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Mean Curvature Flow in a Ricci Flow Background

John Lott*

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720-3840, USA.
E-mail: lott@math.berkeley.edu

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Abstract: Following work of Ecker (Comm Anal Geom 15:1025–1061, 2007), we consider a weighted Gibbons-Hawking-York functional on a Riemannian manifold-with-boundary. We compute its variational properties and its time derivative under Perelman's modified Ricci flow. The answer has a boundary term which involves an extension of Hamilton's differential Harnack expression for the mean curvature flow in Euclidean space. We also derive the evolution equations for the second fundamental form and the mean curvature, under a mean curvature flow in a Ricci flow background. In the case of a gradient Ricci soliton background, we discuss mean curvature solitons and Huisken monotonicity.

1. Introduction

In [3], Ecker found a surprising link between Perelman's $\mathcal{W}$-functional for Ricci flow and Hamilton's differential Harnack expression for mean curvature flow in $\mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, he considered the integral over Ω of Perelman's $\mathcal{W}$-integrand [15, Prop. 9.1], the latter being defined using a positive solution u of the backward heat equation. With an appropriate boundary condition on u, the time-derivative of the integral has two terms. The first term is the integral over Ω of a nonnegative quantity, as in Perelman's work. The second term is an integral over $\partial\Omega$. Ecker showed that the integrand of the second term is Hamilton's differerential Harnack expression for mean curvature flow [6]. Hamilton had proven that this expression is nonnegative for a weakly convex mean curvature flow in $\mathbb{R}^n$.

After performing diffeomorphisms generated by $\nabla \ln u$, the boundary of Ω evolves by mean curvature flow in $\mathbb{R}^n$. Ecker conjectured that his $\mathcal{W}$-functional for Ω is nondecreasing in time under the mean curvature flow of any compact hypersurface in $\mathbb{R}^n$. This conjecture is still open.

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In the present paper we look at analogous relations for mean curvature flow in an arbitrary Ricci flow background. We begin with a version of Perelman's $\mathcal{F}$-functional [15, Sect. 1.1] for a manifold-with-boundary M. We add a boundary term to the interior integral of $\mathcal{F}$ so that the result I_∞ has nicer variational properties. One can think of I_∞ as a weighted version of the Gibbons-Hawking-York functional [5,17], where "weighted" refers to a measure $e^{-f}\, dV_g$. We compute how I_∞ changes under a variation of the Riemannian metric g (Proposition 2). We also compute the time-derivative of I_∞ when g evolves by Perelman's modified Ricci flow (Theorem 2). We derive the evolution equations for the second fundamental form of ∂M and the mean curvature of ∂M under the modified Ricci flow (Theorem 3).

After performing diffeomorphisms generated by $-\nabla f$, the Riemannian metric on M evolves by the standard Ricci flow and ∂M evolves by mean curvature flow.

Theorem 1. *If $u = e^{-f}$ is a solution to the conjugate heat equation*

$$\frac{\partial u}{\partial t} = (-\triangle + R)u \tag{1.1}$$

on M, satisfying the boundary condition

$$H + e_0 f = 0 \tag{1.2}$$

on ∂M, then

$$\begin{aligned}\frac{dI_\infty}{dt} &= 2\int_M |\operatorname{Ric} + \operatorname{Hess}(f)|^2\, e^{-f}\, dV \\ &+ 2\int_{\partial M} \left(\frac{\partial H}{\partial t} - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) \right. \\ &\left. + 2\operatorname{Ric}(e_0, \widehat{\nabla} f) - \frac{1}{2} e_0 R - H \operatorname{Ric}(e_0, e_0) \right) e^{-f}\, dA. \end{aligned} \tag{1.3}$$

Here R is the scalar curvature of M, $\widehat{\nabla}$ is the boundarywise derivative, e_0 is the inward unit normal on ∂M, H is the mean curvature of ∂M and $A(\cdot, \cdot)$ is the second fundamental form of ∂M.

Remark 1. If $g(t)$ is flat Ricci flow on $\mathbb{R}^n$ then the boundary integrand

$$\frac{\partial H}{\partial t} - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + 2\operatorname{Ric}(e_0, \widehat{\nabla} f) - \frac{1}{2} e_0 R - H \operatorname{Ric}(e_0, e_0) \tag{1.4}$$

becomes Hamilton's differential Harnack expression [6, Def. 4.1]

$$Z = \frac{\partial H}{\partial t} + 2\langle V, \widehat{\nabla} H\rangle + A(V, V) \tag{1.5}$$

when the vector field V of (1.5) is taken to be $-\widehat{\nabla} f$. In this flat case, Theorem 1 is the $\mathcal{F}$-version of Ecker's result.

Remark 2. Writing (1.4) as

$$\frac{\partial H}{\partial t} + 2\langle V, \widehat{\nabla} H\rangle - H \operatorname{Ric}(e_0, e_0) + \left\langle e_0, \nabla_V V - \frac{1}{2}\nabla R - 2\operatorname{Ric}(V, \cdot) \right\rangle \tag{1.6}$$

(with $V = -\widehat{\nabla} f$) indicates a link to $\mathcal{L}$-geodesics, since $\nabla_V V - \frac{1}{2}\nabla R - 2\operatorname{Ric}(V, \cdot)$ (with $V = \gamma'$) enters in the Euler-Lagrange equation for the steady version of Perelman's $\mathcal{L}$-length [13, Sect. 4].

Important examples of Ricci flow solutions come from gradient solitons. With such a background geometry, there is a natural notion of a mean curvature soliton. We show that (1.4) vanishes on such solitons (Proposition 7), in analogy to what happens for mean curvature flow in $\mathbb{R}^n$ [6, Lem. 3.2]. On the other hand, in the case of convex mean curvature flow in $\mathbb{R}^n$, Hamilton showed that the shrinker version of (1.5) is nonnegative for all vector fields V [6, Thm. 1.1]. We cannot expect a direct analog for mean curvature flow in an arbitrary gradient shrinking soliton background, since local convexity of the hypersurface will generally not be preserved by the flow.

Magni-Mantegazza-Tsatis [14] showed that Huisken's monotonicity formula for mean curvature flow in $\mathbb{R}^n$ [11, Thm. 3.1] extends to mean curvature flow in a gradient Ricci soliton background. We give two proofs of this result (Proposition 8). The first one is computational and is essentially the same as the proof in [14]; the second one is more conceptual.

In this paper we mostly deal with "steady" quantities : $\mathcal{F}$-functional, gradient steady soliton, etc. There is an evident extension to the shrinking or expanding case.

The structure of the paper is as follows. Section 2 has some preliminary material. In Sect. 3 we define the weighted Gibbons-Hawking-York action and study its variational properties. Section 4 contains the derivation of the evolution equations for the second fundamental form and the mean curvature of the boundary, when the Riemannian metric of the interior evolves by the modified Ricci flow. In Sect. 5 we consider mean curvature flow in a Ricci flow background and prove Theorem 1. In the case of a gradient steady Ricci soliton background, we discuss mean curvature solitons and Huisken monotonicity.

More detailed descriptions appear at the beginnings of the sections.

