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MEAN CURVATURE 1 SURFACES IN HYPERBOLIC 3-SPACE WITH LOW TOTAL CURVATURE II

Dedicated to Professor Katsuei Kenmotsu on his sixtieth birthday

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Abstract. In this work, complete constant mean curvature 1 (CMC-1) surfaces in hyperbolic 3-space with total absolute curvature at most 4π are classified. This classification suggests that the Cohn-Vossen inequality can be sharpened for surfaces with odd numbers of ends, and a proof of this is given.

1. Introduction. This is a continuation (Part II) of the paper [14] (Part I) with the same title. As pointed out in Part I, complete CMC-1 (constant mean curvature 1) surfaces f in the hyperbolic 3-space H^3 have two important invariants. One is the *total absolute curvature* $\mathrm{TA}(f)$, and the other is the *dual total absolute curvature* $\mathrm{TA}(f^\#)$, which is the total absolute curvature of the dual surface $f^\#$. In Part I, we investigated surfaces with low $\mathrm{TA}(f^\#)$. Here we investigate CMC-1 surfaces with low $\mathrm{TA}(f)$.

Classifying CMC-1 surfaces in H^3 with low $\mathrm{TA}(f)$ is more difficult than classifying those with low $\mathrm{TA}(f^\#)$, for the following reasons: $\mathrm{TA}(f)$ equals the area of the spherical image of the (holomorphic) secondary Gauss map g, and g might not be single-valued on the surface. Therefore, $\mathrm{TA}(f)$ is generally not a 4π-multiple of an integer, unlike the case of $\mathrm{TA}(f^\#)$. Furthermore, the Osserman inequality does not hold for $\mathrm{TA}(f)$, also unlike the case of $\mathrm{TA}(f^\#)$. The weaker Cohn-Vossen inequality is the best general lower bound for $\mathrm{TA}(f)$ (with equality never holding [19]). In Section 3, we shall prove the following:

THEOREM 1.1. *Let $f : M^2 \to H^3$ be a complete* CMC-1 *immersion of total absolute curvature* $\mathrm{TA}(f) \leq 4\pi$. *Then f is either*

(1) *a horosphere*,

(2) *an Enneper cousin*,

(3) *an embedded catenoid cousin*,

(4) *a finite δ-fold covering of an embedded catenoid cousin with $M^2 = \boldsymbol{C} \setminus \{0\}$ and secondary Gauss map $g = z^\mu$ for $\mu \leq 1/\delta$, or*

(5) *a warped catenoid cousin with injective secondary Gauss map.*

The horosphere is the only flat (and consequently totally umbilic) CMC-1 surface in H^3. The catenoid cousins are the only CMC-1 surfaces of revolution [3]. The Enneper cousins

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are isometric to minimal Enneper surfaces [3]. The warped catenoid cousins [19] are less well-known and are described in Section 2.

Although this theorem is simply stated, for the reasons stated above the proof is more delicate than it would be if the condition $\mathrm{TA}(f) \leq 4\pi$ were replaced with $\mathrm{TA}(f^{\#}) \leq 4\pi$, or if minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 4\pi$ were considered. CMC-1 surfaces f with $\mathrm{TA}(f^{\#}) \leq 4\pi$ are shown in Part I to be only horospheres, Enneper cousin duals, catenoid cousins, and warped catenoid cousins with embedded ends. It is well-known that the only complete minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 4\pi$ are the plane, the Enneper surface, and the catenoid.

We see from this theorem that any three-ended surface f satisfies $\mathrm{TA}(f) > 4\pi$, and so the Cohn-Vossen inequality is not sharp for such f. On the other hand, the Cohn-Vossen inequality is sharp for catenoid cousins, and a numerical experiment in [15] shows it to be sharp for genus 0 surfaces with 4 ends. This raises the question:

> Which classes of surfaces f have a stronger lower bound for $\mathrm{TA}(f)$ than that given by the Cohn-Vossen inequality?

Pursuing this, in Section 4 we show that stronger lower bounds exist for genus zero CMC-1 surfaces with an odd number of ends.

We extend Theorem 1.1 in a follow-up work [15], to find an inclusive list of possibilities for CMC-1 surfaces with $\mathrm{TA}(f) \leq 8\pi$, and consider which possibilities we can classify or find examples for. (Minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 8\pi$ are classified by Lopez [9]. Those with $\mathrm{TA} \leq 4\pi$ are listed in Table 1 in Section 2.)

2. Preliminaries. Let $f\colon M \to H^3$ be a conformal CMC-1 immersion of a Riemann surface M into H^3. Let ds^2, dA and K denote the induced metric, induced area element and Gaussian curvature, respectively. Then $K \leq 0$ and $d\sigma^2 := (-K)\,ds^2$ is a conformal pseudometric of constant curvature 1 on M. We call the developing map $g\colon \tilde{M} :=$ (the universal cover of M)$\to \boldsymbol{CP}^1$ the *secondary Gauss map* of f, where $\boldsymbol{CP}^1$ is the complex projective line. Namely, g is a conformal map so that its pull-back of the Fubini-Study metric of $\boldsymbol{CP}^1$ equals $d\sigma^2$:

$$d\sigma^2 = (-K)ds^2 = \frac{4dg\,d\bar{g}}{(1+g\bar{g})^2}\,. \tag{2.1}$$

By definition, the secondary Gauss map g of the immersion f is uniquely determined up to transformations of the form

$$g \mapsto a \star g := \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}} \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SU}(2)\,. \tag{2.2}$$

In addition to g, two other holomorphic invariants G and Q are closely related to geometric properties of CMC-1 surfaces. The *hyperbolic Gauss map* $G\colon M \to \boldsymbol{CP}^1$ is holomorphic and is defined geometrically by identifying the ideal boundary of H^3 with $\boldsymbol{CP}^1$: $G(p)$ is the asymptotic class of the normal geodesic of $f(M)$ starting at $f(p)$ and oriented in the mean curvature vector's direction. The *Hopf differential* Q is the symmetric holomorphic

2-differential on M such that $-Q$ is the (2, 0)-part of the complexified second fundamental form. The Gauss equation implies

$$ds^2 \cdot d\sigma^2 = 4\, Q \cdot \bar{Q}\,, \tag{2.3}$$

where $\cdot$ means the symmetric product. Moreover, these invariants are related by

$$S(g) - S(G) = 2Q\,, \tag{2.4}$$

where $S(\cdot)$ denotes the Schwarzian derivative

$$S(h) := \left[\left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2\right] dz^2 \quad \left({}' = \frac{d}{dz}\right)$$

with respect to a complex coordinate z on M.

Since $K \le 0$, we can define the *total absolute curvature* as

$$\mathrm{TA}(f) := \int_M (-K)\, dA \in [0, +\infty]\,.$$

Then $\mathrm{TA}(f)$ is the area of the image in $\boldsymbol{CP}^1$ of the secondary Gauss map. $\mathrm{TA}(f)$ is generally not an integer multiple of 4π — for catenoid cousins [3, Example 2] and their δ-fold covers, $\mathrm{TA}(f)$ admits *any* positive real number.

For each conformal CMC-1 immersion $f : M \to H^3$, there is a holomorphic null immersion $F : \tilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$, the *lift* of f, satisfying the differential equation

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega\,, \quad \omega = \frac{Q}{dg} \tag{2.5}$$

such that $f = FF^*$, where $F^* = {}^t\bar{F}$. Here we consider $H^3 = \mathrm{SL}(2, \boldsymbol{C})/\,\mathrm{SU}(2) = \{aa^* \mid a \in \mathrm{SL}(2, \boldsymbol{C})\}$. If $F = (F_{ij})$, equation (2.5) implies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}\,,$$

and it is shown in [3] that

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}\,.$$

We now assume that the induced metric ds^2 on M is complete and that $\mathrm{TA}(f) < \infty$. Hence there exists a compact Riemann surface $\bar{M}_\gamma$ of genus γ and a finite set of points $\{p_1, \dots, p_n\} \subset \bar{M}_\gamma$ ($n \ge 1$) so that M is biholomorphic to $\bar{M}_\gamma \setminus \{p_1, \dots, p_n\}$. We call the points p_j the *ends* of f. Moreover, the pseudometric $d\sigma^2$ as in (2.1) is an element of $\mathrm{Met}_1(\bar{M}_\gamma)$ ([3, Theorem 4], for a definition of Met_1 see Appendix A).

