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# Mean and variance of single photon counting with deadtime

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**Abstract.** The statistics of photon counting by systems affected by deadtime are potentially important for statistical image reconstruction methods. We present a new way of analysing the moments of the counting process for a counter system affected by various models of deadtime related to PET and SPECT imaging. We derive simple and exact expressions for the first and second moments of the number of recorded events under various models. From our mean expression for a SPECT deadtime model, we derive a simple estimator for the actual intensity of the underlying Poisson process; simulations show that our estimator is unbiased even for extremely high count rates. From this analysis, we study the suitability of the Poisson statistical model assumed in most statistical image reconstruction algorithms. For systems containing 'modules' with several detector elements, where each element can cause deadtime losses for the entire module, such as block PET detectors or Anger cameras, the Poisson statistical model appears to be adequate even in the presence of deadtime losses.

## 1. Introduction

Every photon counting system exhibits a characteristic called *deadtime*. Since the pulses produced by a detector have finite time duration, if a second pulse occurs before the first has disappeared, the two pulses will overlap to form a single distorted pulse (Sorenson and Phelps 1987). Depending on the system, one or both arrivals will be lost. In PET or SPECT scanners, the length of pulse resolving time, often just called 'deadtime', denoted $\tau$, is around $2\mu$s. Counting systems are usually classified into two categories: non-paralysable (type I) or paralysable (type II). In a non-paralysable system, each recorded photon produces a deadtime of length $\tau$; if an arrival is recorded at $t$, then any arrival from $t$ to $t + \tau$ will not be recorded. In a paralysable system, each photon arrival, whether recorded or not, produces a deadtime of length $\tau$; if there is an arrival at $t$, then any arrival from $t$ to $t + \tau$ will not be recorded. In some SPECT systems (Engeland *et al* 1998), we encounter a third model that is similar to the paralysable model: if two photons arrive within $\tau$ of each other, then neither photon will be recorded (e.g. due to pulse pile-up); we call this the type III model. The asymptotic moments of the non-paralysable model are well known (Feller 1968). For the paralysable model, the exact expression for the mean of the number of recorded events from time 0 to $t$, denoted $Y(t)$, has been derived previously (Carloni *et al* 1970). However, for the type III model, only an approximate expression for the mean number of recorded events has been derived (Engeland *et al* 1998). In this paper, we derive the exact mean and variance expressions of $Y(t)$ for both type II and type III models. Figure 1 illustrates the three types of system.

This investigation of deadtime statistics was originally motivated by the goal of finding appropriate statistical models for image reconstruction of PET and SPECT scans with

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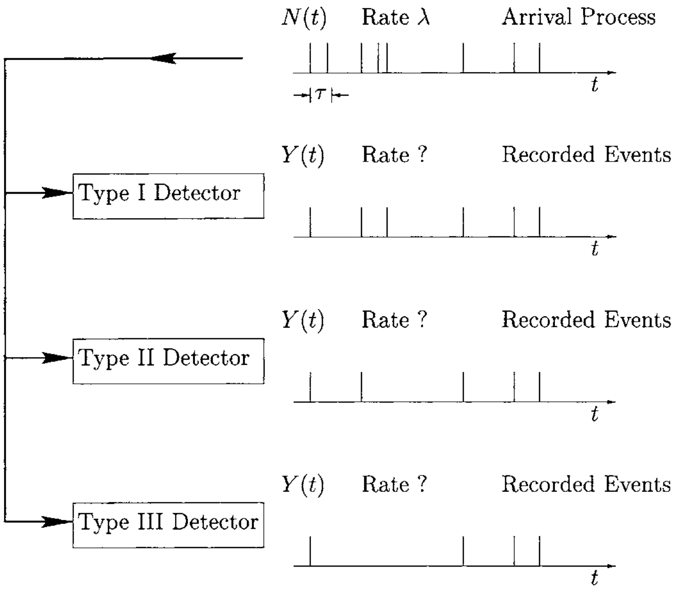



**Figure 1.** Illustration of systems affected by three types of deadtime.

high deadtime losses. There are four natural choices for dealing with deadtime in image reconstruction:

(a) ignore it altogether;
(b) correct the number of recorded events for deadtime losses and plug the corrected data into the reconstruction algorithm;
(c) incorporate deadtime losses into the system matrix of the usual Poisson statistical model;
(d) develop reconstruction algorithms based on the exact statistics of the counting process.

For a quantitatively accurate reconstruction, we must correct for the effect of deadtime to avoid underestimation of source activity. This consideration rules out the first choice. Previous work (Stearns *et al* 1985, Daube-Witherspoon and Carson 1991, Mazoyer *et al* 1985, Yamamoto *et al* 1986, Tai *et al* 1998) in this field usually involves the second choice, i.e. using the method of moments to correct the sinograms for deadtime losses, and reconstructing the image using these corrected counts. In statistical image reconstruction, it is generally assumed that the number of recorded events at a detector is Poisson distributed. However, in the presence of deadtime, the fact that there can be no recorded events within $\tau$ of each other makes the counting process non-Poissonian (Knoll 1989). However, if the process is approximately Poissonian, then a simple modification of the system matrix, i.e. correct the elements of the system matrix, $a_{ij}$, by the deadtime loss factor, should suffice. This is the third choice as listed above, which would yield estimates with lower variance than plugging the *corrected* counts into a statistical reconstruction algorithm with an *uncorrected* system matrix. But simply correcting the number of recorded events or building this as a 'loss factor' in the system model while assuming that the number of recorded events is Poisson distributed may be suboptimal. In this paper, we investigate not only the mean, but also the variance of the number of recorded events. If the mean and variance disagree significantly, then reconstructions based on a Poissonian statistical model would have suboptimally large variances. We discuss this further in section 6 after we derive the exact mean and variance for the counting process.

## 2. Statistical analysis of deadtime

We define a 'photon arrival' to mean a photon interacting with the scintillator with sufficient deposited energy to trigger detection. The photon arrival process $N(t)$ counts the number of arrivals during the time interval $(0, t]$, and the photon recording process $Y(t)$ counts the number of recorded events. For simplicity, we assume that $N(t)$ is a homogeneous Poisson process with constant rate $\lambda$ (photon arrivals per unit time) i.e. we neglect radioisotope decay and other physical or physiological effects that may cause a variable arrival rate (see appendix C for a generalization). We first review a few simple and useful facts about the Poisson process (Feller 1968). The increment $N(t_2) - N(t_1)$, which is the number of photon arrivals during the time interval $(t_1, t_2]$, is Poisson distributed with mean $(t_2 - t_1)\lambda$. $N(t)$ has stationary and independent increments. If $T_n$ denotes the time of the $n$th photon arrival, then the waiting time (or inter-arrival time) $W_n = T_n - T_{n-1}$ is exponentially distributed with mean $1/\lambda$.