2. Preliminaries

In this section we gather some useful formulas about the geometry of a Riemannian manifold-with-boundary. We will use the Einstein summation convention freely.

Let M be a smooth compact n-dimensional manifold-with-boundary. We denote local coordinates for M by $\{x^\alpha\}_{\alpha=0}^{n-1}$. Near ∂M, we take x^0 to be a local defining function for ∂M. We denote the local coordinates for ∂M by $\{x^i\}_{i=1}^n$.

If g is a Riemannian metric on M then we let ∇ denote the Levi-Civita connection on TM and we let $\widehat{\nabla}$ denote the Levi-Civita connection on $T\partial M$. Algorithmically, when taking covariant derivatives we use $\Gamma^\alpha{}_{\beta\gamma}$ to act on indices in $\{0, \dots, n-1\}$ and $\widehat{\Gamma}^i{}_{jk}$ to act on indices in $\{1, \dots, n-1\}$. We let dV denote the volume density on M and we let dA denote the area density on ∂M.

Let e_0 denote the inward-pointing unit normal field on ∂M. For calculations, we choose local coordinates near a point of ∂M so that $\partial_0\big|_{\partial M} = e_0$ and for all $(x^1, \dots, x^{n-1})$, the curve $t \to (t, x^1, \dots, x^{n-1})$ is a unit-speed geodesic which meets ∂M orthogonally. In these coordinates, we can write

$$g = (dx^0)^2 + \sum_{i,j=1}^{n-1} g_{ij}(x^0, x^1 \dots, x^{n-1})\, dx^i\, dx^j. \tag{2.1}$$

We write $A = (A_{ij})$ for the second fundamental form of ∂M, so $A_{ij} = g(e_0, \nabla_{\partial_j}\partial_i)$, and we write $H = g^{ij}A_{ij}$ for the mean curvature. Then on ∂M, we have

$$A_{ij} = \Gamma_{0ij} = -\Gamma_{i0j} = -\Gamma_{ij0} = -\frac{1}{2}\partial_0 g_{ij}. \tag{2.2}$$

The Codazzi-Mainardi equation

$$R_{0ijk} = \widehat{\nabla}_j A_{ik} - \widehat{\nabla}_k A_{ij} \tag{2.3}$$

implies that

$$R_{0j} = \widehat{\nabla}_j H - \widehat{\nabla}_i A^i{}_j. \tag{2.4}$$

For later reference,

$$\nabla_i R_{0j} = \widehat{\nabla}_i R_{0j} - \Gamma^k{}_{0i} R_{kj} - \Gamma^0{}_{ji} R_{00} = \widehat{\nabla}_i R_{0j} + A^k{}_i R_{kj} - A_{ij} R_{00}. \tag{2.5}$$

For any symmetric 2-tensor field v, we have

$$\nabla_0 \left(g^{ij} v_{ij} \right) = g^{ij} \partial_0 v_{ij} + 2A^{ij} v_{ij} = g^{ij} \nabla_0 v_{ij} \tag{2.6}$$

on ∂M. More generally, on ∂M, g^{ij} is covariantly constant with respect to ∇_0.

If $f \in C^\infty(M)$ then on ∂M, we have

$$\nabla_i \nabla_j f = \widehat{\nabla}_i \widehat{\nabla}_j f - \Gamma^0{}_{ji} \nabla_0 f = \widehat{\nabla}_i \widehat{\nabla}_j f - A_{ij} \nabla_0 f \tag{2.7}$$

and

$$\nabla_i \nabla_0 f = \widehat{\nabla}_i \nabla_0 f - \Gamma^k{}_{0i} \widehat{\nabla}_k f = \widehat{\nabla}_i \nabla_0 f + A_i{}^k \, \widehat{\nabla}_k f \tag{2.8}$$

3. Variation of the Weighted Gibbons-Hawking-York Action

In this section we define the weighted Gibbons-Hawking-York action I_∞ and study its variational properties.

In Subsect. 3.1 we list how some geometric quantities vary under changes of the metric. As a warmup, in Subsect. 3.2 we rederive the variational formula for the Gibbons-Hawking action. In Subsect. 3.3 we derive the variational formula for the weighted Gibbons-Hawking action. In Subsect. 3.4 we compute its time derivative under the modified Ricci flow.

3.1. Variations. Let $\delta g_{\alpha\beta} = v_{\alpha\beta}$ be a variation of g. We write $v = g^{\alpha\beta} v_{\alpha\beta}$. We collect some variational equations:

$$\delta R = \nabla_\alpha \nabla_\beta v^{\alpha\beta} - \nabla_\alpha \nabla^\alpha v - v^{\alpha\beta} R_{\alpha\beta}, \tag{3.1}$$

$$\delta (dV) = \frac{v}{2} \, dV, \tag{3.2}$$

$$\delta (e_0) = -\frac{1}{2} v_0{}^0 e_0 - v_0{}^k \partial_k, \tag{3.3}$$

$$\begin{aligned} \delta A_{ij} &= \frac{1}{2} \left(\nabla_i v_{0j} + \nabla_j v_{0i} - \nabla_0 v_{ij} + A_{ij} v_{00} \right) \\ &= \frac{1}{2} \left(\widehat{\nabla}_i v_{0j} + A_{ki} v^k{}_j + \widehat{\nabla}_j v_{0i} + A_{kj} v^k{}_i - \nabla_0 v_{ij} - A_{ij} v_{00} \right), \end{aligned} \tag{3.4}$$

$$\delta H = -v^{ij} A_{ij} + g^{ij} \delta A_{ij} = \widehat{\nabla}_i v_0{}^i - \frac{1}{2} \left(g^{ij} \nabla_0 v_{ij} + H v_{00} \right), \tag{3.5}$$

$$\delta (dA) = \frac{1}{2} \, v_i{}^i \, dA. \tag{3.6}$$

3.2. Gibbons-Hawking-York action.

Definition 1. *The Gibbons-Hawking-York action* [5,17] *is*

$$I_{GHY}(g) = \int_M R\, dV + 2\int_{\partial M} H\, dA. \tag{3.7}$$

If $n = 2$ then $I_{GHY}(g) = 4\pi\chi(M)$.