Unlike the Gauss map for minimal surfaces with $\mathrm{TA} < \infty$ in $\boldsymbol{R}^3$, the hyperbolic Gauss map G of f might not extend to a meromorphic function on $\bar{M}_\gamma$ (as the Enneper cousins show). However, the Hopf differential Q does extend to a meromorphic differential on $\bar{M}_\gamma$ [3]. We say an end p_j ($j = 1, \dots, n$) of a CMC-1 immersion is *regular* if G is meromorphic at p_j. When $\mathrm{TA}(f) < \infty$, an end p_j is regular precisely when the order of Q at p_j is at least -2, and otherwise G has an essential singularity at p_j [19].

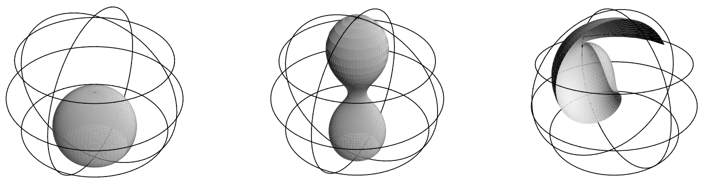

FIGURE 1. A horosphere, a catenoid cousin with $g = z^{\mu}$ ($\mu = 0.8$), and a fundamental piece (one-fourth of the surface with the end cut away) of an Enneper cousin with $g = z$, $Q = (1/2)dz^2$.

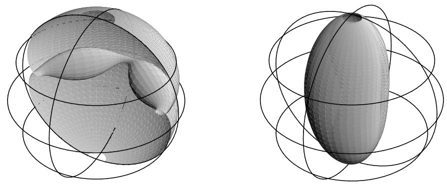

FIGURE 2. Two warped catenoid cousins, the first with $\delta = 1$, $l = 4$, $b = 1/2$ and the second with $\delta = 2$, $l = 1$, $b = 1/2$. (Half of the first surface has been cut away.) Only the second of these two surfaces has $\mathrm{TA}(f) = 4\pi$ (since $l = 1$), even though its ends are not embedded.

Thus the orders of Q at the ends p_j are important for understanding the geometry of the surface, so we now introduce a notation that reflects this. We say a CMC-1 surface is of *type* $\mathbf{\Gamma}(d_1, \dots, d_n)$ if it is given as a conformal immersion $f\colon \bar{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$, where $\mathrm{ord}_{p_j} Q = d_j$ for $j = 1, \dots, n$ (for example, if $Q = z^{-2}dz^2$ at $p_1 = 0$, then $d_1 = -2$). We use $\mathbf{\Gamma}$ because it is the capitalized form of γ, the genus of $\bar{M}_\gamma$. For instance, $\mathbf{I}(-4)$ is the class of surfaces of genus 1 with 1 end so that Q has an order 4 pole at the end, and $\mathbf{O}(-2, -3)$ is the class of surfaces of genus 0 with two ends so that Q has an order 2 pole at one end and an order 3 pole at the other.

We close this section with a description of the warped catenoid cousins. Here is a slightly refined version of Theorem 6.2 in [19]:

THEOREM 2.1. *A complete conformal* CMC-1 *immersion* $f : M = \boldsymbol{C} \setminus \{0\} \to H^3$ *with two regular ends is a* δ*-fold cover of a catenoid cousin* (*which is characterized by* $g = z^{\mu}$ *and* $\omega = (1-\mu^2)z^{-\mu-1}dz/(4\mu)$ *for* $\mu \in \boldsymbol{R}$), *or an immersion* (*or possibly a finite covering of it*), *where* g *and* ω *can be chosen as*

$$g = \frac{\delta^2 - l^2}{4l} z^l + b\,, \qquad \omega = \frac{Q}{dg} = z^{-l-1}dz\,,$$

with $l, \delta \in \mathbf{Z}^+$, $l \neq \delta$, *and* $b \geq 0$.

When $b = 0$, f *is a* δ*-fold cover of a catenoid cousin with* $\mu = l$. *When* $b > 0$, *we call* f *a* warped catenoid cousin, *and its discrete symmetry group is the natural* $\mathbf{Z}_2$ *extension of the*

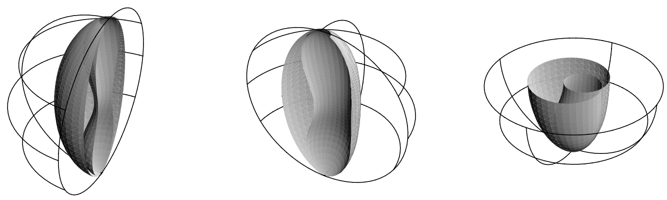

FIGURE 3. Cut-away views of the second warped catenoid cousin in Figure 2.

dihedral group D_l. Furthermore, the warped catenoid cousins can be written explicitly as

$$f = FF^*, \quad F = F_0 B,$$

where

$$F_0 = \sqrt{\frac{\delta^2 - l^2}{\delta}} \begin{pmatrix} \dfrac{1}{l-\delta} z^{(\delta-l)/2} & \dfrac{\delta - l}{4l} z^{(l+\delta)/2} \\ \dfrac{1}{l+\delta} z^{-(l+\delta)/2} & \dfrac{-(l+\delta)}{4l} z^{(l-\delta)/2} \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

PROOF. In [19] it is shown that a complete conformal CMC-1 immersion of $M = \boldsymbol{C} \setminus \{0\}$ with regular ends is a finite cover of a catenoid cousin or an immersion determined by

$$g = az^l + \hat{b}, \qquad \omega = cz^{-l-1}dz,$$

where l is a nonzero integer and $a, \hat{b}$ and c are complex numbers, which satisfy $l^2 + 4acl = \delta^2$ for a positive integer δ and $a, c \neq 0$. (The proof in [19] contains typographical errors: The exponents μ and $-\mu$ in equations (6.13) and (6.14) should be reversed. If $\mu \notin \mathbf{Z}^+$, then the last paragraph of Case 1 is correct. If $\mu \in \mathbf{Z}^+$, then one must consider a possibility that is included in Case 2 in that proof, and the result follows.) Changing z to $1/z$ if necessary, we may assume $l \geq 1$.

Choose θ so that $b := \hat{b}e^{2i\theta} \geq 0$. Doing the SU(2) transformation

$$g \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \star g, \quad \omega \mapsto e^{-2i\theta}\omega,$$

and replacing z with $e^{-2i\theta/l}c^{1/l}z$ produces the same surface, and one has

$$g = acz^l + b, \quad \omega = z^{-l-1}dz, \quad ac = \frac{\delta^2 - l^2}{4l}.$$

Thus g and ω are as desired.

To study the symmetry group of the surface, we consider the transformations

$$\phi_\varrho(z) = e^{2\pi i \varrho/l}\bar{z} \quad (\varrho \in \mathbf{Z}), \quad \text{and} \quad \phi(z) = \left(\frac{16l^2(1+b^2)}{(\delta^2 - l^2)^2}\right)^{1/l} \frac{1}{\bar{z}}$$

of the plane. Then the Hopf differential and secondary Gauss map change as

$$\overline{Q \circ \phi_\varrho} = Q, \quad \overline{g \circ \phi_\varrho} = g, \quad \overline{Q \circ \phi} = Q, \quad \overline{g \circ \phi} = \frac{bg+1}{g-b} = A \star g,$$

TABLE 1. Classification of minimal surfaces in $\boldsymbol{R}^3$ with TA $\leq 4\pi$.

Type	TA	The surface
$\mathbf{O}(0)$	0	Plane
$\mathbf{O}(-4)$	4π	Enneper surface
$\mathbf{O}(-2,-2)$	4π	Catenoid

TABLE 2. Classification of CMC-1 surfaces in H^3 with TA$(f) \leq 4\pi$.

