

# Mean densities and spherical contact distribution function of inhomogeneous Boolean models

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## Abstract

The mean density of a random closed set  $\Theta$  in  $\mathbb{R}^d$  with Hausdorff dimension  $n$  is the Radon-Nikodym derivative of the expected measure  $\mathbb{E}[\mathcal{H}^n(\Theta \cap \cdot)]$  induced by  $\Theta$  with respect to the usual  $d$ -dimensional Lebesgue measure. Starting from an open problem posed by Matheron in [24, p. 50-51], we consider here inhomogeneous Boolean models  $\Xi$  in  $\mathbb{R}^d$  with integer Hausdorff dimension  $n \in \{0, \dots, d\}$ , and we study the mean density of their boundary (which is their mean density if  $n < d$ ) and the differentiability of their spherical contact distribution function  $H_\Xi$ , under general regularity assumptions on the typical grain, related to the existence of its (outer) Minkowski content. In particular, we provide an explicit formula for  $\partial^2 H_\Xi(r, x)/(\partial r^2)$  at  $r = 0$  for a class of Boolean models, whose typical grain has positive reach; known results for stationary Boolean models with convex grains follows then as a particular case. Examples and statistical applications are also discussed.

Keywords: Random sets, Random measures, Boolean models, Mean Densities, Spherical contact distribution, Minkowski content, Geometric measure theory

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## 1 Introduction

As stated in [31, p. 55], a problem of interest is to have explicit formulae for local densities of specific inhomogeneous Boolean models. In particular, about the notion of *mean surface density* of a  $d$ -dimensional random closed set  $\Theta$  in  $\mathbb{R}^d$ , the concept of *specific area of  $\Theta$  at a point  $x \in \mathbb{R}^d$* , defined as the following limit

$$\sigma_\Theta(x) := \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r},$$

provided it exists, has been introduced by Matheron in [24, p. 50]. It is clearly related to the existence of the right partial derivative at  $r = 0$  of the so-called *local spherical contact distribution function*  $H_\Theta$  of  $\Theta$ , the function from  $\mathbb{R}_+ \times \mathbb{R}^d$  to  $[0, 1]$  so defined

$$H_\Theta(r, x) := \mathbb{P}(x \in \Theta_{\oplus r} \mid x \notin \Theta).$$

( $\Theta_{\oplus r}$  denotes here the parallel set of  $\Theta$  at distance  $r$ , i.e.  $\Theta_{\oplus r} := \{x \in \mathbb{R}^d : \text{dist}(x, \Theta) \leq r\}$ .) The existence of  $\sigma_\Theta(x)$  might be related to the existence of the limit of  $\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \setminus \Theta)]/r$  as  $r$  goes to 0, known as *mean outer Minkowski content of  $\Theta$* , introduced in [2]. More precisely, a straightforward application of Fubini's theorem gives

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap B)]}{r} = \lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} dx, \quad (1.1)$$

for all Borel subsets  $B$  of  $\mathbb{R}^d$ ; then it is clear that, whenever  $\Theta$  is stationary,  $\sigma_\Theta$  is constant and given by

$$\sigma_\Theta = \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap [0, 1]^d)]}{r}.$$

Therefore, if furthermore  $\Theta$  is sufficiently regular to admit a (local) Steiner formula, the above limit can be studied in terms of the quermass densities (or Minkowski functionals) associated to  $\Theta$ , and so by means of tools from integral geometry mainly (e.g. see [4, 31] and references therein). Consequently, most of the results available so far in literature about contact distributions and mean surface densities of grain models are proved under the assumption that  $\Theta$  is stationary and with compact convex grains (e.g., see [15, 17, 20, 21, 22, 23, 31]). The passage from stationary to nonstationary random closed sets raises nontrivial problems; anyway, still considering random unions of convex grains, a series of results have been proved (e.g., see [18, 19]). In particular, we mention that the assumption of convexity of the grains plays a fundamental role in order to apply results and techniques from convex and integral geometry in [18]; in such paper, some formulae for contact distributions and mean densities of inhomogeneous germ-grain models are to be taken in *weak form* (e.g., [18, Theorem 4.1]), unless to add further suitable assumptions or to consider special germ-grain models which admit an explicit expression of their capacity functional. As a matter of fact, in order to obtain pointwise convergence results, it is intuitively clear that the exchange between limit and integral in (1.1) has to be proved. (For a more exhaustive discussion about this, we also refer to [1], where results in weak form for the mean density of lower dimensional random closed sets are proved.)

The main goal of the present paper is to provide explicit formulae for  $\sigma_\Theta(x)$  and results concerning the differentiability of the spherical contact distribution function of inhomogeneous germ-grain models. To this end, concepts and recent results from geometric measure theory will play a central role here. As stated in [4], Boolean models are usually considered basic random sets models in stochastic geometry; thus, in order to make the presentation of our results lighter by using the explicit simple formula of the capacity functional of Boolean models, we shall consider here such particular germ-grain models, leaving to subsequent works applications and further generalizations of our techniques to more general germ-grain models. (As it will emerge in the sequel, possible extensions of our results to different germ-grain models might be done under further suitable integrability assumptions which allow to exchange limit and integral in (1.1); a similar problem is discussed in [18, Remark 4.4], for instance.)

The main results of this paper can be summarized as follows:

- (a) In Section 3.2 we find general regularity conditions on the typical grain and on the intensity of an inhomogeneous Boolean model  $\Xi$  in  $\mathbb{R}^d$  such that  $\sigma_\Xi$  exists finite. This answers (for such class of random sets) to the open problem posed by Matheron in [24, p. 50-51] about the existence of  $\sigma_\Xi$ . In particular we provide an explicit formula for  $\sigma_\Xi$  and we show that it differs from the mean density  $\lambda_{\partial\Xi}$  of the topological boundary of  $\Xi$  (i.e., the density of the measure  $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Xi \cap \cdot)]$ ), in general. Then we provide sufficient regularity conditions on the typical grain ensuring  $\sigma_\Xi = \lambda_{\partial\Xi}$ , reobtaining some known results for convex grains as a special case.
- (b) By similar techniques, in Section 3.3 we find an explicit formula for the mean density of inhomogeneous Boolean models with lower dimensional typical grain; in particular, under general regularity assumptions on the typical grain, of the same type as the ones which

guarantee the existence of the Minkowski content of a closed set, we prove that if  $\Xi$  is  $n$ -dimensional, then its mean density  $\lambda_\Xi$  (i.e., the density of the measure  $\mathbb{E}[\mathcal{H}^n(\Xi \cap \cdot)]$ ) is given by

$$\lambda_\Xi(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{b_{d-n} r^{d-n}} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (1.2)$$

Such result can be regarded as the “ $n$ -dimensional counterpart” of  $\sigma_\Xi$ , and it answers to an open problem in [1] in the case of Boolean models. As a by-product, it suggests estimators for  $\lambda_\Xi(x)$  in terms of the empirical capacity functional of  $\Xi$ , which can be considered as the generalization to the case  $0 < n < d$  of the classical density estimation of random variables by means of histograms, in the extreme case  $n = 0$  (Section 6).

- (c) Starting from the explicit formula for  $\sigma_\Xi$ , in Section 4 we study the differentiability of the spherical contact distribution function of inhomogeneous Boolean models with non convex grains, and we provide a formula for the second right partial derivative of  $H_\Xi(r, x)$  at  $r = 0$  for inhomogeneous Boolean models with a typical grain of positive reach.

Simple examples are discussed in Section 5; furthermore, we show how our general formulae for the mean densities and the spherical contact distribution simplify in the special cases in which the Boolean model is assumed to have a deterministic typical grain, or to be stationary. In this last case links with current literature are also provided.

## 2 Basic notation

Throughout the paper  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure,  $dx$  stands for  $\mathcal{H}^d(dx)$ ,  $\mathcal{B}_{\mathbb{R}^d}$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}^d$  and  $\mathcal{H}^n|_A$  denotes the restriction of  $\mathcal{H}^n$  to a  $\mathcal{H}^n$ -measurable set  $A \subset \mathbb{R}^d$  (i.e.  $\mathcal{H}^n|_A(B) = \mathcal{H}^n(A \cap B)$  for all  $B \in \mathcal{B}_{\mathbb{R}^d}$ ).  $B_r(x)$ ,  $b_n$  and  $\mathbf{S}^{d-1}$  will denote the closed ball with centre  $x$  and radius  $r \geq 0$ , the volume of the unit ball in  $\mathbb{R}^n$  and the unit sphere in  $\mathbb{R}^d$ , respectively. We remind that a compact set  $A \subset \mathbb{R}^d$  is said  *$n$ -rectifiable* ( $0 \leq n \leq d - 1$  integer) if it can be written as the image of a compact subset of  $\mathbb{R}^n$  by a Lipschitz map from  $\mathbb{R}^n$  to  $\mathbb{R}^d$ ; more in general, a closed subset  $A$  of  $\mathbb{R}^d$  is said to be *countably  $\mathcal{H}^n$ -rectifiable* if there exist countably many  $n$ -dimensional Lipschitz graphs  $\Gamma_i \subset \mathbb{R}^d$  such that  $A \setminus \cup_i \Gamma_i$  is  $\mathcal{H}^n$ -negligible. (For definitions and basic properties of Hausdorff measure and rectifiable sets see, e.g., [3, 10, 12].)

Let  $\mathbb{F}$  and  $\sigma_{\mathbb{F}}$  be the class of the closed subsets in  $\mathbb{R}^d$  and the  $\sigma$ -algebra generated by the so-called hit-or-miss topology [24], respectively. We say that a random closed set  $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \rightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$  satisfies a certain property (e.g.,  $\Theta$  has Hausdorff dimension  $n$ ) if  $\Theta$  satisfies that property for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Throughout the paper we shall deal with countably  $\mathcal{H}^n$ -rectifiable random closed sets. For a discussion about measurability of  $\mathcal{H}^n(\Theta)$  we refer to [32, 5]. We call *Radon measure* in  $\mathbb{R}^d$  any nonnegative and  $\sigma$ -additive set function  $\mu$  defined on  $\mathcal{B}_{\mathbb{R}^d}$  which is finite on bounded sets, and we write  $\mu \ll \mathcal{H}^n$  to say that  $\mu$  is absolutely continuous with respect to  $\mathcal{H}^n$ . If a  $n$ -dimensional random closed set  $\Theta$  is such that  $\mathbb{E}[\mathcal{H}^n|_\Theta] \ll \mathcal{H}^d$ , we denote by  $\lambda_\Theta$  the Radon-Nikodym derivative of  $\mathbb{E}[\mathcal{H}^n|_\Theta]$  with respect to  $\mathcal{H}^d$ , and we call it the *mean density* of  $\Theta$ .

In particular, we shall consider Boolean models  $\Xi$  in  $\mathbb{R}^d$  [4, 27] with  $n$ -dimensional typical grain  $Z_0$  ( $n \leq d$ , integer); for  $d$ -dimensional typical grains  $Z_0$  we assume that the topological boundary  $\partial Z_0$  has Hausdorff dimension  $d - 1$ . It is well known that Boolean models in  $\mathbb{R}^d$  can be described by marked Poisson point processes on  $\mathbb{R}^d$  with marks in the space of centered compact sets. Since in many examples and applications  $Z_0$  is uniquely determined by a random quantity in a suitable

mark space  $\mathbf{K}$ , (e.g., the length of the radius whenever  $Z_0$  is a random ball centred in the origin, or length and orientation whenever  $Z_0$  is a random segment, and so on), in the sequel we shall consider (inhomogeneous) Boolean models

$$\Xi(\omega) = \bigcup_{(x_i, s_i) \in \Psi(\omega)} x_i + Z_0(s_i),$$

where  $Z_0(s)$  is a compact subset of  $\mathbb{R}^d$  containing the origin, for any  $s \in \mathbf{K}$ , and  $\Psi$  is the marked Poisson point process in  $\mathbb{R}^d$  with marks in  $\mathbf{K}$  associated to  $\Xi$  with intensity measure

$$\Lambda(d(x, s)) = f(x)dxQ(ds).$$

We shall denote by  $\text{diam}(Z_0)$  the (random) diameter of  $Z_0$ . The function  $f$  and the probability measure  $Q$  on  $\mathbf{K}$  are said *intensity of  $\Xi$*  and *mark distribution*, respectively;  $\text{disc}f$  will denote the set of all the points of discontinuity of  $f$ , and  $\mathbb{E}_Q$  the expectation with respect to  $Q$ .

