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MEASURE PRESERVING CONTINUOUS STRAIGHTENING OF FRACTIONAL DIMENSIONAL SETS

Abstract

An s-set in Euclidean space is a set of finite, non-zero, Hausdorff s-dimensional measure. Call an s-set *straight* if its s-measure agrees with its Method I s-outer measure. Examples are given where there is a continuous, one-to-one function f on R^n which is measure preserving on E so that $f(E)$ is straight (such an f will be called a *straightening* of E). It is shown that any s-set can be written as a countable union of sets for which there are straightenings.

The purpose of this paper is to show how to apply a continuous, one-to-one function from Euclidean n space onto itself in such a way as to take a subset E of an s-set onto one with nicer measure properties. First we will need some standard definitions for generating Caratheodory measures: (See e.g., [4] where methods I and II originated for details.)

Method I. Start with a sequential covering class C of a set X; that is, C is a collection of subsets of X with $\phi \in C$ and a sequence of sets $X_i \in C$ so that $X \subset \cup X_i$. Let τ be a non-negative function, possibly infinite, defined on C with $\tau(\phi) = 0$. Then τ generates an outer measure m_I^* on each subset E of X given by

$$m_I^*(E) = \inf\{\sum \tau(E_i) : E \subset \cup E_i \text{ with } E_i \in C\}.$$

A subset E of X is *measurable* if for each set $A \subset X$, $m_I^*(A) = m_I^*(A \cap E) + m_I^*(A \backslash E)$.

Method II. For each $\delta > 0$, suppose C_δ is a sequential covering class of X with $C_\delta \subset C_{\delta'}$ when $\delta < \delta'$. Let τ be a non-negative function defined on

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$C = \cup C_\delta$. Using τ and method I, for each $\delta > 0$, this generates m^*_δ. Then $m^*(E) = \lim_{\delta\to 0} m^*_\delta(E)$ is an outer measure on the set of subsets of X.

In general, when method I and method II generate the same outer measure on E, we will say that the set E is *straight*. A one-to-one map h from X onto X will be called a *straighening* of E if $h(E)$ is straight. Perhaps, the word *flat* would be a better term for sets of dimension larger than 1.

We will be interested here in s-dimensional measure in Euclidean n-space. But, if C_δ is the set of subsets of a metric space each of whose diameter is less than or equal to δ and $\tau(E) = (diam E)^s$, the resulting outer measure is denoted by s-m^*_δ and the outer measure obtained using method II (Hausdorff s-dimensional measure) will be denoted by s-m^*. When a set is measurable, one uses s-m to denote the measure (equal to the outer measure). The outer measure which is obtained with no restriction on δ will be denoted by s-m^*_I. Clearly, for each set E, s-$m^*_I(E) \le s$-$m^*(E)$. (Note that sets of finite, non-zero s-m^*_δ outer measure are in general not s-m^* measurable.) A set is called an s-set if it is measurable with respect to s-m and has finite, non-zero s-measure.

Examples:

1) For Lebesgue measure in n dimensional Euclidean space, C is the collection of 'rectangles' $S = (a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n)$ with $\tau(S) = \Pi(b_i - a_i)$. All subsets are straight because both methods I and II yield Lebesgue measure.

2) For any Caratheodory measure, all sets of measure 0 are straight.

3) The Cantor ternary set with $s = \log 2/\log 3$ is smooth. Indeed, Falconer (See [2] p.16) proves a result which he attributes to R.O. Davies: On the line, if $E_0 = [0, 1]$, $E = \cap E_n$, $E_{n+1} \subset E_n$ and each set E_n is made up of a finite collection of closed intervals I so that $I \cap E_{n+1}$ consists of m intervals of equal length (m may vary with I) $J_1, ..., J_m$ equispaced in I so that $diam(\cup J_i) = diam(I)$ and $\sum(diam(J_i))^s = (diam(I))^s$, then s-$m(E \cap J) \le |J|^s$. That such sets are straight follows from:

Theorem 1. *A necessary and sufficient condition for a set E to be straight with respect to s-measure is that for each set A, (alternately, for each closed convex set A)* s-$m^*(E \cap A) \le (diam(E \cap A))^s$.

