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Mean value results for the approximate functional equation of the square of the Riemann zeta-function

by

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1. Statement of results. Let $s=\sigma+it$ $(0\le\sigma\le 1, t\ge 1)$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and $d(n)$ the number of positive divisors of the integer n. The purpose of this paper is to prove mean value results for the error term $R(s;t/2\pi)$ of the approximate functional equation of $\zeta^2(s)$, defined by

$$R(s;t/2\pi)=\zeta^2(s)-\sum_{n\le t/2\pi}d(n)n^{-s}-\chi^2(s)\sum_{n\le t/2\pi}d(n)n^{s-1},$$

where $\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)$.

It has been shown by Motohashi [4], [6] that

$$\chi(1-s)R(s;t/2\pi)=-\sqrt{2}(t/2\pi)^{-1/2}\Delta(t/2\pi)+O(t^{-1/4}),\tag{1.1}$$

where $\Delta(t/2\pi)$ is the error term in the Dirichlet divisor problem, defined by

$$\Delta(x)={\sum_{n\le x}}' d(n)-x(\log x+2\gamma-1)-1/4.$$

Here γ denotes the Euler constant, and $\sum'$ indicates that the last term is to be halved if x is an integer. We note that Jutila [2] gives another proof of Motohashi's result (1.1). The asymptotic formula

$$\int_1^T\Delta^2(x)\,dx=(6\pi^2)^{-1}\zeta^4(3/2)\zeta^{-1}(3)T^{3/2}+O(T\log^5 T)\tag{1.2}$$

was proved by Tong [8], and the error term has been improved to $O(T\log^4 T)$ by Preissmann [7]. In view of the relation (1.1), we can expect that an analogue of (1.2) can be shown for $|R(s;t/2\pi)|$.

Hereafter we restrict ourselves to the case $s=1/2+it$. Then $|\chi(1-s)|=1$, so it is plausible that

$$\int_1^T |R(1/2+it;t/2\pi)|^2\,dt \sim cT^{1/2}$$

holds with a certain positive constant c. In this paper we verify this asymptotic relation in the following form.

THEOREM 1. *For any $T \ge 1$, we have*

$$\int_1^T |R(1/2+it;t/2\pi)|^2\,dt = \sqrt{2\pi}\Big\{\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2}\Big\}T^{1/2} + O(T^{1/4}\log T)\,,$$

where

$$h(n) = (2/\pi)^{1/2}\int_0^{\infty}(y+n\pi)^{-1/2}\cos(y+\pi/4)\,dy\,.$$

Remark. Theorem 1 includes the fact $|R(1/2+it;t/2\pi)| = \Omega(t^{-1/4})$, but a stronger Ω-result can be deduced from (1.1) and the well-known Ω-result for $\Delta(t/2\pi)$. If the conjecture $\Delta(t/2\pi) \ll t^{1/4+\varepsilon}$ is true, then $|R(1/2+it;t/2\pi)| \ll t^{-1/4+\varepsilon}$ would follow.

To prove Theorem 1, the formula (1.1) is not suitable; the error $O(t^{-1/4})$ is too large. Our starting point is the following "weak form" of the Riemann–Siegel formula for $\zeta^2(s)$, which has been proved in Motohashi [5]:

$$\chi(1-s)R(s;t/2\pi) = (t/2\pi)^{-1/4}\sum_{n=1}^{\infty} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n) + O(t^{-1/2}\log t)\,. \tag{1.3}$$

In the same article, Motohashi announced a stronger approximation formula, and has given a detailed proof in [6]. By using this (rather complicated) "full form" of Motohashi's formula, it might be possible to improve the error estimate in Theorem 1.

Next we consider the mean square of $R(1/2+it;t/2\pi)$ itself. Let $x = t/2\pi$, and $f(x) = 2x - 2x\log x + 1/4$. It follows from Stirling's formula that

$$\chi^2(1/2+it) = \exp(2\pi i f(x)) + O(t^{-1})\,, \tag{1.4}$$

so the χ-factor on the left-hand side of (1.3) can be considered as an "exponential factor". Because of the existence of this factor, it is natural to expect that the integral of $R(1/2+it;t/2\pi)^2$ is smaller than that of $|R(1/2+it;t/2\pi)|^2$. We prove

THEOREM 2. *For any* $\varepsilon>0$, *we have*

$$\int_1^T R(1/2+it;t/2\pi)^2\,dt = O(T^{1/4+\varepsilon})\,.$$

The proof of Theorem 2 is a simple application of well-known upper bounds for exponential integrals. One could obtain a better estimate by a more elaborate analysis of the relevant integrals.

