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Yves Achdou, Italo Capuzzo-Dolcetta

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Mean Field Games: Numerical Methods

Yves Achdou [*], Italo Capuzzo-Dolcetta [†]

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Abstract

Mean field type models describing the limiting behavior of stochastic differential game problems, as the number of players tends to $+\infty$, have recently been introduced by J-M. Lasry and P-L. Lions. Numerical methods for the approximation of stationary and nonstationary such models are proposed. In particular, existence and uniqueness are investigated, as well as bounds on the solutions. Numerical experiments are carried out.

1 Introduction

Mean field type models describing the limiting behavior of stochastic differential game problems as the number of players tends to $+\infty$ have recently been introduced by J-M. Lasry and P-L. Lions [11, 12, 13]. In the stationary setting, a typical model of this kind comprises the following system:

$$-\nu\Delta u + H(x, \nabla u) + \lambda = V[m], \; x \in \mathbb{T}^d, \tag{1}$$

$$-\nu\Delta m - \mathrm{div}\left(m\frac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \; x \in \mathbb{T}^d, \tag{2}$$

$$\int_{\mathbb{T}^d} u = 0, \quad \int_{\mathbb{T}^d} m = 1, \quad m > 0. \tag{3}$$

The unknowns are the scalar functions u, m defined on the $d-$dimensional torus $\mathbb{T}^d$ and the real number λ. The data are a positive number ν, the Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, convex with respect to p and the (nonlinear) mapping V associating to a probability density m a Lipschitz function $V[m]$ on $\mathbb{T}^d$. Typical examples for V include nonlocal smoothing operators.

The time-dependent analogue of system (1)-(3), also considered in [11, 12, 13], is

$$\frac{\partial u}{\partial t} - \nu\Delta u + H(x, \nabla u) = V[m], \quad \text{in } \mathbb{T}^d \times (0, T), \tag{4}$$

$$\frac{\partial m}{\partial t} + \nu\Delta m + \mathrm{div}\left(m\frac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \quad \text{in } \mathbb{T}^d \times (0, T), \tag{5}$$

$$\int_{\mathbb{T}^d} m(x, t)dx = 1, \quad m > 0, \tag{6}$$

$$u(t = 0) = V_0[m(t = 0)], \quad m(t = T) = m_0. \tag{7}$$

[*]UFR Mathématiques, Université Paris 7, Case 7012, 175 rue du Chevaleret, 75013 Paris, France and UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France. achdou@math.jussieu.fr

[†]Dipartimento di Matematica, Università Roma "La Sapienza", Piazzale A. Moro 2, I-00185 Roma, capuzzo@mat.uniroma1.it

We refer to the mentioned papers of J-M. Lasry and P-L. Lions for analytical results concerning system (1)-(3) as well as for its interpretation in stochastic game theory. Let us only mention here that a very important feature of the mean field model above is that uniqueness and stability may be obtained under reasonable assumptions, see [11, 12, 13], in contrast with the Nash system describing the individual behavior of each player, for which uniqueness hardly occurs.

The aim of the present work is to propose discrete approximations by finite difference methods of the mean field model, both in the stationary case (1)-(3) or the non-stationary one (4)-(7). The numerical schemes that we use rely basically on monotone approximations of the Hamiltonian and on a suitable weak formulation of the equation for m. For the sake of simplicity of notations, we will always consider the 2-dimensional case although our approach and results hold for general d.

These schemes have several important features:

- existence and uniqueness for the discretized problems can be obtained by similar arguments as those used in the continuous case,

- they are robust when $\nu \to 0$ (the deterministic limit of the models),

- bounds on the solutions, which are uniform in the grid step, can be proved under reasonable assumptions on the data.

Let us mention in this respect that an important research activity is currently going on about approximation procedures for mean field games. Quite recently, we learned about a different numerical approach, based on the reformulation of the model as an optimization problem, which is restricted however to the case when $V[m](x) = g(m(x))$, see [10]. See also [5] for a very recent work on discrete time, finite state space mean field games.

In Section 2, we present the approximation of the nonlinear operators involved in e.g. (1)-(3) and the main assumptions that are going to be made. The finite difference scheme for the stationary model is discussed in Section 3: emphasis is put on existence, uniqueness and on bounds on the solution; the main difficulty faced there is to obtain bounds on the solution which are uniform in the discretization parameters. An example of a convergence result is also supplied in § 3.

In Section 4 we discuss an implicit in time finite difference method for the following evolution problem comprising two forward parabolic equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) \quad = \quad V[m], \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{8}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - \text{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u)\right) \quad = \quad 0, \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{9}$$

$$\int_{\mathbb{T}^2} m(x, t)dx = 1, \quad m \quad > 0, \tag{10}$$

$$u(x, 0) = u_0(x), \quad m(x, 0) \quad = \quad m_0(x). \tag{11}$$

By analogy with known results long time approximations for the cell problem in homogenization see [14], we expect that there exists some $\overline{\lambda} \in \mathbb{R}$ such that $u(x, t) - \overline{\lambda}t$ and $m(x, t)$ converge, as $t \to +\infty$, to the solution $(\overline{u}(x), \overline{m}(x), \overline{\lambda})$ of the stationary system (1)-(3). The main result in Section 4 is Theorem 5 on the existence for the discrete system.

Section 5 deals with the non-stationary mean field system (4)-(7) and contains results on existence and uniqueness.

Finally, the long time strategy mentioned above and numerical experiments for the stationary models are described in § 6.

A Newton method for the non-stationary problem (4)-(7) will be discussed in a forthcoming work, [2].

2 The finite difference operators

2.1 The proposed schemes

For simplicity, we discuss the approximation of (1)-(3).

Let $\mathbb{T}_h^2$ be a uniform grid on the torus with mesh step h, (assuming that $1/h$ is an integer N_h), and x_{ij} denote a generic point in $\mathbb{T}_h^2$. The values of u and m at $x_{i,j}$ will respectively be approximated by $U_{i,j}$ and $M_{i,j}$.

Hereafter, we will often make the following assumptions on the operator V:

($\mathbf{A_1}$): $V : m \to V[m]$ maps the set of probability measures into a bounded set of Lipschitz functions on $\mathbb{T}^2$.

($\mathbf{A_2}$): if m_n converges weakly to m then $V[m_n]$ converges to $V[m]$ uniformly on $\mathbb{T}^2$.

A possible approximation of $V[m](x_{i,j})$ is

$$(V_h[M])_{i,j} = V[m_h](x_{i,j}), \tag{12}$$

calling m_h the piecewise constant function taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \leq h/2$, and assuming that $V[m_h]$ can be computed in practice.

We introduce the finite difference operators

$$(D_1^+ U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h} \quad \text{and} \quad (D_2^+ U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h}, \tag{13}$$

and the numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$. The finite difference approximation of $H(x, \nabla u)$ is $g\left(x_{i,j}, (D_1^+ U^{n+1})_{i,j}, (D_1^+ U^{n+1})_{i-1,j}, (D_2^+ U^{n+1})_{i,j}, (D_2^+ U^{n+1})_{i,j-1}\right)$. Classically, we choose the discrete version of (1) as

$$\left(\begin{array}{c} \frac{\nu}{h^2}(4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1}) \\ +g\left(x_{i,j}, (D_1^+ U)_{i,j}, (D_1^+ U)_{i-1,j}, (D_2^+ U)_{i,j}, (D_2^+ U)_{i,j-1}\right) + \lambda_h \end{array} \right) = (V_h[M])_{i,j}. \tag{14}$$

With the notations

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1}), \tag{15}$$

and

$$[D_h W]_{i,j} = \left((D_1^+ W)_{i,j}, (D_1^+ W)_{i-1,j}, (D_2^+ W)_{i,j}, (D_2^+ W)_{i,j-1}\right)^T, \tag{16}$$

(14) becomes

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) = (V_h[M])_{i,j}. \tag{17}$$

The following assumptions on the discrete Hamiltonian g: $(q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ will be made

($\mathbf{H_1}$): monotonicity: g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4.

($\mathbf{H_2}$): consistency:

$$g(x, q_1, q_1, q_2, q_2) = H(x, q), \quad \forall x \in \mathbb{T}^2, \forall q = (q_1, q_2) \in \mathbb{R}^2. \tag{18}$$

($\mathbf{H_3}$) Differentiability: g is of class $\mathcal{C}^1$.

The discrete version of (2) is chosen according to the following heuristics:

- when u is fixed, (2) is a linear elliptic equation for m: therefore, when U is fixed, the discrete version of (2) should yield a matrix with the following properties:

  - the diagonal coefficients are positive,
  - the off-diagonal coefficients are nonpositive,

  so that hopefully a discrete maximum principle may be used.

- The argument used by Lasry and Lions for the uniqueness of the continuous problems (1)-(3) and (4)-(7) (see [13, 11, 12]) should hold in the discrete cases. For this reason, the discrete Hamiltonian g introduced above should be used in the discrete version of (2) as well, and we will make another assumption on g:

  $(\mathbf{H_4})$: convexity : the function $(q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ is convex.

The main idea is to consider the weak form of (2): it involves the term

$$- \int_{\mathbb{T}^2} \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u) \right) w = \int_{\mathbb{T}^2} m \frac{\partial H}{\partial p}(x, \nabla u) \cdot \nabla w$$

which will be approximated by

$$h^2 \sum_{i,j} m_{i,j} \nabla_q g(x_{i,j}, [D_h U]_{i,j}) \cdot [D_h W]_{i,j}.$$

This yields the following discrete version of (9):

$$
-\nu(\Delta_h M)_{i,j} \quad
\begin{aligned}
-\frac{1}{h} &\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j}) \\[2mm]
+ M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right) \\[6mm]
-\frac{1}{h} &\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1}) \\[2mm]
+ M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right)
\end{aligned}
\Biggr\} = 0.
\tag{19}
$$

We will also use the more compact but less explicit notation:

$$
\mathcal{B}_{i,j}(U, M) = \frac{1}{h}
\left(
\begin{array}{l}
\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j}) \\[2mm]
+ M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right) \\[8mm]
+
\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1}) \\[2mm]
+ M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right)
\end{array}
\right).
\tag{20}
$$

This yields the shorter form of (19):

$$-\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U, M) = 0. \tag{21}$$

Remark 1 *It is important to realize that the operator $M \mapsto -\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U, M)$ is the adjoint of the linearized version of the operator $U \mapsto -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j})$.*

4

(H₅) a further consistency assumption One has to assume that the discrete operator in (21) is a consistent approximation of the differential operator in (2), i.e. that there exists a positive integer ℓ, a real number δ_0, $\delta_0 \in (0,1)$ and some positive real number r such that for every $v, m \in \mathcal{C}^{\ell,\delta_0}(\mathbb{T}^2)$, there is a constant $\overline{K}$ depending on the norms of v and m in the previously mentioned Schauder spaces, such that for all $h < 1$, calling $\widetilde{V}$ and $\widetilde{M}$ the grid functions defined by $\widetilde{V}_{i,j} = \frac{1}{h^2}\int_{|x-x_{i,j}|_\infty < h/2} v dx$ and $\widetilde{M}_{i,j} = \frac{1}{h^2}\int_{|x-x_{i,j}|_\infty < h/2} m dx$, we have for all i, j,

$$\left| \mathcal{B}_{i,j}(\widetilde{V}, \widetilde{M}) - \mathrm{div}\left(m\frac{\partial H}{\partial p}(x, \nabla v) \right)(x_{i,j}) \right| \leq \overline{K} h^r. \tag{22}$$

This assumption is clearly fulfilled if g satisfies (18) and if g and H are smooth enough.

2.2 Summary

Finally, the finite difference approximation of (1)-(3) is to look for two grid functions U, M on $\mathbb{T}_h^2$ and for a scalar λ such that

$$\forall i, j: \ 0 \leq i, j < N_h, \quad \begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda &= (V_h[M])_{i,j}, \\ -\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U, M) &= 0, \end{cases} \tag{23}$$

$$\sum_{i,j} U_{i,j} = 0, \tag{24}$$

$$h^2 \sum_{i,j} M_{i,j} = 1 \quad \text{and} \quad M_{i,j} \geq 0 \text{ for } 0 \leq i, j < N_h. \tag{25}$$

with $(V_h[M])_{i,j}$ defined in (12), $\mathcal{B}_{i,j}(U, M)$ defined in (20) and the numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$ satisfying at least (**H₁**)-(**H₃**) above.

The same strategy will be used to approximate the nonstationary problems (4)-(7) and (8)-(11) with implicit schemes, see respectively (86)-(88) and (65)-(66) below.

In what follows, for two grid functions W and Z on $\mathbb{T}_h^2$, denote by $(W, Z)_2$ the inner product $(W, Z)_2 = \sum_{i,j} W_{i,j} Z_{i,j}$.

2.3 A useful lemma

We recall a useful lemma, which can be found in e.g. [4]. We give its proof for completeness.