Considering  $d$ -dimensional sets, several notions of boundary of a subset of  $\mathbb{R}^d$  and related concepts will be used. To this end we briefly recall basic (typically standard) definitions from geometric measure theory and integral geometry which will be useful in the sequel. Let  $A$  be a  $\mathcal{H}^d$ -measurable set in  $\mathbb{R}^d$ ; the  *$d$ -dimensional density* (briefly, *density*) of  $A$  at  $x$  is defined as [3]

$$\theta_d(A, x) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A \cap B_r(x))}{b_d r^d},$$

whenever the limit exists. It is clear that  $\theta_d(A, x)$  equals 1 for all  $x$  in the interior of  $A$ , and 0 for all  $x$  in the interior of the complement set of  $A$ , whereas different values can be assumed at its boundary points; in particular, for every  $t \in [0, 1]$  and every  $\mathcal{H}^d$ -measurable set  $A \subset \mathbb{R}^d$ , let

$$A^t := \{x \in \mathbb{R}^d : \theta_d(A, x) = t\}.$$

The set of points  $\partial^*A := \mathbb{R}^d \setminus (A^0 \cup A^1)$  where the density of  $A$  is neither 0 nor 1 is called *essential boundary* of  $A$ . It is proved that all the sets  $A^t$  are Borel sets, and that the  $\mathcal{H}^{d-1}$  measure of  $\partial^*A$  is closely related to the notion of *perimeter*. We remind that if  $A$  is a  $\mathcal{H}^d$ -measurable subset of  $\mathbb{R}^d$ , the *perimeter of  $A$*  in an open set  $E \subseteq \mathbb{R}^d$ , denoted by  $P(A, E)$ , is defined as the total variation  $|D\chi_A|$  of the characteristic function  $\chi_A$  of  $A$  in  $E$ ; more generally, for any Borel set  $B \subset E$ , we define  $P(A, B) := |D\chi_A|(B)$ . In the sequel we shall write  $P(A)$  instead of  $P(A, \mathbb{R}^d)$ . (We refer to [3] for an exhaustive treatment of this subject.)  $A$  is said to have finite perimeter in  $B$  if  $P(A, B) < \infty$ . General theorems on sets with finite perimeter (see [3, §3.5]) guarantee that if  $A$  has finite perimeter in an open set  $E \subset \mathbb{R}^d$ , then the measures  $|D\chi_A|$  and  $\mathcal{H}_{|\partial^*A}^{d-1}$  coincide on the Borel subsets of  $E$ ; as a consequence, the perimeter measure can be computed in terms of the  $\mathcal{H}^{d-1}$  measure, and in particular the following equalities can be proved

$$P(A, B) = \mathcal{H}^{d-1}(\partial^*A \cap B) = \mathcal{H}^{d-1}(A^{1/2} \cap B) \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Note that, being  $\partial^*A \subseteq \partial A$ ,  $P(A) \leq \mathcal{H}^{d-1}(\partial A)$  holds without any regularity or topological assumptions on  $A$ . While the essential boundary of a set  $A \subset \mathbb{R}^d$  is related to the density of the set at its boundary points and to the perimeter measure, the so-called positive boundary of  $A$  is related to the existence of outer normal vectors at points of  $\partial A$ . Namely (e.g., see [11, 20]), for  $A \subset \mathbb{R}^d$  closed let

$$\text{Unp}(A) := \{x \in \mathbb{R}^d : \exists! a \in A \text{ such that } \text{dist}(x, A) = |a - x|\}.$$

The definition of  $\text{Unp}(A)$  implies the existence of a projection mapping  $\xi_A : \text{Unp}(A) \rightarrow A$  which assigns to  $x \in \text{Unp}(A)$  the unique point  $\xi_A(x) \in A$  such that  $\text{dist}(x, A) = |x - \xi_A(x)|$ . Then for all  $x \in \text{Unp}(A)$  with  $\text{dist}(x, A) > 0$  we may define  $u_A(x) := (x - \xi_A(x))/\text{dist}(x, A)$ . The set of all  $x \in \mathbb{R}^d \setminus A$  for which  $\xi_A(x)$  is not defined is called *exoskeleton of A* and it is denoted by  $\text{exo}(A)$ . The *normal bundle of A* is the measurable subset of  $\partial A \times \mathbf{S}^{d-1}$  defined by

$$N(A) := \{(\xi_A(x), u_A(x)) : x \notin A \cup \text{exo}(A)\},$$

whereas the set  $\partial^+ A := \{x \in \partial A : (x, u) \in N(A) \text{ for some } u \in \mathbf{S}^{d-1}\}$  is called the *positive boundary of A*. For any  $x \in \partial^+ A$  we define

$$N(A, x) := \{u \in \mathbf{S}^{d-1} : (x, u) \in N(A)\},$$

and

$$\partial^i A := \{x \in \partial^+ A : \text{card}N(A, x) = i\} \quad \text{for } i = 1, 2.$$

Note that for any  $x \in \partial^1 A$ , the unique element of  $N(A, x)$  is the *outer normal of A at x*; in the sequel we shall denote it by  $n_x$ .

For any closed set  $A \subset \mathbb{R}^d$  there exist uniquely determined signed measures  $\mu_0(A; \cdot), \dots, \mu_{d-1}(A; \cdot)$  on  $N(A)$ , said *support measures of A*, which arise as coefficient measures of a local Steiner formula; in particular the support measure  $\mu_{d-1}(A; \cdot)$  is non-negative and it can be expressed in terms of the  $(d-1)$ -dimensional Hausdorff measure of  $\partial^1 A$  and  $\partial^2 A$ . (We refer to [20] for further details and results.) In Section 4 we shall also consider a class of Boolean models with typical grain having *positive reach*. We remind that the reach of a compact set  $A$  is defined by [11]

$$\text{reach}(A) := \inf_{a \in A} \text{reach}(A, a),$$

where  $\text{reach}(A, a) := \sup\{r > 0 : B_r(a) \subset \text{Unp}(A)\}$  for every  $a \in A$ . If  $\text{reach}(A) > 0$ , the following relationship between the support measure  $\mu_i(A; \cdot)$  and the *curvature measure*  $\Phi_i(A; \cdot)$  associated with  $A$  introduced in [11] holds:

$$\mu_i(A; \cdot \times \mathbf{S}^{d-1}) = \Phi_i(A; \cdot) \quad \forall i = 1, \dots, d-1. \quad (2.1)$$

By using Federer's notation,  $\Phi_i(A) := \Phi_i(A; A)$  will be the total curvature measure of  $A$ .

### 3 Mean densities

Without any other specification, throughout the paper  $\Xi$  is an (inhomogeneous) Boolean model in  $\mathbb{R}^d$  defined as in the previous section. To lighten the notation we set  $Z^x := x - Z_0 \forall x \in \mathbb{R}^d$ , and for any  $x \in \mathbb{R}^d$  and  $r \geq 0$  let  $\mathcal{Z}^{x,r}$  be the subset of  $\mathbb{R}^d \times \mathbf{K}$  so defined

$$\mathcal{Z}^{x,r} = \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : x \in (y + Z_0(s))_{\oplus r}\} = \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : y \in Z_{\oplus r}^x(s)\}.$$

It is clear that  $\mathcal{Z}^{x,r_1} \subset \mathcal{Z}^{x,r_2}$  for all  $r_1 < r_2$ , and that

$$\mathbb{P}(x \notin \Xi_{\oplus r}) = \mathbb{P}(\Psi(\mathcal{Z}^{x,r}) = 0) = \exp\{-\Lambda(\mathcal{Z}^{x,r})\};$$

in particular it follows that

$$\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi) = \mathbb{P}(\{\Psi(\mathcal{Z}^{x,r}) > 0\} \cap \{\Psi(\mathcal{Z}^{x,0}) = 0\}) = e^{-\Lambda(\mathcal{Z}^{x,0})}(1 - e^{-\Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0})}). \quad (3.1)$$

Thus, the problem about the existence of  $\sigma_{\Xi}(x)$  reduces to find conditions ensuring the existence of the following limit

$$\lim_{r \downarrow 0} \frac{\Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0})}{r}. \quad (3.2)$$

Notice that whenever  $\Xi$  is stationary (say  $f \equiv c > 0$ ) with deterministic typical grain,  $\Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0}) = c\mathcal{H}^d(Z_{0_{\oplus r}} \setminus Z_0)$  for all  $x \in \mathbb{R}^d$ , and so the above limit is independent of  $x$ , as expected, and it exists finite if and only if the set  $Z_0$  admits *outer Minkowski content*. In the next section we briefly recall basic definitions and recent results about the (outer) Minkowski content notion, and we provide a generalization which will play a central role in the study of the limit (3.2).

### 3.1 A generalization of the (outer) Minkowski content of sets

We recall that the  $n$ -dimensional *Minkowski content* of a closed set  $A \subset \mathbb{R}^d$  is the quantity

$$\mathcal{M}^n(A) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_{\oplus r})}{b_{d-n} r^{d-n}}$$

whenever the limit exists finite. Well known general results about the existence of the Minkowski content of closed sets in  $\mathbb{R}^d$  are related to rectifiability properties of the involved sets. In particular, the following theorem is proved in [3, p. 110].

**Theorem 3.1** *Let  $A \subset \mathbb{R}^d$  be a countably  $\mathcal{H}^n$ -rectifiable compact set and assume that*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in A, \forall r \in (0, 1) \quad (3.3)$$

*holds for some  $\gamma > 0$  and some Radon measure  $\eta \ll \mathcal{H}^n$  in  $\mathbb{R}^d$ . Then  $\mathcal{M}^n(A) = \mathcal{H}^n(A)$ .*

Condition (3.3) is a kind of quantitative non-degeneracy condition which prevents  $A$  from being too sparse; simple examples show that  $\mathcal{M}^n(A)$  can be infinite, and  $\mathcal{H}^n(A)$  arbitrarily small, when this condition fails [3, 2]. The above theorem extends (see [3, Theorem 2.106]) the well-known Federer's result [12, p. 275] to countably  $\mathcal{H}^n$ -rectifiable compact sets; in particular for any  $n$ -rectifiable compact set  $A \subset \mathbb{R}^d$  there exists a suitable measure  $\eta$  satisfying (3.3) (see [2, Remark 1]).

The right derivative at  $r = 0$  of the volume function  $V(r) := \mathcal{H}^d(A_{\oplus r})$  of a Borel set  $A \subset \mathbb{R}^d$  is also named the *outer Minkowski content of  $A$* , defined as [2]

$$\mathcal{SM}(A) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r},$$

provided that the limit exists finite. Note that if  $A$  has Hausdorff dimension less than  $d$ , then  $\mathcal{SM}(A) = 2\mathcal{M}^{d-1}(A)$ , whereas if  $A$  is a  $d$ -dimensional set, closure of its interior,  $A_{\oplus r} \setminus A$  coincides with the outer Minkowski enlargement at distance  $r$  of  $\partial A$ . It is intuitive that in many situations (e.g., full dimensional convex bodies)  $\mathcal{SM}(A) = \mathcal{M}^{d-1}(\partial A)$ ; instead, it can be proved [29] that the same general conditions which guarantee the existence of  $\mathcal{M}^{d-1}(\partial A)$  imply the existence of the outer Minkowski content of  $A$ , but  $\mathcal{SM}(A)$  differs from  $\mathcal{H}^{d-1}(\partial A)$ , in general. Namely, the following class of sets has been introduced in [29]:

**Definition 3.2 (The class  $\mathcal{O}$ )** *Let  $\mathcal{O}$  be the class of Borel sets  $A$  of  $\mathbb{R}^d$  with countably  $\mathcal{H}^{d-1}$ -rectifiable and bounded topological boundary, such that*

$$\eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial A, \forall r \in (0, 1)$$

*holds for some  $\gamma > 0$  and some probability measure  $\eta \ll \mathcal{H}^{d-1}$  in  $\mathbb{R}^d$ .*

**Theorem 3.3** [29] *The class  $\mathcal{O}$  is stable under finite unions and any  $A \in \mathcal{O}$  admits outer Minkowski content, given by  $\mathcal{SM}(A) = P(A) + 2\mathcal{H}^{d-1}(\partial A \cap A^0)$ .*

**Remark 3.4 (The class  $\mathcal{O}'$ )** In [29] it is also proved that the same conclusions of the above theorem hold for a class of Borel subsets of  $\mathbb{R}^d$ , denoted by  $\mathcal{O}'$ , defined similarly to  $\mathcal{O}$  by replacing the condition of absolute continuity of  $\eta$  with the assumption that  $\mathcal{M}^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A)$ ; it follows that  $\mathcal{O}'$  contains all Borel sets with  $(d-1)$ -rectifiable boundary (and so finite unions of sets with positive reach or with Lipschitz boundary, in particular).