In what follows, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence.

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2. Application of Voronoï's formulas. The classical Voronoï formula asserts (see (15.24) of Ivić [1]) that

$$\Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)+O(x^{-1/4})\,,$$

while the truncated Voronoï formula asserts (see (3.17) of Ivić [1]) that

$$\Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)+E(N;x) \tag{2.1}$$

with

$$E(N;x) = O(x^{\varepsilon}+x^{1/2+\varepsilon}N^{-1/2})\,, \tag{2.2}$$

where $0<N\ll x^A$ for some $A>0$.

Combining these two formulas, we have

$$\sum_{n>N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4) = O(x^{-1/2}+x^{-1/4}|E(N;x)|)\,. \tag{2.3}$$

Let

$$S(N;t) = \sum_{n>N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\,.$$

Integration by parts gives

$$h(n) = -(\pi\sqrt{n})^{-1}+O(n^{-3/2})\,, \tag{2.4}$$

so

$$S(N;t) = -\pi^{-1}\sum_{n>N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)+O(N^{-3/4+\varepsilon})\,,$$

where $x = t/2\pi$. From (2.3), we have

$$S(N;t) = O(x^{-1/2} + x^{-1/4}|E(N;x)| + N^{-3/4+\varepsilon}).$$

Therefore, from (1.3), we have

$$\begin{aligned}(2.5)\quad &\chi(1/2-it)R(1/2+it;t/2\pi)\\ &= (t/2\pi)^{-1/4}\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n) + D(N;t),\end{aligned}$$

with

$$(2.6)\qquad D(N;t) = O(t^{-1/2}|E(N;x)| + t^{-1/2}\log t + t^{-1/4}N^{-3/4+\varepsilon}).$$

If $x \ll N$, then (2.2) implies $E(N;x) = O(x^{\varepsilon})$. In case x is not so close to an integer, Meurman has shown the following sharper estimate.

LEMMA 1 (Meurman [3]). *Denote by $\|x\|$ the distance from x to the nearest integer. If $x \ll N$, then*

$$E(N;x) \ll \begin{cases} x^{-1/4} & \text{if } \|x\| \gg x^{5/2}N^{-1/2},\\ x^{\varepsilon} & \text{otherwise.}\end{cases}$$

3. Proof of Theorem 1. In this section we assume $T \ll N$. From (2.5) we have

$$\begin{aligned}(3.1)\quad &\int_T^{2T} |R(1/2+it;t/2\pi)|^2\,dt = I(N;T)\\ &+ O\Big(\int_T^{2T} t^{-1/4}\Big|\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\Big|\,|D(N;t)|\,dt\Big)\\ &+ O\Big(\int_T^{2T} |D(N;t)|^2\,dt\Big),\end{aligned}$$

where

$$I(N;T) = \int_T^{2T} (t/2\pi)^{-1/2}\Big\{\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\Big\}^2\,dt.$$

By using (2.6) and Lemma 1, the last term on the right-hand side of (3.1) can be estimated as

$$\begin{aligned}(3.2)\qquad &\ll T^{-1}\int_T^{2T} |E(N;x)|^2\,dt + \log^2 T + T^{1/2}N^{-3/2+\varepsilon}\\ &\ll T^{-1}(T^{1/2} + T^{7/2+\varepsilon}N^{-1/2}) + \log^2 T + T^{1/2}N^{-3/2+\varepsilon}\\ &\ll T^{5/2+\varepsilon}N^{-1/2} + \log^2 T.\end{aligned}$$

Hence, by Schwarz's inequality, the second term on the right-hand side of (3.1) is

$$\ll I(N;T)^{1/2}(T^{5/4+\varepsilon}N^{-1/4}+\log T)\,. \tag{3.3}$$

We have

$$\begin{aligned}
I(N;T) &= (\pi/2)^{1/2}\sum_{n\le N} d^2(n)n^{-1/2}h^2(n)\int_T^{2T} t^{-1/2}\,dt \\
&\quad + (\pi/2)^{1/2}\sum_{n\le N} d^2(n)n^{-1/2}h^2(n)\int_T^{2T} t^{-1/2}\sin(4\sqrt{2\pi tn})\,dt \\
&\quad + (\pi/2)^{1/2}\sum_{\substack{m,n\le N\\ m\ne n}}\sum d(m)d(n)(mn)^{-1/4}h(m)h(n) \\
&\qquad\qquad \times \int_T^{2T} t^{-1/2}\sin(2\sqrt{2\pi t}(\sqrt{m}+\sqrt{n}))\,dt \\
&\quad + (\pi/2)^{1/2}\sum_{\substack{m,n\le N\\ m\ne n}}\sum d(m)d(n)(mn)^{-1/4}h(m)h(n) \\
&\qquad\qquad \times \int_T^{2T} t^{-1/2}\cos(2\sqrt{2\pi t}(\sqrt{m}-\sqrt{n}))\,dt \\
&= I_1+I_2+I_3+I_4, \quad \text{say.}
\end{aligned}$$