Lemma 1 *Let V be a grid function on $\mathbb{T}_h^2$ and ρ be a positive parameter. Assume that g satisfies (H₁)-(H₃). There exists a unique grid function U such that*

$$\rho U_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - \nu(\Delta_h U)_{i,j} = V_{i,j}. \tag{26}$$

Proof. Existence for (26) is proved by using Leray-Schauder fixed point theorem: indeed, we consider the mapping $\mathcal{F} : \mathbb{R}^{N_h^2} \mapsto \mathbb{R}^{N_h^2}$,

$$(\mathcal{F}(U))_{i,j} = \frac{1}{\rho}\left(\nu(\Delta_h U)_{i,j} - g(x_{i,j}, [D_h U]_{i,j}) + V_{i,j} \right),$$

and the real number $r = \max_{(i,j)} \frac{1}{\rho}|H(x_{i,j}, 0) - V_{i,j}|$. From the continuity of g, $\mathcal{F}$ is continuous from $B_r = \{U \in \mathbb{R}^{N_h^2} : \|U\|_\infty \leq r\}$ to $\mathbb{R}^{N_h^2}$.

Assuming that $U \in \partial B_r$, there must exist at least one pair of indices (i_0, j_0) such that $U_{i_0,j_0} = \pm r$. Assuming that $U_{i_0,j_0} = r$, we have

$$\nu(\Delta_h U)_{i_0,j_0} - g(x_{i_0,j_0}, [D_h U]_{i_0,j_0}) \leq -H(x_{i_0,j_0}, 0),$$

5

from the monotonicity and the consistency of g. Hence,

$$(\mathcal{F}(U))_{i_0,j_0} \leq \frac{1}{\rho}\left(-H(x_{i_0,j_0},0) + V_{i_0,j_0}\right) \leq r,$$

and $(\mathcal{F}(U))_{i_0,j_0} \neq \lambda U_{i_0,j_0}$ whenever $\lambda > 1$. Similarly, if $U_{i_0,j_0} = -r$, then $(\mathcal{F}(U))_{i_0,j_0} \geq -r$ which implies that $(\mathcal{F}(U))_{i_0,j_0} \neq \lambda U_{i_0,j_0}$. Therefore $\mathcal{F}(U) \neq \lambda U$ for all $\lambda > 1$ and $U \in \partial B_r$. The Leray-Schauder fixed point theorem can be used: there exists a solution of (26) in B_r. Uniqueness for (26) stems from the monotonicity of g. ∎

3 Numerical analysis of the stationary problem (23)-(25)

Existence results for (23)-(25) can be proved under additional assumptions on g and V_h. The proof strategy will be to apply Brouwer theorem to a map χ defined on the compact and convex set

$$\mathcal{K} = \{(M_{i,j})_{0 \leq i,j < N_h} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \geq 0\} \tag{27}$$

which can be viewed as the set of the discrete probability measures.
We will see below that existence can be proved without bounds on U uniform with respect to h since the problem is finite dimensional. However, when possible, we will insist much on obtaining such bounds, for example, equicontinuity with respect to h, because they are important for passing to the limit when $h \to 0$.

We first define a map $\Phi : M \in \mathcal{K} \to U$ where (U, λ) is the unique solution of the first equation in (23) subject to the constraint in (24). The map $M \to \chi(M)$ is then obtained by solving a perturbation of the second equation in (23) with $U = \Phi(M)$, subject to the constraints in (25). The discrete function $U = \Phi(M)$ will be obtained by passing to the limit in the following Hamilton-Jacobi-Bellman equation

$$\rho U_{i,j}^{(\rho)} + g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) - \nu(\Delta_h U^{(\rho)})_{i,j} = (V_h[M])_{i,j}, \tag{28}$$

when the positive parameter ρ tends to 0. Such a strategy is reminiscent of those used for solving the cell problems in the homogenization of Hamilton-Jacobi-Bellman equations, see e.g. [14, 1, 3]. We first need to study (28) and obtain some bounds on U^ρ uniform w.r.t. ρ and M (and possibly uniform w.r.t. h); these will yield bounds on U uniform w.r.t. M (and possibly uniform w.r.t. h).

3.1 Preliminary results

Concerning the continuous problem, one of the assumptions made in [13, 11] was that there exists $\theta \in (0,1)$ such that for $|p|$ large,

$$\inf_{x \in \mathbb{T}^2}\left(\nabla_x H \cdot p + \frac{\theta}{2\nu}H^2\right) > 0. \tag{29}$$

It was useful in order to apply Bernstein's method to (1) and get a bound on $\|\nabla u\|_\infty$.
With assumption (29), we were not able to extend the method of Bernstein to the discrete level. Several other assumptions on H and g can be made. Assumption 1 below will make it possible to use the results of Kuo and Trudinger [9] and [8] on Hölder estimates for the solution of (28).

The slightly stronger assumption 2 will make it possible to apply recent results of Krylov [7]. We will use the following notation:

$$(D_1^c U)_{i,j} = \frac{U_{i+1,j} - U_{i-1,j}}{2h} = \frac{1}{2}\left((D_1^+ U)_{i,j} + (D_1^+ U)_{i-1,j}\right),$$

$$(D_2^c U)_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2h} = \frac{1}{2}\left((D_2^+ U)_{i,j} + (D_2^+ U)_{i,j-1}\right),$$

$$(D_1^2 U)_{i,j} = \frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{h^2} = \frac{1}{h}\left((D_1^+ U)_{i,j} - (D_1^+ U)_{i-1,j}\right),$$

$$(D_2^2 U)_{i,j} = \frac{U_{i,j+1} + U_{i,j-1} - 2U_{i,j}}{h^2} = \frac{1}{h}\left((D_2^+ U)_{i,j} - (D_2^+ U)_{i,j-1}\right).$$

Assumption 1 *a) The Hamiltonian H is of the form*

$$H(x,p) = \max_{\alpha \in \mathcal{A}} \left(p \cdot \alpha - L(x,\alpha)\right), \tag{30}$$

where $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$ and L is a $\mathcal{C}^0$ function on $\mathbb{T}^2 \times \mathcal{A}$. The function H is continuous with respect to x and of class $\mathcal{C}^1$ with respect to p.
b) The discrete Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x,q) \mapsto g(x,q)$, is continuous with respect to x uniformly in h. For all $h \le h_0$, it satisfies $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$.
c) Defining the function $\mathcal{F} : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x, q_1, q_2, s_1, s_2) \mapsto \mathcal{F}(x, q_1, q_2, s_1, s_2)$ by

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) = \mathcal{F}(x_{i,j}, (D_1^c U)_{i,j}, (D_2^c U)_{i,j}, (D_1^2 U)_{i,j}, (D_2^2 U)_{i,j}), \tag{31}$$

we assume that there exist positive constants a_0, a_1 and b_0 such that, for $h = 1/N_h \le h_0$,

$$a_0 \le -\frac{\partial \mathcal{F}}{\partial s_i} \le a_1, \quad and \quad \left|\frac{\partial \mathcal{F}}{\partial q_i}\right| \le b_0. \tag{32}$$

d) There exists a function $g^\infty : \mathbb{R}^4 \to \mathbb{R}$, such that

- $g^\infty(x, 0) = 0$,

- *for all $q \in \mathbb{R}^4$,*

$$\lim_{\epsilon \to 0} \sup_{x \in \mathbb{T}^2} \left| \epsilon g(x, \frac{q}{\epsilon}) - g^\infty(q) \right| = 0, \tag{33}$$

- *g^∞ is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4.*

Example 1 Let H be given by (30) with $\mathcal{A} = \{\alpha \in \mathbb{R}^2, |\alpha| \le 1\}$ and $L(x,\alpha) = L(\alpha) = \frac{1}{\gamma}|\alpha|^\gamma$, with $\gamma > 1$. It can be seen that

$$H(x,p) = H(p) = \begin{cases} \dfrac{\gamma - 1}{\gamma}|p|^{\frac{\gamma}{\gamma-1}}, & |p| \le 1, \\ |p| - \dfrac{1}{\gamma}, & |p| \ge 1, \end{cases}$$

and that with the Godunov scheme

$$g(x, q_1, q_2, q_3, q_4) = H\left(\sqrt{(q_1^-)^2 + (q_3^-)^2 + (q_2^+)^2 + (q_4^+)^2}\right),$$

Assumption 1 holds with in point d), $g^\infty(x, q_1, q_2, q_3, q_4) = \sqrt{(q_1^-)^2 + (q_3^-)^2 + (q_2^+)^2 + (q_4^+)^2}$.

Assumption 2 *a) The Hamiltonian H is of the form (30) where $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$ and L is a $\mathcal{C}^0$ function on $\mathbb{T}^2 \times \mathcal{A}$. The function H is Lipschitz continuous with respect to x uniformly in p and of class $\mathcal{C}^1$ with respect to p.*
b) The discrete Hamiltonian g satisfies point b) in Assumption 1.
c) The discrete Hamiltonian g is of the form

$$g(x, q_1, q_2, q_3, q_4) = \sup_{\beta \in \mathcal{B}} \left(\sum_{\ell=1}^4 (-a_\ell(x, \beta) s_\ell + b_\ell(x, \beta) q_\ell) - f(x, \beta) \right), \tag{34}$$

where

- $\mathcal{B}$ *is a compact set,*

- $s_1 = s_2 = (q_1 - q_2)/h$, $s_3 = s_4 = (q_3 - q_4)/h$,

- $a_1 = a_2 \geq 0$ *and* $a_3 = a_4 \geq 0$,

- *the functions* $a_\ell, b_\ell : \mathbb{T}^2 \times \mathcal{B} \to \mathbb{R}$ *are continuous with respect to β (uniformly w.r.t. h); b_ℓ and $\sqrt{\frac{\nu}{2} + a_\ell}$ are Lipschitz continuous with respect to x (uniformly w.r.t. h).*

- *the function* $f : \mathbb{T}^2 \times \mathcal{B} \to \mathbb{R}$ *is continuous with respect to β and Lipschitz continuous with respect to x.*

- *For all* $h \leq h_0$, $(x, \beta) \in \mathbb{T}^2 \times \mathcal{B}$, $\max\left(hb_1^+(x, \beta) - a_1(x, \beta), hb_2^-(x, \beta) - a_2(x, \beta), hb_3^+(x, \beta) - a_3(x, \beta), hb_4^-(x, \beta) - a_4(x, \beta) \right) \leq 0$.

d) The discrete Hamiltonian g satisfies point d) in Assumption 1.

Example 2 We take H as in Example 1. The Lax-Friedrichs scheme with a large enough artificial viscosity parameter θ satisfies Assumptions 1. It also satisfies Assumptions 2, because

$$g(x, q_1, q_2, q_3, q_4) = H\left(x, \left(\frac{q_1 + q_2}{2}, \frac{q_3 + q_4}{2} \right) \right) - \theta(q_1 - q_2 + q_3 - q_4)$$

$$= \sup_{\alpha \in \mathcal{A}} \left(\alpha_1 \frac{q_1 + q_2}{2} + \alpha_2 \frac{q_3 + q_4}{2} - \theta(q_1 - q_2 + q_3 - q_4) - L(\alpha) \right).$$

Example 3 We give a simple example where H only depends on p_1. Let H be given by (30) with $\mathcal{A} = \{(\alpha, 0), |\alpha| \leq 1\}$ and $L(x, (\alpha, 0)) = \frac{1}{\theta}|\alpha|^\theta$, with $\theta > 1$. It can be seen that $H(x, p) = H(p_1)$ with $H(p_1) = \frac{\theta - 1}{\theta}|p_1|^{\frac{\theta}{\theta - 1}}$ if $|p_1| \leq 1$ and $H(p_1) = |p_1| - \frac{1}{\theta}$ if $|p_1| \geq 1$. Consider the discrete Hamiltonian

$$g(x, q_1, q_2, q_3, q_4) = \max_{|\beta_1| \leq 1, |\beta_2| \leq 1} \left(-\beta_1^- q_1 + \beta_2^+ q_2 - \frac{1}{\theta}|\beta_1|^\theta - \frac{1}{\theta}|\beta_2|^\theta \right),$$

It can be checked that Assumption 2 holds. In particular, $g^\infty(x, q_1, q_2, q_3, q_4) = \max(q_2, -q_1, q_2 - q_1, 0)$.

Proposition 1 *Assume that H and g satisfy Assumption 1 or Assumption 2. Let V be a grid function on $\mathbb{T}_h^2$ (we agree to write V instead of V_h) and ρ be a nonnegative real number. For $h \leq h_0$, let the grid function U on $\mathbb{T}_h^2$ be the solution of*

$$\rho U_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - \nu(\Delta_h U)_{i,j} = V_{i,j}, \tag{35}$$

see Lemma 1.

If Assumption 1 holds and if $\|V\|_\infty$ is bounded uniformly w.r.t $h \leq h_0$ by a constant c_0, then there exist two constants δ, $\delta \in (0,1)$ and C, $C > 0$, both depending on a_0, a_1, b_0, c_0 and on $\|U\|_\infty$ but not on h and ρ, such that for all $h \leq h_0$, $0 < \rho \leq 1$

$$|U(\xi) - U(\xi')| \leq C|\xi - \xi'|^\delta, \qquad \forall \xi, \xi' \in \mathbb{T}_h^2. \tag{36}$$

If Assumption 2 holds and if

$$\|V\|_\infty + \|D_h V\|_\infty \quad \text{is bounded uniformly w.r.t. } h \leq h_0 \text{ by a constant } c_0, \tag{37}$$

there exists a constant C, $C > 0$ depending on $\|U\|_\infty$ but not on h and ρ, such that $\forall h \leq h_0$, $0 < \rho \leq 1$

$$|U(\xi) - U(\xi')| \leq C|\xi - \xi'|, \qquad \forall \xi, \xi' \in \mathbb{T}_h^2. \tag{38}$$

Proof. In the first case, the result is a consequence of a theorem due to Kuo and Trudinger, see formula (3.10) in [9] and also [8], (which makes use of (32)).