Note that if a Radon measure  $\eta$  as in Theorem 3.1 exists, then it can be assumed to be a probability measure, without loss of generality (it is sufficient to consider the measure  $\tilde{\eta}(\cdot) := \eta(W \cap \cdot) / \eta(W)$ , where  $W$  is a compact subset of  $\mathbb{R}^d$  such that  $A_{\oplus 1} \subset W$ ). The next theorem may be seen as a generalization of Theorem 3.1 and of Theorem 3.3.

**Theorem 3.5** *Let  $\mu \ll \mathcal{H}^d$  be a positive measure in  $\mathbb{R}^d$  with locally bounded density  $f$ .*

(a) *Let  $A \subset \mathbb{R}^d$  be a countably  $\mathcal{H}^n$ -rectifiable compact set such that condition (3.3) holds for some  $\gamma > 0$  and some probability measure  $\eta \ll \mathcal{H}^n$  in  $\mathbb{R}^d$ . If  $\mathcal{H}^n(\text{disc}f) = 0$ , then*

$$\lim_{r \downarrow 0} \frac{\mu(A_{\oplus r})}{b_{d-n} r^{d-n}} = \int_A f(x) \mathcal{H}^n(dx).$$

(b) *Let  $A$  belong to  $\mathcal{O}$  (or  $\mathcal{O}'$ ). If  $\mathcal{H}^{d-1}(\text{disc}f) = 0$ , then*

$$\lim_{r \downarrow 0} \frac{\mu(A_{\oplus r} \setminus A)}{r} = \int_{\partial^* A} f(x) \mathcal{H}^{d-1}(dx) + 2 \int_{\partial A \cap A^0} f(x) \mathcal{H}^{d-1}(dx).$$

*Proof.* (a) In [1] an upper bound for the Minkowski content of compact sets in  $\mathbb{R}^d$  and a local version of Theorem 3.1 are proved; namely, if  $S \subset \mathbb{R}^d$  is a countably  $\mathcal{H}^n$ -rectifiable compact set such that condition (3.3) holds for some  $\gamma > 0$  and some finite measure  $\eta \ll \mathcal{H}^n$  in  $\mathbb{R}^d$ , then

$$\frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}} \leq \frac{\eta(\mathbb{R}^d)}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \quad \forall r < 2, \quad (3.4)$$

and

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r} \cap B)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap B)$$

for any  $B \in \mathcal{B}_{\mathbb{R}^d}$  such that  $\mathcal{H}^n(S \cap \partial B) = 0$ .

It is well known that, since  $f \geq 0$ , there exists an increasing sequence  $\{f_k\}_{k \in \mathbb{N}}$  of step functions

$$f_k(x) = \sum_{j=1}^{N(k)} c_j^{(k)} \mathbf{1}_{B_j^{(k)}}(x), \quad c_j^{(k)} \geq 0, \quad N(k) \in \mathbb{N},$$

converging to  $f$ . Being  $\mathcal{H}^n(\text{disc}f) = 0$ , the sequence  $\{f_k\}$  can be chosen such that  $\mathcal{H}^n(A \cap \partial B_j^{(k)}) = 0$  for all  $j, k$ .

Let us define  $g_k(r) := \int_{A_{\oplus r}} f_k(x) dx / (b_{d-n} r^{d-n})$ .  $f_k \uparrow f$  uniformly in  $A_{\oplus r}$  for any  $r > 0$  since  $f$  is bounded in  $A_{\oplus r}$ ; so, for all  $\varepsilon > 0$  and  $k$  sufficiently great,

$$\left| g_k(r) - \frac{\int_{A_{\oplus r}} f(x) dx}{b_{d-n} r^{d-n}} \right| \leq \frac{\int_{A_{\oplus r}} |f_k(x) - f(x)| dx}{b_{d-n} r^{d-n}} < \varepsilon \frac{\mathcal{H}^d(A_{\oplus r})}{b_{d-n} r^{d-n}} \stackrel{(3.4)}{\leq} \varepsilon \frac{1}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \quad \forall r \in (0, 1).$$

Therefore  $g_k(r)$  uniformly converges to  $\int_{A_{\oplus r}} f(x)dx/(b_{d-n}r^{d-n})$  in  $(0,1)$  as  $k$  tends to  $\infty$ . By condition (3.3) we know that

$$\limsup_{r \downarrow 0} \frac{\eta(B_r(x))}{b_n r^n} \geq \frac{\gamma}{b_n} > 0 \quad \forall x \in A;$$

then, a direct application of Theorem 2.56 in [3] implies that  $\mathcal{H}^n(A) \leq b_n/\gamma$ . Then it follows

$$\lim_{r \downarrow 0} g_k(r) = \sum_{j=1}^{N(k)} c_j^{(k)} \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_{\oplus r} \cap B_j^{(k)})}{b_{d-n} r^{d-n}} = \int_A f_k(x) \mathcal{H}^n(dx) \leq \sup_{x \in A} f(x) \mathcal{H}^n(A) < \infty.$$

This, together with the uniform convergence of  $g_k(r)$  in  $(0,1)$ , implies that

$$\lim_{r \downarrow 0} \lim_{k \rightarrow \infty} g_k(r) = \lim_{k \rightarrow \infty} \lim_{r \downarrow 0} g_k(r),$$

and we finally obtain

$$\lim_{r \downarrow 0} \frac{\mu(A_{\oplus r})}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\int_{A_{\oplus r}} f(x)dx}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \lim_{k \rightarrow \infty} g_k(r) = \lim_{k \rightarrow \infty} \int_A f_k(x) \mathcal{H}^n(dx) = \int_A f(x) \mathcal{H}^n(dx).$$

(b) For any  $A$  in  $\mathcal{O}$  (or in  $\mathcal{O}'$ ) it holds [29]

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d((A_{\oplus r} \setminus A) \cap B)}{r} = P(A, B) + 2\mathcal{H}^{d-1}(\partial A \cap A^0 \cap B) \quad (3.5)$$

for any  $B \in \mathcal{B}_{\mathbb{R}^d}$  with  $\mathcal{H}^{d-1}(\partial A \cap \partial B) = 0$ .

We observe that

$$\frac{\mathcal{H}^d(A_{\oplus r} \setminus A)}{r} \leq \frac{\mathcal{H}^d((\partial A)_{\oplus r})}{r} \leq \gamma^{-1} 2^{3d-1} b_d \quad \forall r < 2,$$

and that  $P(A) + 2\mathcal{H}^{d-1}(\partial A \cap A^0) \leq 2\mathcal{H}^{d-1}(\partial A)$ . By proceeding along the same lines of point (a), the assertion follows.  $\square$

### 3.2 Mean surface densities

In this section we consider a general class of  $d$ -dimensional Boolean models, defined by the two following conditions on the intensity  $f$ , and on the typical grain  $Z_0$ .

**Assumptions:** **(A1)**  $\partial Z_0$  is countably  $\mathcal{H}^{d-1}$ -rectifiable and compact, and such that there exist  $\gamma > 0$  and a random closed set  $\Theta \supseteq \partial Z_0$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)] < \infty$  such that, for  $Q$ -a.e.  $s \in \mathbf{K}$ ,

$$\mathcal{H}^{d-1}(\Theta(s) \cap B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial Z_0(s), \forall r \in (0, 1).$$

**(A2)**  $\mathcal{H}^{d-1}(\text{disc}f) = 0$  and  $f$  is locally bounded such that for any compact set  $K \subset \mathbb{R}^d$

$$\sup_{y \in K_{\oplus \text{diam}(Z_0)}} f(y) \leq \xi_K \quad (3.6)$$

for some random variable  $\xi_K$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)\xi_K] < \infty$ .



**Remark 3.6** Condition (3.6) is trivially satisfied whenever  $f$  is bounded, or  $f$  is locally bounded and  $\text{diam}(Z_0) \leq c \in \mathbb{R}_+$   $Q$ -a.s. Assumption (A1) is often fulfilled with  $\Theta = \partial Z_0$  or  $\Theta = \partial Z_0 \cup \tilde{A}$  for some sufficiently regular random closed set  $\tilde{A}$ ; as a matter of fact, it can be seen as the stochastic version of (3.3), which, in many applications, is satisfied with  $\eta(\cdot) = \mathcal{H}^n(\tilde{A} \cap \cdot)$  for some closed set  $\tilde{A} \supseteq A$  (see [3, p. 111], [2]).

**Proposition 3.7** *Let  $\Xi$  be a Boolean model as in the Assumptions. Then, for all  $x \in \mathbb{R}^d$ ,*

$$\sigma_{\Xi}(x) = e^{-\Lambda(\mathcal{Z}^{x,0})} \mathbb{E}_Q \left[ \int_{\partial^* Z^x} f(y) \mathcal{H}^{d-1}(dy) + 2 \int_{\partial Z^x \cap (Z^x)^0} f(y) \mathcal{H}^{d-1}(dy) \right]. \quad (3.7)$$

*Proof.* Let us observe that

$$\frac{\int_{Z_{\oplus r}^x \setminus Z^x(s)} f(y) dy}{r} \leq \frac{\mathcal{H}^d((\partial Z_0(s))_{\oplus r})}{r} \sup_{y \in Z_{\oplus r}^x(s)} f(y) \leq \frac{2^{3d-1} b_d}{\gamma} \mathcal{H}^{d-1}(\Theta(s)) \xi_{B_2(x)}(s) \quad \forall r < 2,$$

where the last inequality follows by (3.4) and the assumption (3.6).

Being  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta) \xi_{B_2(x)}] < \infty$ , the dominated convergence theorem and the part (b) of Theorem 3.5 (with  $\mu = f \mathcal{H}^d$  and  $A = Z^x(s)$ ) imply that

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0})}{r} &= \mathbb{E}_Q \left[ \lim_{r \downarrow 0} \frac{\int_{Z_{\oplus r}^x \setminus Z^x} f(y) dy}{r} \right] \\ &= \mathbb{E}_Q \left[ \int_{\partial^* Z^x} f(y) \mathcal{H}^{d-1}(dy) + 2 \int_{\partial Z^x \cap (Z^x)^0} f(y) \mathcal{H}^{d-1}(dy) \right] \\ &\leq \mathbb{E}_Q \left[ 2 \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] \leq 2 \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0) \xi_{\{x\}}] \in \mathbb{R}_+. \end{aligned}$$

We conclude that  $\lim_{r \downarrow 0} (1 - e^{-\Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0})})/r$  exists finite and, by (3.1), the assertion follows.  $\square$

Dealing with Boolean models, it is commonly assumed that the mean number of grains hitting any compact subset of  $\mathbb{R}^d$  is finite; in terms of  $\Lambda$ , this is equivalent to say that

$$\mathbb{E}[\Psi(\mathcal{Z}^{0,R})] = \int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(dy \times ds) < \infty \quad \forall R > 0. \quad (3.8)$$

**Proposition 3.8** *For any Boolean model  $\Xi$  as in the Assumptions satisfying (3.8),  $\mathbb{E}[\mathcal{H}_{|\partial \Xi}^{d-1}]$  is a Radon measure absolutely continuous with respect to  $\mathcal{H}^d$ .*

*Proof.* For any  $R > 0$

$$\mathbb{E}[\mathcal{H}^{d-1}(\partial \Xi \cap B_R(0))] = \mathbb{E}[\mathbb{E}[\mathcal{H}^{d-1}(\partial \Xi \cap B_R(0)) \mid \Psi(\mathcal{Z}^{0,R})]] \leq \mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)] \mathbb{E}[\Psi(\mathcal{Z}^{0,R})] < \infty,$$

by (3.8) and condition (A1); then  $\mathbb{E}[\mathcal{H}_{|\partial \Xi}^{d-1}]$  is locally finite, and so a Radon measure in  $\mathbb{R}^d$ .

