SIAM J. CONTROL OPTIM.
Vol. 52, No. 5, pp. 3078–3108

MEASURE VALUED DIRECTIONAL SPARSITY FOR PARABOLIC OPTIMAL CONTROL PROBLEMS*

KARL KUNISCH†, KONSTANTIN PIEPER‡, AND BORIS VEXLER‡

Abstract. A directional sparsity framework allowing for measure valued controls in the spatial direction is proposed for parabolic optimal control problems. It allows for controls which are localized in space, where the spatial support is independent of time. Well-posedness of the optimal control problems is established and the optimality system is derived. It is used to establish structural properties of the minimizer. An a priori error analysis for finite element discretization is obtained, and numerical results illustrate the effects of the proposed cost functional and the convergence results.

Key words. optimal control, sparsity, finite elements, error estimates

AMS subject classifications. 35K10, 35Q93, 49K20, 49M25, 65M60, 65M15

DOI. 10.1137/140959055

1. Introduction.

In this paper we analyze an optimal control problem, where we want to minimize

$$\tag{1.1a} \frac{1}{2}\|y - y_d\|^2_{L^2(I\times\Omega_o)} + \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}$$

for a control $u \in \mathcal{M}(\Omega_c, L^2(I))$ in the space of vector measures and a corresponding state y subject to the parabolic equation

$$\tag{1.1b} \begin{aligned} \partial_t y + Ay &= u \quad \text{in } I\times\Omega,\\ y(0) &= y_0. \end{aligned}$$

Here $\Omega \subset \mathbb{R}^d$ for $d \in \{2,3\}$ is a bounded domain, $A = -\nabla\cdot a\nabla$ is an elliptic second order differential operator (to be specified more precisely in section 2.2), $I = (0,T)$ is the time interval, and Ω_c and Ω_o are a control and observation domain, respectively. The cost term (1.1a) involves a standard quadratic tracking of the state variable and the total variation of the vector measure u. The resulting minimization problem (1.1) is convex but not necessarily strictly convex. Our motivation is the following: There are many applications in optimal control and inverse problems where the control enters into the right-hand side of a diffusion equation in a pointwise fashion, see, e.g., [24, 12]. A simple special case involves a right-hand side given by a linear combination of Dirac delta functions, resulting in the equation

$$\tag{1.2} \partial_t y + Ay = \sum_{i=1}^{N} u_i\,\delta_{x_i} \quad \text{in } I\times\Omega$$

*Received by the editors February 28, 2014; accepted for publication (in revised form) July 16, 2014; published electronically October 2, 2014.
http://www.siam.org/journals/sicon/52-5/95905.html

†Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, A-8010 Graz, Austria, and Radon Institute, Austrian Academy of Sciences, 1010 Wien, Austria (karl.kunisch@uni-graz.at).

‡Chair of Optimal Control, Technische Universität München, Boltzmannstraße 3, 85748 Garching bei München, Germany (pieper@ma.tum.de, vexler@ma.tum.de). The second author gratefully acknowledges support from the International Research Training Group IGDK1754, funded by the German Science Foundation (DFG).

with time-dependent intensities $u_i \in L^2(I)$ at fixed positions $x_i \in \Omega_c$. In some problem settings we are additionally interested in finding the optimal locations x_i, which are therefore also subject to optimization. The problem (1.1) is in a sense a generalization of this approach, since any sum $u = \sum_{i=1\dots N} u_i \, \delta_{x_i}$ is an element of the space $\mathcal{M}(\Omega_c, L^2(I))$. Moreover, with the polar decomposition for vector valued measures (see section 2.1), all controls $u \in \mathcal{M}(\Omega_c, L^2(I))$ of the generalized problem are separable into a spatial profile $|u|$ and a space dependent temporal profile u' such that

$$u = u' \, |u|,$$

where $|u| \in \mathcal{M}(\Omega_c)$ is a positive Radon measure and $u'(t,x)$ depends on $t \in I$ and x in the support of $|u|$. An equivalent formulation of (1.1) is therefore to minimize

$$\tfrac{1}{2}\|y - y_d\|^2_{L^2(I\times\Omega_o)} + \alpha|u|(\Omega_c),$$

where the cost term $|u|(\Omega_c) = \||u|\|_{\mathcal{M}(\Omega_c)}$ is the total variation norm for positive measures, subject to the parabolic equation