In the second case, (38) is a particular case of a discrete Lipschitz estimate recently proved by Krylov with a very clever discrete Bernstein method, see [7], Theorem 2.5 and Remark 4.5. ■

Remark 2 *To cast the discrete quasilinear operator into the setting of Theorem 2.5 in [7], one needs to consider the grid function $W_{i,j} = -U_{i,j}$.*

In Proposition 1 the constants C depend on $\|U^\rho\|_\infty$. It is possible to improve this result by realizing that the constants actually depend on $\|U^\rho - U_{0,0}^\rho\|_\infty$ and by showing that this quantity is bounded uniformly w.r.t. ρ and h: the following proposition is due to F. Camilli and C. Marchi, see [3]. We give its proof since [3] is not published yet.

Proposition 2 *1. If Assumption 1 holds and if $\|V\|_\infty$ is bounded uniformly w.r.t $h \leq h_0$ by a constant c_0, then there exist two constants δ, $\delta \in (0,1)$ and C, $C > 0$ both independent of $h \leq h_0$ and ρ such that for all ρ, $1 \geq \rho > 0$, the solution of (35) satisfies*

$$\max_{\xi \neq \xi' \in \mathbb{T}_h^2} \frac{|U^{(\rho)}(\xi) - U^{(\rho)}(\xi')|}{|\xi - \xi'|^\delta} \leq C. \tag{39}$$

2. If Assumption 1 and (37) hold, then there exists a constant $C > 0$ independent of $h \leq h_0$ and ρ, such that for all ρ, $1 \geq \rho > 0$, the solution of (35) satisfies

$$\max_{\xi \neq \xi' \in \mathbb{T}_h^2} \frac{|U^{(\rho)}(\xi) - U^{(\rho)}(\xi')|}{|\xi - \xi'|} \leq C. \tag{40}$$

Proof. We give the proof in the first case only, since the second case is done similarly.

Lemma 1 yields he existence for (35). We also easily obtain a bound on $\|\rho U^{(\rho)}\|_\infty$, namely that

$$\|\rho U^{(\rho)}\|_\infty \leq \max_{i,j} \left(|H(x_{i,j}, 0)| + |V_{i,j}| \right), \tag{41}$$

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so there exists a positive constant C_1 independent of h and ρ such that $\|\rho U^{(\rho)}\|_\infty \le C_1$.

Let us have ρ tend to 0. We set $W^{(\rho)} = U^{(\rho)} - U_{0,0}^{(\rho)}$. Fixing h, assume that there exists a sequence ρ_k such that $\lim_{k \to \infty} \|W^{(\rho_k)}\|_\infty = +\infty$. We use the notation $\epsilon_k = 1/\|W^{(\rho_k)}\|_\infty$. The grid function $Z^{(k)} = \epsilon_k W^{(\rho_k)}$ satisfies

$$Z_{0,0}^{(k)} = 0, \qquad \|Z^{(k)}\|_\infty = 1, \tag{42}$$

$$\frac{\rho_k}{\epsilon_k} Z_{i,j}^{(k)} - \frac{\nu}{\epsilon_k} (\Delta_h Z^{(k)})_{i,j} + g(x_{i,j}, \frac{1}{\epsilon_k}[D_h Z^{(k)}]_{i,j}) + \rho_k U_{0,0}^{(\rho_k)} = V_{i,j}. \tag{43}$$

But (43) is equivalent to

$$\rho_k Z_{i,j}^{(k)} - \nu (\Delta_h Z^{(k)})_{i,j} + \epsilon_k g(x_{i,j}, \frac{1}{\epsilon_k}[D_h Z^{(k)}]_{i,j}) + \rho_k \epsilon_k U_{0,0}^{(\rho_k)} = \epsilon_k V_{i,j}. \tag{44}$$

Note that $-\nu(\Delta_h Z^{(k)})_{i,j} + \epsilon_k g(x_{i,j}, \frac{1}{\epsilon_k}[D_h Z^{(k)}]_{i,j}) = \mathcal{G}(x_{i,j}, (D_1^c Z)_{i,j}, (D_2^c Z)_{i,j}, (D_1^2 Z)_{i,j}, (D_2^2 Z)_{i,j})$ where $\mathcal{G}(x, q_1, q_2, s_1, s_2) = \epsilon \mathcal{F}(x, \frac{q_1}{\epsilon}, \frac{q_2}{\epsilon}, \frac{s_1}{\epsilon}, \frac{s_2}{\epsilon})$, so $\mathcal{G}$ also satisfies estimate (32).

From this and (41), we can apply estimate (36) to $Z^{(k)}$: we get that the grid functions $Z^{(k)}$ are equibounded and equicontinuous. Up to a subsequence, $Z^{(k)}$ converges to Z as k tends to infinity and Z satisfies:

$$Z_{0,0} = 0 \quad \text{and} \quad -\nu(\Delta_h Z)_{i,j} + g^\infty(x_{i,j}, [D_h Z]_{i,j}) = 0, \quad \forall i, j,$$

The assumptions made above on g^∞ and the discrete maximum principle yield that $Z = 0$ which contradicts (42). We have proved that $\|U^{(\rho)} - U_{0,0}^{(\rho)}\|_\infty \le C$ for a constant C independent of ρ, and (39) is a consequence of Proposition 1.

It can be shown by a similar contradiction argument using Ascoli-Arzela theorem that the constant C in the bound $\|U^{(\rho)} - U_{0,0}^{(\rho)}\|_\infty \le C$ does not depend on h, because the unique viscosity solution of $-\nu \Delta z + \sup_{\alpha \in \mathcal{A}} \alpha . \nabla z = 0$ with $z(0) = 0$ is 0. ∎

3.2 Existence for the discrete problem

Theorem 1 *If Assumption 1 is satisfied and if the operator V maps the probability measures to a bounded set of continuous functions on $\mathbb{T}^2$ and satisfies $(\mathbf{A_2})$, then the discrete problem (23)-(25) has at least a solution, and there exist two constants δ, $\delta \in (0,1)$ and $C > 0$ such that for all $h = 1/N_h < h_0$,*

$$\|U\|_\infty + \max_{\xi \ne \xi' \in \mathbb{T}_h^2} \frac{|U(\xi) - U(\xi')|}{|\xi - \xi'|^\delta} \le C. \tag{45}$$

Proof.

Step 1 We consider a mapping $\Phi : M \in \mathcal{K} \to U$, where U is part of the solution of the problem: find a grid function U and a scalar λ such that

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}, \tag{46}$$

with $\sum_{i,j} U_{i,j} = 0$. Indeed, it can be proved that if g satisfies the assumptions mentioned above, then there exist a unique $\lambda \in \mathbb{R}$ and a unique grid function U satisfying (46). To do it, we pass to the limit in (28) as $\rho \to 0$. The existence and uniqueness for (28) stems from Lemma 1. We may apply Proposition 2 since $V_h[M]$ is bounded uniformly w.r.t. h and M. Proposition 2 implies that there exists two constants $C > 0$ and $\delta \in (0,1)$ independent of h, M and ρ such

that (39) holds. This yields that there exists a constant $c(h) \sim h^{\delta-1}$ independent of M and ρ such that

$$\|D_h U^{(\rho)}\|_\infty \le c(h). \tag{47}$$

Up to the extraction of a subsequence we may say that as ρ tends to 0, $U^{(\rho)} - h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to a grid function U such that $\sum_{i,j} U_{i,j} = 0$, and that $\rho h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to $\lambda \in \mathbb{R}$. It is an easy matter to check that (U, λ) satisfies (46) and that the bounds (39) and (47) hold for U.

Uniqueness for λ stems from the following comparison principle: if U is a subsolution of (46) with $\lambda = \lambda_1$ and W is a supersolution of (46) with $\lambda = \lambda_2$, then $\lambda_2 \le \lambda_1$. Uniqueness for U is obtained by repeatedly applying the discrete maximum principle, from the monotonicity of g. We have defined the map $\Phi : M \in \mathcal{K} \to U$, where (U, λ) solve (46) and $\sum_{i,j} U_{i,j} = 0$.

Step 2: continuity of Φ Consider a sequence of grid functions $M^{(k)}$ in $\mathcal{K}$ which tends to $M \in \mathcal{K}$ as k tends to infinity. From the assumptions on V and V_h, $V_h[M^{(k)}]$ converges to $V_h[M]$. Consider λ and U solution of (46) and call $\lambda^{(k)}$, $U^{(k)}$ a solution of (46) with $M = M^{(k)}$. From the estimates above, the sequences $(\lambda^k)_k$ and $(\|U^{(k)}\|_\infty)_k$ are bounded. One can extract a subsequence k' such that $\lambda^{(k)}$ tends to $\widetilde{\lambda}$ and $U^{(k')}$ tends to $\widetilde{U}$ and that

$$-\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \widetilde{\lambda} = (V_h[M])_{i,j},$$

with $\sum_{i,j} \widetilde{U}_{i,j} = 0$.
Uniqueness for (46) implies that $\widetilde{\lambda} = \lambda$ and $\widetilde{U} = U$. The whole sequences $(\lambda^{(k)})_k$, $(U^{(k)})_k$ therefore tend to λ, U.
We have proved that the map Φ is continuous.

Step 3 For $M \in \mathcal{K}$ and $U = \Phi(M)$, consider the following linear problem: find $\widetilde{M}$ such that

$$\mu \widetilde{M}_{i,j} - \nu(\Delta_h \widetilde{M})_{i,j} - \mathcal{B}_{i,j}(U, \widetilde{M}) = \mu M_{i,j}, \tag{48}$$

where μ is a sufficiently large positive number which will be chosen later. This linear problem may be written

$$\mu \widetilde{M} + A\widetilde{M} = \mu M, \tag{49}$$

where A is a linear operator depending on U.
The assumptions of the monotonicity of g imply that $\frac{\partial g}{\partial q_1} \le 0$, $\frac{\partial g}{\partial q_2} \ge 0$, $\frac{\partial g}{\partial q_3} \le 0$ and $\frac{\partial g}{\partial q_4} \ge 0$. This yields that the matrix corresponding to A has positive diagonal entries and nonpositive off-diagonal entries. Furthermore, since g is $\mathcal{C}^1$, (47) implies that there exists a constant C independent of M (but possibly on h) such that for all i, j, $0 \le i, j \le N_h$, and for all $\ell = 1, 2, 3, 4$,

$$\left| \frac{\partial g}{\partial q_\ell}(x_{i,j}, [D_h U]_{i,j}) \right| \le C. \tag{50}$$

From this, we see that for μ large enough depending possibly on h but not on M, the matrix corresponding to $\mu Id + A$ is a M-matrix, and is therefore invertible. The system of linear equations (48) has a unique solution $\widetilde{M}$ and $\widetilde{M}$ is nonnegative since M is nonnegative.
We are left with proving that $h^2(\widetilde{M}, 1)_2 = h^2(M, 1)_2 = 1$. For two grid functions W and Z, let

us compute $(A^n W, Z)_2$: discrete integrations by part lead to

$$
(AW, Z)_2 = \\
\frac{\nu}{h^2} \sum_{i,j} \left((W_{i,j} - W_{i-1,j})(Z_{i,j} - Z_{i-1,j}) + (W_{i,j} - W_{i,j-1})(Z_{i,j} - Z_{i,j-1}) \right)
$$

$$
+ \left(\begin{array}{l} \sum_{i,j} (D_1^+ Z)_{i,j} W_{i,j} \dfrac{\partial g}{\partial q_1} (x_{i,j}, [D_h U]_{i,j}) \\ + \sum_{i,j} (D_1^+ Z)_{i-1,j} W_{i,j} \dfrac{\partial g}{\partial q_2} (x_{i,j}, [D_h U]_{i,j}) \end{array} \right) + \left(\begin{array}{l} \sum_{i,j} (D_2^+ Z)_{i,j} W_{i,j} \dfrac{\partial g}{\partial q_3} (x_{i,j}, [D_h U]_{i,j}) \\ + \sum_{i,j} (D_2^+ Z)_{i,j-1} W_{i,j} \dfrac{\partial g}{\partial q_4} (x_{i,j}, [D_h U]_{i,j}) \end{array} \right),
$$

$$ \tag{51} $$

$$
(AW, Z)_2 = \nu \sum_{i,j} (D_1^+ W)_{i,j} (D_1^+ Z)_{i,j} + \nu \sum_{i,j} (D_2^+ W)_{i,j} (D_2^+ Z)_{i,j}
$$
$$
+ \sum_{i,j} W_{i,j} [D_h Z]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U]_{i,j} \right). \tag{52}
$$

It is easy to check that for all grid functions W, $(AW, 1)_2 = 0$. Therefore, taking the inner product of (49) with the function $Z = 1$, we obtain that $h^2(\widetilde{M}, 1)_2 = h^2(M, 1)_2 = 1$, so $\widetilde{M} \in \mathcal{K}$. We call $\chi : \mathcal{K} \mapsto \mathcal{K}$ the mapping defined by $\chi : M \to \widetilde{M}$.