Type	TA(f)	The surface
$\mathbf{O}(0)$	0	Horosphere
$\mathbf{O}(-4)$	4π	Enneper cousins
$\mathbf{O}(-2,-2)$	$(0, 4\pi]$	Catenoid cousins and their δ-fold covers
$\mathbf{O}(-2,-2)$	4π	Warped catenoid cousins with $l = 1$

where

$$A = \frac{i}{\sqrt{1+b^2}} \begin{pmatrix} b & 1 \\ 1 & -b \end{pmatrix} \in \mathrm{SU}(2)\,.$$

Hence ϕ_Q and ϕ represent isometries of the surface. One can then check that there are no other isometries of the surface, i.e., that there are no other anti-conformal bijections $\hat{\phi}$ of M so that $\overline{Q \circ \hat{\phi}} = Q$ and $\overline{g \circ \hat{\phi}} = A \star g$ for some $A \in \mathrm{SU}(2)$. Thus the symmetry group is $D_l \times \mathbf{Z}_2$.

To see that the warped catenoid cousins have the explicit representation described in the theorem, one needs only to verify that $F = F_0 B$ satisfies (2.5). □

3. Complete CMC-1 surfaces with TA$(f) \leq 4\pi$**.** In this section we will prove Theorem 1.1. First we fix our notation and recall basic facts. For a complete conformal CMC-1 immersion $f\colon M = \bar{M}_\gamma \setminus \{p_1, \ldots, p_n\} \to H^3$, we define μ_j and $\mu_j^\#$ to be the branching orders of the Gauss maps g and G, respectively, at each end p_j. At an irregular end p_j, we have $\mu_j^\# = \infty$. Let $d_j := \mathrm{ord}_{p_j} Q$, the order of Q at p_j. (For an explanation of the notation $\mathrm{ord}_{p_j} Q$, see Section 2.)

If an end p_j is regular, $d_j \geq 2$ holds, and relation (2.4) implies that the Hopf differential Q expands as

$$Q = \left(\frac{1}{2} \frac{c_j}{(z-p_j)^2} + \cdots \right) dz^2\,, \quad c_j = -\frac{1}{2}\mu_j(\mu_j + 2) + \frac{1}{2}\mu_j^\#(\mu_j^\# + 2)\,, \tag{3.1}$$

where z is a local complex coordinate around p_j.

Let $\{q_1, \ldots, q_m\} \subset M$ be the m umbilic points of the surface, and let $\xi_k = \mathrm{ord}_{q_k} Q$. (For example, if $Q = z^m\, dz^2$, then $\mathrm{ord}_0 Q = m$). Then, as in (2.5) of Part I,

$$\sum_{j=1}^{n} d_j + \sum_{k=1}^{m} \xi_k = 4\gamma - 4\,, \quad \text{in particular,} \quad \sum_{j=1}^{n} d_j \leq 4\gamma - 4\,. \tag{3.2}$$

By (2.3) and (2.4), it holds that

(3.3) $$\xi_k = [\text{branch order of } G \text{ at } q_k] = [\text{branch order of } g \text{ at } q_k] = \operatorname{ord}_{q_k} d\sigma^2 .$$

As in (2.4) of Part I, the Gauss-Bonnet theorem implies that

$$\frac{\mathrm{TA}(f)}{2\pi} = \chi(\bar{M}_\gamma) + \sum_{j=1}^{n} \mu_j + \sum_{k=1}^{m} \xi_k ,$$

where χ denotes the Euler characteristic. Combining this with (3.2), we have

(3.4) $$\frac{\mathrm{TA}(f)}{2\pi} = 2\gamma - 2 + \sum_{j=1}^{n} (\mu_j - d_j) .$$

Proposition 4.1 in [19] implies that

(3.5) $$\mu_j - d_j > 1, \qquad \text{in particular,} \qquad \mu_j - d_j \geq 2 \quad \text{if } \mu_j \in \mathbf{Z} .$$

An end p_j is regular if and only if $d_j \geq -2$, and then G is meromorphic at p_j. Thus

(3.6) $$\mu_j^{\#} \text{ is a non-negative integer if } d_j \geq -2 .$$

By Proposition 4 of [3],

(3.7) $$\mu_j > -1 .$$

Hence Equation (3.1) implies that

(3.8) $$\mu_j = \mu_j^{\#} \in \mathbf{Z} \quad \text{if } d_j \geq -1 .$$

Finally, we note that

(3.9) any meromorphic function on a Riemann surface $\bar{M}_\gamma$ of genus $\gamma \geq 1$ has at least three distinct branch points.

To prove this, let φ be a meromorphic function on $\bar{M}_\gamma$ with N branch points $\{q_1, \dots, q_N\}$ of branching order ψ_k at q_k. Then the Riemann-Hurwicz relation implies that

$$2 \deg \varphi = 2 - 2\gamma + \sum_{k=1}^{N} \psi_k .$$

On the other hand, since the multiplicity of φ at q_k is $\psi_k + 1$, $\deg \varphi \geq \psi_k + 1$ ($k = 1, \dots, N$). Thus

$$(N - 2) \deg \varphi \geq 2(\gamma - 1) + N .$$

If $\gamma \geq 1$, then $\deg \varphi \geq 2$, and so $N \geq 3$.

REMARK. Facts (3.4) and (3.5) imply that, for CMC-1 surfaces, the equality never holds in the Cohn-Vossen inequality [19]:

(3.10) $$\frac{\mathrm{TA}(f)}{2\pi} > -\chi(M) = n - 2 + 2\gamma .$$

PROOF OF THEOREM 1.1. By (3.4),

$$2 \geq \frac{\mathrm{TA}(f)}{2\pi} = 2\gamma - 2 + \sum_{j=1}^{n} (\mu_j - d_j)\,. \tag{3.11}$$

Since $\mu_j - d_j > 1$ by (3.5), we have

$$4 > 2\gamma + n\,.$$

Thus the only possibilities are

$$(\gamma, n) = (0,1), \quad (0,2), \quad (0,3), \quad (1,1)\,.$$

THE CASE $(\gamma, n) = (1,1)$. By (3.11) and (3.7), we have $d_1 \geq \mu_1 - 2 > -3$. Thus the end p_1 is regular, and G is meromorphic on $\bar{M}_1$. By (3.2), $d_1 \leq 0$. If $d_1 = -2$, then the end has non-vanishing flux, and the surface does not exist, by Corollary 3 of [13]. If $d_1 = 0$ or -1, then by (3.2) there is at most one umbilic point. Since any branch point of G is at an end or an umbilic point, (3.9) is contradicted. Hence a surface of this type does not exist.

THE CASE $(\gamma, n) = (0,1)$. Here the surface is simply connected, so there is a canonical isometrically corresponding minimal surface in $\boldsymbol{R}^3$ with the same total absolute curvature. We conclude the surface is a horosphere or an Enneper cousin.

THE CASE $(\gamma, n) = (0,2)$. Here, by (3.2), we have $d_1 + d_2 \leq -4$. On the other hand, by (3.11) and (3.7), we have $d_1 + d_2 \geq -4 + (\mu_1 + \mu_2) > -6$. Thus $d_1 + d_2$ is either -4 or -5. We now consider these two cases separately:

The case $d_1 + d_2 = -4$. If $d_1 + d_2 = -4$, then there are no umbilic points, by (3.2). If $d_1, d_2 \geq -2$, then the ends are regular, and Theorem 2.1 implies that the surface is a δ-fold cover of an embedded catenoid cousin with $\delta \leq 1/\mu$, or a warped catenoid cousin with $l = 1$.

Now assume that

$$d_1 \geq -1\,, \quad d_2 \leq -3\,.$$

Then we have $\mu_1 \in \boldsymbol{Z}$ by (3.8). By Proposition A.1 in Appendix A, we cannot have just one $\mu_j \notin \boldsymbol{Z}$, so also $\mu_2 \in \boldsymbol{Z}$. Then g is single-valued on M. Since g and G are both single-valued on M, the lift F is also (see equations (1.6) and (1.7) in [21]), and so the dual immersion $f^\#$ is also single-valued on M. Since $(f^\#)^\# = f$, $f^\#$ is a CMC-1 immersion with dual total absolute curvature 4π and of type $\mathbf{O}(-1,-3)$ (for an explanation of this notation, see Section 2). Such an $f^\#$ cannot exist by Theorem 3.1 of Part I, so such an f does not exist.

