Mean ergodic theorem in locally convex linear topological spaces

by

M. ALTMAN (Warszawa).

KOSAKU YOSIDA has proved the following generalization of the mean ergodic theorem of J. v. NEUMANN:

Let T be a linear transformation which maps a Banach space into itself. Let us further assume that

1^0 *the sequence* $\{T^m(x)\}$ $(n=1,2,\ldots)$ *is bounded for any element* x *and*

2^0 *the transformation* T *is weakly completely continuous.*

Under these assumptions, the sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)\right\}$$

converges strongly to a point y, *and* $T(y)=y$.

Using the same method as in the paper [2] we can prove the above theorem for locally convex linear topological spaces.

Let $\mathfrak{X}$ be a locally convex linear topological space[1]) and let $\bar{\mathfrak{X}}$ be the space of all linear functionals defined in $\mathfrak{X}$.

Definition 1. A sequence of elements $\{x_n\}$ is said to be *weakly convergent to an element* x if for any $X\epsilon\bar{\mathfrak{X}}$ the sequence $\{X(x_n)\}$ converges to $X(x)$.

Definition 2. A transformation T is said to be *weakly completely continuous* if there exists a neighbourhood O of zero such that the image $T(O)$ is sequentially weakly compact, i. e. any infinite part contains a sequence weakly convergent to some element of the space.

It is known (see [2]) that the space $\mathfrak{X}$ is isomorphic with a certain (B_a^*)-space in which is defined a class of pseudonorms $|x|_\vartheta$, where $\vartheta\epsilon\Phi$. The system of neighbourhoods of zero consists of the sets of all elements $x\epsilon\mathfrak{X}$ such that $|x|_{\vartheta_i}<\varepsilon$ $(i=1,2,\ldots,k)$, $\varepsilon>0$. (Φ is an abstract set and $\bar{\bar{\Phi}}=\aleph_a$).

The sequence $\{x_n\}$ is said to be *convergent to zero* if $|x_n|_\vartheta\to 0$ for any $\vartheta\epsilon\Phi$.

Definition 3. A set $Z\subset\mathfrak{X}$ is said to be *bounded* if for any sequence $\{x_n\}\subset Z$ and for every arbitrary sequence of numbers λ_n convergent to zero we have $\lambda_n x_n\to 0$, when $n\to\infty$.

Let T be a linear[2]) weakly completely continuous transformation which maps the neighbourhood of zero O into a sequentially weakly compact set. Then there exists a neighbourhood of zero V such that $V\subset O$, where V is defined by the inequalities $|x|_{\vartheta_i}<\varepsilon$ $(i=1,2,\ldots,k)$. We define the pseudonorm $|x|=\sup_i|x|_{\vartheta_i}$. Then the image of any set bounded in this pseudonorm is sequentially weakly compact and therefore bounded in the sense of the definition 3. Hence it appears the following

Lemma. *If T is a linear weakly completely continuous transformation, then for every ϑ there exists a constant M_ϑ such that $|T(x)|_\vartheta\leqslant M_\vartheta|x|$.*

The proof is the same as in [2] (lemma 1).

Theorem. *Let T be a linear transformation which maps a locally convex linear topological space $\mathfrak{X}$ into itself. Let us further assume that*

1^0 *the sequence* $\{T^m(x)\}$ $(n=1,2,\ldots)$ *is bounded for any* $x\epsilon\mathfrak{X}$,

2^0 *the transformation* T *is weakly completely continuous.*

Under these assumptions, the sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)\right\}$$

converges for any $x\epsilon\mathfrak{X}$ *to a point* y, *and* $T(y)=y$.

Proof. We divide the space $\mathfrak{X}$ into classes and we say that x_1 *and* x_2 *belong to the same class* $\mathfrak{x}$, if $|x_1-x_2|=0$. The set of all elements $x\epsilon\mathfrak{X}$ such that $|x|=0$ constitutes the zero class. We denote by $\mathfrak{X}^*$ the obtained quotient space which is a Banach space with the norm $|\mathfrak{x}|=|x|$.