Step 4: existence of a fixed point of χ From the boundedness and continuity of the mapping Φ, and from the fact that g is $\mathcal{C}^1$ in the variable q, we obtain that χ is continuous. Therefore, we can apply Brouwer fixed point theorem and obtain that χ has a fixed point. ∎

We obtain a better result under Assumption 2 and a stronger assumptions on V:

Theorem 2 *If Assumption 2 holds and if V satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$, then the discrete problem (23)-(25) has at least a solution, and there exists a constant $C > 0$ such that for all $h = 1/N_h < h_0$,*

$$
\|U\|_\infty + \max_{\xi \neq \xi' \in \mathbb{T}_h^2} \frac{|U(\xi) - U(\xi')|}{|\xi - \xi'|} \leq C. \tag{53}
$$

Proof. Similar to that of Theorem 1, using now the second part of Proposition 2. ∎

Since the discrete problem is finite-dimensional, existence for (23)-(25) can be proved without a bound on U uniform with respect to h. Different assumptions on the structure of g can be made; for example:

Theorem 3 *Assume that*

- *g satisfies $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$,*

- *there exist two positive constants $\alpha > 0$ and $\gamma > 1$ and a nonnegative constant C such that*

$$
g(x, q_1, q_2, q_3, q_4) \geq \alpha((q_1^-)^2 + (q_2^+)^2 + (q_3^-)^2 + (q_4^+)^2)^{\gamma/2} - C, \quad \forall x \in \mathbb{T}^2. \tag{54}
$$

- *V maps the probability measures to a bounded set of continuous functions on $\mathbb{T}^2$ and satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$.*

The discrete problem (23)-(25) has at least a solution

Proof. The proof follows the same steps as that of Theorem 1. Only the first step of the proof is modified as follows:

existence and uniqueness for (28) follows from Lemma 1. We also easily obtain a bound on $\|\rho U^{(\rho)}\|_\infty$, namely that

$$\|\rho U^{(\rho)}\|_\infty \leq \max_{i,j} \left| H(x_{i,j}, 0) + (V_h[M])_{i,j} \right|, \tag{55}$$

so there exists a positive constant C_1 independent of M and ρ such that $\|\rho U^{(\rho)}\|_\infty \leq C_1$. From this, we deduce that there exists a positive constant C_2 independent of M and ρ such that

$$g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) - \nu(\Delta_h U^{(\rho)})_{i,j} \leq C_2, \qquad \forall i, j. \tag{56}$$

Using (54), we see that

$$g(x_{i,j}, [D_h W]_{i,j}) - \nu(\Delta_h W)_{i,j}$$
$$\geq \alpha \left(((D_1^+ W)_{i,j}^-)^2 + ((D_1^+ W)_{i-1,j}^+)^2 + ((D_2^+ W)_{i,j}^-)^2 + ((D_2^+ W)_{i,j-1}^+)^2 \right)^{\gamma/2} - C$$
$$- \frac{\nu}{h} \left((D_1^+ W)_{i,j}^+ + (D_1^+ W)_{i-1,j}^- + (D_2^+ W)_{i,j}^+ + (D_2^+ W)_{i,j-1}^- \right).$$

Calling $P_\rho = \|D_h U^{(\rho)}\|_\infty = \max_{i,j} \max(|(D_1^+ U^{(\rho)})_{i,j}|, |(D_2^+ U^{(\rho)})_{i,j}|)$, we deduce from (56) and the previous estimate that there exists a constant C_3 independent of M and ρ such that

$$\alpha P_\rho^\gamma - 4\frac{\nu}{h} P_\rho \leq C_3.$$

This yields (47) for a constant $c(h)$ independent of M and ρ. Up to the extraction of a subsequence we may say that as ρ tends to 0, $U^{(\rho)} - h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to a grid function U such that $\sum_{i,j} U_{i,j} = 0$, and that $\rho h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to $\lambda \in \mathbb{R}$. The limits U and λ satisfy (46). Uniqueness for (46) is proved as above, so Φ is well defined. ∎

Remark 3 *In Theorem 3, we were not able to obtain an estimate on U uniform w.r.t. h.*

Remark 4 *Note that the assumption ($\mathbf{A_2}$) can be relaxed in the discrete case. Indeed, Theorem 1 holds if we replace ($\mathbf{A_2}$) with the assumption that V_h is a continuous map from $\mathcal{K}$ defined in (27) to grid functions bounded by a constant independent of h. Theorem 3 holds if V_h is a continuous map from $\mathcal{K}$ to grid functions.*
These observations lead to existence results when V_h is a local operator, see § 3.4.

3.3 Uniqueness

Proposition 3 *If g satisfies ($\mathbf{H_1}$)-($\mathbf{H_4}$), and if the operator V_h is strictly monotone, i.e.*

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \leq 0 \Rightarrow M = \widetilde{M},$$

then (23)-(25) has at most a solution.

Proof. Let (U, M, λ) and $\widetilde{U}, \widetilde{M}, \widetilde{\lambda}$ be two solutions of (23)-(25). We have

$$-\nu(\Delta_h(U - \widetilde{U}))_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda - \widetilde{\lambda} = \left(V_h[M] - V_h[\widetilde{M}] \right)_{i,j}. \tag{57}$$

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Take a grid function such that $\sum_{i,j} Z_{i,j} = 0$. Multiplying by $Z_{i,j}$ and summing over all i, j leads to

$$-\nu(\Delta_h(U-\widetilde{U}), Z)_2 + \sum_{i,j}(g(x_{i,j}, [D_hU]_{i,j}) - g(x_{i,j}, [D_h\widetilde{U}]_{i,j}))Z_{i,j} = \left(V_h[M] - V_h[\widetilde{M}], Z\right)_2. \quad (58)$$

On the other hand, multiplying the second equation in (23) by $-W_{i,j}$ and summing over all i, j leads to

$$\nu(M, \Delta_h W)_2 - \sum_{i,j} M_{i,j}[D_hW]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_hU]_{i,j}\right) = 0. \quad (59)$$

From this and the similar equation satisfied by $\widetilde{M}$, we obtain

$$\begin{aligned}
0 =& \nu((M - \widetilde{M}), \Delta_h W)_2 \\
& - \sum_{i,j} M_{i,j}[D_hW]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_hU]_{i,j}\right) + \sum_{i,j} \widetilde{M}_{i,j}[D_hW]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h\widetilde{U}]_{i,j}\right).
\end{aligned} \quad (60)$$

Taking $Z = M - \widetilde{M}$ in (58), $W = U - \widetilde{U}$ in (60) and adding the resulting equations leads to

$$\begin{aligned}
&\sum_{i,j} M_{i,j}\left(g\left(x_{i,j}, [D_h\widetilde{U}]_{i,j}\right) - g\left(x_{i,j}, [D_hU]_{i,j}\right) - D_h(\widetilde{U} - U)_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_hU]_{i,j}\right)\right) \\
&+ \sum_{i,j} \widetilde{M}_{i,j}\left(g\left(x_{i,j}, [D_hU]_{i,j}\right) - g\left(x_{i,j}, [D_h\widetilde{U}]_{i,j}\right) - D_h(U - \widetilde{U})_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h\widetilde{U}]_{i,j}\right)\right) \\
&\hspace{6cm} + \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 = 0.
\end{aligned} \quad (61)$$

From the convexity of g and the monotonicity assumption of F, the three terms in (61) must vanish. The strong monotonicity of V_h implies that $M = \widetilde{M}$. A comparison argument similar to that used in the first step of the proof of Theorem 1 yields that $\lambda = \widetilde{\lambda}$ and that $U = \widetilde{U}$. ∎

3.4   The case when V is a local operator

We now aim at relaxing the assumptions on V: we now assume that V is a local operator, i.e. $V[m](x) = F(m(x), x)$, where F is a bounded and $\mathcal{C}^0$ function defined on $\mathbb{R} \times \mathbb{T}^2$.

3.4.1   Existence

From Remark 4, we have the analogues of Theorems 1 and 3:

Proposition 4 *Take V as above. If assumption 1 holds, then the discrete problem (23)-(25) has at least a solution, and there exist two constants δ, $\delta \in (0, 1)$ and $C > 0$ such that (45) is satisfied for all $h = 1/N_h < h_0$.*

Proposition 5 *If g satisfies the same assumptions as in Theorem 3 and if $V[m](x) = F(m(x), x)$ with F is $\mathcal{C}^0$ function defined on $\mathbb{R} \times \mathbb{T}^2$, then the problem (23)-(25) has at least a solution.*

Remark 5 *In Proposition 5, we do not have a bound on U uniform w.r.t. h.*

3.4.2   Uniqueness

We have the corollary of Proposition 3:

Corollary 1 *If g satisfies assumptions (H_1)-(H_4) and if F is strictly monotone, i.e.*

$$(F(m, x) - F(\tilde{m}, x))(m - \tilde{m}) \le 0 \Rightarrow m = \tilde{m},$$

then (23)-(25) has at most a solution.

3.5 A convergence result

It is possible to prove several convergence results. We give the simplest one as an example.

Theorem 4 *We make the same assumptions as in Theorem 1 and we suppose furthermore that $(\mathbf{H_4})$-$(\mathbf{H_5})$ hold and that there exist real numbers c, s, $c > 0$, $s > 0$, such that for all $h < 1$, for all grid functions M and $\widetilde{M}$,*

$$h^2 \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \geq c \|V_h[M] - V_h[\widetilde{M}]\|_\infty^s. \tag{62}$$

Assume that (1)-(3) has a unique solution such that u and m belong to $\mathcal{C}^{\ell,\delta_0}(\mathbb{T}^2) \cap \mathcal{C}^2(\mathbb{T}^2)$, see (22). Calling (U, M, λ_h) the solution of the discrete problem (23)-(25), we have

$$\lim_{h \to 0} \sup_{i,j} |u(x_{i,j}) - U_{i,j}| = 0, \quad \lim_{h \to 0} |\lambda - \lambda_h| = 0.$$

Proof. We call $\widetilde{U}$ and $\widetilde{M}$ the grid functions such that $\widetilde{U}_{i,j} = h^{-2} \int_{|x-x_{i,j}|_\infty < h/2} u \, dx$ and $\widetilde{M}_{i,j} = h^{-2} \int_{|x-x_{i,j}|_\infty < h/2} m \, dx$. From the consistency assumptions, we have that

$$
\begin{cases}
-\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda &= \left(V_h[\widetilde{M}]\right)_{i,j} + o(1), \\
-\nu(\Delta_h \widetilde{M})_{i,j} - \mathcal{B}_{i,j}(\widetilde{U}, \widetilde{M}) &= o(1), \\
\widetilde{M}_{i,j} &\geq 0, \\
h^2 \sum_{i,j} \widetilde{M}_{i,j} = 1, \quad \text{and} \quad \sum_{i,j} \widetilde{U}_{i,j} &= 0,
\end{cases}
\tag{63}
$$

where $o(1)$ means a grid function whose maximum norm tends to 0 as h tends to 0.
On the other hand, from Theorem 1, $\|U\|_\infty$ is bounded by a constant.
We therefore obtain an equation close to (61):

$$
\begin{aligned}
o(h^{-2}) = &\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \\
&+ \sum_{i,j} M_{i,j} \left(g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) - g\left(x_{i,j}, [D_h U]_{i,j}\right) - D_h(\widetilde{U} - U)_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U]_{i,j}\right) \right) \\
&+ \sum_{i,j} \widetilde{M}_{i,j} \left(g\left(x_{i,j}, [D_h U]_{i,j}\right) - g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) - D_h(U - \widetilde{U})_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) \right).
\end{aligned}
\tag{64}
$$

From (64), the convexity of g and (62), we obtain that $\lim_{h\to 0} \|V_h[M] - V_h[\widetilde{M}]\|_\infty = 0$. Thus,

$$
\begin{cases}
-\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda = (V_h[M])_{i,j} + o(1), \\
-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda_h = (V_h[M])_{i,j}.
\end{cases}
$$

The maximum principle at the maximum of $\widetilde{U} - U$ yields that $\lambda - \lambda_h \leq o(1)$. The same argument at the maximum of $U - \widetilde{U}$ yields $\lambda_h - \lambda \leq o(1)$. Therefore, $\lim_{h\to 0} |\lambda - \lambda_h| = 0$.
We know that the family of grid functions u_h is equibounded and equicontinuous. There exists a function u' and a subsequence $u_{h_n} = (U_{i,j}^{(n)})_n \in \mathbb{N}$ such that $\lim_{h\to 0} \sup_{i,j} |u'(x_{i,j}) - U_{i,j}^{(n)}| = 0$. The function u' is a viscosity solution of (1) and is such that $\int_{\mathbb{T}^2} u' dx = 0$. This implies that $u = u'$. Therefore, the whole sequence u_h converges to u. ∎