The case $d_1 + d_2 = -5$. If $d_1 + d_2 = -5$, then the surface has only one umbilic point q_1 with $\xi_1 = 1$, by (3.2), and we can set $\bar{M}_0 = \boldsymbol{C} \cup \{\infty\}$, $p_1 = 0$, $p_2 = \infty$, and $q_1 = 1$.

By (3.11), $\mu_1 + \mu_2 \leq -1$. Then, by (3.7), at least one of μ_1 and μ_2 is not an integer. Hence both are not integers, by Proposition A.1 in Appendix A. Then (3.8) implies that we may assume $d_1 = -2$ and $d_2 = -3$. By Proposition A.2 in Appendix A, the metric $d\sigma^2$ is the pull-back of the Fubini-Study metric on $\boldsymbol{CP}^1$ by the map

$$g = cz^\mu \left(z - \frac{\mu + 1}{\mu} \right) \quad (c \in \boldsymbol{C} \setminus \{0\},\ \mu \in \boldsymbol{R} \setminus \{0, \pm 1\})\,.$$

On the other hand, the Hopf differential Q is of the form

$$Q(z) = q(z)\,dz^2 = \theta \frac{z-1}{z^2}\,dz^2 \quad (\theta \in \boldsymbol{C} \setminus \{0\})\,. \tag{3.12}$$

Thus $\omega = Q/dg$ can be written in the form

$$\omega = w(z)\,dz = \frac{\theta}{c}\frac{1}{\mu+1}\frac{1}{z^{\mu+1}}\,dz\,. \tag{3.13}$$

Consider the equation (which is introduced in [19] as (E.1))

$$X'' + a(z)X' + b(z)X = 0\,, \qquad \left(a(z) := -\frac{w'(z)}{w(z)},\ b(z) := -q(z)\right). \tag{3.14}$$

We expand the coefficients a and b as

$$a(z) = \frac{1}{z}\sum_{j=0}^{\infty} a_j z^j\,, \quad b(z) = \frac{1}{z^2}\sum_{j=0}^{\infty} b_j z^j\,.$$

Then the origin $z = 0$ is a regular singularity of equation (3.14). Let λ and $\lambda + m$ be the solutions of the corresponding indicial equation $t(t-1) + a_0 t + b_0 = 0$ with $m \ge 0$. If the surface exists, then Theorem 2.4 of [19] implies that m must be a positive integer and the log-term coefficient of the solutions of (3.14) must vanish. When $m \in \boldsymbol{Z}^+$, the log-term coefficient vanishes if and only if

$$\sum_{k=0}^{m-1} \{(\lambda + k)a_{m-k} + b_{m-k}\}\eta_k(\lambda) = 0\,,$$

where $\eta_0 = 1$ and $\eta_1, \ldots, \eta_{m-1}$ are given recursively by

$$\eta_j = \frac{1}{j(m-j)}\sum_{k=0}^{j-1} \{(\lambda + k)a_{j-k} + b_{j-k}\}\eta_k$$

as in Proposition A.3 in Appendix A of Part I. Here we have

$$0 = a_1 = a_2 = \cdots\,, \quad 0 = b_2 = b_3 = \cdots\,,$$

and so the log-term coefficient never vanishes at the end p_1, because $b_1 = -\theta \ne 0$. Thus this type of surface does not exist.

THE CASE $(\gamma, n) = (0, 3)$. This is the only remaining case. But this type of surface does not exist, by the following Theorem 3.1. □

THEOREM 3.1. *Let $f\colon M \to H^3$ be a complete* CMC-1 *immersion of genus zero with three ends. Then* $\mathrm{TA}(f) > 4\pi$.

REMARK. The second and third authors proved that $\mathrm{TA}(f) \ge 4\pi$ holds for CMC-1 surfaces of genus 0 with three ends [24, Proposition 2.7]. Then the essential part of Theorem 3.1 is that $\mathrm{TA}(f) = 4\pi$ is impossible.

PROOF OF THEOREM 3.1. We suppose $\mathrm{TA}(f) = 4\pi$, and will arrive at a contradiction. Without loss of generality, we may set $\bar{M}_0 = \boldsymbol{C} \cup \{\infty\}$ and $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty$.

Step 1. Since $\gamma = 0$ and $\mathrm{TA}(f) \leq 4\pi$, (3.4) implies that

$$4 \geq \sum_{j=1}^{3} (\mu_j - d_j)\,. \tag{3.15}$$

Since $\mu_j - d_j > 1$ for all j, (3.15) implies that $\mu_j - d_j < 2$ for all j. Hence $\mu_1, \mu_2, \mu_3 \notin \boldsymbol{Z}$ by (3.5). Then (3.8) implies that $d_j \leq -2$ for all j, and as Equations (3.15) and (3.7) imply that $d_1 + d_2 + d_3 \geq -4 + \mu_1 + \mu_2 + \mu_3 > -7$, we have

$$d_1 = d_2 = d_3 = -2\,, \tag{3.16}$$

and so the ends are regular.

On the other hand, since $\mathrm{TA}(f) = 4\pi$, (3.4) and (3.16) imply that

$$\mu_1 + \mu_2 + \mu_3 = -2\,. \tag{3.17}$$

Then by (3.7), we have

$$-1 < \mu_j < 0 \quad (j = 1, 2, 3)\,, \tag{3.18}$$

and furthermore at least two of the μ_j are less than $-1/2$. We may arrange the ends so that

$$-1 < \mu_1, \mu_2 < -\frac{1}{2} \quad \text{and} \quad -1 < \mu_3 < 0\,. \tag{3.19}$$

Moreover, by Appendix A of [24] (note that the C_j there equal $\pi(\mu_j + 1)$), the metric $d\sigma^2$ is reducible (as defined in Appendix B of the present paper). Then, by Proposition B.1 and the relation (A.3) in the appendices here, the secondary Gauss map g can be expressed in the form

$$g = z^{-(\mu_1+1)}(z-1)^{\beta+1}\frac{a(z)}{b(z)}\,, \tag{3.20}$$

where $a(z)$, $b(z)$ are relatively prime polynomials without zeros at p_1 and p_2, and

$$\beta = \mu_2 \quad \text{or} \quad \beta = -2 - \mu_2\,. \tag{3.21}$$

Note that the order of g at $p_3 = \infty$ is $\pm(\mu_3 + 1)$ and is also $\mu_1 - \beta - \deg a + \deg b$. If $\beta = \mu_2$, then

$$2\mu_1 = \deg a - \deg b - 1 \quad \text{or} \quad 2\mu_2 = \deg b - \deg a - 1$$

holds. Thus either $2\mu_1$ or $2\mu_2$ is an integer, but this contradicts (3.19), so $\beta = -\mu_2 - 2$:

$$g = z^{-\mu_1-1}(z-1)^{-\mu_2-1}\frac{a(z)}{b(z)}\,. \tag{3.22}$$

Thus, by (3.17), we have

$$-\mu_3 - \deg a + \deg b = \pm(\mu_3 + 1)\,.$$

Hence either

$$\deg a - \deg b = 1 \quad \text{and the order of } g \text{ at } \infty \text{ is } -\mu_3 - 1\,, \text{ or} \tag{3.23}$$

$$\mu_3 = -1/2, \deg a = \deg b \quad \text{and the order of } g \text{ at } \infty \text{ is } \mu_3 + 1 \tag{3.24}$$

holds because of (3.19). To get more specific information about $a(z)$ and $b(z)$, we now consider dg:

Step 2. Since Q is holomorphic on $\boldsymbol{C} \setminus \{0, 1\}$ with two zeroes (by (3.2)), (3.1) implies that

$$Q = \frac{1}{2}\left(\frac{c_3 z^2 + (c_2 - c_1 - c_3)z + c_1}{z^2(z-1)^2}\right) dz^2\,, \tag{3.25}$$

with the c_j as in (3.1), as pointed out in [24, page 84]. Note that

$$c_j > 0 \quad (j = 1, 2, 3)\,, \tag{3.26}$$

because $\mu_j^\# \geq 0$ and $-1 < \mu_j < 0$. Let q_1 and q_2 be the two roots of

$$c_3 z^2 + (c_2 - c_1 - c_3)z + c_1 = 0\,. \tag{3.27}$$

In the case of a double root, we write $q := q_1 = q_2$.