For simplicity, we also assume that the deadtime $\tau$ is known and deterministic. Most systems can be adequately modelled to have a constant deadtime, independent of count rate.

### *2.1. Asymptotic analysis via renewal theory*

The counting processes in all three types of systems discussed above are examples of 'renewal processes' (Feller 1968), and renewal theory has been the classical basis for deadtime analysis (Libert 1978, Müller 1973, 1974, Faraci and Pennisi 1983). A renewal process involves recurrent patterns $\mathcal{E}$ after each of which the process starts from scratch. One can view a counting process from this perspective by defining $\mathcal{E}$ to be the state† of 'the counter is ready to record the next photon arrival', and $T_{\mathcal{E}}$ to be the waiting time between one renewal and the next (renewal here means return to $\mathcal{E}$). With $\mathcal{E}$ defined as above, the number of renewals from 0 to $t$ is almost‡ exactly the number of recorded events from 0 to $t$. If $T_{\mathcal{E}}$ has ensemble mean $\mu_{\mathcal{E}}$ and variance $\sigma_{\mathcal{E}}^2$, then the number of renewals from 0 to $t$, $\tilde{Y}(t)$, is asymptotically Gaussian distributed (Cox 1962, Feller 1968) with the following moments:

$$E[\tilde{Y}(t)] \sim t/\mu_{\mathcal{E}} \qquad \mathrm{Var}[\tilde{Y}(t)] \sim t\sigma_{\mathcal{E}}^2/\mu_{\mathcal{E}}^3 \tag{1}$$

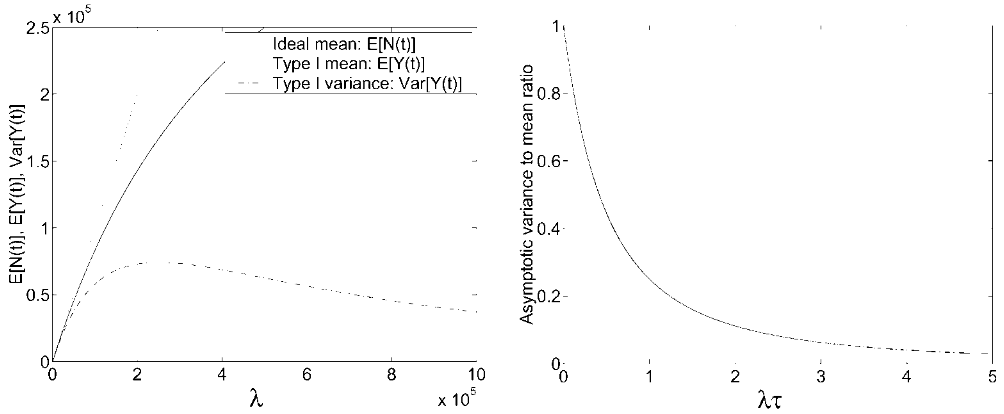
where $\sim$ indicates that the ratio of the two sides tends to unity as $t/\mu_{\mathcal{E}} \to \infty$. We observe that when $\tau = 0$, i.e. no deadtime, $T_{\mathcal{E}}$ is exponentially distributed with mean $1/\lambda$ and variance $1/\lambda^2$; thus $E[\tilde{Y}(t)] \sim \lambda t$ and $\mathrm{Var}[\tilde{Y}(t)] \sim \lambda t$, as expected since $\tilde{Y}(t)$ would be Poisson distributed with mean $\lambda t$ when there is no deadtime. In realistic cases where deadtime loss becomes significant, $\mu_{\mathcal{E}}$ is usually very small when compared to $t$, hence the Gaussian approximation is often very good.

For the non-paralysable deadtime model (type I model), it is easy to derive the asymptotic mean and variance of $\tilde{Y}(t)$ from the moments of $T_{\mathcal{E}}$. After each recording of an event, the 'deadtime' when the system cannot record any incoming arrival is simply $\tau$. Thus $T_{\mathcal{E}} = T + \tau$, where $T$ is an exponentially distributed random variable with mean $1/\lambda$. Hence, $\mu_{\mathcal{E}} = 1/\lambda + \tau = \frac{1+\lambda\tau}{\lambda}$ and $\sigma_{\mathcal{E}} = 1/\lambda$. Thus from (1), the counting process for a non-paralysable (type I) system is asymptotically Gaussian distributed with

$$E[\tilde{Y}(t)] \sim \frac{\lambda t}{1 + \lambda\tau} \qquad \mathrm{Var}[\tilde{Y}(t)] \sim \frac{\lambda t}{(1 + \lambda\tau)^3}. \tag{2}$$

† For type III deadtime, we define renewal as 'return to $\mathcal{E}$ after recording an event'.

‡ Almost since we have to consider photons arriving shortly before time 0 (or $t$) but renewal occurring shortly after time 0 (or $t$). If one redefines the time of a recorded event to be $\tau$ after the photon arrives at the detector, then the number of recorded events and the number of renewals during $(0, t]$ would be exactly the same. For stationary increment processes, which definition one adopts makes absolutely no difference in terms of the statistics of the process.




(a) (b)

**Figure 2.** Mean and variance for non-paralysable (type I) systems, with $t = 1$ s, $\tau = 2$ $\mu$s.

Figure 2 shows the mean and variance of the counting process of systems affected by non-paralysable deadtime. When $\lambda\tau > 0.1$, the mean and variance of $\tilde{Y}(t)$ differ by at least 20%.

For the other two deadtime models, if we try to derive $E[Y(t)]$ from $E[T_{\mathcal{E}}]$, it is much more difficult to obtain a simple closed form expression because if we try to derive $E[T_{\mathcal{E}}]$, we get an infinite sum and it is not easy to obtain every term in this sum, let alone a closed-form expression for $E[T_{\mathcal{E}}]$. The variance of $T_{\mathcal{E}}$ is even more complicated. Therefore, in the following section, we describe a new approach for deriving the moments of counting processes.