Proposition 1.

$$\delta I_{GHY} = -\int_M v^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) dV - \int_{\partial M} v^{ij}\left(A_{ij} - g_{ij}H\right) dA. \tag{3.8}$$

Proof. From (3.1) and (3.2),

$$\delta\int_M R\, dV = -\int_M v^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) dV - \int_{\partial M}\left(\nabla_\alpha v_0{}^\alpha - \nabla_0 v\right) dA. \tag{3.9}$$

On the boundary,

$$\begin{aligned}\nabla_\alpha v_0{}^\alpha - \nabla_0 v &= \nabla_i v_0{}^i - \nabla_0 v_i{}^i \\ &= \widehat{\nabla}_i v_0{}^i - \Gamma^j{}_{0i}\, v_j{}^i + \Gamma^i{}_{0i}\, v_0{}^0 - \nabla_0(g^{ij}v_{ij}) \\ &= \widehat{\nabla}_i v_0{}^i + A^{ij}v_{ij} - Hv_0{}^0 - g^{ij}\nabla_0 v_{ij}.\end{aligned} \tag{3.10}$$

From (3.5) and (3.6),

$$\delta(H\, dA) = \left(\widehat{\nabla}_i v_0{}^i + \frac{1}{2}\left(-g^{ij}\nabla_0 v_{ij} - Hv_{00} + Hv_i{}^i\right)\right) dA. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) gives

$$\begin{aligned}\delta I_{GHY} &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) dV + \int_{\partial M}\widehat{\nabla}_i v_0{}^i\, dA \\ &\quad - \int_{\partial M} v^{ij}\left(A_{ij} - g_{ij}H\right) dA \\ &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) dV - \int_{\partial M} v^{ij}\left(A_{ij} - g_{ij}H\right) dA.\end{aligned} \tag{3.12}$$

This proves the proposition. □

If the induced metric $g_{\partial M}$ is held fixed under the variation then v_{ij} vanishes on ∂M and $\delta I_{GHY} = -\int_M v^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) dV$ is an interior integral. This was the motivation for Gibbons and Hawking to introduce the second term on the right-hand side of (3.7).

Suppose that $n > 2$. We can say that with a fixed induced metric $g_{\partial M}$ on ∂M, the critical points of I_{GHY} are the Ricci-flat metrics on M that induce $g_{\partial M}$. On the other hand, if we consider all variations $v_{\alpha\beta}$ then the critical points are the Ricci-flat metrics on M with totally geodesic boundary.

3.3. Weighted Gibbons-Hawking-York action. Given $f \in C^\infty(M)$, consider the smooth metric-measure space $\mathcal{M} = \left(M, g, e^{-f} dV\right)$. As is now well understood, the analog of the Ricci curvature for $\mathcal{M}$ is the Bakry-Emery tensor

$$\mathrm{Ric}_\infty = \mathrm{Ric} + \mathrm{Hess}(f). \tag{3.13}$$

(There is also a notion of Ric_N for $N \in [1, \infty]$ but we only consider the case $N = \infty$.) As explained by Perelman [15, Sect. 1.3], the analog of the scalar curvature is

$$R_\infty = R + 2\triangle f - |\nabla f|^2. \tag{3.14}$$

Considering the first variation formula for the integral of e^{-f} over a moving hypersurface, one sees that the analog of the mean curvature is

$$H_\infty = H + e_0 f. \tag{3.15}$$

On the other hand, the analog of the second fundamental form is just $A_\infty = A$.

If $\partial M = \emptyset$ then Perelman's $\mathcal{F}$-functional is the weighted total scalar curvature $\mathcal{F} = \int_M R_\infty \, e^{-f} \, dV$.

Definition 2. The weighted Gibbons-Hawking-York action is

$$I_\infty(g, f) = \int_M R_\infty \, e^{-f} \, dV \, + \, 2 \int_{\partial M} H_\infty \, e^{-f} \, dA. \tag{3.16}$$

We write a variation of f as $\delta f = h$. Note that

$$\delta \left(e^{-f} \, dV \right) = \left(\frac{v}{2} - h \right) \, e^{-f} \, dV. \tag{3.17}$$

Proposition 2. *If* $\frac{v}{2} - h = 0$ *then*

$$\begin{aligned} \delta I_\infty = &- \int_M v^{\alpha\beta} \left(R_{\alpha\beta} + \nabla_\alpha \nabla_\beta f \right) \, e^{-f} \, dV \\ &- \int_{\partial M} \left(v^{ij} A_{ij} + v^{00} (H + e_0 f) \right) \, e^{-f} \, dA. \end{aligned} \tag{3.18}$$

Proof. One can check that

$$\begin{aligned} \delta(\triangle f) &= \triangle h - \left(\nabla_\alpha v^{\alpha\beta} \right) \nabla_\beta f - v^{\alpha\beta} \nabla_\alpha \nabla_\beta f + \frac{1}{2} \langle \nabla f, \nabla v \rangle \\ &= \frac{1}{2} \triangle v - \left(\nabla_\alpha v^{\alpha\beta} \right) \nabla_\beta f - v^{\alpha\beta} \nabla_\alpha \nabla_\beta f + \frac{1}{2} \langle \nabla f, \nabla v \rangle \end{aligned} \tag{3.19}$$

and

$$\delta \left(|\nabla f|^2 \right) = 2 \langle \nabla f, \nabla h \rangle - v^{\alpha\beta} \nabla_\alpha f \nabla_\beta f = \langle \nabla f, \nabla v \rangle - v^{\alpha\beta} \nabla_\alpha f \nabla_\beta f. \tag{3.20}$$

Then

$$\delta R_\infty = -v^{\alpha\beta} \left(R_{\alpha\beta} + \nabla_\alpha \nabla_\beta f \right) \, e^{-f} + \nabla_\beta \left(e^{-f} \left(\nabla_\alpha v^{\beta\alpha} - v^{\beta\alpha} \nabla_\alpha f \right) \right). \tag{3.21}$$

Hence

$$\begin{aligned}\delta\left(\int_M R_\infty\, e^{-f}\, dV\right) &= \int_M \delta(R_\infty)\, e^{-f}\, dV \\ &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f\right)\, e^{-f}\, dV \\ &\quad -\int_{\partial M}\left(\nabla_\alpha v^{0\alpha}-v^{0\alpha}\nabla_\alpha f\right)\, e^{-f}\, dA. \end{aligned} \tag{3.22}$$

On the boundary,

$$\nabla_\alpha v^{0\alpha}-v^{0\alpha}\nabla_\alpha f=\nabla_0 v^{00}-v^{00}(H+\nabla_0 f)+\widehat{\nabla}_i v^{0i}-v^{0i}\widehat{\nabla}_i f+v^{ij}A_{ij}. \tag{3.23}$$

Next,

$$\delta(e_0 f)=-\frac{1}{2}v_0{}^0\,\nabla_0 f-v_0{}^i\,\widehat{\nabla}_i f+\nabla_0 h=-\frac{1}{2}v_0{}^0\,\nabla_0 f-v_0{}^i\,\widehat{\nabla}_i f+\frac{1}{2}\nabla_0 v \tag{3.24}$$

and one finds that

$$\begin{aligned}\delta\int_{\partial M} H_\infty\, e^{-f}\, dA &= \int_{\partial M}\delta H_\infty\, e^{-f}\, dA+\int_{\partial M} H_\infty\left(-\frac{1}{2}v+\frac{1}{2}v_i{}^i\right)\, e^{-f}\, dA \\ &= \int_{\partial M}\left(\widehat{\nabla}_i v^{0i}-v^{0i}\widehat{\nabla}_i f-v^{00}(H+e_0 f)+\frac{1}{2}\nabla_0 v^{00}\right)\, e^{-f}\, dA. \end{aligned} \tag{3.25}$$

Combining (3.22), (3.23) and (3.25) gives

$$\begin{aligned}\delta I_\infty &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f\right)\, e^{-f}\, dV-\int_{\partial M}\left(v^{ij}A_{ij}+v^{00}(H+e_0 f)\right)\, e^{-f}\, dA \\ &\quad +\int_{\partial M}\left(\widehat{\nabla}_i v^{0i}-v^{0i}\widehat{\nabla}_i f\right)\, e^{-f}\, dA \\ &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f\right)\, e^{-f}\, dV-\int_{\partial M}\left(v^{ij}A_{ij}+v^{00}(H+e_0 f)\right)\, e^{-f}\, dA \\ &\quad +\int_{\partial M}\widehat{\nabla}_i\left(e^{-f}v^{0i}\right)\, dA \\ &= -\int_M v^{\alpha\beta}\left(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f\right)\, e^{-f}\, dV-\int_{\partial M}\left(v^{ij}A_{ij}+v^{00}(H+e_0 f)\right)\, e^{-f}\, dA. \end{aligned} \tag{3.26}$$

This proves the proposition. □

Remark 3. If $\partial M=\emptyset$ then Proposition 2 appears in [15, Sect. 1.1].