By contradiction, let  $\mathbb{E}[\mathcal{H}_{|\partial \Xi}^{d-1}]$  be not absolutely continuous with respect to  $\mathcal{H}^d$ ; then there exists  $A \subset \mathbb{R}^d$  with  $\mathcal{H}^d(A) = 0$  such that  $\mathbb{E}[\mathcal{H}^{d-1}(\partial \Xi \cap A)] > 0$ . In particular,

$$0 < \mathbb{P}(\mathcal{H}^{d-1}(\partial \Xi \cap A) > 0) \leq \mathbb{P} \left( \sum_{(x_i, s_i) \in \Psi} \mathcal{H}^{d-1}((x_i + \partial Z_0(s_i)) \cap A) > 0 \right) = \mathbb{P}(\Psi(A) > 0), \quad (3.9)$$

where

$$\mathcal{A} := \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : \mathcal{H}^{d-1}((y + \partial Z_0(s)) \cap A) > 0\}.$$

By denoting  $\mathcal{A}_s := \{y \in \mathbb{R}^d : (y, s) \in \mathcal{A}\}$  the section of  $\mathcal{A}$  at  $s \in \mathbf{K}$ , and applying Fubini's theorem we get

$$\int_{\mathcal{A}_s} \mathcal{H}^{d-1}((y + \partial Z_0(s)) \cap A) dy = \int_{\partial Z_0(s)} \left( \int_{\mathcal{A}_s} \mathbf{1}_{A-x}(y) dy \right) \mathcal{H}^{d-1}(dx) = 0,$$

since  $\mathcal{H}^d(A) = 0$ . Being the function  $y \mapsto \mathcal{H}^{d-1}((y + \partial Z_0(s)) \cap A)$  strictly positive in  $\mathcal{A}_s$ , we conclude that  $\mathcal{H}^d(\mathcal{A}_s) = 0$  for all  $s \in \mathbf{K}$ . Then it follows

$$\mathbb{E}[\Psi(\mathcal{A})] = \int_{\mathbf{K}} \left( \int_{\mathcal{A}_s} f(y) dy \right) Q(ds) = 0,$$

which is in contradiction with (3.9).  $\square$

The next theorem tells us that, without any further regularity assumption on  $Z_0$ ,  $\sigma_{\Xi}$  generally differs from  $\lambda_{\partial\Xi}$ .

**Theorem 3.9** *Let  $\Xi$  be a Boolean model as in the Assumptions satisfying (3.8). Then  $\mathbb{E}[\mathcal{H}_{|\partial^*\Xi}^{d-1}]$  and  $\mathbb{E}[\mathcal{H}_{|\partial\Xi \cap \Xi^0}^{d-1}]$  are Radon measures in  $\mathbb{R}^d$  absolutely continuous with respect to  $\mathcal{H}^d$ , and*

$$\sigma_{\Xi}(x) = \lambda_{\partial^*\Xi}(x) + 2\lambda_{\partial\Xi \cap \Xi^0}(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

The proof of the above theorem will be based on the following two main steps:

1. the limit in the left side of (1.1) gives rise (for  $B$  varying in  $\mathcal{B}_{\mathbb{R}^d}$ ) to a Radon measure in  $\mathbb{R}^d$ , namely  $\mathbb{E}[\mathcal{H}_{|\partial^*\Xi}^{d-1} + 2\mathcal{H}_{|\partial\Xi \cap \Xi^0}^{d-1}]$ , absolutely continuous with respect to  $\mathcal{H}^d$ ;
2. limit and integral can be exchanged in the right side of (1.1).

A random compact set  $\Theta$  in  $\mathbb{R}^d$  admits *mean outer Minkowski content* if

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_r \setminus \Theta)]}{r}$$

exists finite. In order to exchange limit and expectation, we shall assume a uniform integrability condition, which can be considered as the stochastic analogous of condition (3.3). In particular, being Boolean models locally finite unions of random sets  $\mathbb{P}$ -a.s., we also need to consider the class  $\mathcal{O}_{\text{loc}}$  of all closed sets  $A$  such that for any  $R > 0$  there exists  $E \in \mathcal{O}$  with  $(A \Delta E) \cap B_R(0) = \emptyset$ , i.e. the sets that locally coincide with sets in  $\mathcal{O}$ . As observed in [29, Prop. 4.13], (3.5) still holds for any  $A \in \mathcal{O}_{\text{loc}}$ . So, let  $W \subset \mathbb{R}^d$  be a compact window and set, for  $\Theta \in \mathcal{O}_{\text{loc}}$ ,

$$\Gamma_W(\Theta) := \max\{\gamma \geq 0 : \exists \text{ a probability measure } \eta \text{ such that} \\ \eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial\Theta \cap W_{\oplus 1}, \forall r \in (0, 1)\}.$$

The following lemma, which gives a local existence result for the mean outer Minkowski content (so that a global version follows as a particular case), invokes an integrability condition on  $1/\Gamma_W(\Theta)$  and the assumption that the process is  $\mathcal{O}_{\text{loc}}$ -valued; as in [1], to avoid the delicate problem of the measurability of  $\Gamma_W(\Theta)$ , we just assume the existence of an integrable random variable  $Y$  bounding  $1/\Gamma_W(\Theta)$  from above (this suffices for most applications). It can be easily proven by proceeding along the lines of the proof of Theorem 4 in [1] and Theorem 22 in [9].

**Lemma 3.10** *Let  $W \subset \mathbb{R}^d$  be a compact window, assume that  $\Theta$  belongs to the class  $\mathcal{O}_{\text{loc}}$  (or  $\mathcal{O}'_{\text{loc}}$ ), and that there exists a random variable  $Y$  with  $\mathbb{E}[Y] < \infty$ , such that  $1/\Gamma_W(\Theta) \leq Y$  almost surely. Then, for any Borel set  $B$  contained in the interior of  $W$  with  $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \partial B)] = 0$ ,*

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap B)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}_{|\partial^* \Theta}(B)] + 2\mathbb{E}[\mathcal{H}^{d-1}_{|\partial\Theta \cap \Theta^0}(B)]. \quad (3.10)$$

*Proof of Theorem 3.9.* Let us show that the hypotheses of Lemma 3.10 are fulfilled.

From assumption (A1) we know that  $\mathcal{H}^{d-1}(\Theta)$  is finite  $Q$ -a.s.; so the probability measure on  $\mathbb{R}^d$

$$\eta^s(\cdot) := \mathcal{H}^{d-1}(\Theta(s) \cap \cdot) / \mathcal{H}^{d-1}(\Theta(s))$$

is well-defined for  $Q$ -a.e.  $s \in \mathbf{K}$ . Then we get that  $\partial Z_0(s)$  belongs to the class  $\mathcal{O}$  for  $Q$ -a.e.  $s \in \mathbf{K}$ , and so  $\Xi \in \mathcal{O}_{\text{loc}}$   $\mathbb{P}$ -a.s. by (3.8).

Let  $W$  be a compact window of  $\mathbb{R}^d$ . Denoting by  $\tilde{\Theta}$  the random closed set

$$\tilde{\Theta}(\omega) := \bigcup_{(x_i, s_i) \in \Psi(\omega)} x_i + \Theta(s_i) \quad \forall \omega \in \Omega,$$

let us consider the probability measure

$$\eta_W(\cdot) := \mathcal{H}^{d-1}(\tilde{\Theta}(\omega) \cap W_{\oplus 2} \cap \cdot) / \mathcal{H}^{d-1}(\tilde{\Theta}(\omega) \cap W_{\oplus 2}).$$

Note that for any  $x \in \partial\Xi(\omega) \cap W_{\oplus 1}$ ,  $B_r(x) \subset W_{\oplus 2}$  for all  $r \in (0, 1)$ , and there exists  $(\bar{x}, \bar{s}) \in \Psi(\omega)$  such that  $x \in \bar{x} + \Theta(\bar{s})$ , being  $\partial\Xi \subseteq \tilde{\Theta}$ ; then we have

$$\eta_W(B_r(x)) \geq \frac{\mathcal{H}^{d-1}(\Theta(\bar{s}) \cap B_r(x - \bar{x}))}{\mathcal{H}^{d-1}(\tilde{\Theta}(\omega) \cap W_{\oplus 2})} \geq \frac{\gamma}{\mathcal{H}^{d-1}(\tilde{\Theta}(\omega) \cap W_{\oplus 2})} r^{d-1}$$

for all  $x \in \partial\Xi(\omega) \cap W_{\oplus 1}$  and  $r \in (0, 1)$ . By noticing that  $\mathbb{E}[\mathcal{H}^{d-1}(\tilde{\Theta} \cap W_{\oplus 2})]$  is finite, Lemma 3.10 applies for any compact window  $W \subset \mathbb{R}^d$  with  $Y = \mathcal{H}^{d-1}(\tilde{\Theta} \cap W_{\oplus 2})/\gamma$ . Being  $\partial^* \Xi$  and  $\partial\Xi \cap \Xi^0$  disjoint subsets of  $\partial\Xi$ , by Proposition 3.8 it follows that  $\mathbb{E}[\mathcal{H}^{d-1}_{|\partial^* \Xi} + 2\mathcal{H}^{d-1}_{|\partial\Xi \cap \Xi^0}]$  is a Radon measure with density  $\lambda_{\partial^* \Xi} + 2\lambda_{\partial\Xi \cap \Xi^0}$ . Thus, by Equation (3.10) and applying Fubini's theorem, we get that

$$\lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi)}{r} dx = \int_B (\lambda_{\partial^* \Xi}(x) + 2\lambda_{\partial\Xi \cap \Xi^0}(x)) dx \quad (3.11)$$

for any bounded Borel set  $B \subset \mathbb{R}^d$  with  $\mathcal{H}^d(\partial B) = 0$ .

Now, by observing that for all  $x \in B$  and  $r < 2$

$$\frac{\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi)}{r} \stackrel{(3.1)}{\leq} \Lambda(\mathcal{Z}^{x,r} \setminus \mathcal{Z}^{x,0}) \leq \mathbb{E}_Q \left[ \int_{(\partial\mathcal{Z}^x)_{\oplus r}} f(y) dy \right] \leq \frac{2^{3d-1} b_d}{\gamma} \mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta) \xi_{B_{\oplus 2}}] < \infty,$$

the dominated convergence theorem implies that limit and integral can be exchanged in (3.11) for any bounded  $B \in \mathcal{B}_{\mathbb{R}^d}$ , which concludes the proof.  $\square$

By assumption (A1) we know that  $Z_0$  has finite perimeter  $Q$ -a.s.; a classical result of geometric measure theory states that any set  $A \subset \mathbb{R}^d$  of finite perimeter has density either 0 or 1 or 1/2 at  $\mathcal{H}^{d-1}$ -almost every point of its boundary (e.g., see [3, Theorem 3.61]), and so

$$\mathcal{H}^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A \cap A^0) + P(A) + \mathcal{H}^{d-1}(\partial A \cap A^1). \quad (3.12)$$

As a consequence,  $\Xi$  is a random set with locally finite perimeter and  $\lambda_{\partial\Xi} = \lambda_{\partial\Xi \cap \Xi^0} + \lambda_{\partial^*\Xi} + \lambda_{\partial\Xi \cap \Xi^1}$ . Then, it follows that  $\sigma_{\Xi}(x) = \lambda_{\partial\Xi}(x) = \lambda_{\partial^*\Xi}(x)$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$  if  $\mathbb{E}[\mathcal{H}_{|\partial\Xi \cap (\Xi^0 \cup \Xi^1)}^{d-1}] = 0$ .

We prove now that  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$  is a sufficient condition for  $\sigma_{\Xi} = \lambda_{\partial\Xi}$ . Taking into account (3.12) and that  $P(A) \leq \mathcal{H}^{d-1}(\partial A) \forall A \subset \mathbb{R}^d$ , it is an easy exercise to prove the next statement.

