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MEAN VALUE THEOREMS FOR VECTOR-VALUED FUNCTIONS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. The object of this paper is to give a generalization to vector-valued functions of the Cauchy mean value theorem of the differential calculus, together with some related results. In the classical Cauchy mean value theorem we have

$$(1.1) \qquad (\phi(b) - \phi(a))\psi'(\xi) = (\psi(b) - \psi(a))\phi'(\xi)$$

for some ξ in $]a, b[$, where $\phi, \psi: [a, b] \to \boldsymbol{R}$ are continuous functions possessing derivatives on $]a, b[$. The counterpart to (1.1) when ϕ is vector-valued is the mean value inequality in Theorem 1 below.

Throughout we suppose that our vector spaces are real. For any function ϕ from an interval $[a, b]$ into a topological vector space Y, we say that an element y of Y is a *right-hand derivative value* of ϕ at the point $t \in [a, b[$ if there exists a sequence (t_n) of points of $]t, b]$ decreasing to the limit t such that $(\phi(t_n) - \phi(t))/(t_n - t) \to y$ in Y as $n \to \infty$. (In particular, if $Y = \boldsymbol{R}$, a right-hand derivative value is finite.)

The use of right-hand derivative values* enables us to avoid the hypothesis that limits in Y are unique. However, if Y is a T_1-space (and therefore Hausdorff), we can define two-sided and one-sided derivatives in the usual way; for example, the right-hand derivative $\phi'_+(t)$ of ϕ at the point $t \in [a, b[$ is the limit

$$\lim_{h \to 0+} \frac{\phi(t + h) - \phi(t)}{h}$$

whenever this limit exists in Y (again it is finite if $Y = \boldsymbol{R}$). Obviously $\phi'_+(t)$ is a right-hand derivative value of ϕ at t.

A sublinear functional on a vector space Y is a function $p: Y \to \boldsymbol{R}$ such that for all $y, z \in Y$ and all $\lambda \geqslant 0$

$$(1.2) \qquad p(y + z) \leqslant p(y) + p(z) \quad \text{and} \quad p(\lambda y) = \lambda p(y) .$$

We note in passing that the first of these relations implies that for all $y, z \in Y$

* The name is used by McLeod [12].

$$p(y) - p(z) \leqslant p(y-z) . \tag{1.3}$$

Following Gál [10], we say that a property $P(t)$ holds for *nearly all* t belonging to a set $E \subset \boldsymbol{R}$, or *nearly everywhere* in E, if there exists a countable subset H of E such that $P(t)$ holds for all $t \in E\backslash H$. The term 'nearly everywhere in E' must be distinguished from the measure-theoretic term 'almost everywhere in E', i.e. except in a subset of E of Lebesgue measure 0.

With this terminology, our main result is as follows.

THEOREM 1. *Let Y be a topological vector space, let p be a continuous sublinear functional on Y, let $\phi\colon [a, b] \to Y$, $\psi\colon [a, b] \to \boldsymbol{R}$ be continuous, let $\psi(a) < \psi(b)$, and for nearly all $t \in [a, b[$ let $\Phi(t)$, $\Psi(t)$ be right-hand derivative values of ϕ, ψ associated with the same sequence (t_n) converging to t. Then there exists a set of positive measure of points ξ such that*

$$p(\phi(b) - \phi(a))\Psi(\xi) \leqslant (\psi(b) - \psi(a))p(\Phi(\xi)) . \tag{1.4}$$

Moreover, either (1.4) *holds with strict inequality for all ξ in some set of positive measure, or equality holds in* (1.4) *for almost all $\xi \in [a, b[$, and in this latter case*

$$p(\phi(b) - \phi(a))(\psi(t) - \psi(a)) = (\psi(b) - \psi(a))p(\phi(t) - \phi(a)) \tag{1.5}$$

for all $t \in [a, b]$.

We note explicitly the form of Theorem 1 for right-hand derivatives.