$$\begin{aligned} \partial_t y + Ay = u'|u| &\quad \text{in } I \times \Omega, \\ y(0) = y_0 &\quad \text{in } \Omega, \end{aligned}$$

and the algebraic constraint

$$\|u'(\cdot, x)\|_{L^2(I)} = 1 \quad \text{for all } x \in \Omega_c.$$

It will turn out that the optimal solutions are sparsely localized in space; see Theorem 2.12. In an important special case, which typically occurs when $\operatorname{dist}(\Omega_o, \Omega_c) > 0$, the spatial profile $|u|$ will consist of a finite sum of Dirac delta functions as described in Corollary 2.13. We also give empirical evidence for this in section 5. The temporal profile u' is continuous in x and will in most cases exhibit additional regularity; see Theorem 3.7.

Directional sparsity in the context of optimal control with PDEs was first proposed in [21], where an additional L^2 regularization term and control constraints allow us to search for a control in the space $L^2(I \times \Omega)$. We will investigate the connection to this problem in section 4. Furthermore, our setting is similar to [5], where an inverse problem is considered involving the norm of $\mathcal{M}(\Omega, \mathbb{R}^n)$ as a regularization term in combination with a general solution operator instead of the solution operator of the parabolic equation (1.1b) considered here. With respect to finite element discretization of optimal control problems governed by parabolic equations, we also refer to [8] on a different sparse control problem and to [15, 23] for a priori error analysis of finite element discretization of pointwise parabolic control problems, i.e., optimal control problems with equations of the form (1.2) with fixed positions x_i. Let us additionally point out [9], where a control problem for the heat equation with measures on a subset of the parabolic cylinder is discussed. In the one-dimensional situation the authors are able to show that the minimizer is given by a finite sum of point sources.

The contribution of this paper is threefold. First of all, we introduce a (well-posed) formulation of an optimal control problem, such that the optimal solution is a measure with respect to the spatial variable and the sparsity pattern does not depend on time. Second, we derive optimality conditions for the optimal control problem under consideration and exploit these conditions to provide additional information (sparsity structure and regularity) about the optimal solution. Moreover, we provide numerical analysis for an appropriate finite element discretization of the problem. We derive an a priori error estimate for the error between the optimal states of the continuous and discretized problems of order $\mathcal{O}(k^{\frac{1}{2}} + h)$ up to a logarithmic factor, where k and h are temporal and spatial discretization parameters. This estimate seems to be optimal at least with respect to h; cf. the discussion in section 5.

The outline of the paper is as follows. In section 2 we summarize the theory for vector measures and describe an appropriate function space setting for (1.1b). We discuss regularity of the parabolic solution for the specific right-hand side and give an optimality condition for the convex minimization problem in Theorem 2.11. In Theorem 2.12 we derive the specific form of the sparsity structure. Section 3 describes a suitable discretization concept for the optimal control problem, where we use finite elements in space and a discontinuous Galerkin method in time. We prove a corresponding error estimate for the solution of the parabolic problem in Theorem 3.15 and for the optimal solutions in Theorem 3.20. Section 4 discusses a practical solution method and section 5 reports on some numerical results. The first numerical example is tailored to illustrate that the convergence results can be achieved in practice and the second example, motivated by an inverse source location problem, gives evidence of the claim that the optimal solutions will be point sources as in (1.2) under appropriate conditions on Ω_o and Ω_c.

2. Analysis.

In this section we discuss a functional analytic framework for (1.1).

2.1. Vector measures.

Let Ω be an open bounded domain in $\mathbb{R}^d$ and $I = (0, T)$. Furthermore the boundary of $\partial\Omega$ is subdivided into a Neumann part $\Gamma \subset \partial\Omega$ and a Dirichlet part $\partial\Omega\setminus\Gamma$, which we require to be closed. In section 2.2 we will impose further restrictions on the regularity of the boundary for the parabolic regularity theory. The control domain is allowed to act in the interior and on the boundary, where

$$\Omega_c \subseteq \Omega \cup \Gamma$$

is any relatively closed subset of $\Omega\cup\Gamma$. The space $\mathcal{M}(\Omega_c, L^2(I))$ contains all countably additive vector measures of bounded total variation $\mu\colon \mathcal{B}(\Omega_c) \to L^2(I)$ on the Borel sets $\mathcal{B}(\Omega_c)$. For $\mu \in \mathcal{M}(\Omega_c, L^2(I))$ the total variation measure $|\mu| \in \mathcal{M}(\Omega_c)$, the space of positive Borel measures, is defined as