### 2.2. *Exact mean and variance of counting processes*

We first consider a general counting process $Y$ where $Y(t_1, t_2)$ denotes the number of recorded events during the time interval $(t_1, t_2]$ and $Y(t)$ is a shorthand for $Y(0, t)$. We define the instantaneous rate $\gamma : \mathbb{R} \to [0, \infty)$ of the process $Y(t)$ as

$$\gamma(s) \triangleq \lim_{\delta \to 0} E[Y(s + \delta) - Y(s)]/\delta \tag{3}$$

and the instantaneous second moment $\alpha : \mathbb{R} \to [0, \infty)$ as

$$\alpha(s) \triangleq \lim_{\delta \to 0} E[(Y(s + \delta) - Y(s))^2]/\delta. \tag{4}$$

We also define the correlation function $\beta : \mathbb{R}^2 \to [0, \infty)$ as

$$\beta(s_1, s_2) \triangleq \lim_{\delta_1, \delta_2 \to 0} E[(Y(s_1 + \delta_1) - Y(s_1))(Y(s_2 + \delta_2) - Y(s_2))]/(\delta_1 \delta_2). \tag{5}$$

We assume that the following regularity conditions hold§:

(a) $\gamma$ and $\alpha$ are well-defined $\mu$-almost everywhere, and $\beta$ is well defined $\mu_2$-almost everywhere, and $\gamma$ and $\beta$ are integrable with respect to $\mu$ and $\mu_2$ over any finite interval and rectangle, respectively.
(b) $E[Y(s, s + \delta)]/\delta$ and $E[Y^2(s, s + \delta)]/\delta$ are uniformly bounded for all $s$ and $\delta \in (0, 1)$.
(c) $E[Y(s_1, s_1 + \delta_1)Y(s_2, s_2 + \delta_2)]/(\delta_1 \delta_2)$ is uniformly bounded for all $s_1, s_2$, and $\delta_1, \delta_2 \in (0, 1)$ such that $(s_1, s_1 + \delta_1) \cap (s_2, s_2 + \delta_2) = \emptyset$.

These assumptions hold for a wide variety of counting processes, including any homogeneous Poisson process with finite intensity. Furthermore, for an arbitrary random process $Y$, if

§ $\mu$ and $\mu_2$ denote Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

$E[Y(s, s+\delta)]/\delta$, $E[Y^2(s, s+\delta)]/\delta$, and $E[Y(s_1, s_1+\delta_1)Y(s_2, s_2+\delta_2)]/(\delta_1\delta_2)$ are respectively uniformly bounded above by those of a homogeneous Poisson process, then assumptions (b) and (c) hold for $Y$. Specifically, if a random process results from some form of selection from a Poisson process with bounded intensity, then assumptions (b) and (c) hold.

For analysis purposes, we artificially divide the time interval $[0, t]$ into $n$ segments of length $\delta$ each, i.e. $t = n\delta$. We have

$$Y(t) = \sum_{i=0}^{n-1} Y(i\delta, (i+1)\delta) \tag{6}$$

$$E[Y(t)] = \sum_{i=0}^{n-1} E[Y(i\delta, (i+1)\delta)] \tag{7}$$

$$= \int_{\mathbb{R}} f_\delta(s)\,\mathrm{d}s \tag{8}$$

where we define the following piecewise constant function:

$$f_\delta(s) \triangleq \begin{cases} E[Y(j\delta, (j+1)\delta)]/\delta & \text{if } s \in (j\delta, (j+1)\delta], 0 \leqslant j \leqslant n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Since $\gamma(t)$ is well defined almost everywhere in the interval $[0, t]$ and $E[Y(s, s+\delta)]/\delta$ is uniformly bounded, by the Lebesgue dominated convergence theorem (LDCT) (Bruckner *et al* 1997)

$$\lim_{\delta\to 0}\int_{\mathbb{R}} f_\delta(s)\,\mathrm{d}\mu(s) = \int_{\mathbb{R}} \lim_{\delta\to 0} f_\delta(s)\,\mathrm{d}\mu(s)$$
$$= \int_0^t \gamma(s)\,\mathrm{d}s. \tag{10}$$

Hence, we have the following simple general expression for the mean of the counting process in terms of its instantaneous rate‖:

$$E[Y(t)] = \int_0^t \gamma(s)\,\mathrm{d}s. \tag{11}$$

We consider the second moment by a similar argument:

$$E[Y^2(t)] = E\left[\left(\sum_{i=0}^{n-1} Y(i\delta, (i+1)\delta)\right)^2\right]$$
$$= \sum_{i=0}^{n-1} E[Y^2(i\delta, (i+1)\delta))] + \sum_{i=0}^{n-1}\sum_{j=0, j\neq i}^{n-1} E[Y(i\delta, (i+1)\delta)Y(j\delta, (j+1)\delta)]$$
$$= \sum_{i=0}^{n-1} E[Y^2(i\delta, (i+1)\delta))]$$
$$+ 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} E[Y(i\delta, (i+1)\delta)Y(j\delta, (j+1)\delta)]$$
$$= \int_{\mathbb{R}} g_\delta(s)\,\mathrm{d}\mu(s) + 2\int_{\mathbb{R}^2} h_\delta(s_1, s_2)\,\mathrm{d}\mu_2(s_1, s_2) \tag{12}$$

‖ If $E[Y(t)]$ is differentiable for all $t$, then $\gamma(t) = \frac{\mathrm{d}E[Y(t)]}{\mathrm{d}t}$, and (11) results from the fundamental theorem of calculus. However, $E[Y(s)Y(t)]$ is not everywhere differentiable even for very simple random processes, e.g. for the Poisson process $N$ with intensity $\lambda$, $E[N(s)N(t)] = \lambda\min(s, t) + \lambda^2 st$. So a similar argument involving the fundamental theorem of calculus runs into difficulties for the second moment.