4   Approximation of the evolution system (8)-(11)

Let N_T be a positive integer and $\Delta t = T/N_T$, $t_n = n\Delta t$, $n = 0, \ldots, N_T$. The values of u and m at $(x_{i,j}, t_n)$ are respectively approximated by $U_{i,j}^n$ and $M_{i,j}^n$. Given $M^0 \in \mathcal{K}$ (the compact set $\mathcal{K}$ is defined in (27) and U^0, the discrete problem is to look for (U^n, M^n), $n = 1, \ldots, N_T$, such that

$$
\begin{aligned}
\forall n, i, j: \\
0 \le n < N_T, \\
0 \le i, j < N_h,
\end{aligned}
\quad
\left\{
\begin{aligned}
\frac{U_{i,j}^{n+1} - U_{i,j}}{\Delta t} - \nu (\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) &= \left(V_h[M^{n+1}] \right)_{i,j}, \\
\frac{M_{i,j}^{n+1} - M_{i,j}}{\Delta t} - \nu (\Delta_h M^{n+1})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, M^{n+1}) &= 0, \\
M_{i,j}^{n+1} &\ge 0,
\end{aligned}
\right.
$$

(65)

with the notations introduced above (in particular $\mathcal{B}_{i,j}$ is defined in (20)), and

$$
h^2 \sum_{i,j} M_{i,j}^{n+1} = 1, \quad \text{for } n = 0, \ldots N_T - 1.
$$

(66)

4.1   The main theorem on existence

Theorem 5 *Assume that*

- *g satisfies ($\mathbf{H_1}$)-($\mathbf{H_3}$) and there exists a constant C such that*

$$
\left| \frac{\partial g}{\partial x}(x, (q_1, q_2, q_3, q_4)) \right| \le C(1 + |q_1| + |q_2| + |q_3| + |q_4|), \qquad \forall x \in \mathbb{T}^2, \quad \forall q_1, q_2, q_3, q_4. \quad (67)
$$

- *V satisfies ($\mathbf{A_1}$) and ($\mathbf{A_2}$).*

If $M^0 \in \mathcal{K}$, then (65) (66) has a solution. If there exists a constant C independent of h such that $\|D_h U^0\|_\infty \le C$, then for all n, $\|D_h U^n\|_\infty \le c$. for a constant c independent of h and δt.

Proof.   The strategy of the proof is similar to that used for Theorem 1. We are going to construct a continuous mapping $\chi : \mathcal{K}^{N_T} \to \mathcal{K}^{N_T}$ and use Brouwer fixed point theorem. We proceed in several steps:

Step 1: a mapping $M \to U$ Given $(U_{i,j}^0)_{0 \le i,j < N_h}$, Consider the map $\Phi : (M^n)_{1 \le n \le N_T} \in \mathcal{K}^{N_T} \to (U^n)_{1 \le n \le N_T}$, solution of the first equation in (65), i.e.

$$
\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} - \nu (\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = \left(V_h[M^{n+1}] \right)_{i,j},
$$

(68)

for $n = 0, \ldots N_T - 1$ and $0 \le i, j < N_h$. The existence and uniqueness of U^{n+1}, $n = 0, \ldots, N_T - 1$ is obtained by induction: at each step, we use Lemma 1 with $\rho = \frac{1}{\Delta t}$ and $V_{i,j} = \frac{U_{i,j}^n}{\Delta t} + \left(V_h[M^{n+1}] \right)_{i,j}$.

Step 2: boundedness and continuity of Φ   Looking at the proof of Lemma 1, we see that

$$
\|U^{n+1}\|_\infty \le \max_{(i,j)} \left| \Delta t \left(H(x_{i,j}, 0) - \left(V_h[M^{n+1}] \right)_{i,j} \right) - U_{i,j}^n \right|,
$$

which implies, from the uniform boundedness assumption on V and of $H(\cdot, 0)$, that there exists a constant C depending on $\|U^0$ but independent of (M^n) such that $\|U^n\|_\infty \le C(1+T)$. Therefore, Φ maps $\mathcal{K}^{N_T}$ to a bounded subset of $(\mathbb{R}^{N_h^2})^{N_T}$. Moreover, by using the assumption on the continuity of V and well known results on continuous dependence on the data for monotone schemes (see e.g. [4]), we see that the mapping Φ is continuous from $\mathcal{K}^{N_T}$ to $(\mathbb{R}^{N_h^2})^{N_T}$.

Step 3: discrete Lipschitz continuity estimates on $\Phi((M^n)_{n=1,\ldots,N_T})$ We proceed by proving discrete Lipschitz continuity estimates on U^{n+1} given by (68). The solution of (68) is noted

$$U^{n+1} = \Psi(U^n, M^{n+1}).$$

Standard arguments on monotone schemes yield that for all $M \in \mathcal{K}$, $U, W \in \mathbb{R}^{N_h^2}$,

$$\|\left(\Psi(U, M) - \Psi(W, M)\right)^+\|_\infty \quad \leq \|(U - W)^+\|_\infty, \tag{69}$$
$$\|\Psi(U, M) - \Psi(W, M)\|_\infty \quad \leq \|U - W\|_\infty. \tag{70}$$

For $(\ell, m) \in \mathbb{Z}^2$, call $\tau_{\ell,m} U$ the discrete function defined by

$$(\tau_{\ell,m} U)_{i,j} = U_{\ell+i,m+j}.$$

It is a simple matter to check that

$$
\begin{aligned}
&\frac{(\tau_{\ell,m} U)_{i,j}^{n+1} - (\tau_{\ell,m} U)_{i,j}^n}{\Delta t} - \nu(\Delta_h(\tau_{\ell,m} U^{n+1}))_{i,j} + g(x_{i,j}, [D_h(\tau_{\ell,m} U^{n+1})]_{i,j}) \\
=\ & \left(V_h[M^{n+1}]\right)_{i,j} + \left(V_h[M^{n+1}]\right)_{i+\ell,j+m} - \left(V_h[M^{n+1}]\right)_{i,j} \\
& -g(x_{i+\ell,j+m}, [D_h(\tau_{\ell,m} U^{n+1})]_{i,j}) + g(x_{i,j}, [D_h(\tau_{\ell,m} U^{n+1})]_{i,j}),
\end{aligned}
$$

and therefore

$$\tau_{\ell,m} U^{n+1} = \Psi(\tau_{\ell,m} U^n + \Delta t E, M^{n+1}),$$

$$E_{i,j} = \left(\begin{array}{c} \left(V_h[M^{n+1}]\right)_{i+\ell,j+m} - \left(V_h[M^{n+1}]\right)_{i,j} \\ -g(x_{i+\ell,j+m}, [D_h(\tau_{\ell,m} U^{n+1})]_{i,j}) + g(x_{i,j}, [D_h(\tau_{\ell,m} U^{n+1})]_{i,j}) \end{array} \right).$$

From the assumptions on V and on g (in particular (67)), there exists a constant C (independent of n, (M^n), h and Δt) such that $\|E\|_\infty \leq C\left(1 + \|D_h U^{n+1}\|_\infty\right) h\sqrt{\ell^2 + m^2}$. We conclude from (70) that

$$\|\tau_{\ell,m} U^{n+1} - U^{n+1}\|_\infty \leq \|\tau_{\ell,m} U^n - U^n\|_\infty + Ch\Delta t\sqrt{\ell^2 + m^2}\left(1 + \|D_h U^{n+1}\|_\infty\right). \tag{71}$$

Thanks to (71),

$$(1 - C\Delta t)\|D_h U^{n+1}\|_\infty \leq \|D_h U^n\|_\infty + C\Delta t.$$

A discrete version of Gronwall's lemma yields that there exists a constant L which only depends on C, T and the initial condition $\|D_h U^0\|_\infty$ such that for all n, $1 \leq n \leq N_T$,

$$\|D_h U^{n+1}\|_\infty \leq L, \tag{72}$$

which is a discrete Lipschitz continuity estimate, uniform with respect to $(M^n)_{1 \leq n \leq N_T}$.

Step 4: a fixed point problem for $(M^n)_{1 \leq n \leq N_T}$ For $(M^n)_{1 \leq n \leq N_T} \in \mathcal{K}^{N_T}$ and $(U^n)_{1 \leq n \leq N_T} = \Phi((M^n)_{1 \leq n \leq N_T})$ and a positive real number μ, consider the following linear problem: find $(\widetilde{M}^n)_{1 \leq n \leq N_T}$ such that

$$\frac{\widetilde{M}_{i,j}^{n+1} - \widetilde{M}_{i,j}^n}{\Delta t} + \mu\widetilde{M}_{i,j}^{n+1} - \nu(\Delta_h \widetilde{M}^{n+1})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, \widetilde{M}^n) = \mu M_{i,j}^{n+1}, \tag{73}$$

with the initial condition $\widetilde{M}^0 = M^0$, with $h^2 \sum_{i,j} M_{i,j}^0 = 1$ and $M^0 \geq 0$.

We are going to prove first that for μ large enough, (73) has a unique solution $(\widetilde{M}^n)_{1 \leq n \leq N_T} \in \mathcal{K}^{N_T}$, then that the mapping $(M^n)_{1 \leq n \leq N_T} \rightarrow (\widetilde{M}^n)_{1 \leq n \leq N_T}$ has a fixed point. Existence for (65) (66) will then be proved.

Step 5: existence for (73) Clearly (73) is a discrete version of a linear parabolic initial value problem. It can be written

$$\widetilde{M}^{n+1} + \Delta t(\mu \widetilde{M}^{n+1} + A^{n+1} \widetilde{M}^{n+1}) = \widetilde{M}^n + \mu \Delta t M^{n+1}, \tag{74}$$

where A^{n+1} is a linear operator depending on U^{n+1}.

As in the proof of Theorem 1, The assumptions on the monotonicity of g imply that the matrix corresponding to $Id + \Delta t A^{n+1}$ has positive diagonal entries and nonpositive off-diagonal entries. Furthermore, since g is $\mathcal{C}^1$, (72) implies that there exists a constant C depending only of $\|D_h U^0\|$ (in particular independent of (M^n)) such that for all n, $1 \leq n \leq N_T$, for all i, j, $0 \leq i, j \leq N_h$, and for all $\ell = 1, 2, 3, 4$,

$$\left| \frac{\partial g}{\partial q_\ell}(x_{i,j}, [D_h U^n]_{i,j}) \right| \leq C. \tag{75}$$

From this, we see that for μ large enough but independent on (M^n), the matrix corresponding to $Id + \Delta t(\mu Id + A^{n+1})$ is a M-matrix, and is therefore invertible. The system of linear equations (74) has a unique solution.

Moreover, since $M^0 \geq 0$ for all $n = 0, \dots, N_T$ and since $Id + \Delta t(\mu Id + A^{n+1})$ is a M-matrix for all n, $1 \leq n \leq N_T$, $\widetilde{M}^n \geq 0$ for all $n = 0, \dots, N_T$.

We are left with proving that $h^2 \sum_{i,j} \widetilde{M}_{i,j}^n = 1$, for all n, $1 \leq n \leq N_T$. As in the proof of Theorem 1, we see that for two grid functions W and Z, we have

$$(A^n W, Z)_2 = \nu \sum_{i,j} (D_1^+ W)_{i,j} (D_1^+ Z)_{i,j} + \nu \sum_{i,j} (D_2^+ W)_{i,j} (D_2^+ Z)_{i,j}$$
$$+ \sum_{i,j} W_{i,j} [D_h Z]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right). \tag{76}$$

From (76) and (74), it can be proved by induction that if $h^2(M^0, 1)_2 = 1$, then the condition $h^2(\widetilde{M}^n, 1)_2 = 1$ holds for all n, $1 \leq n \leq N_T$.

Step 6: existence of a fixed point of χ From the boundedness and continuity of the mapping Φ, and from the fact that g is $\mathcal{C}^1$, we obtain that χ is continuous. Therefore, we can apply Brouwer fixed point theorem and obtain that χ has a fixed point.

Conclusion Assuming that M^0 is such that $M^0 > 0$ and $h^2(M^0, 1)_2 = 1$, we have proved that the mapping χ has a fixed point that we call $(M^n)_{1 \leq n \leq N_T}$. Calling $(U^n)_{1 \leq n \leq N_t} = \Phi((M^n)_{1 \leq n \leq N_t})$, $(M^n)_{n=1\dots,N_T}$ and $(U^n)_{n=1\dots,N_T}$ satisfy (65) and (66). ■

Remark 6 *Existence for problem (65)-(66) can also be obtained without (67) and when V is a local operator, (see §3.4).*

4.2   A different fixed point strategy

In the proof of Theorem 5, keeping the strategy unchanged up to Step 3, it is tempting to take $\mu = 0$ in (73): in Step 4, we now have to find $(\widetilde{M}^n)_{1 \leq n \leq N_T}$ such that

$$\frac{\widetilde{M}_{i,j}^{n+1} - \widetilde{M}_{i,j}^n}{\Delta t} - \nu(\Delta_h \widetilde{M}^{n+1})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, \widetilde{M}^{n+1}) = 0, \tag{77}$$

with the initial condition $\widetilde{M}^0 = M^0$, with $h^2 \sum_{i,j} M_{i,j}^0 = 1$ and $M^0 \geq 0$.

