_i = N_s = r$. Therefore a factor \sqrt{r} is gained:

$$\text{WF}_{\text{dec}}^{\text{QC}}(n, r, t) \geq \frac{\text{WF}_{\text{isd}}(n, r, t)}{\sqrt{r}}.$$

	MDPC	QC-MDPC
Key distinguishing	$\frac{1}{r} \text{WF}_{\text{isd}}(n, n - r, w)$	$\frac{1}{r} \text{WF}_{\text{isd}}(n, n - r, w)$
Key recovery	$\text{WF}_{\text{isd}}(n, n - r, w)$	$\frac{1}{r} \text{WF}_{\text{isd}}(n, n - r, w)$
Decoding	$\text{WF}_{\text{isd}}(n, r, t)$	$\frac{1}{\sqrt{r}} \text{WF}_{\text{isd}}(n, r, t)$

Table 1. Best attacks for code-based encryption schemes using t -error correcting (n, r, w) -MDPC (or QC-MDPC) codes

Thus to compute the cost of each attack, we have considered the non-asymptotic analysis of [6] *decreased* by the possible gains obtained by the DOOM technique. Note that the complex structure of the ISD variant [6] (an increased number of initial lists, pairs of non-disjoint lists and the probability of overlapped positions) might prejudice the maximal gain claimed for DOOM: $N_s/\sqrt{N_i}$. However since the difference of the work-factor obtained for the ISD variant [6] to the work-factor of less complex variants (which achieve the maximal gain for DOOM) is marginal, it is reasonable to use it as a secure lower bound.

Example. Let $n_0 = 2$, $n = 9600$, $r = 4800$, $w = 90$, $t = 84$. The non-asymptotic analysis of [6] gives a cost of $2^{92.70}$ for key-recovery and $2^{87.16}$ for decoding attacks. Decreasing it by the gains of the DOOM setting (a factor of 4800 and $\sqrt{4800}$), the final workfactors are $2^{80.47}$ and $2^{81.04}$.

6 Practical application

In this section, we provide practical parameters and discuss the practical applicability of our scheme. Table 2 summarizes the parameters for our quasi-cyclic variant, the most relevant for practical applications. For each security level, we propose three parameter sets, for $n_0 = 2$, $n_0 = 3$ and $n_0 = 4$, leading to different code rates: $1/2$, $2/3$, $3/4$, respectively.

The security assessment is based on the ISD variant [6] decreased by the possible gains obtained by the DOOM setting [36]. The codes suggested attain decoding failure rates below 10^{-7} for the QC-MDPC case, using our bit-flipping variant. Note that, for the same parameters, the MDPC variant might present an worse error correction capability due to the non-regularity of the column weights, but significant improvements can be obtained with slightly increased code-lengths.

In practice, the MDPC variant obtains huge keys of $r(n - r)$ bits, whilst the QC-MDPC allows for extremely compact keys of $(n - r)$ bits. For $n_0 = 2$, we achieved the smallest key-sizes. Note that, increasing n_0 , better code rates are obtained at the price of less compact key sizes. Table 3 provides a comparison of the key-sizes of our QC-MDPC proposal and the potential⁹ key size of the QC-LDPC variant [2], the key size of the Quasi-Dyadic Goppa McEliece variant [27] and the original McEliece scheme using updated parameters [11]. The column r also gives the syndrome size in bits.

Regarding the complexity efficiency of our proposal, the key-generation step depends only on the generation of random word(s), for the private-key, and on the product of (quasi-cyclic) blocks, for the public key. The encryption reduces to a matrix-vector product and a vectors addition. For decryption, a non-optimized C++ implementation running at an Intel Xeon CPU @3.20GHz decrypts in less than 3 milliseconds for parameters of 80-bits of security. Encryption and key-generation are reduced to simple product of (quasi-cyclic) blocks. We prefer to omit these timings since serious optimizations may lead to much better results.

Note that our system can be scaled to meet arbitrarily large security requirements. It is rather straightforward to prove that the number of errors which can be corrected by the bit flipping algorithm is of order $\frac{n(1+o(1))\ln(w(1-R))}{4w}$, where n is the code-length, w the density of the parity-check matrix, R is the rate of the code. Message recovery attacks and key recovery attacks are of the same order of complexity when w is chosen of the form $(1 + o(1))\sqrt{\frac{n \ln n \ln(1-R)}{\ln R}}$. Thus

⁹ In [2], the use of a CCA-2 secure conversion is not considered. In this case, it is allowed to have public-keys in systematic form. To have a fair comparison, here we recompute their key-sizes assuming matrices in systematic form.

Table 2. Suggested parameters. Syndrome and key size given in bits.