**Lemma 3.11** *Let  $A, B$  be random closed sets in  $\mathbb{R}^d$  of finite perimeter such that  $\mathbb{E}[P(A)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial A)]$  and  $\mathbb{E}[P(B)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial B)]$ .*

*Then  $\mathbb{E}[P(A \cup B)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial(A \cup B))]$  if  $\mathbb{E}[\mathcal{H}^{d-1}(\partial A \cap \partial B)] = 0$ .*

**Proposition 3.12** *Let  $\Xi$  be a Boolean model as in the Assumptions satisfying (3.8), such that  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$ . Then*

$$\sigma_{\Xi}(x) = e^{-\Lambda(\mathcal{Z}^{x,0})} \mathbb{E}_Q \left[ \int_{\partial Z_x} f(y) \mathcal{H}^{d-1}(dy) \right] = \lambda_{\partial\Xi}(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (3.13)$$

*Proof.* Let  $A_i$  be a subset of  $\mathbb{R}^d$  with  $\mathcal{H}^{d-1}(\partial A_i) < \infty$ , for  $i = 1, 2$ ; a straightforward application of Fubini's theorem implies that  $\mathcal{H}^{d-1}((x + \partial A_1) \cap \partial A_2) = 0$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ . It follows that

$$\begin{aligned} & \mathbb{E} \left[ \sum_{(x,s) \in \Psi} \sum_{(x,s) \neq (y,t) \in \Psi} \mathcal{H}^{d-1}(\partial(x + Z_0(s)) \cap \partial(y + Z_0(t))) \right] \\ &= \int \mathcal{H}^{d-1}(\partial(x + Z_0(s)) \cap \partial(y + Z_0(t))) \Lambda(dx, ds) \Lambda(dy, dt) = 0, \end{aligned}$$

being  $\Lambda(dx, dy) = f(x) dx Q(ds)$  and  $\mathcal{H}^{d-1}(\partial(x + Z_0(s)) \cap \partial(y + Z_0(t))) = 0$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ . Lemma 3.11 implies  $\mathbb{E}[\mathcal{H}_{|\partial^*\Theta}^{d-1}] = \mathbb{E}[\mathcal{H}_{|\partial\Theta}^{d-1}]$ , and so, in particular,

$$\lambda_{\partial^*\Xi}(x) = \lambda_{\partial\Xi}(x) \quad \text{and} \quad \lambda_{\partial\Xi \cap \Xi^0}(x) = 0, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Similarly,  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$  implies that  $\mathcal{H}^{d-1}(\partial^* Z_0) = \mathcal{H}^{d-1}(\partial Z_0)$  and  $\mathcal{H}^{d-1}(\partial Z_0 \cap Z_0^0) = 0$   $Q$ -a.s. Thus, Theorem 3.9 and formula (3.7) give the assertion.  $\square$

### 3.3 Mean densities of lower dimensional Boolean models

In many real applications (e.g, see [8] and references therein) it is of interest to study random closed sets at different Hausdorff dimensions and their induced random measure ( $\mathcal{H}_{|\Theta}^n$ , if  $\Theta$  has Hausdorff dimension  $n$ ). We may notice that the expected measure induced by a  $d$ -dimensional random set  $\Theta$  in  $\mathbb{R}^d$  is always absolutely continuous with respect to  $\mathcal{H}^d$ , with density  $\lambda_{\Theta}(x) = \mathbb{P}(x \in \Theta)$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ , whereas whenever  $\Theta = X$  is a random point in  $\mathbb{R}^0$  (i.e.  $\Theta$  is a random variable),  $\mathbb{E}[\mathcal{H}_{|X}^0]$  is just the probability law of  $X$ , and so its mean density is given by the probability density function of  $X$ , provided it exists. Problems arise when  $0 < n < d$ , since it can be more demanding to check that the induced expected measure is absolutely continuous and to evaluate and estimate its density. In this section we observe that the same techniques introduced above also apply to the study of the mean density of lower dimensional Boolean models. The following theorem may be considered as the  $n$ -dimensional counterpart of Proposition 3.7 and Theorem 3.9.

**Theorem 3.13** *Let  $\Xi$  be a Boolean model in  $\mathbb{R}^d$  satisfying the two following conditions on the intensity  $f$  and the typical grain  $Z_0$ :*

(A1')  $Z_0$  is countably  $\mathcal{H}^n$ -rectifiable and compact, and such that there exist  $\gamma > 0$  and a random closed set  $\Theta \supseteq Z_0$  with  $\mathbb{E}_Q[\mathcal{H}^n(\Theta)] < \infty$  such that, for  $Q$ -a.e.  $s \in \mathbf{K}$ ,

$$\mathcal{H}^n(\Theta(s) \cap B_r(x)) \geq \gamma r^n \quad \forall x \in Z_0(s), \forall r \in (0, 1). \quad (3.14)$$

(A2')  $\mathcal{H}^n(\text{disc}f) = 0$  and  $f$  is locally bounded such that, for any compact set  $K \subset \mathbb{R}^d$ ,  $\sup_{y \in K_{\oplus \text{diam}(Z_0)}} f(y) \leq \xi_K$  for some random variable  $\xi_K$  with  $\mathbb{E}_Q[\mathcal{H}^n(\Theta)\xi_K] < \infty$ .

Then  $\mathbb{E}[\mathcal{H}_{|\Xi}^n]$  is a Radon measure in  $\mathbb{R}^d$  absolutely continuous with respect to  $\mathcal{H}^d$ , with density

$$\lambda_{\Xi}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{b_{d-n}r^{d-n}} = \mathbb{E}_Q \left[ \int_{Z^x} f(y) \mathcal{H}^n(dy) \right] \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (3.15)$$

It is intuitive that  $\lambda_{\Xi}$  is related to the existence of its *mean  $n$ -dimensional Minkowski content* of  $Z_0$  (note that (A1') implies  $\mathbb{E}[\mathcal{M}^n(Z_0)] = \mathbb{E}[\mathcal{H}^n(Z_0)]$ ). Condition (A2') plays here the same technical role played by condition (A2) in the previous section. Besides, contrary to the  $d$  dimensional case, (3.8) is now implied by assumptions (A1') and (A2').

**Lemma 3.14** *Condition (3.8) holds for any Boolean model  $\Xi$  under the hypotheses of Theorem 3.13.*

*Proof.* Referring to [1] for details, we mention that the inequality in (3.4) is proved by using covering arguments from geometric measure theory; in particular by showing that  $S_{\oplus r}$  is contained into a finite union  $\bigcup_{i=1}^p B_{4r}(y_i)$  of closed balls with centers in  $S$ , where  $p \leq \eta(\mathbb{R}^d)2^n\gamma^{-1}r^{-n}$ . In the case  $r \geq 2$ , a similar argument leads to  $p \leq \eta(\mathbb{R}^d)2^n\gamma^{-1}$ , and so to

$$\mathcal{H}^d(S_{\oplus r}) \leq \frac{\eta(\mathbb{R}^d)}{\gamma} 2^n 4^d b_d r^d.$$

Then, by taking  $S = Z_0(s)$  and  $\eta = \mathcal{H}_{|\Theta(s)}^n$ , we get that for all  $R > 0$

$$\mathbb{E}[\Psi(\mathcal{Z}^{0,R})] \leq \mathbb{E}_Q \left[ \sup_{y \in Z_{0 \oplus R}} f(y) \mathcal{H}^d(Z_{0 \oplus R}) \right] \leq 2^n 4^d b_d \gamma^{-1} \max\{R^{d-n}; R^d\} \mathbb{E}_Q[\xi_{B_R(0)} \mathcal{H}^n(\Theta)] \stackrel{(A2')}{\leq} \infty.$$

□

**Remark 3.15** Condition (A1) and (A2) are not sufficient to guarantee the validity of (3.8) in the  $d$ -dimensional case, in general. As a simple counterexample consider a stationary Boolean model  $\Xi$  with  $f \equiv c > 0$  and  $Z_0 = B_\rho(0)$ , where  $\rho$  is a random variable greater than 1 such that  $\mathbb{E}[\rho^{d-1}] < \infty$  and  $\mathbb{E}[\rho^d] = \infty$ ; then it is easy to check that  $\Xi$  satisfies the Assumptions with  $\Theta = \partial Z_0$  and  $\xi_K = c$  for all compacts  $K$  in  $\mathbb{R}^d$ , but  $\mathbb{E}[\Psi(\mathcal{Z}^{0,R})] = cb_d \mathbb{E}[(R + \rho)^d] = \infty$ .

By noticing that  $Z_{0 \oplus R} \subseteq (\partial Z_0)_{\oplus (\text{diam}(Z_0) + R)}$ , by the proof of Lemma 3.14 it easily follows that  $\text{diam}(Z_0) \leq D < \infty$   $Q$ -a.s., or, more in general,  $\mathbb{E}[\xi_{B_R(0)} \text{diam}(Z_0)^d \mathcal{H}^n(\Theta)] < \infty$ , is a sufficient condition for the validity of (3.8), whenever  $Z_0$  has nonempty interior.

*Proof of Theorem 3.13.* Taking into account Lemma 3.14, and that the role of Lemma 3.10 is played now by Theorem 4 in [1], by proceeding along the same lines of the proof of Proposition 3.8 and of Theorem 3.9 it can be proved that  $\mathbb{E}[\mathcal{H}_{|\Xi}^n]$  is a Radon measure absolutely continuous with respect to  $\mathcal{H}^d$  and that

$$\mathbb{E}[\mathcal{H}^n(\Xi \cap B)] = \lim_{r \downarrow 0} \frac{\mathcal{H}^d(\Xi_{\oplus r} \cap B)}{b_{d-n}r^{d-n}} = \lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{b_{d-n}r^{d-n}} dx \quad (3.16)$$

for any bounded Borel set  $B \subset \mathbb{R}^d$  with  $\mathcal{H}^d(\partial B) = 0$ . Finally, the same arguments used in the proof of Theorem 3.9 and Proposition 3.7 lead to claim that limit and integral in the last term of the above equation can be exchanged, and that

$$\lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{b_{d-n} r^{d-n}} = \frac{\Lambda(\mathcal{Z}^{x,r})}{b_{d-n} r^{d-n}} = \mathbb{E}_Q \left[ \int_{Z^x} f(y) \mathcal{H}^n(dy) \right],$$

where the last equality follows by part (a) of Theorem 3.5.  $\square$

We point out that the equation  $\lambda_{\Xi}(x) = \mathbb{E}_Q[\int_{Z^x} f(y) \mathcal{H}^n(dy)]$  might also be obtained by means of the well-known Campbell's formula (e.g., see [4, p. 28], and [16] for a similar application), after having shown that the event that different grains overlap in a subset of  $\mathbb{R}^d$  of positive  $\mathcal{H}^n$ -measure has null probability for any Boolean model  $\Xi$  under the hypotheses of Theorem 3.13.

Let us also observe that Theorem 3.13 answers, in the case of Boolean models, to the open problem posed in [1, Remark 8] about the possibility of exchanging limit and integral in (3.16), for statistical purposes (see Section 6). Such limit representation for  $\lambda_{\Xi}$  can be regarded as the  $n$ -dimensional counterpart of  $\sigma_{\Xi}$ . As a matter of fact, if  $Z_0$  is  $(d-1)$ -dimensional, then  $\sigma = 2\lambda_{\Xi}$ , as we observe in the next Remark.

**Remark 3.16 (The case  $n = d - 1$ )** Let  $\Xi$  be a Boolean model satisfying the hypotheses of Theorem 3.13, with  $Z_0$  having Hausdorff dimension  $d - 1$ . Then

$$\lambda_{\Xi}(x) = \mathbb{E}_Q \left[ \int_{Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{2r} = \frac{\sigma_{\Xi}(x)}{2} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

This is in accordance with Proposition 3.7 and Theorem 3.9: it is sufficient to notice that  $Z_0 = \partial Z_0$  (since  $Z_0$  has empty interior) and that  $Z_0$  has null density at any point of  $\mathbb{R}^d$ , and so  $\Xi$  as well.

## 4 On the differentiability of the spherical contact distribution function with respect to $r$

In this section we study the first partial derivative with respect to  $r$ , and the second right partial derivative at  $r = 0$  of the spherical contact distribution function of  $\Xi$ . To lighten the notation, we say that  $H_{\Xi}$  is differentiable with respect to  $r$  at  $r = 0$  if it admits a right partial derivative at  $r = 0$ , and we simply write  $\partial H_{\Xi}(r, x)/(\partial r)|_{r=0}$  (analogously for the second partial derivative). It will emerge how the differentiability of  $H_{\Xi}$  with respect to  $r$  depends on the regularity of both the intensity  $f$  and the boundary of  $Z_0$ .