$$|\mu|(B) = \sup \left\{ \sum_{n=1}^{\infty} \|\mu(B_n)\|_{L^2(I)} \;\middle|\; B_n \in \mathcal{B}(\Omega_c) \text{ disjoint partition of } B \right\}$$

for each $B \in \mathcal{B}(\Omega_c)$ and by $|\mu|(\Omega_c)$ we denote the total variation of μ. It is easy to see that we have

$$\|\mu(B)\|_{L^2(I)} \leq |\mu|(B) \tag{2.1}$$

for all $B \in \mathcal{B}(\Omega_c)$. The space of vector measures $\mathcal{M}(\Omega_c, L^2(I))$ with finite total variation endowed with the norm $\|\mu\|_{\mathcal{M}(\Omega_c, L^2(I))} = \||\mu|\|_{\mathcal{M}(\Omega_c)} = |\mu|(\Omega_c)$ is a Banach space; see, e.g., [22, Chapter 12.3]. Since $\Omega_c \subseteq \Omega\cup\Gamma$, we have the canonical embedding

$$\mathcal{M}(\Omega_c, L^2(I)) \hookrightarrow \mathcal{M}(\Omega \cup \Gamma, L^2(I)).$$

Finite Borel measures on subsets of $\mathbb{R}^d$ are regular (and therefore Radon measures; cf. [25, Theorem 1.10, Corollary 1.11]) and the support of the vector measure $\mu \in \mathcal{M}(\Omega \cup \Gamma, L^2(I))$ defined as

$$\operatorname{supp} \mu = \operatorname{supp}|\mu| = (\Omega \cup \Gamma) \setminus \Big(\bigcup \{ B \text{ open in } \Omega \cup \Gamma \mid |\mu|(B) = 0 \} \Big)$$

is a relatively closed set (and therefore $\operatorname{supp} \mu \in \mathcal{B}(\Omega \cup \Gamma)$). We have the equality

$$\mathcal{M}(\Omega_c, L^2(I)) = \{ \mu \in \mathcal{M}(\Omega \cup \Gamma, L^2(I)) \bigm| \operatorname{supp} \mu \subseteq \Omega_c \}.$$

For each $\mu \in \mathcal{M}(\Omega_c, L^2(I))$ we can define a "polar decomposition," which consists of the total variation measure $|\mu|$ and the function $\mu' \in L^1(\Omega_c, |\mu|, L^2(I))$ such that

$$\mathrm{d}\mu = \mu' \,\mathrm{d}|\mu|, \tag{2.2}$$

which is a short form of $\int \varphi \,\mathrm{d}\mu = \int \varphi \mu' \,\mathrm{d}|\mu|$ in $L^2(I)$ for all $\varphi \in \mathcal{C}_0(\Omega_c)$. The function μ' is the Radon–Nikodym derivative of μ with respect to $|\mu|$; see [22, Corollary 12.4.2] or [10, Corollary IV.1.4]. Certainly, μ is absolutely continuous with respect to $|\mu|$ due to (2.1). In fact we even have $\mu' \in L^\infty(\Omega_c, |\mu|, L^2(I))$ with

$$\begin{aligned} &\|\mu'\|_{L^\infty(\Omega_c, |\mu|, L^2(I))} \le 1 \\ &\quad \text{and } \|\mu'(x)\|_{L^2(I)} = 1 \text{ for } |\mu|\text{-almost all } x \in \Omega_c. \end{aligned} \tag{2.3}$$

The first property is a consequence of

$$\left\| \int_B \mu' \,\mathrm{d}|\mu| \right\|_{L^2(I)} = \left\| \int_B \mathrm{d}\mu \right\|_{L^2(I)} = \|\mu(B)\|_{L^2(I)} \le |\mu|(B),$$

which implies that the $|\mu|$-average of μ' lies in the unit ball of $L^2(I)$. By the averaging lemma [22, Theorem 11.5.15], this implies $\|\mu'(x)\|_{L^2(I)} \le 1$ $|\mu|$-almost everywhere. The second property is implicitly contained in [22, Theorem 12.4.1].