where we define the following piecewise constant functions:

$$g_\delta(s) \triangleq \begin{cases} E[Y^2(j\delta, (j+1)\delta)]/\delta & \text{if } s \in (j\delta, (j+1)\delta] \text{ and } 0 \leqslant j \leqslant n-1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and

$$h_\delta(s_1, s_2) \triangleq \begin{cases} E[Y(i\delta, (i+1)\delta)Y(j\delta, (j+1)\delta)]/\delta^2 & \text{if } s_1 \in (i\delta, (i+1)\delta] \\ & s_2 \in (j\delta, (j+1)\delta] \\ & 0 \leqslant i \leqslant n-2 \\ & \text{and } i+1 \leqslant j \leqslant n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Since $\beta$ is well-defined almost everywhere in $[0, t] \times [0, t]$ and $E[Y(s_1, s_1+\delta)Y(s_2, s_2+\delta)]/\delta^2$ is uniformly bounded, by LDCT and Fubini's theorem (Bruckner *et al* 1997),

$$\begin{aligned} \lim_{\delta\to 0} \int_{\mathbb{R}^2} h_\delta(s_1, s_2)\,\mathrm{d}\mu_2(s_1, s_2) &= \int_{\mathbb{R}^2} \lim_{\delta\to 0} h_\delta(s_1, s_2)\,\mathrm{d}\mu_2(s_1, s_2) \\ &= \int_0^t \int_{s_1}^t \beta(s_1, s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_0^t \int_{s_2}^t \beta(s_1, s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2. \end{aligned} \tag{15}$$

Similarly, one can show that

$$\lim_{\delta\to 0} \int_{\mathbb{R}} g_\delta(s)\,\mathrm{d}\mu(s) = \int_0^t \alpha(s)\,\mathrm{d}s. \tag{16}$$

Thus using (12), (15) and (16), we have the following general expression for the second moment of $Y(t)$:

$$E[Y^2(t)] = \int_0^t \alpha(s)\,\mathrm{d}s + 2\int_0^t \int_{s_1}^t \beta(s_1, s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{17}$$

In the context of counting processes with deadtime, which includes all random processes considered in this paper, the process satisfies this additional assumption:

(d) there exists a positive $\delta_0$ such that $\forall\delta \in (0, \delta_0)$, $Y(s, s+\delta) \leqslant 1$.

If we pick $\delta_0 < \tau$, then assumption (d) holds. For $\delta < \delta_0$, since $0^2 = 0$ and $1^2 = 1$

$$E[Y^2(s, s+\delta)] = E[Y(s, s+\delta)] \tag{18}$$

so

$$\alpha(s) = \gamma(s). \tag{19}$$

Thus we obtain the following corollary of (17) for random processes satisfying assumptions (a) to (d):

$$E[Y^2(t)] = E[Y(t)] + 2\int_0^t \int_{s_1}^t \beta(s_1, s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{20}$$

Furthermore, if $Y(t)$ has stationary increments, then $\gamma(s)$ is constant and $\beta(s_1, s_2) = \beta(0, s_2 - s_1)$ and we can further simplify the results (11) and (20) to the following:

$$E[Y(t)] = \gamma t \tag{21}$$

$$E[Y^2(t)] = \gamma t + 2\int_0^t (t-s)\beta(0, s)\,\mathrm{d}s. \tag{22}$$

The above general approach used to find the second moment of $Y(t)$ could be extended to higher-order moments. However, as the order gets higher, the expressions get more complicated.

## 3. Single photon counting

### *3.1. Mean and variance of recorded singles counts, model type II*

First we consider the paralysable model: if the waiting time for a photon arrival is less than $\tau$, then this photon is not recorded. We derive the mean and variance of $Y(t)$, the number of recorded events from time 0 to time $t$. We observe that $Y(t)$ inherits the stationary increment property of the arrival process $N(t)$. We first derive $E[Y(0, \delta)]$, where we pick $\delta < \tau$ such that the number of recorded events during $(0, \delta]$ is either 0 or 1. Let $T_1$ denote the time of the first photon arrival after time 0; it is exponentially distributed. If there is an arrival at $T_1 = s$, $0 < s < \delta$, and there is no arrival between $s - \tau$ and $s$ (in fact, we only need to make sure there is no arrival between $s - \tau$ and 0, i.e. $N(0) - N(s - \tau) = 0$, since the first arrival after 0 occurs at $s$), then there will be a recorded event during the interval $(0, \delta]$. Thus

$$\begin{aligned} E[Y(0,\delta)] &= P[Y(0,\delta) = 1] \\ &= \int_0^\infty P[Y(0,\delta) = 1 | T_1 = s] f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\delta P[\text{no arrival during } (s-\tau, 0) | T_1 = s] f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\delta P[N(s-\tau, 0) = 0 | T_1 = s] f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\delta \mathrm{e}^{-\lambda(\tau - s)} \lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s = \int_0^\delta \lambda\,\mathrm{e}^{-\lambda\tau}\,\mathrm{d}s = \lambda\delta\,\mathrm{e}^{-\lambda\tau}. \end{aligned} \tag{23}$$

Hence by the definition given in (3), the instantaneous rate of $Y(t)$ is

$$\gamma = \lambda\,\mathrm{e}^{-\lambda\tau} \tag{24}$$

and by (21), we easily obtain the following result (e.g. Sorenson and Phelps 1987),

$$E[Y(t)] = \lambda t\,\mathrm{e}^{-\lambda\tau} \tag{25}$$

i.e. the recorded/arrival ratio for type II systems, denoted $\xi_2$, is

$$\xi_2 \triangleq \frac{E[Y(t)]}{E[N(t)]} = \mathrm{e}^{-\lambda\tau}. \tag{26}$$

The variance of $Y(t)$ for the type II model is (see appendix A):

$$\mathrm{Var}[Y(t)] = \lambda t\,\mathrm{e}^{-\lambda\tau}[1 - (2\lambda\tau - \lambda\tau^2/t)\,\mathrm{e}^{-\lambda\tau}]. \tag{27}$$