The variations in Proposition 2 all fix the measure $e^{-f}\,dV$. If we also fix an induced metric $g_{\partial M}$ on ∂M then the critical points of I_∞ are gradient steady solitons on M that satisfy $H+e_0 f=0$ on ∂M. On the other hand, if we allow variations that do not fix the boundary metric then the critical points are gradient steady solitons on M with totally geodesic boundary and for which f satisfies Neumann boundary conditions.

3.4. Time-derivative of the action.

Assumption 1. *Hereafter we assume that $H + e_0 f = 0$ on ∂M.*

Then on ∂M, we have

$$\nabla_i \nabla_j f = \widehat{\nabla}_i \widehat{\nabla}_j f + H A_{ij} \tag{3.27}$$

and

$$\nabla_i \nabla_0 f = -\widehat{\nabla}_i H + A_i^{\ k} \widehat{\nabla}_k f. \tag{3.28}$$

Theorem 2. *Under the assumptions*

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} + \mathrm{Hess}(f)) \tag{3.29}$$

and

$$\frac{\partial f}{\partial t} = -\triangle f - R, \tag{3.30}$$

we have

$$\begin{aligned} \frac{dI_\infty}{dt} = 2 \int_M |\mathrm{Ric} + \mathrm{Hess}(f)|^2 \, e^{-f} \, dV + 2 \int_{\partial M} \Big(\widehat{\triangle} H - 2 \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle \\ + A(\widehat{\nabla} f, \widehat{\nabla} f) + A^{ij} A_{ij} H + A^{ij} R_{ij} + 2 R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i} \Big) \, e^{-f} \, dA. \end{aligned} \tag{3.31}$$

If $\left(R_{ij} + \nabla_i \nabla_j f \right) \Big|_{\partial M} = 0$ *and* $\left(R_{i0} + \nabla_i \nabla_0 f \right) \Big|_{\partial M} = 0$ *then*

$$\widehat{\triangle} H - 2 \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + A^{ij} A_{ij} H + A^{ij} R_{ij} + 2 R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i} = 0. \tag{3.32}$$

Proof. Equations (3.29) and (3.30) imply that $e^{-f(t)} dV_{g(t)}$ is constant in t. Then Proposition 2 implies that

$$\frac{dI_\infty}{dt} = 2 \int_M |\mathrm{Ric} + \mathrm{Hess}(f)|^2 \, e^{-f} \, dV + 2 \int_{\partial M} A^{ij} \left(R_{ij} + \nabla_i \nabla_j f \right) e^{-f} \, dA. \tag{3.33}$$

Lemma 1. *On ∂M,*

$$\begin{aligned} A^{ij} \left(R_{ij} + \nabla_i \nabla_j f \right) e^{-f} - \widehat{\nabla}_i \left(\left(R^{i0} + \nabla^i \nabla^0 f \right) e^{-f} \right) \\ = \left(\widehat{\triangle} H - 2 \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + A^{ij} A_{ij} H + A^{ij} R_{ij} + 2 R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i} \right) e^{-f}. \end{aligned} \tag{3.34}$$

Proof. As

$$A^{ij} (\widehat{\nabla}_i \widehat{\nabla}_j f) e^{-f} = \widehat{\nabla}_i \left(A^{ij} (\widehat{\nabla}_j f) e^{-f} \right) - \left(\widehat{\nabla}_i A^{ij} \right) \widehat{\nabla}_j f e^{-f} + A(\widehat{\nabla} f, \widehat{\nabla} f) e^{-f}, \tag{3.35}$$

we have

$$\begin{aligned} & A^{ij}\left(R_{ij}+\nabla_i\nabla_j f\right)\, e^{-f} \\ &= \left(A^{ij}R_{ij}+A^{ij}\widehat{\nabla}_i\widehat{\nabla}_j f+A^{ij}A_{ij}H\right)\, e^{-f} \\ &= \left(A^{ij}R_{ij}-\left(\widehat{\nabla}_i A^{ij}\right)\widehat{\nabla}_j f+A(\widehat{\nabla} f,\widehat{\nabla} f)+A^{ij}A_{ij}H\right)\, e^{-f} \\ &\quad +\widehat{\nabla}_i\left(A^{ij}(\widehat{\nabla}_j f)e^{-f}\right). \end{aligned} \tag{3.36}$$

Adding

$$0=\left(\widehat{\triangle}H-\langle\widehat{\nabla} f,\widehat{\nabla}H\rangle\right)e^{-f}-\widehat{\nabla}_i\left(e^{-f}\widehat{\nabla}^i H\right) \tag{3.37}$$

and using (2.4) gives

$$\begin{aligned} & A^{ij}\left(R_{ij}+\nabla_i\nabla_j f\right)\, e^{-f} \\ &= \left(\widehat{\triangle}H-\langle\widehat{\nabla} f,\widehat{\nabla}H\rangle+A^{ij}R_{ij}-\left(\widehat{\nabla}_i A^{ij}\right)\widehat{\nabla}_j f+A(\widehat{\nabla} f,\widehat{\nabla} f)+A^{ij}A_{ij}H\right)\, e^{-f} \\ &\quad +\widehat{\nabla}_i\left(\left(A^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\right)e^{-f}\right) \\ &= \left(\widehat{\triangle}H-2\langle\widehat{\nabla} f,\widehat{\nabla}H\rangle+A^{ij}R_{ij}+R^{0j}\widehat{\nabla}_j f+A(\widehat{\nabla} f,\widehat{\nabla} f)+A^{ij}A_{ij}H\right)\, e^{-f} \\ &\quad +\widehat{\nabla}_i\left(\left(A^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\right)e^{-f}\right). \end{aligned} \tag{3.38}$$

Using (3.28),

$$\begin{aligned} \widehat{\nabla}_i\left(\left(A^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\right)e^{-f}\right) &= \widehat{\nabla}_i\left(\left(R^{i0}+\nabla^i\nabla^0 f\right)e^{-f}\right)-\widehat{\nabla}_i\left(R^{0i}e^{-f}\right) \\ &= \widehat{\nabla}_i\left(\left(R^{i0}+\nabla^i\nabla^0 f\right)e^{-f}\right) \\ &\quad +\left(-\widehat{\nabla}_i R^{0i}+R^{0i}\widehat{\nabla}_i f\right)e^{-f}. \end{aligned} \tag{3.39}$$

The lemma follows. □

Equation (3.31) follows from (3.33), along with integrating both sides of (3.34) over ∂M. If $\left(R_{ij}+\nabla_i\nabla_j f\right)\Big|_{\partial M}=0$ and $\left(R_{i0}+\nabla_i\nabla_0 f\right)\Big|_{\partial M}=0$ then (3.32) follows from (3.34). □

4. Evolution Equations for the Boundary Geometry Under a Modified Ricci Flow

In this section we consider a manifold-with-boundary whose Riemannian metric evolves by the modified Ricci flow. We derive the evolution equations for the second fundamental form and the mean curvature of the boundary.