Let $\mathcal{C}_c(\Omega \cup \Gamma, L^2(I))$ be the space of continuous functions on $\Omega \cup \Gamma$ with values in $L^2(I)$ which are compactly supported in $\Omega \cup \Gamma$ and let $\mathcal{C}_c(\Omega_c, L^2(I))$ be the subset consisting of canonical restrictions of such functions to Ω_c. Then we define

$$\mathcal{C}_0(\Omega_c, L^2(I)) = \overline{\mathcal{C}_c(\Omega_c, L^2(I))},$$

where the closure is with respect to the supremum norm. We recall that this is equivalent to

$$\mathcal{C}_0(\Omega_c, L^2(I)) = \{ \varphi \in \mathcal{C}(\overline{\Omega_c}, L^2(I)) \bigm| \varphi(x) = 0 \text{ for } x \in \partial\Omega \setminus \Gamma \}.$$

With the pairing, defined for $\mu \in \mathcal{M}(\Omega_c, L^2(I))$ and $v \in \mathcal{C}_0(\Omega_c, L^2(I))$,

$$\langle \mu, v \rangle = \int_\Omega (\mu'(x), v(x))_{L^2(I)} \,\mathrm{d}|\mu|(x)$$

we have a natural injection into the dual space $\mathcal{M}(\Omega_c, L^2(I)) \hookrightarrow \mathcal{C}_0(\Omega_c, L^2(I))^*$. In fact this is an isometric isomorphism. This identification (for a more general setting) is known as Singer's representation theorem; see, e.g., [27] and the references therein. In the following we will identify $\mathcal{M}(\Omega_c, L^2(I))$ with the dual space of $\mathcal{C}_0(\Omega_c, L^2(I))$.

Up to now we have always considered μ as an object depending on the spatial variable $x \in \Omega_c$. Now we want to switch the point of view to consider μ as a variable of $t \in I$. We recall from [8] and the references therein the definition of the Bochner space $L^2(I, \mathcal{M}(\Omega_c))$ (of weakly measurable $\mathcal{M}(\Omega_c)$-valued functions which are square integrable in time) and the identification of the dual $L^2(I, \mathcal{C}_0(\Omega_c))^* = L^2(I, \mathcal{M}(\Omega_c))$. We note that

$$\mathcal{M}(\Omega_c, L^2(I)) \hookrightarrow L^2(I, \mathcal{M}(\Omega_c)), \tag{2.4}$$

which follows from the dense embedding $L^2(I, \mathcal{C}_0(\Omega_c)) \hookrightarrow \mathcal{C}_0(\Omega_c, L^2(I))$. Therefore, for each $\mu \in \mathcal{M}(\Omega_c, L^2(I))$ the expression $\mu(t) \in \mathcal{M}(\Omega_c)$ for $t \in I$ is well-defined in an almost everywhere in I sense. Conversely, due to (2.3) the polar decomposition μ' is an element of the Hilbert space $L^2(\Omega_c, |\mu|, L^2(I))$ which is isometrically isomorphic to $L^2(I, L^2(\Omega_c, |\mu|))$. This explains the expression $\mu'(t) \in L^2(\Omega_c, |\mu|)$ in an almost everywhere in I sense. We can now check that

$$\mathrm{d}\mu(t) = \mu'(t)\,\mathrm{d}|\mu| \quad \text{for almost all } t \in I \tag{2.5}$$

holds independently of the equivalence representations chosen for the functions $\mu' \colon I \to L^2(\Omega_c, |\mu|)$ and $\mu \colon I \to \mathcal{M}(\Omega_c)$.

2.2. Well-posedness of the state equation. We define a suitable solution to (1.1b) with the well-known method of transposition. The construction of solutions to elliptic equations with measure valued data by this technique goes back to the classical paper of Stampacchia [30]. In the parabolic case the combination of this technique with a result on maximal parabolic regularity is fairly straightforward. Nevertheless, we give a derivation for the sake of completeness. We employ the notation $(\cdot\,,\,\cdot)$ for the inner product in $L^2(\Omega)$ and the notation $(\cdot\,,\,\cdot)_I$ for the inner product in $L^2(I, L^2(\Omega))$.