If (77) has a unique solution $(\widetilde{M}^n)_{1 \leq n \leq N_T} \in \mathcal{K}^{N_T}$, then we have a new mapping $(M^n)_{1 \leq n \leq N_T} \rightarrow$

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$(\widetilde{M}^n)_{1 \leq n \leq N_T}$ which may have a fixed point. If it is the case, existence for (65) (66) will then be proved. We are going to show that this strategy is indeed feasible, under a mild assumption on the time step. In spite of this further assumption, this strategy is interesting because it uses a discrete weak maximum principle which will be useful for studying uniqueness.

Clearly (77) can be written

$$\widetilde{M}^{n+1} + \Delta t A^{n+1} \widetilde{M}^{n+1} = \widetilde{M}^n, \tag{78}$$

with the notation defined above. We introduce the semi-norm $|||W|||$:

$$|||W|||^2 = \sum_{i,j} \left((D_1^+ W)_{i,j}^2 + (D_2^+ W)_{i,j}^2 \right).$$

From (75) and (51) we deduce that there exists a nonnegative constant σ depending only of $\|D_h U^0\|_\infty$ (in particular independent of $(M^n)_{n=1,\dots,N_T}$ and n) such that for all grid function W,

$$\left| \sum_{i,j} W_{i,j} [D_h W]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right) \right| \leq \frac{\nu}{2} |||W|||^2 + \sigma \|W\|_2^2, \tag{79}$$

$$(A^n W, W)_2 \geq \frac{\nu}{2} |||W|||^2 - \sigma \|W\|_2^2.$$

The latter estimate is a discrete Gårding inequality and implies that if $\sigma \Delta t < 1$ then the system of linear equations (78) has a unique solution.

We now have to prove that $\widetilde{M}^n \geq 0$, for all n, $1 \leq n \leq N_T$. For a grid function W, denote by W^+ (resp. W^-) the grid function obtained by taking the positive (resp. negative) part of W: $(W^+)_{i,j} = (W_{i,j})^+$. Let us take $Z = -W^-$ in (51): we have

$$-(A^n W, W^-)_2 = \nu |||W^-|||^2 + \frac{\nu}{h^2} \sum_{i,j} (W_{i,j} W_{i-1,j})^- + \frac{\nu}{h^2} \sum_{i,j} (W_{i,j} W_{i,j-1})^-$$
$$+ \sum_{i,j} W_{i,j}^- [D_h W^-]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right) \tag{80}$$
$$- \sum_{i,j} W_{i,j}^+ [D_h W^-]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right).$$

From (79), there exists a nonnegative constant σ such that, for all grid function W,

$$\left| \sum_{i,j} W_{i,j}^- [D_h W^-]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right) \right| \leq \frac{\nu}{2} |||W^-|||^2 + \sigma \|W^-\|_2^2. \tag{81}$$

On the other hand, from the monotonicity of g, easy algebra shows that for all grid function W,

$$-\sum_{i,j} W_{i,j}^+ [D_h W^-]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U^n]_{i,j} \right) \geq 0. \tag{82}$$

From (80), (81) and (82), we see that for all grid functions W,

$$-(A^n W, W^-)_2 \geq \frac{\nu}{2} |||W^-|||^2 - \sigma \|W^-\|_2^2. \tag{83}$$

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Similarly,

$$
-\left((W^{n+1}-W^n),(W^{n+1})^-\right)_2 = \left(((W^{n+1})^- -(W^n)^-),(W^{n+1})^-\right)_2 + \sum_{i,j}(W_{i,j}^{n+1})^-(W_{i,j}^n)^+
$$

$$
= \frac{1}{2}\left\|(W^{n+1})^-\right\|_2^2 + \frac{1}{2}\left\|(W^{n+1})^- -(W^n)^-\right\|_2^2 - \frac{1}{2}\left\|(W^n)^-\right\|_2^2 + \sum_{i,j}(W_{i,j}^{n+1})^-(W_{i,j}^n)^+
$$

$$
\geq \frac{1}{2}\left\|(W^{n+1})^-\right\|_2^2 + \frac{1}{2}\left\|(W^{n+1})^- -(W^n)^-\right\|_2^2 - \frac{1}{2}\left\|(W^n)^-\right\|_2^2 .
$$

$$(84)$$

We deduce from (78), (83) and (84) that

$$
\left\|(\widetilde{M}^{n+1})^-\right\|_2^2 + \nu\Delta t |||(\widetilde{M}^{n+1})^-|||^2 - 2\sigma\Delta t \left\|(\widetilde{M}^{n+1})^-\right\|_2^2 \leq \left\|(\widetilde{M}^n)^-\right\|_2^2 . \tag{85}
$$

If $2\sigma\Delta t < 1$, then (85) implies that $\widetilde{M}^n \geq 0$ for all n, $1 \leq n \leq N_T$ as soon as $M^0 \geq 0$.

Remark 7 *We have used a weak discrete maximum principle which does not require that the matrices corresponding to $Id + \Delta t A^n$, $n = 1\ldots, N_T$ be diagonal dominant. The latter may require a much more restrictive assumption on the time step, namely that $\alpha\frac{\Delta t}{h} < 1$ for some positive α. On the other hand, if the matrices corresponding to $Id + \Delta t A^n$, $n = 1\ldots, N_T$ are diagonal dominant, they are irreducible M-matrices, and we get the stronger property that $M^0 > 0$ implies $\widetilde{M}^n > 0$ for all n, $1 \leq n \leq N_T$.*

The proof that $h^2 \sum_{i,j} \widetilde{M}_{i,j}^n = 1$, for all n, $1 \leq n \leq N_T$ is obtained exactly as above. We thus have a new mapping $\chi : \mathcal{K}^{N_T} \mapsto \mathcal{K}^{N_T}$, $(M^n)_{1\leq n \leq N_T} \to (\widetilde{M}^n)_{1\leq n \leq N_T}$. The existence of a fixed point for χ is proved as above.

5 A finite difference scheme for problem (4)-(7)

5.1 Description of the scheme

Given M^{N_T}, we choose the following scheme:

$$
\begin{cases}
\dfrac{U_{i,j}^{n+1}-U_{i,j}^n}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j},[D_h U^{n+1}]_{i,j}) &=& \left(V_h[M^{n+1}]\right)_{i,j}, \\[2mm]
\dfrac{M_{i,j}^{n+1}-M_{i,j}^n}{\Delta t} + \nu(\Delta_h M^n)_{i,j} + \mathcal{B}_{i,j}(U^n,M^n) &=& 0, \\[2mm]
M_{i,j}^n &\geq& 0,
\end{cases} \tag{86}
$$

for $n = 0,\ldots N_T - 1$ and $0 \leq i,j < N_h$, with

$$
h^2 \sum_{i,j} M_{i,j}^n = 1, \quad \text{for } n = 0,\ldots N_T - 1, \tag{87}
$$

and

$$
U_{i,j}^0 = \left(V_{0,h}(M^0)\right)_{i,j} \equiv V_0[m_h^0](x_{i,j}), \tag{88}
$$

where m_h^0 is the piecewise constant function taking the value $M_{i,j}^0$ in the square $|x-x_{i,j}|_\infty \leq h/2$. We have the analogue of Theorem 5:

Theorem 6 *We make the same assumptions as in Theorem 5, and we also assume that V_0 satisfies ($\mathbf{A_1}$) and ($\mathbf{A_2}$). If $M^{N_T} \geq 0$ and $\sum_{i,j} M_{i,j}^{N_T} = 1$, then (86)-(88) has a solution. There exists a constant C independent of h and Δt such that $\|D_h U^n\|_\infty \leq C$.*

Proof. Similar to that of Theorem 5. ∎

5.2 Uniqueness

We slightly modify (86)-(88) in order to obtain uniqueness: we either replace the second equation in (86) for $n = 0$ by $M^0 = M^1$, or we replace (88) by

$$U_{i,j}^0 = \left(V_{0,h}(M^1)\right)_{i,j}. \tag{89}$$

These modifications do not affect the consistency of the scheme. For brevity, we only discuss the second modification. Let $(U^n, M^n)_{n=0,\dots,N_T}$ and $(\widetilde{U}^n, \widetilde{M}^n)_{n=0,\dots,N_T}$ be two solutions of (86), (87), (89). We have

$$\frac{(U^{n+1} - \widetilde{U}^{n+1})_{i,j} - (U^n - \widetilde{U}^n)_{i,j}}{\Delta t} - \nu(\Delta_h(U^{n+1} - \widetilde{U}^{n+1}))_{i,j}$$
$$+ g(x_{i,j}, [D_h U^{n+1}]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}) = \left(V_h[M^{n+1}] - V_h[\widetilde{M}^{n+1}]\right)_{i,j}, \tag{90}$$

for $n = 0, \dots N_T - 1$. Multiplying by $Z_{i,j}^{n+1}$ and summing over all i, j leads to

$$\frac{((U^{n+1} - \widetilde{U}^{n+1}) - (U^n - \widetilde{U}^n), Z)_2}{\Delta t} - \nu(\Delta_h(U^{n+1} - \widetilde{U}^{n+1}), Z)_2$$
$$+ \sum_{i,j} (g(x_{i,j}, [D_h U^{n+1}]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j})) Z_{i,j} \tag{91}$$
$$= \left(V_h[M^{n+1}] - V_h[\widetilde{M}^{n+1}], Z\right)_2, \qquad n = 0, \dots N_T - 1.$$

On the other hand, multiplying the second equation in (86) by $W_{i,j}$ and summing over all i, j leads to

$$\frac{(M^{n+1} - M^n, W)_2}{\Delta t} + \nu(M^n, \Delta_h W)_2 - \sum_{i,j} M_{i,j}^n [D_h W]_{i,j} \cdot \nabla_q g(x_{i,j}, [D_h U^n]_{i,j}) = 0, \tag{92}$$

for $n = 0, \dots N_T - 1$. From this and the similar equation satisfied by $\widetilde{M}^n$, we obtain

$$0 = \frac{((M^{n+1} - M^n) - (\widetilde{M}^{n+1} - \widetilde{M}^n), W)_2}{\Delta t} + \nu((M^n - \widetilde{M}^n), \Delta_h W)_2$$
$$- \sum_{i,j} M_{i,j}^n [D_h W]_{i,j} \cdot \nabla_q g(x_{i,j}, [D_h U^n]_{i,j}) + \sum_{i,j} \widetilde{M}_{i,j}^n [D_h W]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}^n]_{i,j}\right), \tag{93}$$

for $n = 0, \dots N_T - 1$. Taking $Z = M^{n+1} - \widetilde{M}^{n+1}$ in (91) for $n = 0, \dots N_T - 1$ and $W = U^n - \widetilde{U}^n$ for $n = 1, \dots, N_T - 1$ and summing leads to

$$0 = \sum_{n=1}^{N_T-1} \left(V_h[M^n] - V_h[\widetilde{M}^n], M^n - \widetilde{M}^n\right)_2 + \frac{1}{\Delta t}\left(V_{0,h}[M^1] - V_{0h}[\widetilde{M}^1], M^1 - \widetilde{M}^1\right)_2$$

$$+ \sum_{n=1}^{N_T-1} \sum_{i,j} M_{i,j}^n \left(g(x_{i,j}, [D_h \widetilde{U}^n]_{i,j}) - g(x_{i,j}, [D_h U^n]_{i,j}) - [D_h(\widetilde{U}^n - U^n)]_{i,j} \cdot \nabla_q g(x_{i,j}, [D_h U^n]_{i,j})\right)$$

$$+ \sum_{n=1}^{N_T-1} \sum_{i,j} \widetilde{M}_{i,j}^n \left(g(x_{i,j}, [D_h U^n]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}^n]_{i,j}) - [D_h(U^n - \widetilde{U}^n)]_{i,j} \cdot \nabla_q g(x_{i,j}, [D_h \widetilde{U}^n]_{i,j})\right), \tag{94}$$

from $M^{N_T} = \widetilde{M}^{N_T}$ and (89).

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Theorem 7 *We make the same assumptions as in Theorem 6. We assume furthermore that g is convex w.r.t. to (q_1, q_2, q_3, q_4), i.e. (**H$_4$**), and that the operators V_h and $V_{0,h}$ are strictly monotone, i.e.*

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \leq 0 \Rightarrow V_h[M] = V_h[\widetilde{M}],$$

$$\left(V_{0,h}[M] - V_{0,h}[\widetilde{M}], M - \widetilde{M} \right)_2 \leq 0 \Rightarrow V_{0,h}[M] = V_{0,h}[\widetilde{M}].$$

There exists a constant $\sigma \geq 0$ independent of h such that if $2\sigma \Delta t < 1$, then (86), (87), (89) has a unique solution.