From (2.4) we see that

$$h(n)=O(n^{-1/2})\,, \tag{3.4}$$

so

$$\begin{aligned}
I_1 &= (2\pi)^{1/2}(\sqrt{2T}-\sqrt{T})\Big\{\sum_{n=1}^{\infty} d^2(n)n^{-1/2}h^2(n)+O\Big(\sum_{n>N} d^2(n)n^{-3/2}\Big)\Big\} \\
&= (2\pi)^{1/2}\Big\{\sum_{n=1}^{\infty} d^2(n)n^{-1/2}h^2(n)\Big\}(\sqrt{2T}-\sqrt{T})+O(T^{1/2}N^{-1/2+\varepsilon})\,.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_T^{2T} t^{-1/2}\exp(2iu\sqrt{2\pi t})\,dt \\
&\qquad = (iu\sqrt{2\pi})^{-1}\{\exp(2iu\sqrt{4\pi T})-\exp(2iu\sqrt{2\pi T})\}\ll u^{-1}\,,
\end{aligned} \tag{3.5}$$

we see that $I_2 = O(1)$ and

$$I_3 \ll \sum_{m,n\le N}\sum d(m)d(n)(mn)^{-1} \ll \log^4 N\,,$$

by using (3.4), the inequality $2(mn)^{1/4} \le \sqrt{m}+\sqrt{n}$ and the estimate

$$\sum_{n\le N} d(n)n^{-1} \ll \log^2 N\,.$$

The estimate $I_4 = O(\log^4 N)$ follows from (3.4), (3.5), the inequality

$$\sum_{n\le N} d^2(n)n^{-1} \ll \log^4 N\,,$$

and the following

LEMMA 2 (Corollary of Preissmann [7]). *Suppose that a_n, b_n, and c_n $(1 \le n \le M)$ denote real numbers. Then*

$$\Big|\sum_{\substack{m,n\le M\\ m\ne n}}\sum a_m a_n (mn)^{-1/4}(\sqrt{m}-\sqrt{n})^{-1}\exp(i(b_m-c_n))\Big| \ll \sum_{n\le M} a_n^2\,.$$

Therefore, we obtain

$$\begin{aligned}(3.6)\qquad I(N;T) = (2\pi)^{1/2}\Big\{\sum_{n=1}^{\infty} d^2(n)n^{-1/2}h^2(n)\Big\}(\sqrt{2T}-\sqrt{T})\\ + O(T^{1/2}N^{-1/2+\varepsilon} + \log^4 N)\,.\end{aligned}$$

Now we put $N = T^{\lambda}$, with the parameter $\lambda \ge 1$. Then (3.6) implies $I(N;T) = O(T^{1/2})$, so (3.3) is estimated by

$$\ll T^{3/2-\lambda/4+\varepsilon} + T^{1/4}\log T\,.$$

Substituting this estimate, (3.2) and (3.6) into (3.1), we have

$$\begin{aligned}\int_T^{2T} |R(1/2+it;t/2\pi)|^2\,dt = (2\pi)^{1/2}\Big\{\sum_{n=1}^{\infty} d^2(n)n^{-1/2}h^2(n)\Big\}(\sqrt{2T}-\sqrt{T})\\ + O(T^{3/2-\lambda/4+\varepsilon} + T^{5/2-\lambda/2+\varepsilon} + T^{1/4}\log T)\,,\end{aligned}$$

and the error term can be written as $O(T^{1/4}\log T)$, if we choose a sufficiently large value of λ. This completes the proof of Theorem 1.

Remark. If we content ourselves with the error $O(T^{1/4+\varepsilon})$ in Theorem 1, then we do not need Meurman's lemma; the estimate (2.2) suffices.