COROLLARY. *Let Y be a T_1-topological vector space, let p be a continuous sublinear functional on Y, let $\phi\colon [a, b] \to Y$, $\psi\colon [a, b] \to \boldsymbol{R}$ be continuous functions whose right-hand derivatives $\phi'_+(t)$, $\psi'_+(t)$ exist for nearly all $t \in [a, b[$, and let $\psi(a) < \psi(b)$. Then there exists a set of positive measure of points $\xi \in [a, b[$ such that*

$$p(\phi(b) - \phi(a))\psi'_+(\xi) \leqslant (\psi(b) - \psi(a))p(\phi'_+(\xi)) . \tag{1.6}$$

Moreover, either (1.6) *holds with strict inequality for all ξ in some set of positive measure, or equality holds in* (1.6) *for almost all $\xi \in [a, b[$ and* (1.5) *holds for all $t \in [a, b]$.*

The case of Theorem 1 in which $\psi(t) = t$ is a known result of McLeod [12, Theorem 5]. Both our proof and that of McLeod use Lemma 1 below, but in other respects our proof (and indeed our formulation of the theorem) is simpler than McLeod's.

The first result of this type, in which the classical mean value equality is replaced by an inequality holding for some intermediate ξ, is due to Aziz and Diaz [2, 3]. Their theorem is included in McLeod's result, as is also a subsequent generalization of the Aziz-Diaz theorem by Aziz, Diaz and Mlak [4], namely that if ϕ is a continuous function from $[a, b]$ into a normed space Y whose right-hand derivative $\phi'_+(t)$ exists nearly everywhere in $[a, b[$, then there exists $\xi \in]a, b[$ such that

$$|| \phi(b) - \phi(a) || \leqslant (b - a) || \phi'_+(\xi) || .^*$$

None of these authors consider the question of equality.

Our arguments give also a simplified proof of the following theorem of McLeod [12, Theorem B], which is the increment inequality corresponding to Theorem 1.

THEOREM 2. *Suppose that the hypotheses of Theorem* 1 *are satisfied, with the omission of the condition that* $\psi(a) < \psi(b)$, *and that in addition* $p(\Phi(t)) \leqslant \Psi(t)$ *for almost all* $t \in [a, b[$. *Then*

$$p(\phi(b) - \phi(a)) \leqslant \psi(b) - \psi(a) . \tag{1.7}$$

Moreover, there is equality in (1.7) *if and only if*

$$p(\phi(t) - \phi(a)) = \psi(t) - \psi(a)$$

for all $t \in [a, b]$ (*and there is strict inequality if* $p(\Phi(t)) < \Psi(t)$ *for at least one* $t \in [a, b[$).

The well-known increment inequality of Bourbaki [5, p. 22] (see also Dieudonné [7, p. 153] and Cartan [6, p. 42]) is a special case of this result.

The proofs of Theorems 1 and 2 are given in § 3. In § 2 we note some applications of Theorem 1 and its corollary to various versions of Taylor's theorem. The paper concludes with a number of variants of Theorems 1 and 2, and with a brief survey of other extensions of the mean value theorem for vector-valued functions.

2. As might be expected, Theorem 1 and its corollary give versions of Taylor's theorem for vector-valued functions. We consider explicitly only results expressed in terms of derivatives, for which we employ Theorem 1, Corollary; corresponding results in terms of derivative values can be obtained by similar arguments from Theorem 1.

It has been pointed out by Dieudonné [7, Chapter 8] that increment inequalities (such as Theorem 2 above) serve equally well for this purpose, but (perhaps because of familiarity with the classical forms) the mean value inequalities seem more suggestive.

Theorem 3 below is a generalization of a mean value equality for real-valued functions given by the author [8, p. 169]. Its interest lies in

* Our argument has points of similarity with that of Aziz, Diaz and Mlak, but gives more with less effort.

the fact that it implies a Taylor theorem of W. H. Young's type for both vector-valued functions of a real variable and for Fréchet-differentiable functions of a vector variable.

THEOREM 3. *Let Y be a T_1 topological vector space, let p be a continuous sublinear functional on Y, let n be a positive integer, and let $\phi\colon [a, b] \to Y$ be a function whose $(n-1)$-th derivative $\phi^{(n-1)}(t)$ exists for all $t \in [a, b]$ and whose n-th (right-hand) derivative $\phi^{(n)}(a)$ exists. Then there exists a set of positive measure of points $\xi \in [a, b[$ such that*

$$(2.1) \qquad p\Big(\phi(b) - \phi(a) - (b-a)\phi'(a) - \cdots - \frac{(b-a)^n}{n!}\phi^{(n)}(a)\Big) \leqslant \frac{(b-a)^n}{n!}\, p\Big(\frac{\phi^{(n-1)}(\xi) - \phi^{(n-1)}(a)}{\xi - a} - \phi^{(n)}(a)\Big) .$$

This is obtained by $n-1$ successive applications of Theorem 1, Corollary, starting with the pair of functions

$$t \mapsto \phi(t) - \phi(a) - (t-a)\phi'(a) - \cdots - \frac{(t-a)^n}{n!}\phi^{(n)}(a)$$

and $t \mapsto (t-a)^n/n!$.