Theorem 3. *Under the assumptions*

$$\frac{\partial g}{\partial t}=-2(\mathrm{Ric}+\mathrm{Hess}(f)) \tag{4.1}$$

and

$$\frac{\partial f}{\partial t} = -\triangle f - R, \tag{4.2}$$

on ∂M we have

$$\frac{\partial g_{ij}}{\partial t} = -(\mathcal{L}_{\widehat{\nabla} f} g)_{ij} - 2R_{ij} - 2H A_{ij}, \tag{4.3}$$

$$\begin{aligned}\frac{\partial A_{ij}}{\partial t} &= (\widehat{\triangle} A)_{ij} - (\mathcal{L}_{\widehat{\nabla} f} A)_{ij} - A^k{}_i \, R^l{}_{klj} - A^k{}_j \, R^l{}_{kli} + 2A^{kl} R_{kilj} \\ &\quad -2H A_{ik} A^k{}_j + A^{kl} A_{kl} A_{ij} + \nabla_0 R_{0i0j}\end{aligned} \tag{4.4}$$

and

$$\frac{\partial H}{\partial t} = \widehat{\triangle} H - \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + 2A^{ij} R_{ij} + A^{ij} A_{ij} H + \nabla_0 R_{00}. \tag{4.5}$$

Proof. Using (3.27),

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= -2R_{ij} - 2\nabla_i \nabla_j f \\ &= -2R_{ij} - 2\widehat{\nabla}_i \widehat{\nabla}_j f - 2H A_{ij} \\ &= -(\mathcal{L}_{\widehat{\nabla} f} g)_{ij} - 2R_{ij} - 2H A_{ij}.\end{aligned} \tag{4.6}$$

Next, from (3.4),

$$\begin{aligned}\frac{\partial A_{ij}}{\partial t} &= -\nabla_i \left(R_{j0} + \nabla_j \nabla_0 f\right) - \nabla_j \left(R_{i0} + \nabla_i \nabla_0 f\right) + \nabla_0 \left(R_{ij} + \nabla_i \nabla_j f\right) \\ &\quad -A_{ij} \left(R_{00} + \nabla_0 \nabla_0 f\right).\end{aligned} \tag{4.7}$$

Now

$$\nabla_0 \nabla_i \nabla_j f - \nabla_j \nabla_i \nabla_0 f = \nabla_0 \nabla_j \nabla_i f - \nabla_j \nabla_0 \nabla_i f = -R^k{}_{i0j} \widehat{\nabla}_k f - R^0{}_{i0j} \nabla_0 f \tag{4.8}$$

and

$$\begin{aligned}\nabla_i \nabla_j \nabla_0 f &= \widehat{\nabla}_i \nabla_j \nabla_0 f - \Gamma^0{}_{ji} \nabla_0 \nabla_0 f - \Gamma^k{}_{0i} \nabla_j \nabla_k f \\ &= \widehat{\nabla}_i \left(\widehat{\nabla}_j \nabla_0 f + A^k{}_j \widehat{\nabla}_k f\right) - A_{ij} \nabla_0 \nabla_0 f + A^k{}_i \left(\widehat{\nabla}_j \widehat{\nabla}_k f + H A_{jk}\right) \\ &= -\widehat{\nabla}_i \widehat{\nabla}_j H + \left(\widehat{\nabla}_i A^k{}_j\right) \widehat{\nabla}_k f + A^k{}_j \, \widehat{\nabla}_i \widehat{\nabla}_k f - A_{ij} \nabla_0 \nabla_0 f \\ &\quad + A^k{}_i \, \widehat{\nabla}_j \widehat{\nabla}_k f + H A^k{}_i A_{jk}.\end{aligned} \tag{4.9}$$

Then

$$\begin{aligned}\frac{\partial A_{ij}}{\partial t} &= -\nabla_i R_{j0} - \nabla_j R_{i0} + \nabla_0 R_{ij} - A_{ij}\,(R_{00} + \nabla_0\nabla_0 f) - R^k{}_{i0j}\,\widehat{\nabla}_k f + H R^0{}_{i0j}\\ &\quad + \widehat{\nabla}_i\widehat{\nabla}_j H - \left(\widehat{\nabla}_i A^k{}_j\right)\widehat{\nabla}_k f - A^k{}_j\,\widehat{\nabla}_i\widehat{\nabla}_k f + A_{ij}\nabla_0\nabla_0 f\\ &\quad - A^k{}_i\,\widehat{\nabla}_j\widehat{\nabla}_k f - H A^k{}_i A_{jk}\\ &= \widehat{\nabla}_i\widehat{\nabla}_j H - \left(\widehat{\nabla}_k A_{ij}\right)\widehat{\nabla}_k f - A^k{}_j\,\widehat{\nabla}_i\widehat{\nabla}_k f - A^k{}_i\,\widehat{\nabla}_j\widehat{\nabla}_k f\\ &\quad - \nabla_i R_{j0} - \nabla_j R_{i0} + \nabla_0 R_{ij} - A_{ij} R_{00} + H R^0{}_{i0j} - H A^k{}_i A_{jk}\\ &= \widehat{\nabla}_i\widehat{\nabla}_j H - \left(\mathcal{L}_{\widehat{\nabla} f} A\right)_{ij} - \nabla_i R_{j0} - \nabla_j R_{i0} + \nabla_0 R_{ij}\\ &\quad - A_{ij} R_{00} + H R^0{}_{i0j} - H A^k{}_i A_{jk} \end{aligned} \tag{4.10}$$

A form of Simons' identity [16, Thm. 4.2.1] says that

$$\begin{aligned}\widehat{\nabla}_i\widehat{\nabla}_j H &= (\widehat{\triangle} A)_{ij} + \widehat{\nabla}_i R_{j0} + \widehat{\nabla}_j R_{i0} - \nabla_0 R_{ij}\\ &\quad + A^k{}_i R_{0k0j} + A^k{}_j R_{0k0i} - A_{ij} R_{00} + 2A^{kl} R_{kilj}\\ &\quad - H R_{0i0j} - H A^k{}_i A_{jk} + A^{kl} A_{kl} A_{ij} + \nabla_0 R_{0i0j}. \end{aligned} \tag{4.11}$$

(As a check, one can easily show that the contraction of both sides of (4.11) with g^{ij} is the same.) Then using (2.5),

$$\begin{aligned}\frac{\partial A_{ij}}{\partial t} &= (\widehat{\triangle} A)_{ij} - \left(\mathcal{L}_{\widehat{\nabla} f} A\right)_{ij} - \left(\nabla_i R_{j0} - \widehat{\nabla}_i R_{j0}\right) - \left(\nabla_j R_{i0} - \widehat{\nabla}_j R_{i0}\right)\\ &\quad - 2A_{ij} R_{00} + A^k{}_i R_{0k0j} + A^k{}_j R_{0k0i} + 2A^{kl} R_{kilj}\\ &\quad - 2H A^k{}_i A_{jk} + A^{kl} A_{kl} A_{ij} + \nabla_0 R_{0i0j}\\ &= (\widehat{\triangle} A)_{ij} - (\mathcal{L}_{\widehat{\nabla} f} A)_{ij} - A^k{}_i R^l{}_{klj} - A^k{}_j R^l{}_{kli} + 2A^{kl} R_{kilj}\\ &\quad - 2H A_{ik} A^k{}_j + A^{kl} A_{kl} A_{ij} + \nabla_0 R_{0i0j}. \end{aligned} \tag{4.12}$$

This proves the evolution equation for A.