Proof. Note that in order to compute the outer measure of E it suffices to cover E with closed convex sets. Suppose the condition holds; that is, suppose that for each set A, s-$m^*(E \cap A) \le (diam(E \cap A))^s$. Then

$$\begin{aligned} s\text{-}m^*(E) &= \lim_{\delta\to 0} \inf\{\textstyle\sum(diam(E_i))^s : E = \cup E_i \ , diam(E_i) < \delta\} \\ &\ge \inf\{\textstyle\sum(diam(E_i))^s : E = \cup E_i\} \\ &\ge \inf\{\textstyle\sum s\text{-}m^*(E_i) : E = \cup E_i\} = s\text{-}m^*(E). \end{aligned}$$

Since $\inf\{\sum(diam(E_i))^s : E = \cup E_i\} = s$-$m^*(E)$, the equality s-$m^*(E) = s$-$m^*_I(E)$ follows and E is smooth. For the converse, suppose s-$m^*(E) = s$-

$m_I^*(E)$. If there were a set A so that $s\text{-}m^*(E \cap A) > (diam(E \cap A))^s$, then A could be chosen to be closed and convex so that, since A is then measurable with respect to $s\text{-}m^*$,

$$s\text{-}m^*(E) = s\text{-}m^*(E \cap A) + s\text{-}m^*(E \backslash A)$$
$$> diam(E \cap A)^s + s\text{-}m^*(E \backslash A) \geq s\text{-}m_I^*(E) \ ,$$

a contradiction to the fact that the two measures are equal.

Note: Subsets of straight sets are straight. Given a set E and real number k, let $k \cdot E = \{k \cdot x : x \in E\}$. Then with respect to a given s-measure, $k \cdot E$ is straight if E is. In general, most sets are not straight. For example, any curve other than a line segment is not straight with respect to 1-measure. Because of theorem 1, the upper s-density of each straight s-set is less than or equal to 1 at each of its points. It is well known that the upper density of s-sets is less than or equal to 1 at almost every point with respect to s-measure (see e.g., [2] p.24). The original paper establishing density properties is [3].

Examples:

4) If $F(x)$ is absolutely continuous on $[a, b]$, the graph of F can be straightened with respect to 1-measure; indeed, the map taking each point in the plane of the form $(x, F(x) + y)$ to the corresponding point $(\int_a^x \sqrt{1 + F'(t)^2}dt,\ y)$ is a smoothing of the graph.

5) If $F(x, y)$ is continuously differentiable,

$$(x, y, z + F(x, y)) \to (x, \int_0^y (1 + F_x^2(x, v) + F_y^2(x, v))^{1/2} dv \ ,\ z)$$

is a straightening of the graph of F. Here, the graph is 'rolled out' onto the (x, y) plane in one direction. It is easy to see that this is a straightening by comparing the area measure of the graph over rectangles to the area of the image and noting that the two are the same.

6) The shape of a letter T cannot be straightened with respect to 1-measure. (However, it is the union of two sets which are straight.)

7) In R^2 , let $S = \{(x, y) : 0 \leq y \leq f(x) \text{ with } x \in [0, 1]\}$ where $f(x) = 2^{-2n+1}$ if $x = m \cdot 2^{-n}$; $f(x) = 0$, otherwise. Then S has length 2 and S is not a finite union of sets which can be straightened. (It is a countable union of straight sets, obviously.)

The theorem that follows gives an affirmative answer to the following question: Is each s-set $E \subset R^n$ the countable union of sets which can be straightened? (with continuous, one-to-one maps of R^n onto R^n which are measure preserving on each E_i where $E = \cup E_i$) The general proof will only be sketched. An affirmative answer to an interesting question would make the results somewhat moot.

Problem 1: Is every s-set the countable union of straight sets? Alternatively, if E is an s-set, is $E = \cup F_i \cup Z$ where $s\text{-}m^*(Z) = 0$ and each F_i is a compact, straight set?