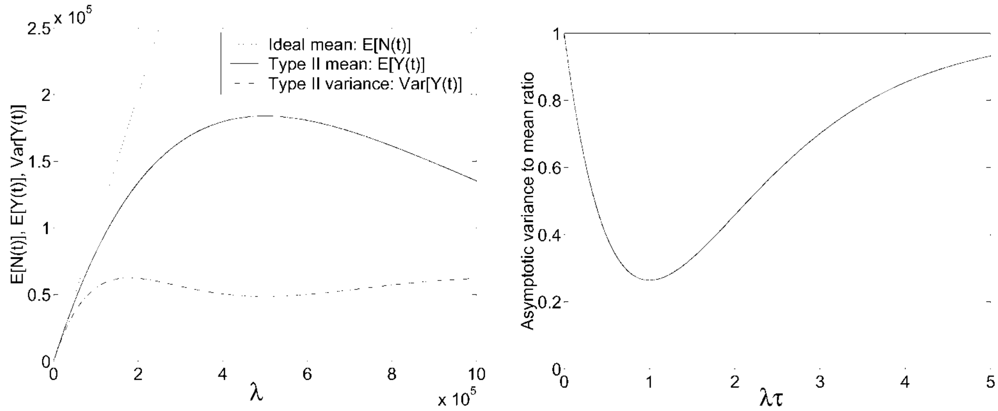
We can compute numerically that $\max_{\lambda\tau}(2\lambda\tau\,\mathrm{e}^{-\lambda\tau}) \approx 0.74$, hence $\mathrm{Var}[Y(t)]$ will always be positive. To compare the variance and the mean, we note that

$$\lim_{t\to\infty} \frac{\mathrm{Var}[Y(t)]}{E[Y(t)]} = 1 - 2\lambda\tau\,\mathrm{e}^{-\lambda\tau} = 1 - 2\xi_2 \log \xi_2. \tag{28}$$

Figure 3 shows the mean and variance of the singles count for a detector affected by deadtime of type II. Since the mean and variance can differ greatly, $Y(t)$ is not Poisson.

### *3.2. Mean and variance of recorded singles counts, model type III*

Now we turn to the type of system described in Engeland *et al* (1998): if the waiting time for a photon arrival is less than $\tau$, then neither this photon nor the previous photon will be recorded. We again observe that $Y(t)$ inherits the stationary increment property of the arrival




(a) (b)

**Figure 3.** Mean and variance for paralysable (type II) systems, with $t = 1$s, $\tau = 2\mu$s.

process $N(t)$. We first derive $E[Y(0, \delta)]$, where we pick $\delta < \tau$ such that the number of recorded events during $(0, \delta]$ is still either 0 or 1. Hence

$$\begin{aligned} E[Y(0,\delta)] &= P[Y(0,\delta) = 1] \\ &= \int_0^\delta P[Y(0,\delta) = 1 \mid T_1 = s] f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\delta P[N(s-\tau, 0) = 0] P[(s, s+\tau) = 0] f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\delta \mathrm{e}^{-\lambda(\tau - s)}\,\mathrm{e}^{-\lambda\tau}\lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s = \int_0^\delta \lambda\,\mathrm{e}^{-\lambda 2\tau}\,\mathrm{d}s = \lambda\delta\,\mathrm{e}^{-\lambda 2\tau}. \end{aligned} \quad (29)$$

Hence for this system, the instantaneous rate as defined in (3) is

$$\gamma = \lambda\,\mathrm{e}^{-\lambda 2\tau} \quad (30)$$

and by (21), the expected number of recorded events for a type III system is exactly

$$E[Y(t)] = \lambda t \mathrm{e}^{-\lambda 2\tau}. \quad (31)$$

The type III system was analysed using approximations in Engeland *et al* (1998). To compare our exact result (31) with the approximate analysis presented in Engeland *et al* (1998), we note that the mean waiting time between recorded events is

$$\mu_{\mathcal{E}} = t/E[Y(t)] = \frac{1}{\lambda}\,\mathrm{e}^{\lambda 2\tau} \quad (32)$$

$$= \frac{1}{\lambda}\left[1 + 2\lambda\tau + 2(\lambda\tau)^2 + \frac{4}{3}(\lambda\tau)^3 + \frac{2}{3}(\lambda\tau)^4 + O(\lambda\tau)^5\right]. \quad (33)$$

Comparing this exact expansion to the approximate mean waiting time derived in Engeland *et al* (1998, equation (16)), we find that the approximation in Engeland *et al* (1998) is accurate to second order.

The variance of $Y(t)$ for the type III model is (see appendix B):

$$\mathrm{Var}[Y(t)] = \lambda t\,\mathrm{e}^{-\lambda 2\tau} + 2\,\mathrm{e}^{-3\lambda\tau}(\lambda t - \lambda\tau - 1) + \mathrm{e}^{-4\lambda\tau}(4\lambda^2\tau^2 - 4\lambda^2 t\tau + 2 - 2\lambda t + 4\lambda\tau). \quad (34)$$

To compare the variance and the mean, we observe that

$$\lim_{t\to\infty} \frac{\mathrm{Var}[Y(t)]}{E[Y(t)]} = 1 - 2(1 + 2\lambda\tau - \mathrm{e}^{\lambda\tau})\,\mathrm{e}^{-2\lambda\tau}. \quad (35)$$

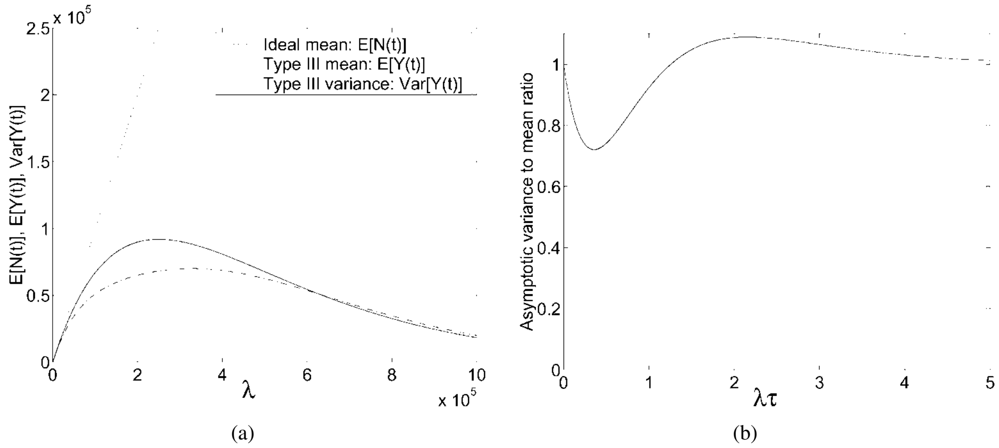



**Figure 4.** Mean and variance for type III systems, with $t = 1$ s, $\tau = 2$ $\mu$s.

To simplify this expression, we observe that when $\lambda\tau \ll 1$, $\mathrm{e}^{\lambda\tau} - 1 \approx \lambda\tau$, and

$$\lim_{t\to\infty} \frac{\mathrm{Var}[Y(t)]}{E[Y(t)]} \approx 1 - 2\lambda\tau\,\mathrm{e}^{-2\lambda\tau} = 1 - \xi_3 \log \xi_3 \tag{36}$$

where $\xi_3 \triangleq E[Y(t)]/E[N(t)] = \mathrm{e}^{-2\lambda\tau}$. Figure 4 shows the (exact) mean and variance of the singles count $Y(t)$ for type III systems. Again $Y(t)$ is not Poissonian, but the difference between the variance and the mean is much smaller than for type I or type II systems.