Then

$$\begin{aligned}\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t}\left(g^{ij} A_{ij}\right) = 2\left(R_{ij} + \widehat{\nabla}_i\widehat{\nabla}_j f + H A_{ij}\right) A^{ij} + g^{ij}\frac{\partial A_{ij}}{\partial t}\\ &= 2A^{ij} R_{ij} + \widehat{\triangle} H - \left(g^{ij}(\mathcal{L}_{\widehat{\nabla} f} A)_{ij} - 2A^{ij}\widehat{\nabla}_i\widehat{\nabla}_j f\right) + A^{ij} A_{ij} H + g^{ij}\nabla_0 R_{0i0j}\\ &= \widehat{\triangle} H - \langle\widehat{\nabla} f, \widehat{\nabla} H\rangle + 2A^{ij} R_{ij} + A^{ij} A_{ij} H + \nabla_0 R_{00}. \end{aligned} \tag{4.13}$$

This proves the theorem. □

Proposition 3. *Under the assumptions*

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} + \mathrm{Hess}(f)) \tag{4.14}$$

and

$$\frac{\partial f}{\partial t} = -\triangle f - R, \tag{4.15}$$

we have

$$\frac{dI_\infty}{dt} = 2\int_M |\operatorname{Ric} + \operatorname{Hess}(f)|^2 \, e^{-f} \, dV$$
$$+ 2\int_{\partial M} \left(\frac{\partial H}{\partial t} - \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i} \widehat{\nabla}_i f - \frac{1}{2} \nabla_0 R - H R_{00} \right) e^{-f} \, dA. \tag{4.16}$$

If $\left(R_{ij} + \nabla_i \nabla_j f\right)\Big|_{\partial M} = 0$ *and* $\left(R_{i0} + \nabla_i \nabla_0 f\right)\Big|_{\partial M} = 0$ *then*

$$\frac{\partial H}{\partial t} - \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i} \widehat{\nabla}_i f - \frac{1}{2} \nabla_0 R - H R_{00} = 0. \tag{4.17}$$

Proof. From Theorem 3,

$$\widehat{\triangle} H - 2\langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + A^{ij} A_{ij} H + A^{ij} R_{ij} + 2R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i}$$
$$= \frac{\partial H}{\partial t} - \langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) - A^{ij} R_{ij} + 2R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i} - \nabla_0 R^{00}. \tag{4.18}$$

From the second contracted Bianchi identity,

$$\frac{1}{2} \nabla^0 R = \nabla_i R^{i0} + \nabla_0 R^{00} = \widehat{\nabla}_i R^{i0} + A^{ij} R_{ij} - H R^{00} + \nabla_0 R^{00}. \tag{4.19}$$

The proposition now follows from Theorem 2. □

5. Hypersurfaces in a Ricci Flow Background

In this section we consider mean curvature flow in a Ricci flow background. Mean curvature flow in a fixed Riemannian manifold was considered in [10].

In Subsect. 5.1 we translate the results of the previous sections from a fixed manifold-with-boundary, equipped with a modified Ricci flow, to an evolving hypersurface in a Ricci flow solution.

Starting in Subsect. 5.2, we consider mean curvature flow in a gradient Ricci soliton background. We define what it means for a hypersurface to be a mean curvature soliton. We show that the differential Harnack-type expression vanishes on mean curvature solitons.

In Subsect. 5.3 we give two proofs of the monotonicity of a Huisken-type functional. The first proof, which is calculational, is essentially the same as the one in [14]. The second proof is noncalculational.

5.1. Mean curvature flow in a general Ricci flow background. Let M be a smooth n-dimensional manifold and let $g(\cdot)$ satisfy the Ricci flow equation $\frac{\partial g}{\partial t} = -2 \operatorname{Ric}$. Given an $(n-1)$-dimensional manifold Σ, let $\{e(\cdot)\}$ be a smooth one-parameter family of immersions of Σ in M. We write Σ_t for the image of Σ under $e(t)$, and consider $\{\Sigma_t\}$ to be a 1-parameter family of parametrized hypersurfaces in M. Suppose that $\{\Sigma_t\}$ evolves by the mean curvature flow

$$\frac{dx}{dt} = H e_0. \tag{5.1}$$

Proposition 4. *We have*

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2HA_{ij}, \tag{5.2}$$

$$\frac{\partial A_{ij}}{\partial t} = (\widehat{\triangle} A)_{ij} - A^k{}_i R^l{}_{klj} - A^k{}_j R^l{}_{kli} + 2A^{kl} R_{kilj} - 2HA_{ik}A^k{}_j + A^{kl}A_{kl}A_{ij} + \nabla_0 R_{0i0j} \tag{5.3}$$

and

$$\frac{\partial H}{\partial t} = \widehat{\triangle} H + 2A^{ij}R_{ij} + A^{ij}A_{ij}H + \nabla_0 R_{00}. \tag{5.4}$$

Proof. Suppose first that $\Sigma_t = \partial X_t$ with each X_t compact. Given a time interval $[a, b]$, find a positive solution on $\bigcup_{t\in[a,b]}(X_t \times \{t\}) \subset M \times [a, b]$ of the conjugate heat equation

$$\frac{\partial u}{\partial t} = (-\triangle + R)u, \tag{5.5}$$

satisfying the boundary condition $e_0 u = Hu$, by solving it backwards in time from $t = b$. (Choosing diffeomorphisms from $\{X_t\}$ to X_a, we can reduce the problem of solving (5.5) to a parabolic equation on a fixed domain.) Define f by $u = e^{-f}$.

Let $\{\phi_t\}_{t\in[a,b]}$ be the one-parameter family of diffeomorphisms generated by $\{-\nabla_{g(t)} f(t)\}_{t\in[a,b]}$, with $\phi_a = \mathrm{Id}$. Then $\phi_t(X_a) = X_t$ for all t. Put $\widehat{g}(t) = \phi_t^* g(t)$ and $\widehat{f}(t) = \phi_t^* f(t)$. Then

- $\widehat{g}(t)$ and $\widehat{f}(t)$ are defined on X_a,
- $\frac{\partial \widehat{g}}{\partial t} = -2(\mathrm{Ric}_{\widehat{g}} + \mathrm{Hess}(\widehat{f}))$,
- $e_0 \widehat{f} + \widehat{H} = 0$ and
- the measure $e^{-\widehat{f}(t)}\, dV_{\widehat{g}(t)}$ is constant in t.