As a corollary to Proposition 3.7 we have that the spherical contact distribution function  $H_{\Xi}$  of a Boolean model  $\Xi$  satisfying the Assumptions is differentiable at  $r = 0$  for all  $x \in \mathbb{R}^d$ , and

$$\frac{\partial}{\partial r} H_{\Xi}(r, x)|_{r=0} = \frac{\sigma_{\Xi}(x)}{\mathbb{P}(x \notin \Xi)} \stackrel{(3.7)}{=} \mathbb{E}_Q \left[ \int_{\partial^* Z^x} f(y) \mathcal{H}^{d-1}(dy) + 2 \int_{\partial Z^x \cap (Z^x)^0} f(y) \mathcal{H}^{d-1}(dy) \right]. \quad (4.1)$$

In accordance with Remark 3.16,  $\frac{\partial}{\partial r} H_{\Xi}(r, x)|_{r=0} = 2\lambda_{\Xi}$   $\mathcal{H}^d$ -a.e. if  $Z_0$  has Hausdorff dimension  $d - 1$ . If  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$ , Equation (4.1) simplifies

$$\frac{\partial}{\partial r} H_{\Xi}(r, x)|_{r=0} \stackrel{(3.13)}{=} \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] = \mathbb{E}_Q \left[ \int_{\partial Z_0} f(x - y) \mathcal{H}^{d-1}(dy) \right],$$

in accordance, for instance, with [18, Remark 4.15] in the case of convex grains.

The following theorem provides an explicit formula for the second right partial derivative of  $H_{\Xi}$

at  $r = 0$  for a class of Boolean models with typical grain satisfying  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$ . We remind that if  $A \subset \mathbb{R}^d$  is a compact set with positive reach, such that

$$\mathcal{H}^0(N(A, x)) = 1 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial A, \quad (4.2)$$

then  $\mathcal{H}^{d-1}(\partial A) = P(A) = \mathcal{SM}(A) = 2\Phi_{d-1}(A)$  (see [2]).

**Theorem 4.1** *Let  $\Xi$  be a Boolean model with  $Z_0$  as in the assumption (A1) with  $\text{reach}(Z_0) > R$  for some  $R > 0$  and satisfying condition (4.2). Moreover we assume that*

$$\mathbb{E}_Q[|\Phi_i|(Z_0)] < \infty \quad \forall i = 1, \dots, d-1, \quad (4.3)$$

where  $|\Phi_i|(Z_0)$  is the total variation of the measure  $\Phi_i(Z_0; \cdot)$ , and that the intensity  $f$  is bounded, Lipschitz, and differentiable at  $\mathcal{H}^{d-1}$ -a.e.  $x \in \mathbb{R}^d$ . Then, for all  $x \in \mathbb{R}^d$ ,

$$\frac{\partial}{\partial r} H_\Xi(r, x) = \frac{\mathbb{P}(x \notin \Xi_{\oplus r})}{\mathbb{P}(x \notin \Xi)} \mathbb{E}_Q \left[ \int_{\partial Z_{\oplus r}^x} f(y) \mathcal{H}^{d-1}(dy) \right] \quad \forall r \in [0, R), \quad (4.4)$$

$$\begin{aligned} \frac{\partial^2}{\partial r^2} H(r, x)|_{r=0} &= \mathbb{E}_Q \left[ 2\pi \int_{\mathbb{R}^d} f(y) \Phi_{d-2}(Z^x; dy) + \int_{\partial^1 Z^x} D_{n_y} f(y) \mathcal{H}^{d-1}(dy) \right] \\ &\quad - \left( \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] \right)^2, \end{aligned} \quad (4.5)$$

where  $D_{n_y} f$  is the directional derivative of  $f$  along  $n_y \in \mathbf{S}^{d-1}$ .

**Remark 4.2** For the sake of simplicity, we assumed  $f$  bounded; actually, the assertion of the theorem still holds with  $f$  locally bounded and satisfying a technical assumption similar to (A2), in order to guarantee an application of the dominated convergence theorem in the proof. We may also notice that if  $Z_0$  has a bounded diameter (i.e.,  $\text{diam}(Z_0)(s) \leq C \in \mathbb{R}$  for  $Q$ -a.e.  $s \in \mathbf{K}$ ), then  $f$  might be taken locally bounded, and condition (4.3) is trivially satisfied (see [11, Remark 5.10]). By (4.4) and (3.13) it follows that, for  $\mathcal{H}^{d-1}$ -a.e.  $x \in \mathbb{R}^d$ ,

$$\frac{\partial}{\partial r} H_\Xi(r, x) = \lambda_{\partial \Xi_{\oplus r}}(x) / \mathbb{P}(x \notin \Xi) \quad \forall r \in [0, R).$$

## 4.1 Proof of Theorem 4.1

In order to prove Theorem 4.1, we need to prove some preliminary results.

**Proposition 4.3** *Let  $A \subset \mathbb{R}^d$  be a compact set with positive reach. If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Lipschitz function such that the directional derivative  $D_u f(x)$  of  $f$  along the vector  $u$  exists for all  $u \in \mathbf{S}^{d-1}$  for  $\mathcal{H}^{d-1}$ -a.e.  $x \in \partial A$ , then*

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\int_{\mathbb{R}^d} f(x) \Phi_{d-1}(A_{\oplus r}; dx) - \int_{\mathbb{R}^d} f(x) \Phi_{d-1}(A; dx)}{r} \\ = \pi \int_{\mathbb{R}^d} f(x) \Phi_{d-2}(A; dx) + \int_{N(A)} D_u f(x) \mu_{d-1}(A; d(x, u)). \end{aligned} \quad (4.6)$$

*Proof.* Let  $R := \text{reach}(A)$ . By (2.1) and Corollary 4.4 in [20] we get that

$$\Phi_{d-1}(A_{\oplus r}; \cdot) = \sum_{i=0}^{d-1} r^{d-1-i} \frac{(d-i)b_{d-i}}{2} \int_{N(A)} \mathbf{1}\{x + ru \in \cdot\} \mu_i(A; d(x, u)) \quad \forall r \in (0, R);$$

besides it is known that the measures  $\Phi_i(A; \cdot)$  and  $\mu_i(A; \cdot)$  have bounded total variation, being  $A$  compact with positive reach (e.g., see [11, 20]). As a consequence it is not hard to see that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) \Phi_{d-1}(A_{\oplus r}; dx) &= \sum_{i=0}^{d-1} r^{d-1-i} \frac{(d-i)b_{d-i}}{2} \int_{N(A)} f(x+ru) \mu_i(A; d(x, u)) \\ &= \int_{N(A)} f(x+ru) \mu_{d-1}(A; d(x, u)) + r\pi \int_{N(A)} f(x+ru) \mu_{d-2}(A; d(x, u)) + o(r). \end{aligned} \quad (4.7)$$

Finally, by relation (2.1),

$$\int_{N(A)} f(x) \mu_i(A; d(x, u)) = \int_{\partial A} \int_{N(A, x)} f(x) \mu_i(A; d(x, u)) = \int_{\partial A} f(x) \Phi_i(A; dx) \quad \forall i = 1, \dots, d-1,$$

and we get that

$$\begin{aligned} &\lim_{r \downarrow 0} \frac{\int_{\mathbb{R}^d} f(x) \Phi_{d-1}(A_{\oplus r}; dx) - \int_{\mathbb{R}^d} f(x) \Phi_{d-1}(A; dx)}{r} \\ &= \lim_{r \downarrow 0} \left( \int_{N(A)} \frac{f(x+ru) - f(x)}{r} \mu_{d-1}(A; d(x, u)) + \pi \int_{N(A)} f(x+ru) \mu_{d-2}(A; d(x, u)) + \frac{o(r)}{r} \right) \\ &= \int_{N(A)} D_u f(x) \mu_{d-1}(A; d(x, u)) + \pi \int_{\mathbb{R}^d} f(x) \Phi_{d-2}(A; dx), \end{aligned}$$

where the last equality follows by applying the dominated convergence theorem, as  $f$  is bounded on compact sets and  $|f(x+ru) - f(x)|/r \leq \text{Lip}(f) \in \mathbb{R}_+$  for all  $(x, u) \in \mathbb{R}^d \times \mathbf{S}^{d-1}$ .  $\square$

**Corollary 4.4** *Under the same assumptions of Proposition 4.3, if furthermore  $A$  satisfies condition (4.2), then*

$$\lim_{r \downarrow 0} \frac{\int_{\partial A_{\oplus r}} f(x) \mathcal{H}^{d-1}(dx) - \int_{\partial A} f(x) \mathcal{H}^{d-1}(dx)}{r} = 2\pi \int_{\mathbb{R}^d} f(x) \Phi_{d-2}(A; dx) + \int_{\partial^1 A} D_{n_x} f(x) \mathcal{H}^{d-1}(dx), \quad (4.8)$$

where  $n_x$  is the outer normal of  $A$  at  $x$ .

*Proof.* By Proposition 4.1 in [20] we get that  $\mu_{d-1}(A; \cdot \times \mathbf{S}^{d-1}) = \frac{1}{2} \mathcal{H}_{|\partial^1 A}^{d-1}(\cdot) + \mathcal{H}_{|\partial^2 A}^{d-1}(\cdot)$ .

Eqn. (4.8) follows directly by (4.6), by noticing that condition (4.2) implies  $\mathcal{H}^{d-1}(\partial^1 A) = \mathcal{H}^{d-1}(\partial A)$ ,  $2\Phi_{d-1}(A; \cdot) = \mathcal{H}_{|\partial A}^{d-1}(\cdot)$  and

$$\int_{N(A)} D_u f(x) \mu_{d-1}(A; d(x, u)) = \int_{\partial^1 A} D_{n_x} f(x) \Phi_{d-1}(A; dx).$$

$\square$

**Lemma 4.5** *Let  $Z_0$  be compact with  $\text{reach}(Z_0) > R$  for some  $R > 0$ . Then, for any  $r \in (0, R)$ , there exist  $\gamma > 0$  such that for  $Q$ -a.e.  $s \in Q$ ,*

$$\mathcal{H}^{d-1}(\partial(Z_{0 \oplus r}(s)) \cap B_\rho(x)) \geq \gamma \rho^{d-1} \quad \forall x \in \partial(Z_{0 \oplus r}(s)), \forall \rho \in (0, 1).$$



*Proof.* The case  $d = 1$  is trivial. Let  $d \geq 2$  and  $E^x$  be the connected component of  $Z_{0_{\oplus r}}(s)$  containing  $x \in \partial(Z_{0_{\oplus r}}(s))$ .

If  $\partial E^x \cap \partial B_\rho(x) = \emptyset$ , then either  $E^x \subset B_\rho(x)$ , or  $\mathbb{R}^d \setminus E^x \subset B_\rho(x)$ , and

$$\mathcal{H}^{d-1}(\partial(Z_{0_{\oplus r}}(s)) \cap B_\rho(x)) \geq \mathcal{H}^{d-1}(\partial E^x \cap B_\rho(x)) = \mathcal{H}^{d-1}(\partial E^x) \geq P(E^x).$$

By the isoperimetric inequality (e.g., [3, p. 149]) we have that

$$P(E^x) \geq \gamma(d)(\min\{\mathcal{H}^d(E^x), \mathcal{H}^d(\mathbb{R}^d \setminus E^x)\})^{(d-1)/d}$$

for some dimensional constant  $\gamma(d) > 0$ . By noticing that  $\mathcal{H}^d(E^x) \geq b_d r^d > b_d r^d \rho^{d-1}$  and  $\mathcal{H}^d(\mathbb{R}^d \setminus E^x) \geq b_d R^d > b_d R^d \rho^{d-1}$  for all  $\rho \in (0, 1)$ , we conclude that there exists a constant  $\gamma = \gamma(d, R, r) > 0$  (and so independent of  $s$ ) such that  $P(E^x) > \gamma \rho^{d-1}$ .

If  $\partial E^x \cap \partial B_\rho(x) \neq \emptyset$ , a similar conclusion is obtained by projecting on suitable hyperplanes, taking into account that  $\partial(Z_{0_{\oplus r}}(s))$  is contained into a finite union of Lipschitz manifolds.  $\square$

We are now ready to prove the main Theorem of the section.