Conjecture: Yes. Motivation: One can observe that the circumference of the circle is not straight with respect to 1-measure because for each point on the circumference and each ball of radius r about the point, the measure (length of the circumference) inside the ball is larger than the diameter of the ball. However, consider a quarter circle of radius 1. By removing an arc of length 1/2 from the middle of this set, then removing two arcs of length 1/8 from the middle of the remaining arcs and, in general, removing 2^n arcs of length $4^{-n}/2$ from each of the remaining arcs at the *nth* stage, the perfect set which remains can be shown to be straight; the remaining set has positive 1-measure equal to $\pi/2 - 1$. Thus the circumference contains a straight set of positive 1-measure. By taking an appropriate countable collection of rotations of this set along with a set of length 0, the circumference can be seen to be a countable union of straight sets.

Perhaps, by 'punching holes' in an s-set, one can always leave behind a straight subset of positive measure. Then continuing inductively through the ordinal numbers one would obtain an at most countable collection of straight sets of positive measure; the set of measure 0 that remains is necessarily straight and it would follow that the set is a countable union of straight sets.

Every s-set contains a set of positive s-m measure which can be straightened. We first prove that straightening can occur on the line; there a slightly more general result is easily obtained. For this purpose and in what follows, for each $s \in (0,1)$, we will need the following subset of $[0,1]$: let P_s be the symmetric perfect set of dimension s which for each natural number n is contained in 2^n intervals of size $2^{-n/s}$. (This is constructed by the method given in Example 3) above using $m = 2$; Davies result implies that P_s is straight.)

Theorem 2. *Let F be a compact s-set which is a subset of the real numbers. Then if $s\text{-}m(c \cdot P_s) > s\text{-}m(F)$, there is a continuous, increasing function on the real line which is measure preserving on F and takes a perfect subset of F having the same measure as F to a subset of $c \cdot P_s$.*

Proof. It suffices to consider $F \subset [0,1]$. Remove from F all open intervals I for which $s\text{-}m(F \cap I) = 0$. The remaining set P is perfect and has the same s-measure as F. In each interval contiguous to P select a point x_n and define a measure m on $E = P \cup \cup_n \{x_n\}$ so that $m(P \cap I) = s\text{-}m(P \cap I)$ and $m(\{x_n\}) = a_n$ where a_n are chosen so that $\sum a_n = s\text{-}m(c \cdot P_s) - s\text{-}m(P)$. On E, let $f(x) = y$ if $m(E \cap [0,x)) = s\text{-}m(P_s \cap [0,y))$. Then f is increasing on P and contiguous intervals of P correspond to jumps of $f|_P$. Thus when f is extended linearly on the intervals contiguous to P, the resulting function is continuous, one-to-one and measure preserving and takes $F \backslash P$ onto a set of s-measure 0 and P onto a subset of $c \cdot P_s$ so that it is a straightening of F . This completes the proof.

We now turn to consider subsets of k-dimensional space. In R^k suppose $s < k$ and let $Q_s = P_{s/k} \times P_{s/k} \times ... \times P_{s/k}$. Then Q_s is an s-set. It will then be shown that if E is an s-set, there is a compact subset $F \subset E$ with $s\text{-}m^*(F) > 0$ and a continuous, one-to-one map h from R^k onto R^k so that h is measure preserving on F and $h(F) = c \cdot Q_s$ where c is chosen so that $s\text{-}m^*(F) = s\text{-}m^*(c \cdot Q_s)$.

That Q_s is straight is due to an application of theorems which appear in the literature. First of all, Q_s is easily seen to be a self-similar set according to the definition of Falconer given in [2] p.119. It is also shown there that such sets are s-sets where s is the similarity dimension. And s is easily seen to be the similarity dimension of Q_s. Then proposition 3 in [1], asserts that if α is the similarity dimension of a self-similar set A (here, a slightly more general definition of self-similarity is used), then A satisfies $\alpha\text{-}m^*(B) = \alpha\text{-}m^*_I(B)$ for each α-measurable subset B of A. It follows that all self-similar sets as defined in [2] are straight s-sets.