Level security	n_0	n	r	w	t	QC-MDPC key-size
80	2	9600	4800	90	84	4800
80	3	10752	3584	153	53	7168
80	4	12288	3072	220	42	9216
128	2	19712	9856	142	134	9856
128	3	22272	7424	243	85	14848
128	4	27200	6800	340	68	20400
256	2	65536	32768	274	264	32768
256	3	67584	22528	465	167	45056
256	4	81920	20480	644	137	61440

Table 3. Key-size comparison. Key-sizes given in bits.

Level security	QC-MDPC	QC-LDPC [2]	QD-Goppa [27]	Goppa [11]
80	4800	12096	20480	460 647
128	9856	–	32768	1 537 536
256	32768	–	65536	7 667 855

choosing an $(n, (1 - R)n, w)$ -code with w of this form allows to reach arbitrarily large security, when n goes to infinity.

7 Conclusion

We propose two McEliece variants: one from Moderate Density Parity-Check codes (MDPC) and another from quasi-cyclic MDPC codes. These codes are LDPC codes of higher density than what is usually adopted for telecommunication solutions, providing an worse correction capability. However, in cryptography, we are not necessarily interested in correcting many errors, but only a number which ensures an adequate security level, a condition satisfied by our codes. The benefits of their employment are many.

The first one refers to the security of the scheme. The McEliece cryptosystem security relies on two problems: the indistinguishability of the code family and the hardness of decoding random linear codes. Under the reasonable assumption that distinguishing a (quasi-cyclic) MDPC code from a (quasi-cyclic) random linear code amounts to being able to ascertain the existence of low weight code-words in its dual code, we show that our proposal reduces the key-distinguishing problem to the problem of decoding random linear codes. Thus the security of our McEliece variant relies only on a well studied coding-theory problem. This provides a strong argument in favor of our scheme and must be compared to the scenario for Goppa codes at the moment. Distinguishing Goppa codes is not necessarily a hard problem, as shown in [15], where a method for distinguishing high-rate Goppa codes is proposed. Although this does not represent an effective attack, it scratches the confidence on the use of algebraic codes in cryptography.

Another benefit comes from the use of a quasi-cyclic structure. In this case, our variant provides extremely compact keys, e.g. only 4800 bits for 80-bits of security. Note that the state of the art indicates that a quasi-cyclic structure, by itself, does not imply a significant improvement for an adversary. All previous attacks on McEliece schemes are based on the *combination* of a quasi-cyclic/dyadic structure with some algebraic code information. Considering the manner we generate our codes, this last ingredient simply does not exist for our variant.

Regarding its complexity efficiency, our work presents another benefit due to the use of very simple operations. The key-generation basically depends on the generation of random word(s), the encryption is performed through cyclic-block product and addition and the decryption performed by our bit-flipping variant is very simple and achieves very low complexity. In summary, MDPC codes seem to be very interesting for cryptographic purposes.

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A Computing the threshold for the Bit-Flipping algorithm

A way for estimating the waterfall threshold for the bit-flipping algorithm is considering the probability of a bit to be in error after a given number of iterations of the algorithm. When such probability converges to zero, reliable error correction can be achieved. Below we discuss the weak bound presented in [19] based on this probability.

We denote by P_i the probability of a bit be in error after i iterations of the decoding algorithm. When the code length is supposed to be infinite and that there are no cycles of length less than or equal to $2i$ in the Tanner graph associated to the parity-check matrix, this probability does not depend on a particular position [33]. These conditions can be relaxed and a finite analysis of the decoding process can be obtained, but this is beyond the scope of this work. Furthermore, practical parameters can be achieved from the refinement provided by the exhaustive simulation, ensuring an adequate decoding failure rate.

We denote by H the parity-check matrix of an (n, r, w) -MDPC code. Suppose we are verifying the convergence of P_i , when messages containing t errors are

received (thus $P_0 = \frac{t}{n}$). To describe how p_i evolves, we have to introduce some additional notation. Let m be the total number of entries equal to 1 in H . Let m_i be the total number of entries equal to 1 of H which appear in a column of weight i and let $\lambda_i \stackrel{\text{def}}{=} \frac{m_i}{m}$. Notice that m_i is also equal to i times the number of columns of weight i in H . In the quasi-cyclic case, note that $m = rw$ and $m_i = \sum_{j=0}^{n_0-1} w_j^2 \mathbf{1}_{w_j=i}$, where $\mathbf{1}_{w_j=i}$ stands for the indicator of the event $w_j = i$ (i.e. it is equal to 1 if $w_j = i$ and 0 otherwise). With this notation we have

$$p_{i+1} = p_0 - p_0 \sum_d \lambda_d \sum_{l=b_d}^{d-1} \binom{d-1}{l} \left[\frac{1 + (1-2p_i)^{w-1}}{2} \right]^l \left[\frac{1 - (1-2p_i)^{w-1}}{2} \right]^{d-l-1} \\ + (1-p_0) \sum_d \lambda_d \sum_{l=b_d}^{d-1} \binom{d-1}{l} \left[\frac{1 - (1-2p_i)^{w-1}}{2} \right]^l \left[\frac{1 + (1-2p_i)^{w-1}}{2} \right]^{d-l-1}$$