Using (3.3) and Proposition B.1 in Appendix B, dg has only the following four possibilities:

$$dg = C\frac{z^{-\mu_1-2}(z-1)^{-\mu_2-2}(z-q_1)(z-q_2)}{\prod_{k=1}^r (z-a_k)^2}\, dz\,, \tag{3.28}$$

$$dg = C\frac{z^{-\mu_1-2}(z-1)^{-\mu_2-2}(z-q_1)}{(z-q_2)^3 \prod_{k=1}^r (z-a_k)^2}\, dz \qquad (q_1 \neq q_2)\,, \tag{3.29}$$

$$dg = C\frac{z^{-\mu_1-2}(z-1)^{-\mu_2-2}}{(z-q_1)^3(z-q_2)^3 \prod_{k=1}^r (z-a_k)^2}\, dz \qquad (q_1 \neq q_2)\,, \tag{3.30}$$

or

$$dg = C\frac{z^{-\mu_1-2}(z-1)^{-\mu_2-2}}{(z-q)^4 \prod_{k=1}^r (z-a_k)^2}\, dz \qquad (q = q_1 = q_2)\,, \tag{3.31}$$

where r is a non-negative integer and the points $a_k \in \boldsymbol{C} \setminus \{0, 1, q_1, q_2\}$ are mutually distinct. In the first case (3.28), the order of dg at infinity ($z = p_3 = \infty$) is given by

$$\mu_1 + \mu_2 + 2r = 2r - 2 - \mu_3 = \mu_3 \text{ or } -\mu_3 - 2\,.$$

So $2r - 2 = 2\mu_3 \in (-2, 0)$ or $2r - 2 - \mu_3 = -\mu_3 - 2$. Hence $r = 0$ and the order of dg at ∞ is $-\mu_3 - 2$ in the first case.

In the other three cases (3.29), (3.30) and (3.31), the orders of dg at infinity are

$$\mu_1 + \mu_2 + (2 \text{ or } 6 \text{ or } 4) + 2r + 2 \geq 2 - \mu_3 + 2r > 2\,,$$

respectively. These orders must equal either $\mu_3 < 0$ or $-\mu_3 - 2 < 0$, so none of these three cases can occur. We conclude that dg is of the form

$$dg = Cz^{-\mu_1-2}(z-1)^{-\mu_2-2}(z-q_1)(z-q_2)\,dz \qquad (C \in \boldsymbol{C} \setminus \{0\})\,. \tag{3.32}$$

Since the order of dg at ∞ is $\mu_1 + \mu_2 = -\mu_3 - 2 < 0$, (3.23) holds.

Step 3. Now we determine the polynomials $a(z)$, $b(z)$ in the expression (3.22). Differentiating (3.22), we have

$$dg = \frac{z^{-\mu_1-2}(z-1)^{-\mu_2-2}}{b^2(z)} f(z)dz\,, \tag{3.33}$$

where

$$f(z) = -(1+\mu_1)(z-1)ab - (1+\mu_2)zab + z(z-1)(a'b - ab')\,. \tag{3.34}$$

Since $a(z)$ and $b(z)$ are relatively prime, $b(z)$ does not divide $f(z)$ when $\deg b \geq 1$. But (3.32) and (3.33) imply that $b^2(z)$ divides $f(z)$, so $b(z)$ is constant, and we may assume $b = 1$. Here, as seen in the previous step, (3.23) holds, and then, $\deg a = 1$. Thus we have

$$a(z) = a_1 z + a_0 \quad \text{and} \quad b = 1 \quad (a_1 \neq 0)\,. \tag{3.35}$$

Step 4. By (3.32), (3.33), (3.34) and (3.35) we have

$$\begin{aligned} -a_1(\mu_1+\mu_2+1)z^2 + \{\mu_1 a_1 - (\mu_1+\mu_2+2)a_0\}\,z + (1+\mu_1)a_0 \\ = C(z-q_1)(z-q_2)\,. \end{aligned} \tag{3.36}$$

Equation (3.27) also has roots q_1 and q_2, so

$$q_1 q_2 = \frac{a_0}{a_1}\frac{1+\mu_1}{1+\mu_3} = \frac{c_1}{c_3}, \quad q_1 + q_2 = -\frac{\mu_3 a_0 + \mu_1 a_1}{a_1(1+\mu_3)} = \frac{c_1}{c_3} + 1 - \frac{c_2}{c_3}\,. \tag{3.37}$$

By (3.7), (3.26) and the first equation of (3.37), we have $a_0/a_1 > 0$. Substituting the first equation of (3.37) into the second, we have

$$\frac{c_2}{c_3} = -\frac{1+\mu_2}{1+\mu_3}\left(\frac{a_0}{a_1} + 1\right).$$

Since $a_0/a_1 > 0$, (3.7) implies that $c_2/c_3 < 0$, contradicting (3.26) and completing the proof. □

4. Improvement of the Cohn-Vossen Inequality. For a complete CMC-1 immersion f into H^3, the equality in the Cohn-Vossen inequality never holds ([19, Theorem 4.3]). In particular, when f is of genus 0 with n ends,

$$\mathrm{TA}(f) > 2\pi(n-2)\,. \tag{4.1}$$

For $n = 2$, the catenoid cousins show that (4.1) is sharp. But Theorem 3.1 implies that

$$\mathrm{TA}(f) > 4\pi \quad \text{for} \quad n = 3\,,$$

which is stronger than the Cohn-Vossen inequality (4.1). The following theorem gives a sharper inequality than that of Cohn-Vossen, when n is any odd integer:

THEOREM 4.1. *Let* $f : \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \ldots, p_{2l+1}\} \to H^3$ *be a complete conformal* CMC-1 *immersion of genus* 0 *with* $2l+1$ *ends*, $l \in \boldsymbol{Z}$. *Then*

$$\mathrm{TA}(f) \geq 4\pi l \,.$$

To show this, we first prove two lemmas and a proposition.

LEMMA 4.2. *Let* $\theta_1, \theta_2, \theta_3 \in [0, \pi]$ *be three real numbers such that*

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + 2\cos\theta_1\cos\theta_2\cos\theta_3 \leq 1 \,. \tag{4.2}$$

Then the following inequalities hold:

$$\theta_1 + \theta_2 + \theta_3 \geq \pi \,, \tag{4.3}$$

$$\theta_2 - \theta_1 \leq \pi - \theta_3 \,. \tag{4.4}$$

REMARK. It is well-known that the inequality

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + 2\cos\theta_1\cos\theta_2\cos\theta_3 < 1$$

is a necessary and sufficient condition for the existence of a spherical triangle $\mathcal{T}$ with angles θ_1, θ_2 and θ_3. Then (4.3) follows directly from the Gauss-Bonnet formula, and (4.4) is the triangle inequality for the polar triangle of $\mathcal{T}$, and the lemma follows. ($\mathcal{T}$'s polar triangle is the one whose vertices are the centers of the great circles containing the edges of $\mathcal{T}$.) However, we give an alternative proof:

PROOF OF LEMMA 4.2. We set

$$E := \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + 2\cos\theta_1\cos\theta_2\cos\theta_3 - 1 \leq 0 \,.$$

Then

$$\begin{aligned} E = 4\cos\left(\frac{\theta_1+\theta_2+\theta_3}{2}\right)\cos\left(\frac{-\theta_1+\theta_2+\theta_3}{2}\right) \\ \times \cos\left(\frac{\theta_1-\theta_2+\theta_3}{2}\right)\cos\left(\frac{\theta_1+\theta_2-\theta_3}{2}\right). \end{aligned}$$

If $\theta_1+\theta_2+\theta_3 < \pi$, then we have $|\pm\theta_1 \pm \theta_2 \pm \theta_3| < \pi$, and so

$$\cos\left(\frac{\pm\theta_1 \pm \theta_2 \pm \theta_3}{2}\right) > 0 \,,$$

implying $E > 0$, a contradiction. This proves (4.3). Now, since

$$E = \cos^2\theta_1 + \cos^2(\pi-\theta_2) + \cos^2(\pi-\theta_3) + 2\cos\theta_1\cos(\pi-\theta_2)\cos(\pi-\theta_3) - 1$$

and $E \leq 0$ and $\theta_1, \pi-\theta_2, \pi-\theta_3 \in [0,\pi]$, (4.3) implies that

$$\theta_1 + (\pi-\theta_2) + (\pi-\theta_3) \geq \pi \,,$$

that is, (4.4) holds. □

For a matrix $a \in \mathrm{SU}(2)$, there is a unique $C \in [0, \pi]$ such that a has eigenvalues $\{-e^{\pm iC}\}$. We define the *rotation angle* of a as

$$\theta(a) := 2C\,.$$

Indeed, if one considers the matrix acting on the unit sphere as an isometry (Möbius action on $\boldsymbol{CP}^1$ with the Fubini-Study metric), $\theta(a)$ is exactly the angle of rotation.