Proof. From the convexity of g and the monotonicity assumptions on V_h and $V_{0,h}$, the four terms in the right hand side of (94) vanish, and $V_h[M^n] = V_h[\widetilde{M}^n]$ for all $n = 1, \ldots, N_T$, and $V_{0,h}[M^1] = V_{0,h}[\widetilde{M}^1]$. Since the scheme for U^n is monotone, this implies that $U^n = \widetilde{U}^n$ for all $n = 0, \ldots, N_T$. Let σ be the constant of the Gårding inequality (79). Uniqueness for (78) (given U^n) yields that $M^n = \widetilde{M}^n$ for all $n = 0, \ldots, N_T$. ∎

6 Numerical simulations

6.1 Long time approximation of the stationary problem

As mentioned in the introduction, we consider a solution $(\widetilde{u}, \widetilde{m})$ of (8)-(11) with the Cauchy data $\widetilde{m}_0$ and $\widetilde{u}_0$ defined on $\mathbb{T}^2$, m_0 being a probability measure. We e expect that there exist a C^2 function u on $\mathbb{T}^2$, a function m in $W^{1,p}(\mathbb{T}^2)$ and a scalar λ such that

$$\lim_{t \to \infty} \widetilde{u}(t, x) - \lambda t = u(x), \qquad \lim_{t \to \infty} \widetilde{m}(t, x) = m(x),$$

and $\int_{\mathbb{T}^2} u = 0$. If so, then (u, m, λ) is a solution of (1)-(3).

Such long time approximations have been justified for the cell problem in the homogenization of Hamilton-Jacobi or Hamilton-Jacobi-Bellman equations, see for example [14], [1], [3]. This approach is close to the so-called *eductive strategy* in economy. In [6], O. Guéant studies the *eductive stability* on some examples where V has not the monotony property used in Proposition 3 and justifies the approach.

The same long time approximation method may be used at the discrete level. The discrete version of (8)-(11) is (65)-(66) if an implicit Euler scheme is chosen. A semi-implicit linearized scheme can be used as well. Both schemes require the numerical solution of a linearized problem. Linearizing (65)-(66) needs careful computations and is not always possible. In such cases, an explicit method can be used.

Remark 8 *In tests 1, 2, and 4, the Hamiltonian is of the form $H(x, p) = |p|^2 + g(x)$. In such cases, as observed in [13], the system (1)-(3) is equivalent to a generalized Hartree equation. Indeed, introducing $\phi = \exp\left(-\frac{u}{\nu} \right) \left(\exp\left(-\frac{2u}{\nu} \right) \right)^{-\frac{1}{2}}$ and taking $m = \phi^2$, the system reduces to $-\nu^2 \Delta \phi - g\phi + \phi V[\phi^2] = \lambda \phi$ and m can be written as a function of u.*

6.2 Results

In all the problems considered below, the Hamiltonian is of the form $H(x, p) = \psi(x, |p|)$ and the discrete Hamiltonian is obtained via a Godunov scheme, i.e

$$g(x, q_1, q_2, q_3, q_4) = \psi\left(x, \sqrt{(q_1^-)^2 + (q_3^-)^2 + (q_2^+)^2 + (q_4^+)^2} \right).$$

A semi-implicit linearized scheme is used and the time step is progressively increased: when the asymptotic regime is reached, very long time steps ($\Delta t \sim 1000$) have been used. On the other hand, if the condition $m^{n+1} \geq 0$ is violated, we start back from (u^n, m^n) and decrease the time step δt^m.

Table 1 contains the data of the problems simulated below, i.e. the real number ν, the Hamiltonian H and the operator V.

Table 1: the real number ν, the hamiltonian H and the operator V

Test	ν	$H(x,p)$	$V[m](x)$
1	1	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2$	$m^2(x)$
2	0.01	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2$	$m^2(x)$
3	0.1	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^\alpha$ $\alpha = 1.5, 3, 6, 9$	$m^2(x)$
4	0.1	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2$	$-\log(m(x))$
5	0.1	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2}$	$200\left((1-\Delta)^{-1}(1-\Delta)^{-1}m\right)(x)$
6	0.001	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2}$	$200\left((1-\Delta)^{-1}(1-\Delta)^{-1}m\right)(x)$
7	0.001	$\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1)$ $+(0.6 + 0.59\cos(2\pi x))|p|^{3/2}$	$200\left((1-\Delta)^{-1}(1-\Delta)^{-1}m\right)(x)$
8	0.001	$|p|^2$	$4\cos(4\pi x) + m(x)$

The contours of the potential $x \mapsto \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1)$ are displayed on Figure 1.

6.2.1 Test 1

See Table 1 for the data of the problem. We first check that the long time approximation yields the expected asymptotic behavior. In Figure 2, we plot the graph of $\frac{h^2}{T}\sum_{i,j} U_{i,j}(t = T)$, when the mesh step is $h = 1/50$; as T tends to infinity, this quantity tends to a constant λ_h, as expected. Here, we find that $\lambda_h \simeq 0.9784$.

Here $V[m](x) = F(m(x))$ with $F(y) = y^2$ is a nondecreasing function. Such a function is used
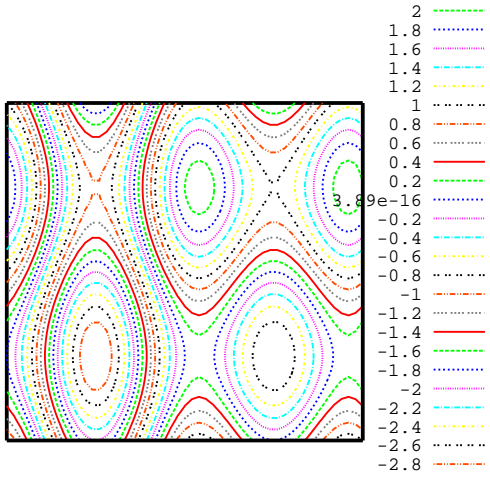
Figure 1: the contours of the potential $x \mapsto \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1)$ used in tests 1-6.
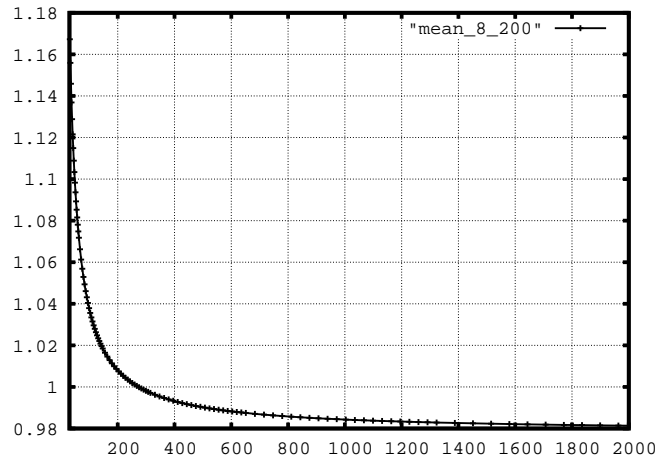

Figure 2: Test 1: graph of $\frac{h^2}{T} \sum_{i,j} U_{i,j}(t = T)$ for $T \leq 2000$, with $N_h = 50$

to model repulsive cases when the players do not like to share their position with others. If ν is not too small, then the players positions should be well distributed, and the density m should not be strongly localized. In Figure 3, we plot the contours of u_h and m_h; we see indeed that m_h is supported in the whole domain $\mathbb{T}^2$. Moreover, from Remark 8 and since ν is large enough, m is a smooth decreasing function of u. This explains why the contour plots of u_h and m_h have the same aspect.
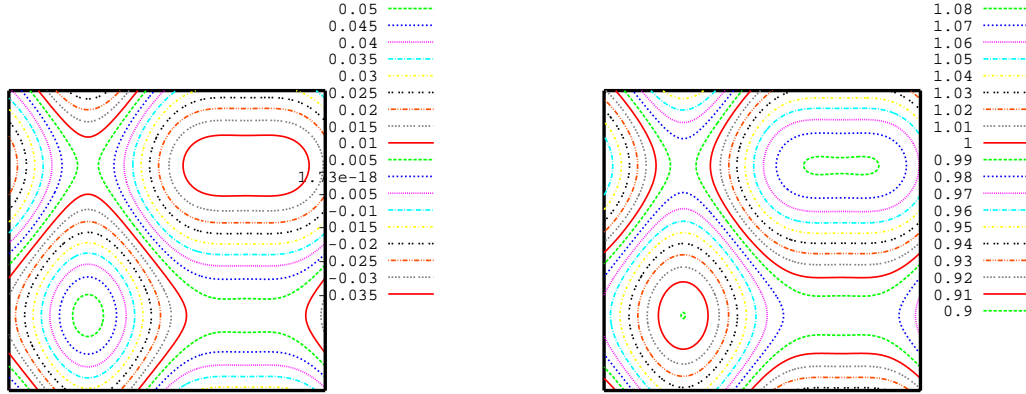
Figure 3: Test 1: the contours of u_h(left) and m_h(right), with $N_h = 200$

Finally, in order to estimate the rate of convergence as h tends to 0, we compare the solutions with the one obtained with $N_h = 200$ (considered as a fine grid). We compute the errors in the maximum norm. The graphs of the errors are displayed in Figure 4. The convergence looks linear.

6.2.2 Test 2

See Table 1 for the data of the problem. Here the value of $\nu = 0.01$ is small, so the case is close to the deterministic limit.

As in Test 1, the solution of the discrete non stationary problem has the expected behavior for large times. In Figure 5, we plot the graph of $\frac{h^2}{T} \sum_{i,j} U_{i,j}(t = T)$, when the mesh step is $h = 1/50$; as T tends to infinity, this quantity tends to a constant λ_h, as expected. Here, we find that $\lambda_h \simeq 1.1885$.

In Figure 6, we plot the contours of u_h and m_h. Note that the supports of ∇u_h and of m_h tend to be disjoint for such a small values of ν. This is coherent with the results concerning the deterministic limit in [13], see also § 6.2.8 below for more details. This test confirms the fact that the method is robust for small values of ν.

6.2.3 Test 3

In Figure 7, we compare m_h for different values of α (see Table 1). We see that the variations of m_h become stiffer as α grows.
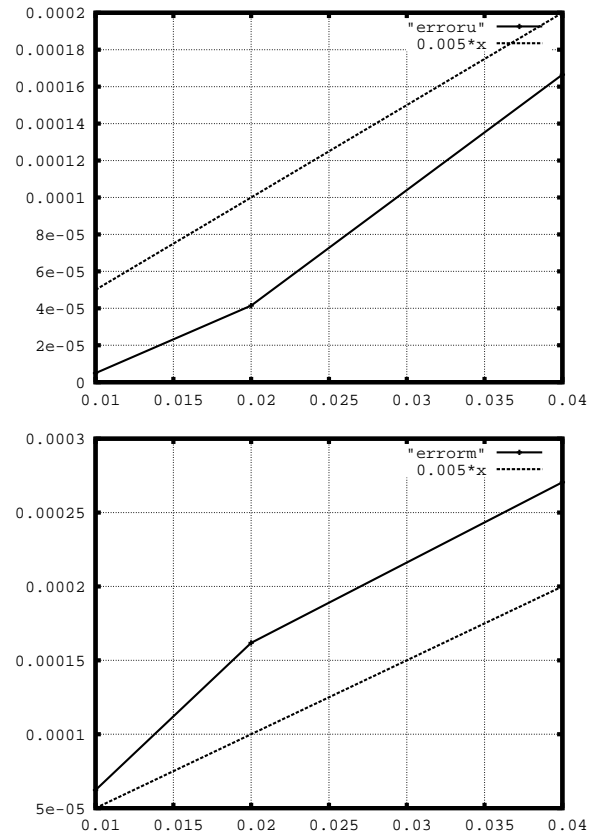
Figure 4: Test 1: error with respect to the discrete solutions obtained on a 200×200 grid as a function of h. Top: error for u, bottom: error for m
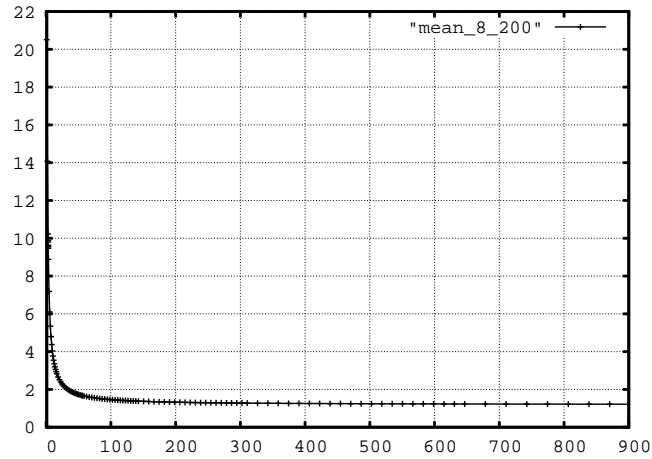