In [19], the integer b_d is chosen as an integer between $d-1$ and $d/2$ which aims at minimizing the function p_{i+1} .

$$\frac{1-p_0}{p_0} \leq \left[\frac{1 + (1-2p_i)^{w-1}}{1 - (1-2p_i)^{w-1}} \right]^{2b_d-d+1}$$

The waterfall *threshold* of an (n, r, w) -MDPC code for the original bit-flipping algorithm is obtained as the maximal integer t such that $p_0 = t/n$ and p_i converges to 0.

B Computing the work-factor of the ISD variant [6].

Let $H \in \mathbb{F}_2^{r \times n}$, $s \in \mathbb{F}_2^r$ and $k = n - r$. We are interested in finding a vector $e \in \mathbb{F}_2^n$ of weight w such that $He^T = s$. Equivalently we want to find a linear combination of w columns of H which when added to s gives a 0-vector. Below we briefly describe the algorithm proposed in [6] for solving this problem. The algorithm is divided in two steps: the setup and the search step. The former consists in randomly permute the columns of H and it proceeds with a partial Gaussian elimination on the rows of H . More precisely, let l be an optimal algorithm parameter, we compute the matrix:

$$H' = \left[\begin{array}{c|c} I^{(r-l) \times (r-l)} & Q^{r \times (k+l)} \\ \hline 0^{l \times (r-l)} & \end{array} \right]$$

where I stands to an identity block and 0 to a zero block. The second step depends on the algorithm parameter $p < w$. The value of p defines the error pattern of the sought error vector, which is vectors of: weight $w - p$ in the first $r - l$ positions and of weight p in the last $k + l$ positions. A valid strategy for finding solutions is: compute all possible linear combinations of p columns in Q and select those one which sums up to a vector coinciding in the last l positions of the syndrome. We have found a solution when the sum of such combination

plus the syndrome gives a vector of weight $w - p$. Note that the sum of each combination plus the syndrome gives a vector of weight 0 in the last l positions. Thus the weight of each combination plus the syndrome will be concentrated in the first $r - l$ positions. When this part has weight exactly $w - p$, we can add the $w - p$ columns from the identity part of H' which erase these positions. In summary, we have selected $w - p$ columns from the first $r - l$ columns of H' plus p columns from the last $k + l$ columns of H' , therefore it is a solution.

An improvement is achieved using a Meet-In-The-Middle strategy. It is convenient to compute two lists $\mathcal{L}_1, \mathcal{L}_2$ of all possible linear combinations of $p/2$ columns in Q , instead of computing all possible linear combinations of p columns in Q , taking advantage from the Birthday Paradox. Then we select the sums $\{a + b | a \in \mathcal{L}_1, b \in \mathcal{L}_2\}$ which have weight exactly p . Note that the fact of \mathcal{L}_1 and \mathcal{L}_2 be not disjoint might lead to multiple representations of the same solution. The main improvement presented in [6] is that they allow elements in \mathcal{L}_1 and \mathcal{L}_2 of weight $p/2 + \epsilon$, for some small integer ϵ . This generalizes the previous approaches. Basically they are considering also the case when ϵ positions of a are erased by ϵ positions of b (i.e. $1 + 1 = 0$ for binary codes), which still gives a sum of weight p . Actually, the authors propose to apply this strategy not only once. This leads to an algorithm which can be divided in 4 layers, we label it from 3 (the initial) until 0 (the final layer). The third layer has 4 pairs of two disjoint lists each one. The second layer has two pairs of lists. The first layer has one pair and the layer 0 has the final list. Next we describe the algorithm along with the cost for each step.

Let $p, l, p_1, p_2, \epsilon_1, \epsilon_2, r_1, r_2$ be optimal algorithm parameters such that: $p_1 = p/2 + \epsilon_1, p_2 = p_1/2 + \epsilon_2$ and $l > r_1 > r_2$. In the initial layer, we produce 4 pairs of 2 disjoint lists each one. Each list has the linear combination of $p_2/2$ columns of Q . Thus the size of each list is: $S_3 = \binom{(k+l)/2}{p_2/2}$. We develop the discussion for a pair of lists $\mathcal{L}_{3,1}$ and $\mathcal{L}_{3,2}$, but the same apply for the other three pairs.