For vector-valued functions of a real variable, Theorem 3 gives

THEOREM 4. *Let Y be a locally convex T_1 topological vector space, let n be a positive integer, and let $\phi\colon [a, b] \to Y$ be a function whose $(n-1)$-th derivative $\phi^{(n-1)}(t)$ exists for all $t \in [a, b]$ and whose n-th (right-hand) derivative $\phi^{(n)}(a)$ exists. Then*

$$(2.2) \qquad h^{-n}\Big\{\phi(a+h) - \phi(a) - h\phi'(a) - \cdots - \frac{h^n}{n!}\phi^{(n)}(a)\Big\} \to 0 \text{ as } h \to 0+ .$$

Let $q(h)$ denote the expression on the left of (2.2). Since any neighbourhood of 0 in Y contains a closed convex neighbourhood of 0, it is enough to show that if V is a closed convex neighbourhood of 0, there exists $\delta > 0$ such that $q(h) \in V$ whenever $0 < h < \delta$.

Let p be the Minkowski functional of V. Then p is a continuous sublinear functional on Y, and $y \in V$ if and only if $p(y) \leqslant 1$ (see, for example, Taylor [15, p. 135]). Since $\phi^{(n)}(a)$ exists, we can find $\delta > 0$ such that

$$\frac{1}{n!}\Big\{\frac{\phi^{(n-1)}(\xi) - \phi^{(n-1)}(a)}{\xi - a} - \phi^{(n)}(a)\Big\} \in V$$

whenever $a < \xi < a + \delta$. It then follows from (2.1) that $q(h) \in V$, as required.

The corresponding result for Fréchet-differentiable functions of a vector variable is as follows. We use the terminology and notation of Dieudonné [7, Chapter 8].

THEOREM 5. *Let X, Y be normed spaces, let n be a positive integer, and let f be a function from a subset of X into Y that is n-times Fréchet-differentiable at the point x. Then*

$$h^{-n}\left\{f(x+h)-f(x)-f'(x)\cdot h-\frac{1}{2!}f''(x)\cdot h^{(2)}-\cdots-\frac{1}{n!}f^{(n)}(x)\cdot h^{(n)}\right\}\to 0 \quad \text{as } h\to 0\,.$$

By hypothesis, we can find an open ball B in X with centre x and radius η such that $f^{(n-1)}(z)$ exists for all $z\in B$. Let $x+h\in B$. Then $x+th\in B$ for $0\leqslant t\leqslant 1$, and we define

$$\phi(t)=f(x+th) \qquad (0\leqslant t\leqslant 1)\,.$$

By the chain rule (cf. Dieudonné [7, p. 186]),

$$\phi^{(r)}(t)=f^{(r)}(x+th)\cdot h^{(r)}$$

for $r=1,\cdots,n$, and hence, by Theorem 3 with $a=0$, $b=1$, there exists $\xi\in\,]0,1[$ such that

$$\text{(2.3)} \qquad n!\left\|f(x+h)-f(x)-f'(x)\cdot h-\cdots-\frac{1}{n!}f^{(n)}(x)\cdot h^{(n)}\right\| \leqslant\left\|\frac{f^{(n-1)}(x+\xi h)\cdot h^{(n-1)}-f^{(n-1)}(x)\cdot h^{(n-1)}}{\xi}-f^{(n)}(x)\cdot h^{(n)}\right\|\,.$$

But since f is n-times differentiable at x, given $\varepsilon>0$ we can find δ satisfying $0<\delta<\eta$ such that

$$\|f^{(n-1)}(x+k)-f^{(n-1)}(x)-f^{(n)}(x)\cdot k\|\leqslant\varepsilon\|k\|$$

whenever $\|k\|\leqslant\delta$ (where the norm on the left is now that of $\mathscr{L}(X, Y^{(n-1)})$). This in turn implies that the expression on the right of (2.3) does not exceed $\varepsilon\|h\|^n$ whenever $\|h\|<\delta$, and this is the required result.

An alternative proof of Theorem 4 for a normed space Y, using induction, is given by Bourbaki [5, p. 33]. An alternative proof of Theorem 5, making use of this same induction argument, has been given by Cartan [6, p. 78] (see also [8, p. 379]). The result of Theorem 5 for Euclidean spaces was proved by W. H. Young himself, employing a very similar induction argument [19, p. 27].