Theorem 3. *Let E be a measurable set of σ-finite s-measure which is a subset of R^k. Then $E = \cup E_i \cup Z$ where there are straightenings f_i of E_i and Z is a set of* s-measure 0.

Proof. Without loss of generality, we may suppose that E is a compact set of finite positive s-measure. We may also suppose that each intersection of E with any hyperplane in R^k is of s-measure 0. For otherwise, by induction, subsets can be founded in the intersection of E with a hyperplane for which there are straightenings. We may also suppose that every open set which intersects E meets it in a set of positive s-measure (otherwise such intersections can be removed from E and adjoined to Z). Given any number p between 0 and 1, it will then be possible to construct a compact subset F of E with $s\text{-}m(F) = p \cdot s\text{-}m(E)$ and a straightening h of F which takes F onto a set of the form $c \cdot Q_s$ where c is chosen to make $s\text{-}m(c \cdot Q_s) = p \cdot s\text{-}m(E) = a$.

The construction will be sketched for a compact subset of R^2 . Fix $p \in (0, 1)$ and let $a = p \cdot s\text{-}m(E)$. Determine x_0 and x_1 so that

$$E_0 = E \cap \{(x, y) : x \le x_0\} \text{ and } E_1 = E \cap \{(x, y) : x \ge x_1\}$$

are disjoint and $s\text{-}m(E_i) = a/2 + \delta_1$ with $\delta_1 < 1/2$. Determine $y_{00}, y_{01}, y_{10}, y_{11}$ so that

$$E_{i_1,0} = E_{i_1} \cap \{(x, y) : y \le y_{i_1,0}\} \text{ and } E_{i_1,1} = E_{i_1} \cap \{(x, y) : y \ge y_{i_1,1}\}$$

are disjoint and of s-measure equal to $a/4 + \delta_2$ with $\delta_2 < 1/4$. Enclose the sets E_{i_1,i_2} in four rectangles $R^{'}_{i_1,i_2}$ so that the sets E_{i_1,i_2} have points on each edge of each corresponding rectangle. These rectangles can in turn be enclosed in four slightly larger, pairwise disjoint rectangles R_{i_1,i_2} . Then $c \cdot Q_s = c \cdot (P_{s/2} \times P_{s/2})$ is contained in four squares S'_{i_1,i_2} of side length $2^{-4/s} \cdot c$. These squares are contained in four slightly larger squares S_{i_1,i_2} which are also pairwise disjoint. A map h_1 is then defined so that it takes the complement of the four slightly larger rectangles onto the complement of the four slightly larger squares, takes the smaller rectangles onto the smaller squares, takes sides of rectangles to the corresponding sides of squares, and takes the remaining points onto the remaining ones in a continuous one-to-one manner.

Continuing, each of the sets E_{i_1,i_2} is divided into four sets which are each enclosed in four rectangles and four slightly larger rectangles each contained in a rectangle R'_{i_1,i_2} . Also $c \cdot Q_s$ is contained in sixteen squares which are contained in sixteen slightly larger pairwise disjoint squares. Then h_2 can be defined as above with the restriction that it agrees with h_1 on the complement of the union of the rectangles R_{i_1,i_2} from the previous stage. If this is continued, it results in the desired compact set F and the sequence of functions $\{h_n\}$ which converge uniformly to a one-to-one continuous function h defined on R^2 which is measure preserving on F. That is, h is a straightening of F, as required.

We conclude with some additional problems.

Problems.

2. Under what conditions are cross products of straight sets straight (with appropriate dimension)?

3. When can one 'roll out' a set into a straight set? (By 'roll out' is meant find a straightening h in which at least one of the coordinates of R^k is fixed.)

4. Given a totally disconnected compact subset A of R^k, is there a continuous measure preserving straightening of A ?

5. Do such results hold in an abstract setting? E.g., if τ is non-decreasing and, say $\limsup m^*(E_n)/\tau(E_n) = 1$ at m^* a.e. $x \in X$ where the lim sup is over all sequences with $\tau(E_n) \to 0$ and $x \in E_n$, then is every method II measurable set a countable union of straight sets?

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