## 4. Recorded singles counts by block detectors

In many photon counting systems, several detectors are grouped into a 'block'; examples include block PET detectors and Anger cameras. When a photon arrives at any detector in the block, the whole block goes dead for $\tau$, i.e. no detector in the block can record any photon for $\tau$. For analysis purposes, we can initially treat the block of detectors as a single big detector. Let $\lambda_1, \ldots, \lambda_l$ denote the incident photon arrival rates for each of the $l$ detectors in the block. Let $Y_j(t)$ denote the number of events recorded by the $j$th detector, and let $Z(t)$ denote the total number of events recorded by all detectors in the block ($Z = \sum_{j=1}^{l} Y_j$). We have derived above the exact first and second moments of $Z(t)$ for detector blocks affected by type II and type III deadtime, and in each case the mean and the variance of $Z(t)$ can differ greatly. However, what is of greater interest in image reconstruction is the mean and variance of the number of events recorded by each detector in the block. Given that $Z(t)$ events are recorded by the entire block, the conditional distribution of the number of events recorded by any individual detector is multinomial where the fraction of events allotted to the $j$th detector is $\eta_j \triangleq \lambda_j/\lambda$. Thus from Barrett and Swindell (1981, p 99)

$$E[Y_j(t)] = \eta_j E[Z(t)] \tag{37}$$

$$\mathrm{Var}[Y_j(t)] = \eta_j(1 - \eta_j)E[Z(t)] + \eta_j^2\,\mathrm{Var}[Z(t)]. \tag{38}$$

We observe that the variance to mean ratio is

$$\frac{\mathrm{Var}[Y_j(t)]}{E[Y_j(t)]} = 1 - \eta_j(1 - \mathrm{Var}[Z(t)]/E[Z(t)]) \tag{39}$$

$$\geqslant 1 - \eta_j. \tag{40}$$

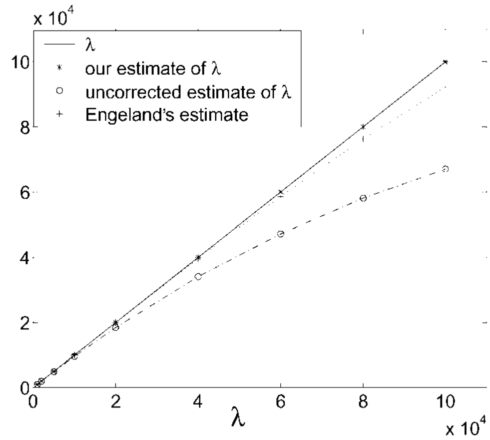



**Figure 5.** 20 realizations, with $t = 10$ s, $\tau = 2$ $\mu$s.

For a system with, say, 64 detectors in a block, $\eta_j \approx 1/64$ (assuming that the count rates $\lambda_j$ are nearly uniform), so from (40) the mean and the variance of the number of recorded events by a single detector will differ by less than 2%, regardless of count rates and deadtime losses. Furthermore, since $E[Z(t)]$ must be quite large for deadtime to have a significant effect, when $\eta_j$ is small, the distribution of $Y_j(t)$ will be approximately Poissonian by the usual binomial argument. The only case where the variance to mean ratio is significantly less than 1 would be when $\eta_j$ is large (i.e. the count rates $\lambda_j$ are very heterogeneous) and $\text{Var}[Z(t)]/E[Z(t)]$ is small (i.e. the total count rate $\sum_{j=1}^{l} \lambda_j$ is large). In all other cases, the mean and the variance would be approximately equal. However, the *covariance* between the measurements recorded by different elements within the block can be non-zero (Barrett and Swindell 1981, p 101):

$$\text{Cov}(Y_i(t), Y_j(t)) = \eta_i \eta_j (\text{Var}[Z(t)] - E[Z(t)]). \tag{41}$$

Thus in the presence of deadtime, the assumption that the measurements are independent (which is made ubiquitously in statistical reconstruction methods) is incorrect. However, when $\eta_i$ and $\eta_j$ are small, so is the covariance between individual detector elements, so the impact of this dependence may be small.

## 5. Count rate correction for system type III

For a quantitatively accurate reconstruction, we must correct for the effect of deadtime to avoid underestimation of source activity. For type III systems, Engeland *et al* (1998) proposed the following correction formula

$$\hat{\lambda} = \frac{Y}{t}\left(1 + \frac{2Y}{t}\tau + \frac{6Y}{t^2}\tau^2\right) \tag{42}$$

which they obtained by solving an approximate mean waiting time expression up to second order in $\tau$ by means of the expansion $\lambda = a + b\tau + c\tau^2$. We propose to estimate the true count rate by solving numerically our exact expression (31), i.e. solve

$$\frac{Y}{t} = \hat{\lambda}\, e^{-2\hat{\lambda}\tau} \tag{43}$$

for $\hat{\lambda}$ given $Y$ and $t$. One could solve analytically the exact mean waiting time, expression (32), up to second order in $\tau$, which yields exactly the same estimator as (42), but this estimator does not solve (32) exactly. Figure 5 compares our new estimator (43) and the estimator proposed in Engeland *et al* (1998). It shows that our new estimator is unbiased even at very high count rates. The error bars are not shown in the figure as they are smaller than the plotting symbols. When $t$ is large, the standard deviation is very small when compared with the mean of $Y(t)$, thus these estimates have extremely small standard deviations. By solving (43) numerically, we obtain essentially perfect deadtime correction for a type III system.