We suppose that Ω is a Lipschitz domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with boundary $\partial\Omega$ and that $\Gamma \subset \partial\Omega$ such that $\Omega \cup \Gamma$ is regular in the sense of Gröger; see [18, Definition 2], [19, Definition 3.1], or the alternative characterization [19, Theorems 5.2, 5.4]. In the following we consider elliptic operators of the form

$$A = -\nabla \cdot a\nabla,$$

given in weak formulation with bounded, symmetric, and uniformly elliptic coefficients $a \in L^\infty(\Omega, \mathbb{R}^{d\times d})$. For mixed Dirichlet and Neumann boundary conditions we define the space $W^{1,q}_\Gamma(\Omega)$ with $1 < q < \infty$ in the usual way as the closure of the set $\{u|_\Omega \mid u \in \mathcal{C}^\infty_c(\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\backslash\Gamma\}$ in $W^{1,q}(\Omega)$. We denote by

$$W^{-1,q}_\Gamma(\Omega) = W^{1,q'}_\Gamma(\Omega)^*, \text{ where } \tfrac{1}{q} + \tfrac{1}{q'} = 1,$$

the corresponding dual spaces. Since a is symmetric the formal adjoint A^* is identical to A. It is clear that A^* continuously maps the spaces $W^{1,q}_\Gamma(\Omega)$ into the spaces $W^{-1,q}_\Gamma(\Omega)$ for any q. We also define for $q \geq 2$ the space $D^*_q \supset W^{1,q}_\Gamma(\Omega)$ as the domain of A^*, endowed with the graph norm in $W^{-1,q}_\Gamma(\Omega)$,

$$D^*_q = \left\{\, v \in W^{1,2}_\Gamma(\Omega) \;\middle|\; A^*v \in W^{-1,q}_\Gamma(\Omega) \,\right\} \text{ and } \|v\|_{D^*_q} = \|A^*v + v\|_{W^{-1,q}_\Gamma(\Omega)}.$$

With an elliptic regularity result (see, e.g., [30, 11, 19] and the references therein), we obtain that these spaces embed into spaces of Hölder continuous functions for $q > d$.

THEOREM 2.1 (Theorem 3.3 in [19]). *Under the above conditions and for $q > d$ we have the continuous embedding*

$$D_q^* \hookrightarrow \mathcal{C}^\beta(\Omega)$$

for some $\beta > 0$.

Remark 2.2. It can be shown that there is a $\bar{q} = \bar{q}(\Omega, \Gamma, a) \in (2, \infty]$ such that $D_q^* = W_\Gamma^{1,q}(\Omega)$ holds for all $2 \le q < \bar{q}$; see [18]. For the following construction we will only need the embedding

$$D_q^* \hookrightarrow W_\Gamma^{1,2}(\Omega) \cap \mathcal{C}^\beta(\Omega) \hookrightarrow \mathcal{C}_0(\Omega \cup \Gamma),$$

which holds without further smoothness assumptions on Ω, Γ, or a.

We consider now the dual equation to (1.1b), which is the backwards in time parabolic equation

$$\begin{aligned} -\partial_t \varphi + A^* \varphi &= f \quad \text{in } L^2(I, W_\Gamma^{-1,s'}(\Omega)), \\ \varphi(T) &= 0 \end{aligned} \tag{2.6}$$

for a given right-hand side $f \in L^2(I, W_\Gamma^{-1,s'}(\Omega))$ with $s' > d$, where the time derivative is interpreted as the distributional derivative [2, Chapter III.1]. We can apply a result on maximal parabolic regularity from [20] to characterize the solutions of (2.6).

THEOREM 2.3 (Theorem 5.4 in [20]). *Suppose that $s' \ge 2$ with $s' < \infty$ in two dimensions and $s' < \frac{3\bar{q}}{3-\bar{q}}$ in three dimensions, where $\bar{q}$ is the constant from Remark* 2.2. *Then the solution to* (2.6) *lies in the space*

$$X^{s'} = \left\{ v \in L^2(I, D_{s'}^*) \cap H^1(I, W_\Gamma^{-1,s'}(\Omega)) \;\middle|\; v(T) = 0 \right\}$$

with the corresponding a priori estimate

$$\|\varphi\|_{X^{s'}} \le c_s \|f\|_{L^2(I, W_\Gamma^{-1,s'}(\Omega))}. \tag{2.7}$$

Remark 2.4. In particular we can choose $s' \le 6$ in three spatial dimensions.