LEMMA 4.3. *Let a_0, a_1, a_2, a_3 be four matrices in* SU(2) *satisfying $a_1a_2a_3 = a_0$. Then it holds that*

$$\theta(a_1) + \theta(a_2) + \theta(a_3) \geq \theta(a_0)\,.$$

PROOF. Setting $b := a_3(a_0)^{-1} = (a_1a_2)^{-1}$, we have $a_1a_2b = \mathrm{id}$. Then Appendix A of [24] implies that

$$\cos^2\frac{\theta(a_1)}{2} + \cos^2\frac{\theta(a_2)}{2} + \cos^2\frac{\theta(b)}{2} + 2\cos\frac{\theta(a_1)}{2}\cos\frac{\theta(a_2)}{2}\cos\frac{\theta(b)}{2} \leq 1\,.$$

So by Lemma 4.2 we have

$$\frac{\theta(a_1)}{2} + \frac{\theta(a_2)}{2} + \frac{\theta(b)}{2} \geq \pi\,. \tag{4.5}$$

On the other hand, we have $a_3^{-1}ba_0 = \mathrm{id}$. Again Appendix A of [24] implies that

$$\cos^2\frac{\theta(a_0)}{2} + \cos^2\frac{\theta(a_3)}{2} + \cos^2\frac{\theta(b)}{2} + 2\cos\frac{\theta(a_0)}{2}\cos\frac{\theta(a_3)}{2}\cos\frac{\theta(b)}{2} \leq 1\,,$$

since $\theta(a_3^{-1}) = \theta(a_3)$. By (4.4) of Lemma 4.2, we have

$$\frac{\theta(a_0)}{2} - \frac{\theta(a_3)}{2} \leq \pi - \frac{\theta(b)}{2}\,. \tag{4.6}$$

By (4.5) and (4.6), we get the assertion. □

PROPOSITION 4.4. *Let $a_1, \ldots, a_{2m+1}$ be matrices in* SU(2) *satisfying*

$$a_1a_2\cdots a_{2m+1} = \mathrm{id}\,.$$

Then it holds that

$$\sum_{j=1}^{2m+1}\theta(a_j) \geq 2\pi\,.$$

REMARK. This result does not hold for an even number of matrices: Suppose $a_1, \ldots,$ $a_{2m} \in \mathrm{SU}(2)$ satisfy $a_1a_2\cdots a_{2m} = \mathrm{id}$. Then the inequality $\sum_{j=1}^{2m}\theta(a_j) \geq 0$ is sharp. In fact, the equality will hold if all $a_j = -\mathrm{id}$.

PROOF OF PROPOSITION 4.4 We argue by induction. If $m = 1$, the result follows from Lemma 4.3 with $a_0 = \mathrm{id}$. Now suppose that the result always holds for $m - 1 (\geq 1)$. Set

$$b := a_1a_2a_3\,.$$

Then, by Lemma 4.3,

$$\theta(a_1) + \theta(a_2) + \theta(a_3) \geq \theta(b) . \tag{4.7}$$

On the other hand, we have $ba_4 \cdots a_{2m+1} = \mathrm{id}$, so by the inductive assumption,

$$\theta(b) + \sum_{j=4}^{2m+1} \theta(a_j) \geq 2\pi . \tag{4.8}$$

By (4.7) and (4.8), we get the assertion. □

We now apply Proposition 4.4 to the monodromy representation of pseudometrics in $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ (see Appendices A and B):

COROLLARY 4.5. *Let* $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ *with divisor*

$$D = \sum_{j=1}^{s} \beta_j p_j + \sum_{k=1}^{n} \xi_k q_k , \quad \beta_j > -1 , \quad \xi_k \in \boldsymbol{Z}^+ ,$$

where the $p_1, \dots, p_s, q_1, \dots, q_n$ *are mutually distinct points in* $\boldsymbol{C} \cup \{\infty\}$.

If $s + \xi_1 + \cdots + \xi_n$ *is an odd integer, then* $\beta_1 + \cdots + \beta_s \geq 1 - s$.

PROOF. Let g be a developing map of $d\sigma^2$ with the monodromy representation $\rho_g : \pi_1(M) \to \mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ on $M = \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_s, q_1, \dots, q_n\}$.

ρ_g can be lifted to an SU(2) representation $\tilde{\rho}_g : \pi_1(M) \to \mathrm{SU}(2)$ so that the following properties hold:

(1) Let T_j $(j = 1, \dots, s)$ and S_k $(k = 1, \dots, n)$ be deck transformations on $\tilde{M}$ corresponding to loops about p_j and q_k, respectively. Then it holds that

$$\tilde{\rho}_g(T_1) \cdots \tilde{\rho}_g(T_s) \tilde{\rho}_g(S_1) \cdots \tilde{\rho}_g(S_n) = \mathrm{id} .$$

(2) The eigenvalues of the matrix $\tilde{\rho}_g(T_j)$ (resp. $\tilde{\rho}_g(S_k)$) are $\{-e^{\pm i\pi(\beta_j+1)}\}$ (resp. $\{-e^{\pm i\pi(\xi_k+1)}\}$).

This is proven in [24, Lemma 2.2] for $s = 3, n = 0$, and the same argument will work for general s and n. We include an outline of the argument here: One chooses a solution $\tilde{F}$ to equation (2.12) in [24] (with $G = z$ and $Q = S(g)/2$). Then $\tilde{F}$ has a monodromy representation $\rho_{\tilde{F}} : \pi_1(M) \to \mathrm{SU}(2)$, where $\tilde{F} \to \tilde{F} \cdot \rho_{\tilde{F}}(\gamma)$ about loops $\gamma \in \pi_1(M)$. Then $\rho_g = \pm \rho_{\tilde{F}}$, and we simply choose the lift $\tilde{\rho}_g$ so that $\tilde{\rho}_g = +\rho_{\tilde{F}}$. The first property is then clear.

To show the second property, we note that when β_j and ξ_k are all given the value 0, then Q is identically 0 and so $\tilde{F}$ is constant and all $\rho_{\tilde{F}} = +\mathrm{id}$. Hence the eigenvalues $\{\pm e^{\pm i\pi(\beta_j+1)}\}$ (resp. $\{\pm e^{\pm i\pi(\xi_k+1)}\}$) of $\tilde{\rho}_g(T_j)$ (resp. $\tilde{\rho}_g(S_k)$) are $\{-e^{\pm i\pi(\beta_j+1)}\}$ (resp. $\{-e^{\pm i\pi(\xi_k+1)}\}$) in this case. Then, as β_j and ξ_k are deformed back to their original values, the matrices $\tilde{\rho}_g(T_j)$ (resp. $\tilde{\rho}_g(S_k)$) change analytically and so the sign of the eigenvalues cannot change, showing the second property.