A Taylor theorem with a remainder of Lagrange-Cauchy-Schlömilch form is more easily obtained. Thus for functions of a real variable we have

THEOREM 6. *Let Y be a T_1 topological vector space, let p be a continuous sublinear functional on Y, let n be a positive integer, let $r > 0$, and let $\phi: [a, b] \to Y$ be a function whose $(n-1)$-th derivative $\phi^{(n-1)}(t)$ exists for all $t \in [a, b]$ and whose n-th derivative $\phi^{(n)}(t)$ exists nearly everywhere in $[a, b[$. Then there exists a set of positive measure of points $\xi \in [a, b[$ such that*

$$(2.4)\qquad p\Big(\phi(b) - \phi(a) - (b-a)\phi'(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!}\phi^{(n-1)}(a)\Big) \leqslant \frac{(b-\xi)^{n-r}(b-a)^r}{r(n-1)!}\, p(\phi^{(n)}(\xi)) \,.$$

Moreover, either (2.4) *holds with strict inequality for all ξ in some set of positive measure, or equality holds in* (2.4) *for almost all $\xi \in [a, b[$.*

Here we simply apply Theorem 1, Corollary, to the pair of functions

$$t \mapsto -\phi(b) + \phi(t) + (b-t)\phi'(t) + \cdots + \frac{(b-t)^{n-1}}{(n-1)!}\phi^{(n-1)}(t)$$

and $t \mapsto -(b-t)^r$.

The corresponding result for Fréchet-differentiable functions follows from (2.4) applied to $\phi(t) = f(x + th)$, as in Theorem 5.

3. We consider now the proof of Theorems 1 and 2, and for these we require the following two lemmas. As usual, D_+ denotes the lower right-hand Dini derivative, infinite values being allowed.

LEMMA 1. *Let $\sigma: [a, b] \to \boldsymbol{R}$ be a continuous function such that $D_+\sigma(t) \leqslant 0$ almost everywhere and $D_+\sigma(t) < \infty$ nearly everywhere in $[a, b[$. Then σ is decreasing.*

This result is well-known (see, for example, Saks [14, p. 204, Theorem (7.3)]; a proof using only basic ideas of elementary analysis is given by Gál [10]).

LEMMA 2. *Let p be continuous sublinear functional on a topological vector space Y, let $\phi: [a, b] \to Y$ be a given function, and suppose that y is a right-hand derivative value of ϕ at t associated with the sequence (t_n). Then*

$$(3.1)\qquad \liminf_{n\to\infty} \frac{p(\phi(t_n) - \phi(a)) - p(\phi(t) - \phi(a))}{t_n - t} \leqslant p(y) \,.$$

By (1.3) and (1.2),

$$\frac{p(\phi(t_n)-\phi(a))-p(\phi(t)-\phi(a))}{t_n-t} \leqslant \frac{p(\phi(t_n)-\phi(t))}{t_n-t} = p\left(\frac{\phi(t_n)-\phi(t)}{t_n-t}\right),$$

and this implies (3.1), since p is continuous.

Suppose now that p, ϕ, ψ satisfy the hypotheses of Theorem 1, and let σ: $[a, b] \to \boldsymbol{R}$ be given by

$$\sigma(t) = (\psi(b)-\psi(a))p(\phi(t)-\phi(a)) - p(\phi(b)-\phi(a))(\psi(t)-\psi(a)) .$$

Clearly σ is continuous, and $\sigma(b) = \sigma(a) = 0$. Also, if t is a point of $[a, b[$ for which $\Phi(t)$, $\Psi(t)$ exist, and (t_n) is the associated sequence, then, by Lemma 2.

$$\begin{aligned} \text{(3.2)} \qquad D_+\sigma(t) &= \liminf_{s\to t+} \frac{\sigma(s)-\sigma(t)}{s-t} \leqslant \liminf_{n\to\infty} \frac{\sigma(t_n)-\sigma(t)}{t_n-t} \\ &\leqslant (\psi(b)-\psi(a))p(\Phi(t)) - p(\phi(b)-\phi(a))\Psi(t) . \end{aligned}$$

In particular, this implies that $D_+\sigma(t) < \infty$ nearly everywhere.