*Proof of Theorem 4.1.* Corollary 4.9 in [11] tells us that  $\text{reach}(Z_{0_{\oplus r}}) \geq R - r$  for all  $r \in [0, R)$ . It is easy to see that  $Z_{0_{\oplus r}}$  has density 1/2 at any point of its boundary, and so  $P(Z_{0_{\oplus r}}) = \mathcal{H}^{d-1}(\partial Z_{0_{\oplus r}})$ , for any  $r \in (0, R)$ . By assumption (4.2) we also have  $P(Z_0) = \mathcal{H}^{d-1}(\partial Z_0)$ . We know (see [11, Remark 5.8]) that

$$\Phi_{d-1}(Z_{0_{\oplus r}}) = \sum_{j=0}^{d-1} (d-j)r^{d-1-j} b_{d-j} \Phi_j(Z_0)/2 \quad \forall r \in [0, R).$$

Thus we can claim that  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_{0_{\oplus r}})] = 2\mathbb{E}_Q[\Phi_{d-1}(Z_{0_{\oplus r}})] < \infty$  for all  $r \in [0, R)$ . This and Lemma 4.5 imply that  $Z_{0_{\oplus r}}$  satisfies the assumption (A1). Being  $f$  bounded and Lipschitz, the assumption (A2) is easily checked. Therefore, for all  $r \in [0, R)$ ,  $\Xi_{\oplus r}$  is a Boolean model with intensity measure  $\Lambda(dy \times ds) = f(y)dyQ(y, ds)$  and typical grain  $Z_{0_{\oplus r}}$ , satisfying the Assumptions with  $\mathbb{E}_Q[P(Z_{0_{\oplus r}})] = \mathbb{E}[\mathcal{H}^{d-1}(\partial Z_{0_{\oplus r}})]$ . Then we get that

$$\frac{\partial^+}{\partial r} H_{\Xi}(r, x) = \frac{\sigma_{\Xi_{\oplus r}}}{\mathbb{P}(x \notin \Xi)} \stackrel{(3.13)}{=} \frac{\mathbb{P}(x \notin \Xi_{\oplus r})}{\mathbb{P}(x \notin \Xi)} \mathbb{E}_Q \left[ \int_{\partial Z_{\oplus r}^x} f(y) \mathcal{H}^{d-1}(dy) \right] \quad \forall r \in [0, R). \quad (4.9)$$

Proposition 4.11 in [29] implies

$$\lim_{h \downarrow 0} \frac{\mathcal{H}^d(Z_{0_{\oplus r}} \setminus Z_{0_{\oplus r-h}})}{h} = \mathcal{H}^{d-1}(\partial Z_{0_{\oplus r}}) \quad \forall r \in (0, R);$$

then, by proceeding along the same lines of Theorem 3.5, an analogous result holds for the ‘‘inner’’ Minkowski content as well, i.e.

$$\lim_{h \downarrow 0} \int_{Z_{\oplus r}^x \setminus Z_{\oplus r-h}^x(s)} f(y) dy / h = \int_{\partial Z_{\oplus r}^x(s)} f(y) \mathcal{H}^{d-1}(dy).$$

Similarly, it can be proved that the left partial derivative of  $H_{\Xi}$  at  $r \in (0, R)$  exists equal to (4.9).

Since  $\mu_{d-1}(A; \cdot)$  is a positive measure and  $C := \sup_{(x,u) \in N(A)} |f(x + ru)| < \infty$ , we have that

$$\left| \frac{\int_{\partial Z_{\oplus r}^x(s)} f(x) \mathcal{H}^{d-1}(dx) - \int_{\partial Z^x(s)} f(x) \mathcal{H}^{d-1}(dx)}{r} \right| \stackrel{(4.7)}{\leq} \text{Lip}(f) \mathcal{H}^{d-1}(Z_0(s)) + C \sum_{i=0}^{d-2} r^{d-2-i} (d-i) \frac{b_{d-i}}{2} |\Phi_i|(Z_0(s)).$$

The differentiability of  $f$  implies that it admits directional derivative for all  $u \in \mathbf{S}^{d-1}$  at  $\mathcal{H}^{d-1}$ -a.e.  $x \in \partial Z_0(s)$  for  $Q$ -a.e.  $s \in \mathbf{K}$ ; thus, by Corollary 4.4 and the dominated convergence theorem,

$$\lim_{r \downarrow 0} \frac{\mathbb{E}_Q \left[ \int_{\partial Z_{\oplus r}^x} f(y) \mathcal{H}^{d-1}(dy) \right] - \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right]}{r} \stackrel{(4.8)}{=} \mathbb{E}_Q \left[ 2\pi \int_{\mathbb{R}^d} f(y) \Phi_{d-2}(Z^x; dy) + \int_{\partial^1 Z^x} D_{n_y} f(y) \mathcal{H}^{d-1}(dy) \right]. \quad (4.10)$$

Finally, the following chain of equalities holds:

$$\begin{aligned} \frac{\partial^2}{\partial r^2} H_{\Xi}(r, x)|_{r=0} &= \lim_{r \downarrow 0} \frac{1}{r} \left( \frac{\mathbb{P}(x \notin \Xi_{\oplus r})}{\mathbb{P}(x \notin \Xi)} \mathbb{E}_Q \left[ \int_{\partial Z_{\oplus r}^x} f(y) \mathcal{H}^{d-1}(dy) \right] - \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] \right) \\ &= \lim_{r \downarrow 0} \frac{\mathbb{P}(x \notin \Xi_{\oplus r})}{\mathbb{P}(x \notin \Xi)} \frac{\mathbb{E}_Q \left[ \int_{\partial Z_{\oplus r}^x} f(y) \mathcal{H}^{d-1}(dy) - \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right]}{r} \\ &\quad + \lim_{r \downarrow 0} \frac{\mathbb{P}(x \notin \Xi_{\oplus r}) - \mathbb{P}(x \notin \Xi)}{r \mathbb{P}(x \notin \Xi)} \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] \\ &\stackrel{(4.10)}{=} \mathbb{E}_Q \left[ 2\pi \int_{\mathbb{R}^d} f(y) \Phi_{d-2}(Z^x; dy) + \int_{\partial^1 Z^x} D_{n_y} f(y) \mathcal{H}^{d-1}(dy) \right] \\ &\quad - \frac{\mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right]}{\mathbb{P}(x \notin \Xi)} \frac{\partial}{\partial r} H_{\Xi}(r, x)|_{r=0} = (4.5). \end{aligned}$$

## 5 Particular cases and examples

In this section some particular cases and examples are discussed; we start showing how some of the above results simplify whenever  $\Theta$  is stationary, or the typical grain  $Z_0$  is a deterministic subset of  $\mathbb{R}^d$  satisfying the hypotheses of Theorem 3.1.

### 5.1 Deterministic typical grain

**Corollary 5.1** *Let  $\Xi$  be a Boolean model in  $\mathbb{R}^d$  with locally bounded intensity  $f$  and deterministic typical grain  $Z_0$ .*

- (i) *If  $Z_0$  is a compact set in  $\mathcal{O}$  (or in  $\mathcal{O}'$ ) and  $\mathcal{H}^{d-1}(\text{disc}f) = 0$ , then the results stated in Proposition 3.7, Proposition 3.8, Theorem 3.9 and Proposition 3.12 hold.*
- (ii) *If  $\mathcal{H}^n(\text{disc}f) = 0$  and  $Z_0$  is a countably  $\mathcal{H}^n$ -rectifiable compact set such that condition (3.3) holds for some  $\gamma > 0$  and some probability measure  $\eta \ll \mathcal{H}^n$  in  $\mathbb{R}^d$ , then the assertion of Theorem 3.13 holds.*

A proof of Corollary 5.1 (i) (point (ii) follows similarly) might be obtained by proceeding along the same lines of the proofs given in Section 3.2, by noticing that:

- By the definition of the class  $\mathcal{O}$  (and of  $\mathcal{O}'$ ), there exists a probability measure  $\eta$  in  $\mathbb{R}^d$  such that  $\eta(B_r(x)) \geq \gamma r^{d-1}$  for all  $x \in Z_0$  and  $r \in (0, 1)$ , which plays here the same role of  $\mathcal{H}^{d-1}(\Theta)$  in the assumption (A1). Furthermore, as observed in the proof of Theorem 3.5, we also have that  $\mathcal{H}^{d-1}(\partial Z_0) < \infty$ .

- $\text{diam}(Z_0) < \infty$  since  $Z_0$  is compact; then assumption (A2) and condition (3.8) are easily checked.
- The role of the measure  $\eta_W$  in the proof of Theorem 3.9 is played now by the measure

$$\tilde{\eta}_W(\cdot) := \frac{\sum_{x_i \in \Psi(\omega)} \eta(\cdot - x_i) \mathbf{1}_{(x_i + \partial Z_0) \cap W_{\oplus 2} \neq \emptyset}}{\text{card}\{x_i \in \Psi(\omega) : (x_i + \partial Z_0) \cap W_{\oplus 2} \neq \emptyset\}}.$$

It is easily seen that  $\tilde{\eta}_W$  is a probability measure and that

$$\tilde{\eta}_W(B_r(x)) \geq \frac{\gamma}{\text{card}\{x_i \in \Psi(\omega) : (x_i + Z_0) \cap W_{\oplus 2} \neq \emptyset\}} r^{d-1} \quad \forall x \in \partial \Xi(\omega) \cap W_{\oplus 1}, \quad \forall r \in (0, 1),$$

with  $\mathbb{E}[\text{card}\{x_i \in \Psi(\omega) : (x_i + Z_0) \cap W_{\oplus 1} \neq \emptyset\}] < \infty$  by (3.8).

About the differentiability of  $H_{\Xi}$ , Theorem 4.1 simplifies as follows, by taking into account Remark 3.4 and Remark 4.2.

**Corollary 5.2** *Let  $\Xi$  be a Boolean model in  $\mathbb{R}^d$  with deterministic typical grain  $Z_0$  and intensity  $f$ . If  $Z_0$  is a compact set with positive reach satisfying condition (4.2) and  $f$  is locally bounded, Lipschitz and differentiable at  $\mathcal{H}^{d-1}$ -a.e.  $x \in \mathbb{R}^d$ , then (4.5) holds.*

## 5.2 Stationary case

We observe now how our results simplify in the stationary case, in accordance with the available results in current literature in the special case of convex grains. Let us notice that if  $\Xi$  is stationary with  $f \equiv c > 0$ , then only the regularity assumption (A1) on the typical grain  $Z_0$  and the usual condition (3.8) are required (only (A1') in the lower dimensional case, by Lemma 3.14). Then  $\sigma_{\Xi}$  is now independent of  $x$ , as expected, given by

$$\sigma_{\Xi} = e^{-c\mathbb{E}_Q[\mathcal{H}^d(Z_0)]} c\mathbb{E}_Q[\mathcal{SM}(Z_0)],$$

(where  $\mathcal{SM}(Z_0)$  exists finite equal to  $P(Z_0) + 2\mathcal{H}^{d-1}(Z_0^0 \cap \partial Z_0)$  as a consequence of (A1)), and so

$$H'_{\Xi}(0) = c\mathbb{E}_Q[P(Z_0) + 2\mathcal{H}^{d-1}(Z_0^0 \cap \partial Z_0)].$$

All the other results of Section 3 simplify similarly; in particular the next statement is the stationary counterpart of Theorem 4.1. Note that the integrability condition (4.3) is here weakened.

**Proposition 5.3** *Let  $\Xi$  be a stationary Boolean model in  $\mathbb{R}^d$  with intensity  $f \equiv c$  and typical grain  $Z_0$  as in the assumption (A1), such that  $\text{reach}(Z_0) > R$  for some  $R > 0$ , condition (4.2) is fulfilled, and  $\mathbb{E}_Q[|\Phi_i(Z_0)|] < \infty$  for all  $i = 1, \dots, d-1$ . Then*

$$H''(0) = 2\pi c\mathbb{E}_Q[\Phi_{d-2}(Z_0)] - (c\mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)])^2. \quad (5.1)$$

Note that (5.1) coincides with Equation (1.4) in [23], since  $\mathcal{H}^{d-1}(\partial Z_0) = 2\Phi_{d-1}(Z_0)$  by (4.2).

Besides, under the assumptions of Theorem 4.1, if  $\mathcal{H}^{d-1}(\partial Z_0) = 0$  (e.g.,  $\Xi$  segment Boolean model in  $\mathbb{R}^3$ ) Equation (4.5) gives

$$\frac{\partial^2}{\partial r^2} H_{\Xi}(r, x)|_{r=0} = 2\pi\mathbb{E}_Q \left[ \int_{\mathbb{R}^d} f(y) \Phi_{d-2}(Z^x; dy) \right];$$

if furthermore  $\Xi$  is stationary with  $f \equiv c$ , then  $H''(0) = 2\pi c\mathbb{E}[\Phi_{d-2}(Z_0)] = 2\pi\mathbb{E}[\Phi_{d-2}(\Xi)]$ , in accordance with Corollary 7.3 in [20].