We have

$$\theta(\tilde{\rho}_g(T_j)) \leq 2\pi(\beta_j + 1) ,$$

and since ξ_k is an integer, we have

$$\tilde{\rho}_g(S_k) = (-1)^{\xi_k}\,\mathrm{id}\ . \tag{4.9}$$

Assume $s = 2m+1$ is an odd number. Then, by the assumption, $\xi_1 + \cdots + \xi_n$ is an even integer, and by (4.9) above we have $\tilde{\rho}_g(T_1)\cdots\tilde{\rho}_g(T_{2m+1}) = \mathrm{id}$, so by Proposition 4.4,

$$2\pi \sum_{j=1}^{2m+1} (\beta_j + 1) \geq \sum_{j=1}^{2m+1} \theta(\tilde{\rho}_g(T_j)) \geq 2\pi\ ,$$

proving the corollary when s is odd.

Now suppose that $s = 2m$ is even. We have $\tilde{\rho}_g(S_1)\cdots\tilde{\rho}_g(S_n) = -\,\mathrm{id}$, because $\xi_1 + \cdots + \xi_n$ is odd. Hence $\tilde{\rho}_g(T_1)\cdots\tilde{\rho}_g(T_{2m})(-\,\mathrm{id}) = \mathrm{id}$, and since $\theta(-\,\mathrm{id}) = 0$, Proposition 4.4 implies that

$$2\pi \sum_{j=1}^{2m}(\beta_j + 1) \geq \sum_{j=1}^{2m} \theta(\tilde{\rho}_g(T_j)) + \theta(-\,\mathrm{id}) \geq 2\pi\ ,$$

proving the corollary when s is even. □

PROOF OF THEOREM 4.1. Suppose that $\mu_1 \in \mathbf{Z}$. Then by (3.4) and (3.5),

$$\frac{\mathrm{TA}(f)}{2\pi} \geq -2 + (\mu_1 - d_1) + \sum_{j=2}^{2m+1} (\mu_j - d_j) > -2 + 2 + 2m = 2m\ , \tag{4.10}$$

proving the theorem when $\mu_1 \in \mathbf{Z}$.

Next, suppose that $d_1 \leq -3$. In this case, $\mu_1 - d_1 > -1 + 3 = 2$. Hence again by (3.4) and (3.5), we have (4.10), and the theorem follows.

Thus we may assume $\mu_j \notin \mathbf{Z}$ and $d_j \geq -2$ at all ends. Then, by (3.8), we have all $d_j = -2$. So, by (3.2) and (3.3), the corresponding pseudometric $d\sigma^2$ has divisor

$$\sum_{j=1}^{2m+1} \mu_j p_j + \sum_{k=1}^{l} \xi_k q_k\ , \quad \sum_{k=1}^{l} \xi_k = 4m - 2 \in 2\mathbf{Z}\ ,$$

where $\xi_k = \mathrm{ord}_{q_k} Q$ at each umbilic point q_k $(k = 1, \ldots, l)$. Then by Corollary 4.5,

$$\mu_1 + \mu_2 + \cdots + \mu_{2m+1} \geq -2m\ ,$$

and so (3.4) implies the theorem. □

REMARK. When $m = 1$, we know the lower bound $4\pi m$ in Theorem 4.1 is sharp. However, we do not know if it is sharp for general m. For CMC-1 surfaces of genus 0 with an even number $n \geq 4$ of ends, we do not know if there exists any stronger lower bound than that of the Cohn-Vossen inequality.

In [15], it is shown numerically that there exist CMC-1 surfaces of genus 0 with four ends whose total absolute curvature gets arbitrarily close to 4π.

Appendix A. For a compact Riemann surface $\bar{M}$ and points $p_1, \dots, p_n \in \bar{M}$, a conformal metric $d\sigma^2$ of constant curvature 1 on $M := \bar{M} \setminus \{p_1, \dots, p_n\}$ is an element of $\mathrm{Met}_1(\bar{M})$ if there exist real numbers $\beta_1, \dots, \beta_n > -1$ so that each p_j is a conical singularity of order β_j, that is, if $d\sigma^2$ is asymptotic to $c_j|z - p_j|^{2\beta_j}\, dz \cdot d\bar{z}$ at p_j, for $c_j \neq 0$ and z a local complex coordinate around p_j. We call the formal sum

$$D := \sum_{j=1}^{n} \beta_j p_j \tag{A.1}$$

the *divisor* corresponding to $d\sigma^2$. For a pseudometric $d\sigma^2 \in \mathrm{Met}_1(\bar{M})$ with divisor D, there is a holomorphic map $g : \tilde{M} \to \boldsymbol{CP}^1$ such that $d\sigma^2$ is the pull-back of the Fubini-Study metric of $\boldsymbol{CP}^1$. This map, called the *developing map* of $d\sigma^2$, is uniquely determined up to Möbius transformations $g \mapsto a \star g$ for $a \in \mathrm{SU}(2)$.

For a conical singularity p_j of $d\sigma^2$, there exists a developing map g and a local coordinate z of $\bar{M}$ around p_j such that

$$g(z) = (z - p_j)^{\tau_j}\hat{g}(z) \quad (\tau_j \in \boldsymbol{R} \setminus \{0\}),$$

where $\hat{g}(z)$ is holomorphic in a neighborhood of p_j and $\hat{g}(p_j) \neq 0$. Here, the order β_j of $d\sigma^2$ at p_j is

$$\beta_j = \begin{cases} \tau_j - 1 & \text{if } \tau_j > 0\,, \\ -\tau_j - 1 & \text{if } \tau_j < 0\,. \end{cases} \tag{A.2}$$

In other words, if $dg = (z - p_j)^{\beta}\hat{h}(z)\, dz$, where $\hat{h}(z)$ is holomorphic near p_j and $\hat{h}(p_j) \neq 0$, then the order β_j is expressed as

$$\beta_j = \begin{cases} \beta & \text{if } \beta > -1\,, \\ -\beta - 2 & \text{if } \beta < -1\,. \end{cases} \tag{A.3}$$

The following proposition gives an obstruction to the existence of certain pseudometrics in $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$.

PROPOSITION A.1. *For any non-integer $\beta > -1$, there is no pseudometric $d\sigma^2$ in* $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ *with the divisor*

$$\beta p_1 + \sum_{j=2}^{n} m_j p_j \quad (m_2, \dots, m_n \in \boldsymbol{Z}),$$

where $p_1, \dots, p_n$ are mutually distinct points in $\boldsymbol{C} \cup \{\infty\}$.

When $n = 1$ (i.e., when $\sum_{j=2}^{n} m_j p_j$ is removed), this nonexistence of a “tear-drop" has been pointed out in [17] and [4].

PROOF. We may set $p_1 = \infty$. Since the $m_j \in \boldsymbol{Z}$, the developing map g of $d\sigma^2$ is well-defined on $\boldsymbol{C}$, and so g is meromorphic on $\boldsymbol{C}$. As $d\sigma^2$ has finite total curvature, g extends to $z = \infty$ as a holomorphic mapping. In particular, $\beta \in \boldsymbol{Z}$. □

REMARK. When a Riemann surface $\bar{M}_\gamma$ has genus $\gamma > 0$, there is a pseudometric in $\mathrm{Met}_1(\bar{M}_\gamma)$ with only one singularity that has order less than 0, by [18].

PROPOSITION A.2. *Suppose a pseudometric $d\sigma^2$ in $\mathrm{Met}_1(\boldsymbol{C}\cup\{\infty\})$ has divisor*

$$\beta_1 p_1 + \beta_2 p_2 + p_3 \qquad (\beta_1, \beta_2 > -1 \text{ and } \beta_1, \beta_2 \notin \boldsymbol{Z}),$$

where $p_1 := 0$, $p_2 := \infty$, and $p_3 := 1$. Then $d\sigma^2$ has a developing map g of the form

$$g = cz^\mu \left(z - \frac{\mu+1}{\mu} \right) \qquad (c \in \boldsymbol{C},\ \mu \in \boldsymbol{R}), \tag{A.4}$$

where $\beta_1 = |\mu| - 1$ and $\beta_2 = |\mu+1| - 1$.