The proposition now follows from applying $(\phi_t^*)^{-1}$ to Eqs. (4.3), (4.4) and (4.5) (the latter three being written in terms of $\widehat{g}$ and $\widehat{f}$).

As the result could be derived from a local calculation on Σ_t, it is also valid without the assumption that Σ_t bounds a compact domain. □

Example 1. If $M = \mathbb{R}^n$ and $g(t) = g_{\mathrm{flat}}$ then Eqs. (5.2), (5.3) and (5.4) are the same as [9, Lem. 3.2, Thm. 3.4 and Cor. 3.5]

Proposition 5. *If $u = e^{-f}$ is a solution to the conjugate heat equation (5.5) then*

$$\frac{dI_\infty}{dt} = 2\int_M |\,\mathrm{Ric}+\mathrm{Hess}(f)|^2\, e^{-f}\, dV + 2\int_{\partial M}\left(\frac{\partial H}{\partial t} - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + A(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R - HR_{00}\right) e^{-f}\, dA. \tag{5.6}$$

Proof. This follows from Proposition 3. □

Example 2. If $M = \mathbb{R}^n$, and $g(t) = g_{\mathrm{flat}}$ then Proposition 5 is the same as [3, Props. 3.2 and 3.4], after making the change from the $\mathcal{F}$-type functional considered in this paper to the $\mathcal{W}$-type functional considered in [3].

We will need the next lemma later.

Lemma 2. *We have*

$$\frac{d}{dt}(dA) = -\left(R_i{}^i + H^2\right) dA. \tag{5.7}$$

Proof. Using (5.2),

$$\frac{d}{dt}(dA) = \frac{1}{2}\left(g^{ij}\,\frac{\partial g_{ij}}{\partial t}\right) dA = -\left(R_i{}^i + H^2\right) dA. \tag{5.8}$$

This proves the lemma. □

Using Lemma 2, we prove that a mean curvature flow of two-spheres, in a three-dimensional immortal Ricci flow solution, must have a finite-time singularity.

Proposition 6. *Suppose that $(M, g(\cdot))$ is a three-dimensional Ricci flow solution that is defined for $t \in [0, \infty)$, with complete time slices and uniformly bounded curvature on compact time intervals. If $\{\Sigma_t\}$ is a mean curvature flow of two-spheres in $(M, g(\cdot))$ then the mean curvature flow has a finite-time singularity.*

Proof. We estimate the area of Σ_t, along the lines of Hamilton's area estimate for minimal disks in a Ricci flow solution [7, Sect. 11], [12, Lem. 91.12]. Let $\mathcal{A}(t)$ denote the area of Σ_t. From Lemma 2,

$$\frac{d\mathcal{A}}{dt} = -\int_{\Sigma_t} (R_i{}^i + H^2)\, dA. \tag{5.9}$$

Now

$$R_i{}^i = \frac{1}{2}\left(R + R_{ij}{}^{ij}\right) = \frac{1}{2}\left(R + \widehat{R} - H^2 + A^{ij}A_{ij}\right), \tag{5.10}$$

where $\widehat{R}$ denotes the scalar curvature of Σ_t. From a standard Ricci flow estimate [12, (B.2)],

$$R(x,t) \geq -\frac{3}{2t}. \tag{5.11}$$

Then

$$\frac{d\mathcal{A}}{dt} = -\frac{1}{2}\int_{\Sigma_t}\left(R + \widehat{R} + H^2 + A^{ij}A_{ij}\right) dA \leq \frac{3}{4t}\mathcal{A}(t) - 2\pi\chi(\Sigma_t) = \frac{3}{4t}\mathcal{A}(t) - 4\pi. \tag{5.12}$$

It follows that for any time $t_0 > 0$ at which the mean curvature flow exists, we would have $\mathcal{A}(t) \leq 0$ for $t \geq t_0\left(1 + \frac{\mathcal{A}(t_0)}{16\pi t_0}\right)^4$. Thus the mean curvature flow must go singular. □

Remark 4. The analog of Proposition 6, in one dimension lower, is no longer true, as can been seen by taking a closed geodesic in a flat 2-torus. This contrasts with the fact that any compact mean curvature flow in $\mathbb{R}^n$ has a finite-time singularity.

5.2. Mean curvature solitons. Suppose that $(M, g(\cdot), \overline{f}(\cdot))$ is a gradient soliton solution to the Ricci flow. We recall that this means

(1) $(M, g(\cdot))$ is a Ricci flow solution
(2) At time t we have

$$R_{\alpha\beta} + \nabla_\alpha \nabla_\beta \overline{f} = \frac{c}{2t} g_{\alpha\beta}, \tag{5.13}$$

where $c = 0$ in the steady case (for $t \in \mathbb{R}$), $c = -1$ in the shrinking case (for $t \in (-\infty, 0)$) and $c = 1$ in the expanding case (for $t \in (0, \infty)$), and
(3) The function $\overline{f}$ satisfies

$$\frac{\partial \overline{f}}{\partial t} = |\nabla \overline{f}|^2. \tag{5.14}$$

Definition 3. *At a given time t, a hypersurface Σ_t is a mean curvature soliton if*

$$H + e_0 \overline{f} = 0. \tag{5.15}$$

Equation (5.15) involves no choice of local orientations.
When restricted to Σ_t, the equations $R_{ij} + \nabla_i \nabla_j \overline{f} = 0$ and $R_{i0} + \nabla_i \nabla_0 \overline{f} = 0$ become

$$\begin{aligned} R_{ij} + \widehat{\nabla}_i \widehat{\nabla}_j \overline{f} + H A_{ij} &= 0, \\ R_{i0} - \widehat{\nabla}_i H + A_i{}^k \widehat{\nabla}_k \overline{f} &= 0. \end{aligned} \tag{5.16}$$

Example 3. If $M = \mathbb{R}^n$ and $g(t) = g_{\text{flat}}$, let L be a linear function on $\mathbb{R}^n$. Put $\overline{f} = L + t|\nabla L|^2$, so that $\overline{f}$ satisfies (5.14). Then after changing $\overline{f}$ to $-f$, the equations in (5.16) become

$$\begin{aligned} \widehat{\nabla}_i \widehat{\nabla}_j f &= H A_{ij}, \\ \widehat{\nabla}_i H + A_i{}^k \widehat{\nabla}_k f &= 0, \end{aligned} \tag{5.17}$$

which appear on [6, p. 219] as equations for a translating soliton.

If $(M, g(\cdot), \overline{f}(\cdot))$ is a gradient steady soliton, let $\{\phi_t\}$ be the one-parameter family of diffeomorphisms generated by the time-independent vector field $-\nabla_{g(t)} \overline{f}(t)$, with $\phi_0 = \mathrm{Id}$. If Σ_0 is a mean curvature soliton at time zero then its ensuing mean curvature flow $\{\Sigma_t\}$ consists of mean curvature solitons, and $\{\Sigma_t\}$ differs from $\{\phi_t(\Sigma_0)\}$ by hypersurface diffeomorphisms.