4. Proof of Theorem 2. From (2.5) and Schwarz's inequality, it follows that

$$\int_T^{2T} R(1/2+it;t/2\pi)^2\,dt \tag{4.1}$$

$$= J(N;T) + O\Big(I(N;T)^{1/2}\Big(\int_T^{2T} |D(N;t)|^2\,dt\Big)^{1/2}\Big) + O\Big(\int_T^{2T} |D(N;t)|^2\,dt\Big),$$

where

$$J(N;T) = \int_T^{2T} (t/2\pi)^{-1/2}\chi^2(1/2+it) \times\Big\{\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\Big\}^2\,dt\,.$$

In this section we put $N = T^{1-\varepsilon}$. Then, from (3.6) we have $I(N;T) = O(T^{1/2})$, and from (2.2) and (2.6) we have $D(N;t) = O(t^{-1/2+\varepsilon})$. Substituting these estimates into (4.1), we obtain

$$\int_T^{2T} R(1/2+it;t/2\pi)^2\,dt = J(N;T) + O(T^{1/4+\varepsilon})\,. \tag{4.2}$$

By using (1.4), we have

$$J(N;T) = J^*(N;T) + O(J^{**}(N;T))\,,$$

where

$$J^*(N;T) = \int_T^{2T} (t/2\pi)^{-1/2}\exp(2\pi i f(x)) \times\Big\{\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\Big\}^2\,dt\,,$$

$$J^{**}(N;T) = \int_T^{2T} t^{-3/2}\Big\{\sum_{n\le N} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h(n)\Big\}^2\,dt\,.$$

By using the truncated Voronoï formula (2.1) and the classical estimate $\Delta(x) = O(x^{1/3}\log^2 x)$, we can prove $J^{**}(N;T) = O(T^{-1/3}\log^4 T)$. For our purpose, however, the trivial estimate

$$J^{**}(N;T) = O(T^{\varepsilon}) \tag{4.3}$$

is sufficient.

Similarly to the case of $I(N;T)$, we have

$$
\begin{aligned}
(4.4)\qquad J^*(N;T) = (\pi/2)^{1/2} \sum_{n\le N} d^2(n) n^{-1/2} h^2(n) \int_T^{2T} t^{-1/2} \exp(2\pi i f(x))\, dt \\
+ (\pi/2)^{1/2} \sum_{n\le N} d^2(n) n^{-1/2} h^2(n) \\
\times \int_T^{2T} t^{-1/2} \exp(2\pi i f(x)) \sin(4\sqrt{2\pi t n})\, dt \\
+ (\pi/2)^{1/2} \sum_{\substack{m,n\le N \\ m\ne n}}\sum d(m)d(n)(mn)^{-1/4} h(m)h(n) \\
\times \int_T^{2T} t^{-1/2} \exp(2\pi i f(x)) \sin(2\sqrt{2\pi t}(\sqrt{m}+\sqrt{n}))\, dt \\
+ (\pi/2)^{1/2} \sum_{\substack{m,n\le N \\ m\ne n}}\sum d(m)d(n)(mn)^{-1/4} h(m)h(n) \\
\times \int_T^{2T} t^{-1/2} \exp(2\pi i f(x)) \cos(2\sqrt{2\pi t}(\sqrt{m}-\sqrt{n}))\, dt\,.
\end{aligned}
$$

The right-hand side of (4.4) can be estimated by using the following well-known

LEMMA 3 ((2.3) of Ivić [1]). *Let $F(x)$ be real differentiable, $F'(x)$ monotonic, $F'(x) \ge m > 0$ or $\le -m < 0$ in $[a,b]$. Let $G(x)$ be positive monotonic, $|G(x)| \le M$ in $[a,b]$. Then*

$$
\Big| \int_a^b G(x) \exp(iF(x))\, dx \Big| \ll M/m\,.
$$

Let $F(x) = 2\pi(f(x) + 2u\sqrt{x})$, with $|u| \le 2\sqrt{N}$. Then $|F'(x)| \gg \log T$, so Lemma 3 implies

$$
\int_{T/2\pi}^{T/\pi} x^{-1/2} \exp(2\pi i (f(x) + 2u\sqrt{x}))\, dx \ll T^{-1/2} (\log T)^{-1}\,.
$$

From the cases $u = 0$ and $u = \pm 2\sqrt{n}$, it follows that the first and the second sums on the right-hand side of (4.4) are

$$
\ll T^{-1/2} (\log T)^{-1} \sum_{n\le N} d^2(n) n^{-3/2} \ll T^{-1/2} (\log T)^{-1}\,,
$$

and from the cases $u=\pm(\sqrt{m}\pm\sqrt{n})$, it follows that the third and the fourth sums are

$$\ll T^{-1/2}(\log T)^{-1}\Big\{\sum_{n\le N} d(n)n^{-3/4}\Big\}^2 \ll T^{-1/2}N^{1/2}\log T \ll 1\,.$$

Hence we have $J^*(N;T)=O(1)$, and with (4.2) and (4.3), we obtain the assertion of Theorem 2.

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