## 6. Discussion

We have analysed the mean and variance of the recorded singles counts for three distinct models of deadtime. In all three cases, the variance can be significantly less than the mean, indicating that the counting statistics are not Poissonian in the presence of deadtime. Deadtime losses can be significant in practical SPECT and PET systems, particularly in fully 3D PET imaging and in SPECT transmission measurements with a scanning line source. The count rates for a detector block (PET) or detector zone (SPECT) can be significant enough to yield non-Poissonian statistics for the total counts recorded by the block or zone. However, in the practical situations that we are aware of, the count rates for individual detector elements within the block or zone are usually not high enough to correspond to significant differences between the mean and the variance. As we have shown in section 4, even though the variance of the counts recorded by a block can be significantly lower than the mean, the variance of the counts recorded by an individual detector within a block is nevertheless quite close to the mean and likely to be well approximated by a Poisson distribution. Furthermore, the correlation between individual detectors will be fairly small. Thus it appears that statistical image reconstruction based on Poisson models, while certainly not optimal, should be adequate in practice even under fairly large deadtime losses, provided the deadtime loss factor is included in the system matrix. We must add one caveat to this conclusion, however. Although pairs of individual detectors have small correlation, the correlation coefficient between the *sum* of one group of detectors and the *sum* of all other detectors in a block may not be small in the presence of deadtime. The effect of such correlations on image reconstruction algorithms is unknown and may deserve further investigation. Another natural extension of this work would be to consider systems with random resolving times $\tau$. As long as the minimum resolving time is greater than zero, assumption (d) would still hold and the derivations would be similar.

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## Appendix A

We derive the variance of $Y(t)$ for deadtime model II, the paralysable model. We first derive $\beta(0, s)$. We consider two cases:

- Case 1: $0 < s < \tau$: We pick $\delta$ such that $0 < \delta < s < s + \delta < \tau$. Two recorded events cannot correspond to photons that arrived within $\tau$ of each other. Hence for $0 < s < \tau$, $E[Y(0, \delta)Y(s, s + \delta)] = 0$, and, by the definition given in (3), $\beta(0, s) = 0$.
- Case 2: $\tau < s < t$: We pick $\delta$ such that $\delta < \tau$ and $s + \delta < t$ and $\delta < s - \tau$. For $s > \tau$, $Y(0, \delta)$ and $Y(s, s + \delta)$ are statistically independent, since the event ‘there is an

arrival during $(0, \delta]$' is statistically independent from the event 'there is an arrival during $(s, s+\delta]$', because they are at least $\tau$ apart in time¶. Hence by (23),

$$E[Y(0,\delta)Y(s,s+\delta)] = E^2[Y(0,\delta)] = (\lambda\delta\, \mathrm{e}^{-\lambda\tau})^2 \tag{44}$$

and

$$\beta(0,s) = (\lambda\, \mathrm{e}^{-\lambda\tau})^2. \tag{45}$$

Combining the above two cases and using (22) yields

$$\begin{aligned} E[Y^2(t)] &= \gamma t + 2\int_\tau^t (t-s)(\lambda\, \mathrm{e}^{-\lambda\tau})^2\, \mathrm{d}s \\ &= \lambda t\, \mathrm{e}^{-\lambda\tau} + [(t-\tau)(\lambda\, \mathrm{e}^{-\lambda\tau})]^2. \end{aligned} \tag{46}$$

Using $\mathrm{Var}[Y(t)] = E[Y^2(t)] - E^2[Y(t)]$ with (25) and (46), and simplifying, yields (27).

## Appendix B

We derive the variance of $Y(t)$ for the type III deadtime model. Again, we first derive the correlation function $\beta(0,s)$. This derivation is more complicated than the type II model, due to the fact that if two photons arrive at times $s_1$ and $s_2$ respectively and $\tau < s_2 - s_1 < 2\tau$, then $(s_1-\tau, s_1+\tau) \cap (s_2-\tau, s_2+\tau) \neq \emptyset$ and $Y(s_1, s_1+\delta)$ and $Y(s_2, s_2+\delta)$ would both depend on what happens during $(s_2-\tau, s_1+\tau)$.

- Case 1: $0 < s < \tau$: We pick $\delta$ such that $0 < \delta < s < s+\delta < \tau$. Two recorded events cannot correspond to photons that arrived within $\tau$ of each other. Hence for $0 < s < \tau$, $E[Y(0,\delta)Y(s,s+\delta)] = 0$, and $\beta(0,s) = 0$.
- Case 2: $\tau < s < 2\tau$: We pick $\delta$ such that $s+\delta < 2\tau$ (hence $\delta < \tau$) and $\delta < s-\tau$. As discussed above, for $\tau < s < 2\tau$, $Y(0,\delta)$ and $Y(s,s+\delta)$ will be statistically dependent. If there is exactly one photon arrival each during $(0,\delta]$ and $(s,s+\delta]$ at time $s_1$ and $s_2$ respectively, then both events will be recorded if and only if there is no arrival during $(s_1-\tau, s_1)$, $(s_1, s_2)$, or $(s_2, s_2+\tau]$ (since $\tau < s_2 - s_1 < 2\tau$, $(s_1, s_1+\tau] \cup (s_2-\tau, s_2) = (s_1, s_2)$.) Hence

$$\begin{aligned} E[Y(0,\delta)Y(s,s+\delta)] &= P[Y(0,\delta)=1, Y(s,s+\delta)=1] \\ &= \int_0^\delta\int_0^\delta P[\text{no arrival during}(s_1-\tau,0),\ \text{or } (s_1,s),\ \text{or } (s_2,s_2+\tau]] \\ &\quad\times f_{T_1}(s_1) f_{T_1}(s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2 \\ &= \int_0^\delta\int_0^\delta \mathrm{e}^{-\lambda(\tau-s_1)}\, \mathrm{e}^{-\lambda(s-s_1)}\, \mathrm{e}^{-\lambda\tau}\lambda\, \mathrm{e}^{-\lambda s_1}\, \lambda \mathrm{e}^{-\lambda s_2}\, \mathrm{d}s_1\, \mathrm{d}s_2 \\ &= \mathrm{e}^{-\lambda 2\tau}\lambda^2\, \mathrm{e}^{-\lambda s}\int_0^\delta\int_0^\delta \mathrm{e}^{\lambda s_1}\, \mathrm{e}^{-\lambda s_2}\, \mathrm{d}s_1\, \mathrm{d}s_2 \\ &= (\mathrm{e}^{\lambda\delta}-1)^2\, \mathrm{e}^{-\lambda(2\tau+s+\delta)} \end{aligned} \tag{47}$$

and

$$\beta(0,s) = \lambda^2\, \mathrm{e}^{-\lambda(2\tau+s)}. \tag{48}$$

¶ If there is one arrival each during $(0,\delta]$, $(s/2, s/2+\delta]$, and $(s, s+\delta]$, then $Y(0,\delta)Y(s,s+\delta) = 0$; but loss of the photon that arrived during $(s, s+\delta]$ is due to the arrival during $(s/2, s/2+\delta]$; whether there is any arrival during $(0,\delta]$ is independent of whether the arrival during $(s, s+\delta]$ is recorded.