Figure 5: Test 2: graph of $\frac{h^2}{T} \sum_{i,j} U_{i,j}(t = T)$ for $T \leq 900$, with $N_h = 50$
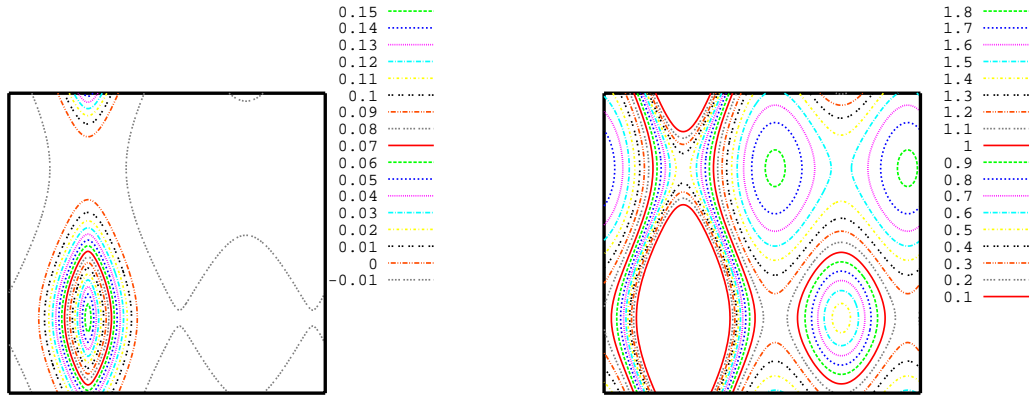
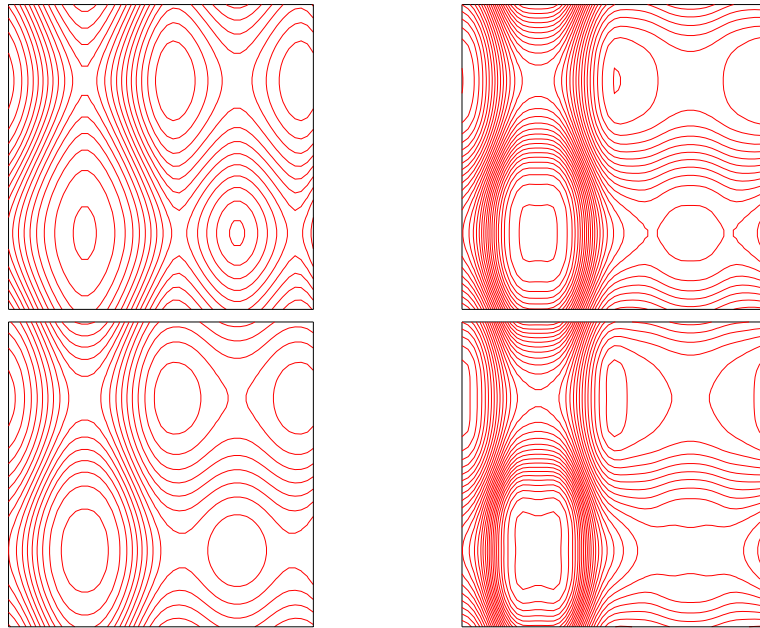Figure 6: Test 2: the contours of u_h(left) and m_h(right), with $N_h = 200$

Figure 7: Test 3: contours of m_h: comparison for $\alpha = 1.5$ (top-left), $\alpha = 3$ (bottom-left), $\alpha = 6$ (top-right), $\alpha = 9$ (bottom-right)

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6.2.4 Test 4

See Table 1 for the data of the problem. Here $V[m](x) = F(m(x))$ with $F(y) = -\log(y)$. By contrast with tests 1 and 2, F is a decreasing function. Such a function F is used to describe situations when the agents are gregarious, i.e. they like to all be in the same position. O. Guéant proved results concerning the eductive stability in close cases, see [6]. Indeed, as shown in Figure 8, the solution of the discrete non stationary problem has the expected behavior for large times.
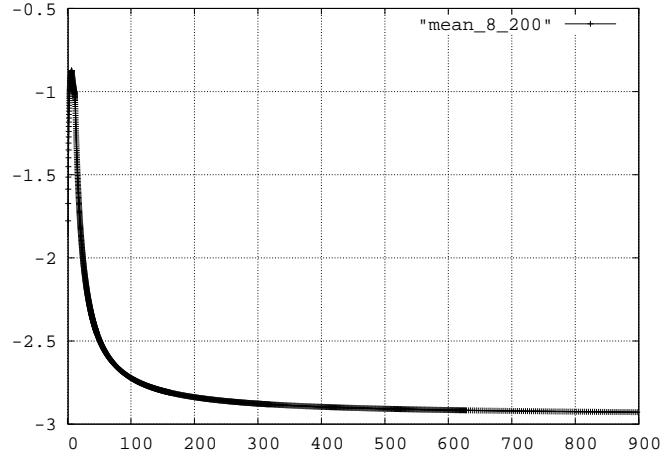
Figure 8: Test 4: graph of $\frac{h^2}{T} \sum_{i,j} U_{i,j}(t = T)$ for $T \leq 900$, with $N_h = 100$

In Figure 9, we plot the contours of u_h and m_h. Note that the measure m_h concentrates near the minimum of u_h, i.e. the players take positions close to each other.
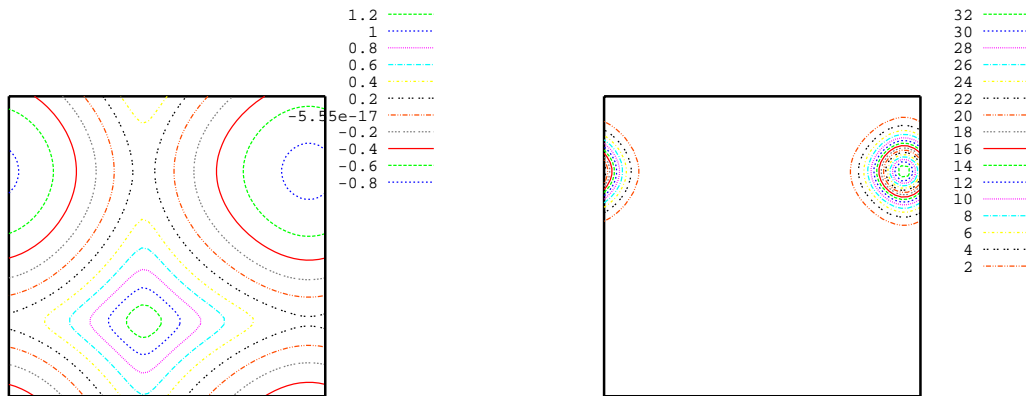
Figure 9: Test 4: the contours of u_h(left) and m_h(right), with $N_h = 100$

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6.2.5　Test 5

See Table 1 for the data of the problem. By contrast with the previous cases, the operator V is nonlocal. At the discrete level, applying V is done by solving a system of linear equations. Alternatively, a method based on fast Fourier transform could be used. In Figure 10, we plot the contours of u_h and m_h.
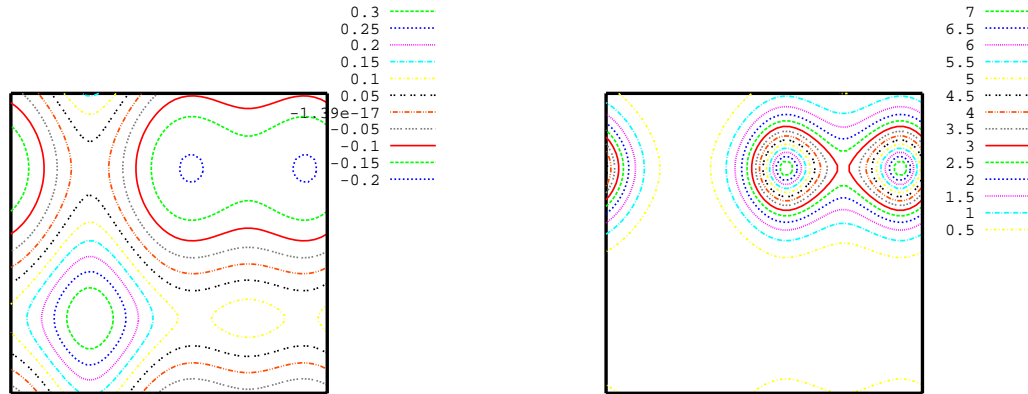

Figure 10: Test 5: the contours of u_h(left) and m_h(right), with $N_h = 100$

6.2.6　Test 6

See Table 1 for the data of the problem. Compared to Test 5, we keep everything unchanged, except that we now take $\nu = 0.001$. In Figure 11, we plot the graphs of u_h and m_h. We see that u_h is not better than Lipschitz continuous and that m_h is close to a sum of two Dirac masses located at the minima of u_h.
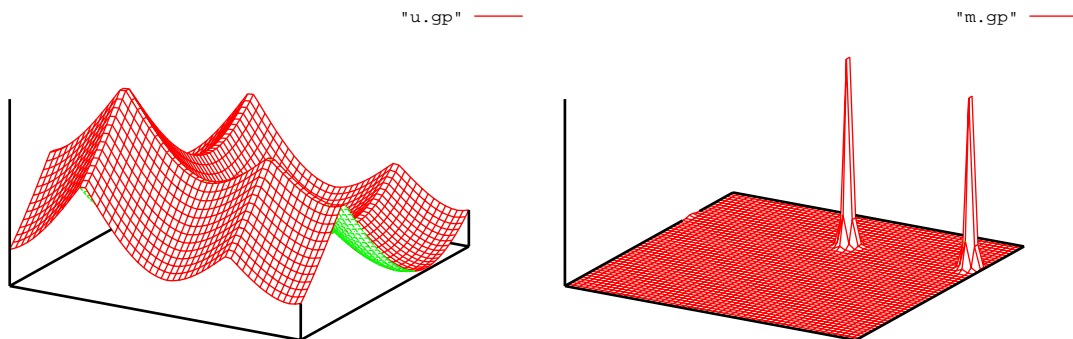

Figure 11: Test 6: the graphs of u_h(left) and m_h(right)

6.2.7 Test 7

See Table 1 for the data of the problem. Compared to Test 5, we keep everything unchanged, except that the Hamiltonian strongly couples x and p. In Figure 12, we plot the contours of u_h and m_h.
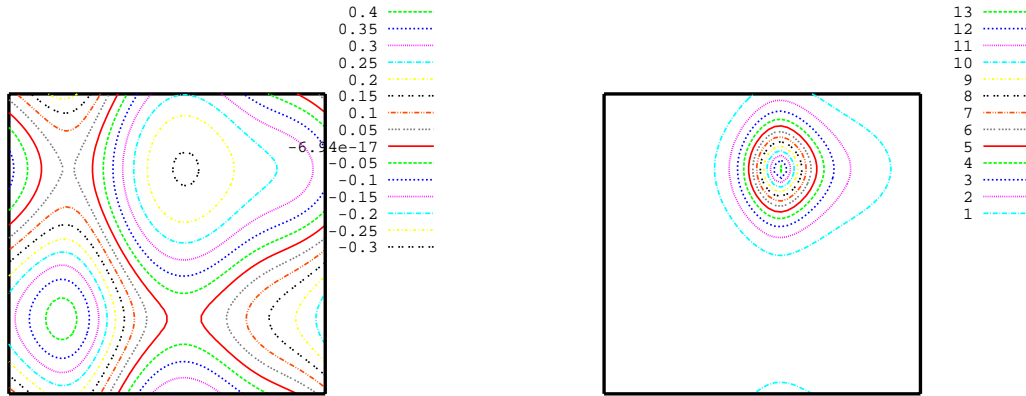

Figure 12: Test 7: the contours of u_h(left) and m_h(right), with $N_h = 100$

6.2.8 Test 8: the deterministic limit

See Table 1 for the data of the problem. In [13], it is proved that if $H(x, p) \geq H(x, 0) = 0$ and if $V[m] = F(m) + f_0(x)$ where $F' > 0$, then calling λ_ν u_ν, $m_n u$ the solution of (1), (2), (3),

$$\lim_{\nu \to 0} (\lambda_\nu, m_\nu) = (\widetilde{\lambda}, \widetilde{m}),$$

where

$$\widetilde{m}(x) = \left(F^{-1}(\widetilde{\lambda} - f_0(x)) \right)^+ \qquad \text{and} \qquad \int_{\mathbb{T}^2} \widetilde{m} dx = 1.$$

To illustrate this property, we choose

$$\nu = 0.001, \quad H(x, p) = |p|^2, \quad V[m](x) = 4\cos(4\pi x) + m.$$

In Figure 13, we plot the graphs of u_h and m_h. We obtain that $\widetilde{m}(x) \approx (\widetilde{\lambda} - 4\cos(4\pi x))^+$, as expected. We also see that the supports of $\nabla \widetilde{u}$ and of $\widetilde{m}$ are disjoint.

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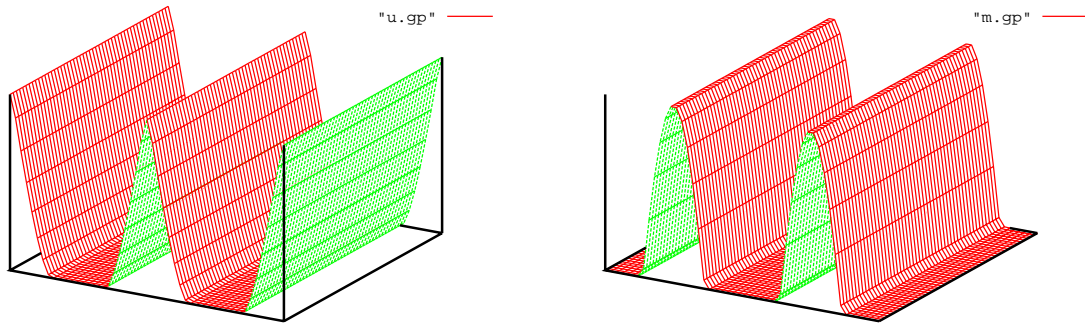
Figure 13: Test 8: the contours of u_h(left) and m_h(right), with $N_h = 100$

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