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[1]) Concerning the definition see [2].

[2]) We consider the continuity in the sense of Heine, i. e. $x_n\to 0$ implies that $T(x_n)$ converges to zero.

The transformation $y=T(x)$ defines a transformation $\mathfrak{y}=\mathfrak{T}(\mathfrak{x})$ in the space $\mathfrak{X}^*$, where $x\epsilon\mathfrak{x}, y\epsilon\mathfrak{y}$. It follows from the lemma that $\mathfrak{T}$ is a linear transformation. It is obvious that $\mathfrak{T}$ is a weakly completely continuous transformation.

We denote by $\mathfrak{X}'$ the completion of the space $\mathfrak{X}^*$. The transformation $\mathfrak{T}$ can be extended on the whole space $\mathfrak{X}'$. We shall show that the range of this extension is contained in $\mathfrak{X}^*$. Let $\mathfrak{x}$ be an arbitrary element of $\mathfrak{X}'$. There exists a sequence $\{\mathfrak{x}_n\}\subset\mathfrak{X}^*$ such that $\mathfrak{x}_n\to\mathfrak{x}$. Since the sequence $\{\mathfrak{x}_n\}$ is convergent and therefore bounded, the sequence $\mathfrak{T}(\mathfrak{x}_n)$ is weakly compact in $\mathfrak{X}^*$ and convergent to $\mathfrak{y}\,\epsilon\,\mathfrak{X}^*$, hence $\mathfrak{y}=\mathfrak{T}(\mathfrak{x})$. From the condition 1^0 it follows that the sequence $\{\mathfrak{T}^n(\mathfrak{x})\}$ is bounded for any $\mathfrak{x}\,\epsilon\,\mathfrak{X}^*$. Since the range of the transformation $\mathfrak{T}$ is contained in $\mathfrak{X}^*$, the sequence $\{\mathfrak{T}^n(\mathfrak{x})\}$ is bounded for any $\mathfrak{x}\,\epsilon\,\mathfrak{X}'$, and the theorem of Kosaku Yosida is valid for the transformation $\mathfrak{T}$.

Consider now the sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)\right\}.$$

The sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1}\mathfrak{T}^i(\mathfrak{x})\right\},\quad \text{where}\quad x\,\epsilon\,\mathfrak{x},$$

converges to some element $\mathfrak{y}\,\epsilon\,\mathfrak{X}'$, and $\mathfrak{y}=\mathfrak{T}(\mathfrak{y})\,\epsilon\,\mathfrak{X}^*$. On account of the lemma and since

$$\left|\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)-\bar{y}\right|=\left|\frac{1}{n}\sum_{i=0}^{n-1}\mathfrak{T}^i(\mathfrak{x})-\mathfrak{y}\right|,\quad \text{where}\quad \bar{y}\,\epsilon\,\mathfrak{y},$$

the sequence

$$\left\{T\left(\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)\right)\right\}=\left\{\frac{1}{n}\sum_{i=1}^{n}T^i(x)\right\}$$

converges to $y=T(\bar{y})$. Since

$$\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)=\frac{1}{n}\,x+\sum_{i=1}^{n}T^i(x)-\frac{1}{n}\,T^n(x),$$

the sequences $\{x/n\}$ and $\{T^n(x)/n\}$ converge to 0, the sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)\right\}$$

converges to y. Then the sequence

$$\left\{\frac{1}{n}\sum_{i=1}^{n}T^i(x)\right\}$$

converges to $T(y)$. Since

$$\frac{1}{n}\sum_{i=1}^{n}T^i(x)=\frac{1}{n}\sum_{i=0}^{n-1}T^i(x)+\frac{1}{n}\,T^n(x)-\frac{1}{n}\,x,$$

we have $T(y)=y$.

References.

[1] Kosaku Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Japan 14 (1938), p. 292 - 294.

[2] M. Altman, *On linear functional equations in locally convex linear topological spaces*, Studia Mathematica, this volume.

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