### 5.3 Some examples

The results presented in the previous sections involve regularity properties of the boundary of the typical grain and of the intensity measure of the Poisson process, which may apply to a great variety of Boolean models. In the following examples we extend known results for Boolean models of balls and of segments, often considered in literature, and discuss a couple of situations relevant in real applications, namely, birth-and-growth processes (e.g., see [8]) and fibre processes.

**Example 1** In [31, Section 6] a planar ( $d = 2$ ) Boolean model  $\Xi$  of discs with intensity  $f$  and radius distribution  $G$  is considered (with  $f$  and  $G$  sufficiently regular); in particular, in [31, Theorem 7] it is proved that

$$\lambda_{\partial\Xi}(x) = \exp \left\{ - \int_0^\infty \int_{B_r(0)} f(x-y) dy G(dr) \right\} \int_0^\infty \left( \int_{\partial B_1(0)} f(x-ry) \mathcal{H}^1(dy) \right) r G(dr). \quad (5.2)$$

More in general, let  $\Xi$  be in  $\mathbb{R}^d$  ( $d \geq 2$ ), then Proposition 3.12 applies (see [28, Ch. 4] for a discussion on Assumption (A1) for this kinds of models) and (5.2) follows by (3.1), by noticing that

$$\Lambda(\mathcal{Z}^{x,0}) = \int_0^\infty \int_{B_r(x)} f(y) dy G(dr) = \int_0^\infty \int_{B_r(0)} f(x-y) dy G(dr)$$

and

$$\begin{aligned} \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] &= \int_0^\infty \int_{\partial B_r(x)} f(y) \mathcal{H}^{d-1}(dy) G(dr) \\ &= \int_0^\infty \left( \int_{\partial B_1(0)} f(x-ry) \mathcal{H}^{d-1}(dy) \right) r^{d-1} G(dr). \end{aligned}$$

In [7, Remark 7(3)] a class of birth-and-growth processes with constant growth rate is considered; the resulting time dependent random closed set turns out to be a Boolean model of balls with random radius, and so of the same kind as in the above example. We also mention that birth-and-growth processes such that the nucleation is site-saturated (i.e., it takes place at time  $t = 0$ ), according to an inhomogeneous spatial Poisson point process, and such that the shape of the grains is preserved during the process, are considered in real applications (e.g., [30]); for this kind of processes, the growing region at any fixed time  $t$  is a Boolean model, and so our results apply in the study of the mean densities, whenever the shape of the grains is sufficiently regular.

**Example 2 (Segment Boolean model)** Let  $\Xi$  be a homogeneous, say  $f \equiv c > 0$ , Boolean model of segments with random length  $L$  and orientation; then Equation (3.15) gives the well known result  $\lambda_\Xi(x) = c\mathbb{E}[L] \forall x \in \mathbb{R}^d$  (see [6, p. 42]). Such result can be generalized to the inhomogeneous case. For the sake of simplicity we consider  $d = 2$ , but a similar example can be done in  $\mathbb{R}^d$  with  $d > 2$ . So, let  $\mathbf{K} = \mathbb{R}_+ \times [0, 2\pi]$ , and for all  $s = (l, \alpha) \in \mathbf{K}$  let

$$Z_0(s) := \{(u, v) \in \mathbb{R}^2 : u = \tau \cos \alpha, v = \tau \sin \alpha, \tau \in [0, l]\}$$

be the segment with length  $l$  and orientation  $\alpha$ . Let  $\mathbb{P}_L(dl)$  be the probability law of the random length  $L$  of  $Z_0$ , and let  $\Psi$  be the marked Poisson process in  $\mathbb{R}^d \times \mathbf{K}$  having intensity measure  $\Lambda(dy \times ds) = f(y) dy Q(ds)$ , with  $f(u, v) = u^2 + v^2$  and  $Q(ds) = \frac{1}{2\pi} d\alpha \mathbb{P}_L(dl)$  such that  $\int_{\mathbb{R}_+} l^3 \mathbb{P}_L(dl) < \infty$ . (This last assumption is sufficient to guarantee condition (A2'); of course, for different intensities

$f$  we might have different conditions on the moments of  $L$ .) Then, Theorem 3.13 applies and for  $\mathcal{H}^2$ -a.e.  $x = (x_1, x_2) \in \mathbb{R}^2$  we get

$$\lambda_{\Xi}(x_1, x_2) = \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \int_0^l f(x_1 - \tau \cos \alpha, x_2 - \tau \sin \alpha) d\tau d\alpha \mathbb{P}_L(dl) = (x_1^2 + x_2^2)\mathbb{E}[L] + \frac{1}{3}\mathbb{E}[L^3].$$

In the above example  $Z_0$  was assumed to be a random segment, i.e. the most regular 1-dimensional curve in  $\mathbb{R}^2$ ; clearly Theorem 3.13 applies also in the case of much less regular curves. For instance, if  $Z_0$  is a rectifiable curve in  $\mathbb{R}^d$ , then the corresponding Boolean model is called *fibre process* or *system*, and it is of interest in many real applications (see [6] and reference therein); general results which extend the stationary case might be obtained as a direct application of (3.15).

## 6 A statistical application: estimation of the mean density of lower dimensional Boolean models

We know that whenever  $\Theta$  is a  $d$ -dimensional random closed set, then  $\lambda_{\Theta}(x) = \mathbb{P}(x \in \Theta)$  for  $\mathcal{H}^d$ -a.e.  $x$ , and so it can be easily estimated by means of the empirical capacity functional; whereas whenever  $\Theta$  is a random point, the problem of the estimation of its mean density, which coincides with its probability density function, has been largely solved since long in nowadays standard literature by means of either histograms or kernel estimators. Problems arise with nontrivial lower dimensional random sets; then, by noticing also that (3.13) tells us that  $\lambda_{\partial\Xi}(x) = (1 - \mathbb{P}(x \in \Xi))\lambda_{\Xi}(x)$  if  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$ , where  $\tilde{\Xi}$  is the Boolean model with the same intensity measure but with typical grain  $\partial Z_0$ , we consider in this section Boolean models with lower dimensional typical grain.

Even if inhomogeneous random closed sets appear frequently in real applications, the problem of the mean density estimation has been widely examined only in the stationary case (e.g., see [6, 27] and reference therein), in which  $\Xi$  is often assumed to have unknown constant intensity  $c > 0$  and known mark distribution  $Q$ , so that only the parameter  $c$  has to be estimated, being  $\lambda_{\Xi} = c\mathbb{E}_Q[\mathcal{H}^n(Z_0)]$  in this case. results about the estimation of the intensity  $c$  of the underlying Poisson point process associated to  $\Xi$ , related to the estimation of  $\lambda_{\Xi}$ , can be found in [6, §3.4](see also [25, 27]). The inhomogeneous case has been mainly faced by assuming local stationarity or gradient structures (e.g., see [14]), so that known results in the homogeneous case might be applied to estimate a stepwise approximation of the mean density.

Whenever it is possible to estimate the intensity  $f$  and the mark distribution  $Q$  of the typical grain  $Z_0$ , an estimation of  $\lambda_{\Xi}$  might be obtained by the explicit formulae proved in the sections above; actually, the estimation of  $f$  and  $Q$  as well as the evaluation of the mean density  $\lambda_{\Xi}(x)$  at a given point  $x \in \mathbb{R}^d$ , might be fairly hard. Thus, the aim of the present section is to provide estimators for the mean density of lower dimensional Boolean models in the general case of  $f$  and  $Q$  unknown.

Let  $\Xi$  be a Boolean model as in the assumptions of Theorem 3.13; then by (3.15) and noticing that  $\mathbb{P}(x \in \Xi_{\oplus r}) = T_{\Xi}(B_r(x))$ , where  $T_{\Xi}$  is the capacity (or hitting) functional of  $\Xi$  [24], a natural estimator of  $\lambda_{\Xi}(x)$  can be given in terms of the *empirical capacity functional* of  $\Xi$ , defined as [13]

$$\hat{T}_{\Xi}^N(K) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\Xi_i \cap K \neq \emptyset}, \quad \forall \text{ compact } K \subset \mathbb{R}^d,$$

for any i.i.d. random sample  $\Xi_1, \dots, \Xi_N$  of  $\Xi$ .

**Proposition 6.1** *Let  $\Xi$  be a Boolean model as in the assumptions of Theorem 3.13 and  $\{\Xi_i\}_{i \in \mathbb{N}}$  be a sequence of random closed sets i.i.d. as  $\Xi$ ; then the estimator  $\lambda_{\Xi}^N(x)$  of  $\lambda_{\Xi}(x)$  so defined*

$$\widehat{\lambda}_{\Xi}^N(x) := \frac{\sum_{i=1}^N \mathbf{1}_{\Xi_i \cap B_{R_N}(x) \neq \emptyset}}{N b_{d-n} R_N^{d-n}} = \frac{\widehat{T}_{\Xi}^N(B_{R_N}(x))}{b_{d-n} R_N^{d-n}}, \quad (6.1)$$

is asymptotically unbiased and weakly consistent for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ , if  $R_N$  is such that

$$\lim_{N \rightarrow \infty} R_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} N R_N^{d-n} = \infty, \quad (6.2)$$

*Proof.* The law of large numbers and (3.15) imply the asymptotic unbiasedness for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ . Note that  $\lim_{N \rightarrow \infty} \mathbb{P}(x \in \Xi_{\oplus R_N}) = \mathbb{P}(x \in \Xi) = 0$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$  since  $n < d$ ; so

$$\lim_{N \rightarrow \infty} \text{var}(\widehat{\lambda}_{\Xi}^N(x)) = \lim_{N \rightarrow \infty} \frac{N \mathbb{P}(x \in \Xi_{\oplus R_N})(1 - \mathbb{P}(x \in \Xi_{\oplus R_N}))}{(N b_{d-n} R_N^{d-n})^2} = 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

and we conclude that  $\widehat{\lambda}_{\Xi}^N(x)$  converges to  $\lambda_{\Xi}(x)$  in probability for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .  $\square$

Then, a problem of statistical interest could be to find the optimal *width*  $R_N$  satisfying condition (6.2) which minimizes the mean square error of  $\widehat{\lambda}_{\Xi}^N(x)$  (i.e.  $\mathbb{E}[(\widehat{\lambda}_{\Xi}^N(x) - \lambda_{\Xi}(x))^2]$ ). To investigate this problem is not the aim of the present paper and we leave this as open problem for further developments; we point out here that Proposition 6.1 apply to any random closed set  $\Xi$  in  $\mathbb{R}^d$ , not necessarily a Boolean model, such that (1.2) holds, and that  $\widehat{\lambda}_{\Xi}^N$  can be seen as the generalization to the case of  $n$ -dimensional random closed sets of the well known estimator of the probability density of a random point. Indeed, even if the particular case  $n = 0$  can be handle with much more elementary tools, if in particular  $\Xi = X$  is a random point in  $\mathbb{R}^d$  with probability density function  $f_X$ , then Equation (1.2) holds with  $\lambda_X = f_X$  (see also [1, Remark 8]), the assertion of Proposition 6.1 still holds, and the estimator  $\widehat{\lambda}_X^N$  turns out to be closely related to the well known definition of histogram (see [26] and [28] for details): for sake of simplicity, let us consider the case  $d = 1$  with  $X$  random variable with density  $f_X$ ; then, if  $\{X_i\}_{i \in \mathbb{N}}$  is a sequence of random variables i.i.d. as  $X$ ,

$$\widehat{f}_X(x) := \widehat{\lambda}_X^N(x) \stackrel{(6.1)}{=} \frac{\sum_{i=1}^N \mathbf{1}_{B_{R_N}(x)}(X_i)}{N b_1 R_N} = \frac{\text{card}\{i : X_i \in I_x\}}{N |I_x|},$$

where  $I_x$  is the interval in  $\mathbb{R}$  centered in  $x$  with length  $|I_x| = 2R_N$  with the usual condition

$$|I_x| \longrightarrow 0 \quad \text{and} \quad N |I_x| \longrightarrow \infty \quad \text{as } N \rightarrow \infty.$$

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