- Case 3: $2\tau < s < t$: We pick $\delta$ such that $\delta < 2\tau$ and $s + \delta < t$ and $\delta < s - 2\tau$. For $2\tau < s < t$, $Y(0, \delta)$ and $Y(s, s + \delta)$ are statistically independent, since the event 'there is an arrival during $(0, \delta]$' is statistically independent from the event 'there is an arrival during $(s, s + \delta]$', because they are at least $2\tau$ apart in time. Thus

$$E[Y(0, \delta)Y(s, s + \delta)] = E^2[Y(0, \delta)] = (\lambda\delta\, \mathrm{e}^{-\lambda 2\tau})^2 \tag{49}$$

and

$$\beta(0, s) = (\lambda\, \mathrm{e}^{-\lambda 2\tau})^2. \tag{50}$$

Combining the above three cases and using (22) yields

$$\begin{aligned} E[Y^2(t)] &= \gamma t + 2\int_{\tau}^{2\tau} (t - s)\lambda^2\, \mathrm{e}^{-\lambda(2\tau+s)}\, \mathrm{d}s + 2\int_{2\tau}^{t} (t - s)(\lambda\, \mathrm{e}^{-\lambda 2\tau})^2\, \mathrm{d}s \\ &= \lambda t\, \mathrm{e}^{-\lambda 2\tau} + 2\, \mathrm{e}^{-4\lambda\tau}(1 - \lambda t + 2\lambda\tau) + 2\, \mathrm{e}^{-3\lambda\tau}(\lambda t - \lambda\tau - 1) \\ &\quad + [(t - 2\tau)(\lambda\, \mathrm{e}^{-\lambda 2\tau})]^2. \end{aligned} \tag{51}$$

Simple algebra leads to (34).

## Appendix C

Due to the decay of an isotope photon source, the photon arrival process is not exactly homogeneous. In medical imaging, the arrival rates are inhomogeneous due to radiotracer dynamics. In this appendix, we derive $E[Y(t)]$ for the paralysable deadtime model[+], assuming only that the instantaneous photon arrival rate $\lambda(t)$ is continuous. This relaxes the assumption made in section 2 that $\lambda$ is constant. For an inhomogeneous process, $E[Y(s, s + \delta)] \neq E[Y(0, \delta)]$ in general. First we observe that the waiting time for the first photon arrival after time $s$, denoted $T_1$, has the following distribution:

$$\begin{aligned} F_{T_1}(r) &= P[T_1 \leqslant r] = 1 - P[T_1 > r] = 1 - P[N(s, r) = 0] \\ &= 1 - \mathrm{e}^{-\int_s^r \lambda(q)\, \mathrm{d}q}. \end{aligned} \tag{52}$$

Hence for $r > s$,

$$f_{T_1}(r) = \frac{\mathrm{d}}{\mathrm{d}r} F_{T_1}(r) = \lambda(r)\, \mathrm{e}^{-\int_s^r \lambda(q)\, \mathrm{d}q}. \tag{53}$$

For $0 < \delta < \tau$, we have:

$$\begin{aligned} E[Y(s, s + \delta)] &= P[Y(s, s + \delta) = 1] \\ &= \int_s^{s+\delta} P[Y(s, s + \delta) = 1 \mid T_1 = r] f_{T_1}(r)\, \mathrm{d}r \\ &= \int_s^{s+\delta} P[N(r - \tau, s) = 0] f_{T_1}(r)\, \mathrm{d}r \\ &= \int_s^{s+\delta} \mathrm{e}^{-\int_{r-\tau}^{s} \lambda(q_1)\, \mathrm{d}q_1} \lambda(r)\, \mathrm{e}^{-\int_s^r \lambda(q_2)\, \mathrm{d}q_2}\, \mathrm{d}r \\ &= \int_s^{s+\delta} \lambda(r)\, \mathrm{e}^{-\int_{r-\tau}^{r} \lambda(q)\, \mathrm{d}q}\, \mathrm{d}r. \end{aligned} \tag{54}$$

Since $\lambda$ is continuous, and $\mathrm{e}^{-\int_{r-\tau}^{r} \lambda(q)\, \mathrm{d}q}$ is continuous in $r$, we conclude:

$$\gamma(s) = \lambda(s)\, \mathrm{e}^{-\int_{s-\tau}^{s} \lambda(q)\, \mathrm{d}q}. \tag{55}$$

[+] Extension to the type III deadtime model is straightforward.

Hence*

$$E[Y(t)] = \int_0^t \lambda(s)\,\mathrm{e}^{-\int_{s-\tau}^{s} \lambda(q)\,\mathrm{d}q}\,\mathrm{d}s. \tag{56}$$

If $\tau$ is small relative to variations in $\lambda$, then $\int_{s-\tau}^{s} \lambda(q)\,\mathrm{d}q \approx \lambda(s)\tau$, so

$$E[Y(t)] \approx \int_0^t \lambda(s)\,\mathrm{e}^{-\lambda(s)\tau}\,\mathrm{d}s. \tag{57}$$

This approximation can be applied to other deadtime models as well. Similarly, the second moment of $Y$ is:

$$E[Y^2(t)] = E[Y(t)] + 2\int_0^t \int_{s_1+\tau}^{t} \gamma(s_1)\gamma(s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{58}$$

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* In fact, it is unnecessarily restrictive to limit $\lambda$ to continuous functions; all we need is that $\lambda$ is integrable and bounded over $[0, t]$. If $\lambda$ is integrable on $[0, t]$, then $\lambda_1(r) = \lambda(r)\,\mathrm{e}^{-\int_{r-\tau}^{r} \lambda(q)\,\mathrm{d}q}$ is integrable on $[0, t]$; then almost every point of $[0, t]$ is a Lebesgue point of $\lambda_1$ (Bruckner *et al* 1997, Theorem 7.40); and if $s$ is a Lebesgue point of $\lambda_1$, then $\lim_{\delta\to 0} \frac{\int_s^{s+\delta} \lambda_1(q)\,\mathrm{d}q}{\delta} = \lambda_1(s)$ (Bruckner *et al* 1997, Theorem 7.39).