Now let A, B be respectively the sets of points $\xi \in [a, b[$ for which

$$p(\phi(b)-\phi(a))\Psi(\xi) < (\psi(b)-\psi(a))p(\Phi(\xi))$$

and

$$p(\phi(b)-\phi(a))\Psi(\xi) \geqslant (\psi(b)-\psi(a))p(\Phi(\xi)) .$$

If A has positive measure, the first alternative of Theorem 1 holds. If A has measure 0, so has $[a, b]\backslash B$. Also, by (3.2),

$$\text{(3.3)} \qquad D_+\sigma(\xi) \leqslant (\psi(b)-\psi(a))p(\Phi(\xi)) - p(\phi(b)-\phi(a))\Psi(\xi) \leqslant 0$$

for all $\xi \in B$, and therefore for almost all $\xi \in [a, b[$. By Lemma 1, σ is decreasing, and since $\sigma(b) = \sigma(a) = 0$, it follows that $\sigma(t) = 0$ for all $t \in [a, b]$, and that $D_+\sigma(t) = 0$ for all $t \in [a, b[$. Hence equality holds in (3.3) for all $\xi \in B$, i.e. for almost all ξ, and therefore the second alternative of Theorem 1 holds.

The proof of Theorem 2 is simpler. We set

$$\sigma(t) = p(\phi(t)-\phi(a)) - (\psi(t)-\psi(a)) .$$

By Lemma 2,

$$D_+\sigma(t) \leqslant p(\Phi(t)) - \Psi(t)$$

whenever $\Phi(t)$, $\Psi(t)$ exist, and therefore $D_+\sigma(t) < \infty$ nearly everywhere and $D_+\sigma(t) \leqslant 0$ almost everywhere in $[a, b[$. By Lemma 1, σ is decreasing, and $\sigma(b) = 0$ if and only if σ is identically 0.

4. The known proofs of Lemma 1 are not easy, and it is worth while to see what can be obtained by easier means. For instance, we can use in place of Lemma 1 the following well-known lemma of Zygmund (see Saks [14, p. 204]), which has an elegantly simple proof.

LEMMA 3. *Let* $\sigma\colon [a, b] \to \boldsymbol{R}$ *be a continuous function such that* $D_+\sigma(t) \leqslant 0$ *nearly everywhere in* $[a, b[$. *Then* σ *is decreasing.*

Arguing as in § 3, we then obtain a theorem with the hypotheses of Theorem 1, but with the conclusion that there exists an uncountable set of points $\xi \in [a, b[$ such that (1.4) holds, and, moreover, either (1.4) holds with strict inequality for uncountably many ξ, or equality holds in (1.4) nearly everywhere and ϕ, ψ satisfy (1.5) for all t.

This result suffices for most applications. The case in which Y is normed, p is the norm function on Y, and $\psi(t) = t$, contains the mean value theorems of Aziz and Diaz [2, 3] and Aziz, Diaz and Mlak [4] mentioned in § 1.

We also obtain a simpler analogue of Theorem 2, in which the hypothesis that $p(\Phi(t)) \leqslant \Psi(t)$ is now supposed to hold nearly everywhere, and in which the conclusion remains as before. The case of this result in which Y is normed and p is the norm function on Y contains the increment inequality of Bourbaki [5, p. 22].

These weaker versions of Theorems 1 and 2 avoid measure-theoretic ideas. We may also go to the opposite extreme and introduce further measure-theoretic ideas, by replacing Lemma 1 by the following familiar result (a proof independent of integration theory is given by Gál [10]).

LEMMA 4. *Let* $\sigma\colon [a, b] \to \boldsymbol{R}$ *be an absolutely continuous function such that* $D_+\sigma(t) \leqslant 0$ *almost everywhere in* $[a, b[$. *Then* σ *is decreasing.*

Using Lemma 4 in place of Lemma 1, we see easily that if Y is normed and p is either the norm function or a continuous linear functional on Y, and if ϕ, ψ are absolute continuous,* $\psi(a) < \psi(b)$, and $\Phi(t)$, $\Psi(t)$ are assumed to exist only almost everywhere, then the conclusion of Theorem 1 holds.

6. If in addition to the hypotheses of Theorem 1 we suppose that ψ is strictly increasing and that $\Psi(t) > 0$ almost everywhere, we can write the inequality (1.4) in the form

$$p\left(\frac{\phi(b) - \phi(a)}{\psi(b) - \psi(a)}\right) \leqslant p\left(\frac{\Phi(t)}{\Psi(t)}\right).$$

* The absolute continuity of ϕ is defined in terms of $\Sigma \,||\, \phi(t_i) - \phi(t_{i-1}) \,||$, exactly as for real-valued functions.