For the next layer, we select all sums $\{a + b | a \in \mathcal{L}_{3,1}, b \in \mathcal{L}_{3,2}\}$ of weight $p_2 = p_1/2 + \epsilon_2$ and which coincide with the syndrome in the last r_2 positions. Thus the size of each list is: $S_2 = \frac{(S_3)^2}{2^{r_2}}$. Let the result be $\mathcal{L}_{2,1}$ and let $\mathcal{L}_{2,2}$ be the merge from another pair in the previous layer.

For the next layer, we select all sums $\{a + b | a \in \mathcal{L}_{2,1}, b \in \mathcal{L}_{2,2}\}$ of weight $p_1 = p/2 + \epsilon_1$ and which coincide with the syndrome in the last r_1 positions. Since all elements already coincide in the last r_2 positions, and $r_1 > r_2$, we have to discard only $2^{r_1 - r_2}$ from all possibilities obtained from $\mathcal{L}_{2,1} \times \mathcal{L}_{2,2}$. Thus the cost of merging these lists is $C_2 = \frac{(S_2)^2}{2^{r_1 - r_2}}$. Since $\mathcal{L}_{2,1}$ and $\mathcal{L}_{2,2}$ are not disjoint, we can obtain multiple representations of the same partial solution. We must proceed with a single representation of each solution. The rate of distinct solutions can be measured by:

$$\mu_2 = \frac{\binom{k+l}{\epsilon_2} \binom{k+l-\epsilon_2}{p_2-\epsilon_2} \binom{k+l-p_2}{p_2-\epsilon_2}}{\binom{k+l}{p_2}^2}$$

The maximal size of this list is $S_1^{max} = \frac{\binom{k+l}{p_1}}{2^{r_1}}$. Thus the size of the list of distinct solutions is $S_1 = \min(\mu_2 C_2, S_1^{max})$. Let the result be $\mathcal{L}_{1,1}$ and consider $\mathcal{L}_{1,2}$ be the result from the other pair in the second layer. Finally, we select all sums $\{a+b|a \in \mathcal{L}_{1,1}, b \in \mathcal{L}_{1,2}\}$ of weight p and which coincide with the syndrome in the last l positions. Since all elements already coincide in the last r_1 positions, and $l > r_1$, we have to discard only 2^{l-r_1} from all possibilities obtained from $\mathcal{L}_{1,1} \times \mathcal{L}_{1,2}$. Thus the cost of merging these lists is $C_1 = \frac{(S_1)^2}{2^{l-r_1}}$. Again, since $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ are not disjoint, we can obtain multiple representations of the same solution. We must consider a single representation of each solution. The rate of distinct solutions can be measured by:

$$\mu_1 = \frac{\binom{k+l}{\epsilon_1} \binom{k+l-\epsilon_1}{p_1-\epsilon_1} \binom{k+l-p_1}{p_1-\epsilon_1}}{\binom{k+l}{p_1}^2}$$

The maximal size of the final list is $S_0^{max} = \frac{\binom{k+l}{p}}{2^l}$. Thus the size of the final list of distinct solutions is $S_0 = \min(\mu_1 C_1, S_0^{max})$. Considering the cost for the Gaussian elimination as $K_0 = \frac{(n+1)(n-k)}{\log_2(n+1)}$ [1] and the cost of merging two lists being twice the cost of building a list (we use coefficients $K_1 = 1$ and $K_2 = 2$ to make this adjustment), the cost of each iteration (an attempt of the algorithm in finding a solution) is:

$$WF^{\text{iteration}}(n, r, w, p, l, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2) = K_0 + 8S_3K_1 + 4C_3K_2 + 2C_2K_2 + C_1K_2$$

The number of iterations that the algorithm must perform until find a solution depends on the probability of finding an error vector with the sought error pattern: vectors of weight $w-p$ in the first $r-l$ positions and p in the last $k+l$ positions. This probability is

$$P(n, r, w, p, l, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2) = \frac{\binom{n-k-l}{w-p} \binom{k+l}{p} \frac{S_0}{S_0^{max}}}{\binom{n}{w}} = \frac{\binom{n-k-l}{w-p} S_0 2^l}{\binom{n}{w}}$$

Therefore given $l, p, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2$:

$$\begin{aligned} WF(n, r, w, p, l, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2) &= P^{-1} \cdot WF^{\text{iteration}}(n, k, w, p, l, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2) \\ &= P^{-1}(K_0 + 8S_3K_1 + 4C_3K_2 + 2C_2K_2 + C_1K_2) \end{aligned}$$

There are several ways for choosing the parameters $l, p, r_1, r_2, \epsilon_1, \epsilon_2, p_1, p_2$. With some heuristic approaches, we succeeded to find good parameters, providing lower work-factors than what is obtained for the other ISD variants. Note however, for the parameters presented in Section 6, these values are still quite close from what is obtained for much simpler ISD variants.