PROOF. Since $d\sigma^2$ has only two non-integral conical singularities, it is reducible, and Proposition B.1 in Appendix B shows that the map g is written in the form

$$g = z^\mu \frac{a(z)}{b(z)} \qquad (\mu \notin \boldsymbol{Z}),$$

where $a(z)$ and $b(z)$ are relatively prime polynomials with $a(0) \neq 0$ and $b(0) \neq 0$. Note that $b(z)$ can have a multiple root only at a conical singularity of $d\sigma^2$, hence only at $z = 1$. Thus $b'(z_0) \neq 0$ for all roots $z_0 \in \boldsymbol{C} \setminus \{0, 1\}$ of b.

Since the change $g \mapsto 1/g$ preserves $d\sigma^2$, we may assume that $\deg a \geq \deg b$. By a direct calculation, we have

$$dg(z) = \frac{z^{\mu-1}}{b(z)^2} h(z) dz, \text{ with } h(z) = \mu a(z)b(z) + za'(z)b(z) - za(z)b'(z).$$

Note that $h(0) = \mu a(0)b(0) \neq 0$.

Let $z_0 \in \boldsymbol{C} \setminus \{0, 1\}$. If $b(z_0) \neq 0$, then $g(z_0) \neq \infty$, and since z_0 is not a singularity of $d\sigma^2$, we have $dg(z_0) \neq 0$, and hence $h(z_0) \neq 0$. If $b(z_0) = 0$, then $a(z_0) \neq 0$ and $b'(z_0) \neq 0$, so $h(z_0) \neq 0$. Hence the only root of the polynomial $h(z)$ is 1:

$$h(z) = k\,(z-1)^m, \quad m \in \boldsymbol{Z}^+, \quad k \in \boldsymbol{C} \setminus \{0\}.$$

We claim that $m = 1$. If $b(1) \neq 0$, then g (or $d\sigma^2$) having order 1 at $p_3 = 1$ means that $m = 1$, by (A.3) and the above form of $dg(z)$. Suppose $b(1) = 0$. Then we have $b(z) = (z-1)^l \hat{b}(z)$, where $\hat{b}(z)$ is a polynomial in z with $\hat{b}(1) \neq 0$ and $l \in \boldsymbol{Z}^+$. Furthermore, $h(z) = (z-1)^{l-1}\hat{h}(z)$, where $\hat{h}(z)$ is a polynomial with $\hat{h}(1) \neq 0$, since $a(1) \neq 0$. So $m = l - 1$. Then, by (A.3), we have $m = 1$.

Suppose that $\deg b \geq 1$. Since $\deg a \geq \deg b$, the top term of $h(z)$ must vanish. Thus we have $\mu = \deg b - \deg a \in \boldsymbol{Z}$, contradicting that $\beta_1, \beta_2 \notin \boldsymbol{Z}$. So $b(z)$ is constant. Similarly, if $\deg a \geq 2$, then $\mu = -\deg a \in \boldsymbol{Z}$. Hence $\deg a = 1$, and g is as in (A.4). $\beta_1 = |\mu| - 1$ and $\beta_2 = |\mu+1| - 1$ follow from (A.3). □

Appendix B. Consider $d\sigma^2 \in \mathrm{Met}_1(\bar{M})$ with divisor D as in (A.1) in Appendix A and developing map g. Since the Fubini-Study metric of $\boldsymbol{CP}^1$ is invariant under the deck transformation group $\pi_1(M)$ of $M := \bar{M} \setminus \{p_1, \dots, p_n\}$, there is a representation

$$\rho_g : \pi_1(M) \to \mathrm{SU}(2)$$

such that

$$g \circ T^{-1} = \rho_g(T) \star g \quad (T \in \pi_1(M)) .$$

The metric $d\sigma^2$ is called *reducible* if the image of ρ_g is a commutative subgroup in SU(2), and is called *irreducible* otherwise. Since the maximal abelian subgroup of SU(2) is U(1), the image of ρ_g for a reducible $d\sigma^2$ lies in a subgroup conjugate to U(1), and this image might be simply the identity. We call a reducible metric $d\sigma^2$ $\mathcal{H}^3$*-reducible* if the image of ρ_g is the identity, and $\mathcal{H}^1$*-reducible* otherwise (for more on this, see [12, Section 3]).

Let $p_1, \dots, p_{n-1}$ be distinct points in $\boldsymbol{C}$ and $p_n = \infty$. We set

$$M_{p_1,\dots,p_n} := \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, p_2, \dots, p_n\} \qquad (p_n = \infty) ,$$

and $\tilde{M}_{p_1,\dots,p_n}$ its universal cover.

The following assertion was needed in the proof of Theorem 1.1.

PROPOSITION B.1. *Let* $p_1, \dots, p_{n-1}$ *be mutually distinct points of* $\boldsymbol{C}$, *and let* $d\sigma^2$ *be a metric of constant curvature* 1 *defined on* $M_{p_1,\dots,p_n}$ $(p_n = \infty)$ *which has a conical singularity at each* p_j. *Suppose that* $d\sigma^2$ *is reducible and* $\beta_j := \mathrm{ord}_{p_j} d\sigma^2$ *satisfy*

$$\beta_1, \dots, \beta_m \notin \boldsymbol{Z} , \quad \beta_{m+1}, \dots, \beta_{n-1} \in \boldsymbol{Z} , \quad \beta_n \notin \boldsymbol{Z} ,$$

for some $m \le n-1$. *Then the metric* $d\sigma^2$ *has a developing map* $g : \tilde{M}_{p_1,\dots,p_n} \to \boldsymbol{C} \cup \{\infty\}$ *given by*

$$g = (z-p_1)^{\tau_1} \cdots (z-p_m)^{\tau_m} r(z) \quad (\tau_1, \dots, \tau_m \in \boldsymbol{R} \setminus \boldsymbol{Z}) ,$$

where $r(z)$ *is a rational function on* $\boldsymbol{C} \cup \{\infty\}$ *and*

$$(z-p_1)^{\tau_1} \cdots (z-p_m)^{\tau_m} := \exp\left(\sum_{j=1}^{m} \tau_j \int_{z_0}^{z} \frac{dz}{z-p_j} \right) \quad (z \in M_{p_1,\dots,p_n})$$

for some base point $z_0 \in M_{p_1,\dots,p_n}$.

PROOF. $d\sigma^2$ is reducible only if the image of the representation ρ_g is simultaneously diagonalizable, so we may choose a developing map $g : \tilde{M}_{p_1,\dots,p_n} \to \boldsymbol{CP}^1$ such that

$$\rho_g(T) = \begin{pmatrix} e^{i\theta_T} & 0 \\ 0 & e^{-i\theta_T} \end{pmatrix} . \tag{B.1}$$

Thus we have

$$\log(g \circ T^{-1}) = \log(g) + 2i\theta_T .$$

Differentiating this gives

$$d\log(g \circ T^{-1}) = d\log(g) ,$$

which implies that $d\log(g)$ is single-valued on $M_{p_1,\dots,p_n}$.

On the other hand, by Proposition 4 in [3], there is a complex coordinate w around each end p_j such that

$$a_j \star g = (w - p_j)^{\tau_j} \quad (\tau_j \in \boldsymbol{R} \setminus \{0, \pm 1\}) \tag{B.2}$$

for some $a_j \in \mathrm{SU}(2)$ $(j = 1, \ldots, n)$. Let T_j be the deck transformation of $\tilde{M}_{p_1,\ldots,p_n}$ corresponding to a loop surrounding p_j. Then

$$\rho_g(T_j) \neq \pm\,\mathrm{id} \qquad \text{for } j = 1, \ldots, m \text{ and } j = n\,.$$

Hence $\tau_j \notin \boldsymbol{Z}$ when $j \leq m$ and $j = n$. By (B.1), a_j in (B.2) is diagonal, so

$$g(p_j) = 0 \quad \text{or} \quad \infty \qquad (j = 1, \ldots, m, n)\,.$$

Hence $d\log(g)$ has poles of order 1 at $p_1, \ldots, p_m$, and thus

$$d\log(g) = \frac{dg}{g} = \frac{\tau_1\, dz}{z - p_1} + \cdots + \frac{\tau_m\, dz}{z - p_m} + u(z)\, dz\,,$$

where $u(z)$ is meromorphic. Integrating this gives the assertion. □

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