However, when Theorem 1 is cast in this form, it can be strengthened by the use of relative derivatives. For this purpose we use the following analogue of Lemma 1, which can be proved by a straightforward adaptation of Gál's proof of Lemma 1.

LEMMA 5. *Let* $\psi\colon [a, b] \to \boldsymbol{R}$ *be a strictly increasing continuous function, and let* μ *be the Lebesgue-Stieltjes measure induced on* $[a, b]$ *by* ψ.* *Let also* $\sigma\colon [a, b] \to \boldsymbol{R}$ *be continuous, let*

$$D_{\psi+}\sigma(t) = \liminf_{s \to t+} \frac{\sigma(s) - \sigma(t)}{\psi(s) - \psi(t)} \qquad (a \leqslant t < b) ,$$

where infinite values are allowed, and suppose that $D_{\psi+}\sigma(t) \leqslant 0$ μ*-almost everywhere and* $D_{\psi+}\sigma(t) < \infty$ *nearly everywhere in* $[a, b]$. *Then* σ *is decreasing.*

Using Lemma 5 in place of Lemma 1, we obtain:

THEOREM 7. *Suppose that* ψ, μ *satisfy the conditions of Lemma* 5, *that* Y *is a topological vector space, that* p *is a continuous sublinear functional on* Y, *and that* $\phi\colon [a, b] \to Y$ *is continuous. Suppose also that, for nearly all* $t \in [a, b[$, $\Phi_{\psi}(t)$ *is a right-hand derivative value of* ϕ *relative to* ψ, i.e. *there exists a sequence* (t_n) *of points of* $]t, b]$ *decreasing to the limit* t *such that* $(\phi(t_n) - \phi(t))/(\psi(t_n) - \psi(t)) \to \Phi_{\psi}(t)$ *in* Y. *Then there exists a set of positive* μ*-measure of points* $\xi \in [a, b[$ *such that*

$$p(d) \leqslant p(\Phi_{\psi}(\xi)) , \tag{6.1}$$

where $d = (\phi(b) - \phi(a))/(\psi(b) - \psi(a))$. *Moreover, either* (6.1) *holds with strict inequality in some set of positive* μ*-measure, or equality holds in* (6.1) μ*-almost everywhere, and in the latter case* ϕ, ψ *satisfy* (1.5) *for all* $t \in [a, b]$.

If in addition $p(\Phi_{\psi}(t)) \leqslant M$ μ*-almost everywhere, then* $p(d) \leqslant M$, *with equality if and only if* $p\{(\phi(t) - \phi(a))/(\psi(t) - \psi(a))\} = M$ *for all* $t \in [a, b]$.

7. We remark in conclusion that a number of authors (Ważewski [16, 17, 18], Mlak [13], McLeod [12], Averbukh and Smolyanov [1], and Frölicher and Bucher [9]) have proved more geometrical versions of the mean value theorem involving closed convex sets. The most complete results are those of McLeod, who proves that if Y is a locally convex topological space, $\phi\colon [a, b] \to Y$ is a continuous function that has nearly everywhere a right-hand derivative value $\Phi(t)$, and $\Phi(t)$ belongs to a closed convex set $E \subset Y$ for almost all $t \in [a, b]$, then

** See, for example, Halmos [11, p. 67].

(i) $q = (\phi(b) - \phi(a))/(b - a) \in E$,

(ii) either the interior E° of E is empty, or $q \in E^\circ$, or there exists a proper closed subspace X of Y such that $\phi(t) - \phi(a) - (t - a)q \in X$ for all $t \in [a, b]$, and that $\Phi(t) \in q + X$ whenever $\Phi(t)$ exists.

The first part of this result implies trivially that q belongs to the closed convex hull of the set of values of Φ. The second part leads to McLeod's striking theorem that if Y is finite-dimensional, then q belongs to the convex hull of the set of values of Φ.

The other authors mentioned confine their attention to a result of type (i). Averbukh and Smolyanov are concerned with ordinary derivatives, while Ważewski and Mlak use derivative values relative to a function ψ, as in Theorem 7.

The increment inequality of Theorem 7 gives a comprehensive result that includes most of the results mentioned; we have only to replace $\Phi(t)$ in McLeod's result by $\Phi_\psi(t)$, replace 'almost all' by 'μ-almost all', and write $\psi(b) - \psi(a)$ and $\psi(t) - \psi(a)$ instead of $b - a$ and $t - a$. The proof uses the Hahn-Banach theorem, and is essentially the same as that of McLeod.

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