There is a similar description of the mean curvature flow of a mean curvature soliton if $(M, g(\cdot), \overline{f}(\cdot))$ is a gradient shrinking soliton or a gradient expanding soliton.

Proposition 7. *If $(M, g(\cdot), \overline{f}(\cdot))$ is a gradient steady soliton and $\{\Sigma_t\}$ is the mean curvature flow of a mean curvature soliton then*

$$\frac{\partial H}{\partial t} - 2\langle \widehat{\nabla} \overline{f}, \widehat{\nabla} H \rangle + A(\widehat{\nabla} \overline{f}, \widehat{\nabla} \overline{f}) + 2R^{0i} \widehat{\nabla}_i \overline{f} - \frac{1}{2} \nabla_0 R - H R_{00} = 0. \tag{5.18}$$

Proof. We clearly have $\left(R_{ij} + \nabla_i \nabla_j \overline{f}\right)\Big|_{\Sigma_t} = 0$ and $\left(R_{i0} + \nabla_i \nabla_0 \overline{f}\right)\Big|_{\Sigma_t} = 0$. The proposition now follows from Proposition 3, along the lines of the proof of Proposition 4. □

Example 4. Suppose that $M = \mathbb{R}^n$, $g(t) = g_{\text{flat}}$, L is a linear function on $\mathbb{R}^n$ and $\overline{f} = L + t|\nabla L|^2$. After putting $V(t) = -\widehat{\nabla} \overline{f}$, Proposition 7 is the same as [6, Lem. 3.2].

5.3. Huisken monotonicity.

Proposition 8 [14]. *If $\{\Sigma_t\}$ is a mean curvature flow of compact hypersurfaces in a gradient steady Ricci soliton $(M, g(\cdot), \overline{f}(\cdot))$ then $\int_{\Sigma_t} e^{-\overline{f}(t)}\, dA$ is nonincreasing in t. It is constant in t if and only if $\{\Sigma_t\}$ are mean curvature solitons.*

Proof 1. Using the mean curvature flow to relate nearly Σ_t's, Lemma 2 gives

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t} e^{-\overline{f}(t)}\, dA &= -\int_{\Sigma_t}\left(\frac{d\overline{f}}{dt} + R_i{}^i + H^2\right) e^{-\overline{f}(t)}\, dA \\ &= -\int_{\Sigma_t}\left(\frac{\partial \overline{f}}{\partial t} + He_0\overline{f} + R_i{}^i + H^2\right) e^{-\overline{f}(t)}\, dA \\ &= -\int_{\Sigma_t}\left(|\nabla \overline{f}|^2 + He_0\overline{f} + R_i{}^i + H^2\right) e^{-\overline{f}(t)}\, dA. \qquad (5.19)\end{aligned}$$

From the soliton equation,

$$0 = R_i{}^i + \nabla_i \nabla^i \overline{f} = R_i{}^i + \widehat{\nabla}_i \widehat{\nabla}^i \overline{f} + \Gamma^i{}_{0i} \nabla^0 \overline{f} = R_i{}^i + \widehat{\triangle}\overline{f} - He_0\overline{f}. \qquad (5.20)$$

Then

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t} e^{-\overline{f}(t)}\, dA &= -\int_{\Sigma_t}\left(-\widehat{\triangle}\overline{f} + |\widehat{\nabla}\overline{f}|^2 + |e_0\overline{f}|^2 + 2He_0\overline{f} + H^2\right) e^{-\overline{f}(t)}\, dA \\ &= -\int_{\Sigma_t}\left(H + e_0\overline{f}\right)^2 e^{-\overline{f}(t)}\, dA. \qquad (5.21)\end{aligned}$$

The proposition follows. □

Proof 2. Let $\{\phi_t\}$ be the one-parameter family of diffeomorphisms considered after Example 3. Put $\widehat{g}(t) = \phi_t^* g(t)$ and $\widehat{f}(t) = \phi_t^* \overline{f}(t)$. Then for all t, we have $\widehat{g}(t) = g(0)$ and $\widehat{f}(t) = \overline{f}(0)$. Put $\widehat{\Sigma}_t = \phi_t^{-1}(\Sigma_t)$. In terms of $g(0)$ and $\overline{f}(0)$, the surfaces $\widehat{\Sigma}_t$ satisfy the flow

$$\frac{dx}{dt} = He_0 + \nabla\overline{f}(0) = (H + e_0\overline{f}(0))e_0 + \widehat{\nabla}\overline{f}(0), \qquad (5.22)$$

which differs from the flow

$$\frac{dx}{dt} = (H + e_0\overline{f}(0))e_0 \qquad (5.23)$$

by diffeomorphisms of the hypersurfaces. The flow (5.23) is the negative gradient flow of the functional $\widehat{\Sigma} \to \int_{\widehat{\Sigma}} e^{-\overline{f}(0)} dA$. Hence $\int_{\widehat{\Sigma}_t} e^{-\overline{f}(0)} dA$ is nonincreasing in t, and more precisely,

$$\frac{d}{dt}\int_{\widehat{\Sigma}_t} e^{-\overline{f}(0)} dA = -\int_{\widehat{\Sigma}_t}\left(H + e_0\overline{f}(0)\right)^2 e^{-\overline{f}(0)} dA. \qquad (5.24)$$

The proposition follows. □

Remark 5 There are evident analogs of Proposition 8, and its proofs, for mean curvature flows in gradient shrinking Ricci solitons and gradient expanding Ricci solitons. For the shrinking case, where $t \in (-\infty, 0)$, put $\tau = -t$. Then $\tau^{-(n-1)/2} \int_{\Sigma_t} e^{-\overline{f}}\, dA$ is non-increasing in t. When $M = \mathbb{R}^n$, $g(\tau) = g_{\text{flat}}$ and $\overline{f}(x, \tau) = \frac{|x|^2}{4\tau}$, we recover Huisken monotonicity [11, Thm. 3.1].

Remark 6 With reference to the second proof of Proposition 8, the second variation of the functional $\Sigma \to \int_\Sigma e^{-f} dA$ was derived in [1]; see [4,8] for consequences. The second variation formula also plays a role in [2, Sect. 4].

Remark 7 As a consequence of the monotonicity statement in Remark 5, we can say the following about singularity models; compare with [11, Thm. 3.5]. Suppose that $(M, g(\cdot), \overline{f}(\cdot))$ is a gradient shrinking Ricci soliton, defined for $t \in (-\infty, 0)$. Let $\{\phi_t\}_{t\in(-\infty,0)}$ be the corresponding 1-parameter family of diffeomorphisms, with $\phi_t^* g(t) = g(-1)$. Suppose that $\{\Sigma_t\}$ is a mean curvature flow in the Ricci soliton. Suppose that for a sequence $\{t_i\}_{i=1}^\infty$ approaching zero from below, a smooth limit $\lim_{i\to\infty} \phi_{t_i}^{-1}\left(\Sigma_{t_i}\right)$ exists and is a compact hypersurface Σ_∞ in $(M, g(-1))$. Then Σ_∞ is a time -1 mean curvature soliton.

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