We denote the corresponding solution operator by $\varphi = S^\star(f)$. With Theorem 2.3 it is an isomorphism on the spaces

$$S^\star \colon L^2(I, W_\Gamma^{-1,s'}(\Omega)) \to X^{s'}.$$

Moreover, since $D_{s'}^* \hookrightarrow \mathcal{C}^\beta(\Omega)$ for some $\beta > 0$, we additionally obtain the embedding

$$X^{s'} \hookrightarrow L^2(I, \mathcal{C}_0(\Omega \cup \Gamma)).$$

A very weak solution of the state equation (1.1b) can now be given in the following way: Consider dual exponents $s' \in (d, \frac{2d}{d-2}]$ and exponents s with $\frac{1}{s} + \frac{1}{s'} = 1$. For any control $u \in \mathcal{M}(\Omega_c, L^2(I))$ the state $y \in L^2(I, W_\Gamma^{1,s}(\Omega))$ is sought as the solution of the very weak formulation

$$\langle y, -\partial_t \varphi + A^* \varphi \rangle_I = (y_0, \varphi(0)) + \langle u, \chi_{\Omega_c} \varphi \rangle \quad \text{for all } \varphi \in X^{s'}. \tag{2.8}$$

Here, by $\langle \cdot\,, \cdot \rangle_I$ we denote the duality pairing between $L^2(I, W_\Gamma^{1,s}(\Omega))$ and its dual $L^2(I, W_\Gamma^{-1,s'}(\Omega))$ and χ_{Ω_c} is the canonical embedding

$$\chi_{\Omega_c} \colon X^{s'} \hookrightarrow \mathcal{C}_0(\Omega_c, L^2(I)),$$

which explains the duality product between u and $\chi_{\Omega_c}\varphi$. With the embedding $X^{s'} \hookrightarrow X^2 \hookrightarrow \mathcal{C}(I, L^2(\Omega))$ the point evaluation $\varphi(0)$ is well-defined and continuous in $X^{s'}$. Therefore for any $f \in L^2(I, W_\Gamma^{-1,s'}(\Omega))$ and the corresponding $\varphi = S^\star(f)$ we obtain from the definition (2.6) that

$$\langle y, f\rangle_I = (y_0, \varphi(0)) + \langle u, \chi_{\Omega_c}\varphi\rangle \leq c \left(\|u\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{L^2(\Omega)}\right) \|\varphi\|_{X^{s'}}.$$

By reflexivity of the space $L^2(I, W_\Gamma^{1,s}(\Omega))$ we now see that the very weak formulation has a unique solution and with (2.7) we obtain

$$\|y\|_{L^2(I, W_\Gamma^{1,s}(\Omega))} \leq c_s \left(\|u\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.9}$$

By choosing appropriate test functions in (2.8) we derive that $\partial_t y = -Ay + \chi_{\Omega_c}^* u$ holds in the distributional sense and we obtain $\partial_t y \in L^2(I, W_\Gamma^{-1,s}(\Omega))$. With this and the integration by parts formula it follows

$$\langle y(0), \varphi(0)\rangle = -\langle \partial_t y, \varphi\rangle_I - \langle y, \partial_t \varphi\rangle_I = (y_0, \varphi(0)) \leq \|y_0\|_{L^2(\Omega)} \|\varphi(0)\|_{L^2(\Omega)} \tag{2.10}$$

for all $\varphi \in H^1(I, W_\Gamma^{1,s'}(\Omega))$ with $\varphi(T) = 0$. We can choose φ with $\varphi(0) \in W_\Gamma^{1,s'}(\Omega)$ arbitrarily and conclude $y(0) = y_0$ by density. Finally, we obtain the following result.

THEOREM 2.5. *The state equation, given in the weak formulation*

$$\begin{aligned} \langle \partial_t y, \varphi\rangle_I + \langle Ay, \varphi\rangle_I &= \langle \chi_{\Omega_c}^* u, \varphi\rangle \quad \text{for all } \varphi \in L^2(I, D_{s'}^*), \\ y(0) &= y_0 \end{aligned} \tag{2.11}$$

with $s' > d$ possesses a unique solution y in the space

$$Y^s = L^2(I, W_\Gamma^{1,s}(\Omega)) \cap H^1(I, W_\Gamma^{-1,s}(\Omega)),$$

where $1 \leq s < \frac{d}{d-1}$, with the corresponding estimate

$$\|y\|_{Y^s} \leq c_s \left(\|u\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.12}$$

Proof. We take the unique solution $y \in Y^s$ for $s \in [\frac{2d}{d+2}, \frac{d}{d-1})$ of the very weak formulation (2.8), which fulfills (2.12) by (2.9) and the representation of the time derivative. The regularity for all $s < \frac{d}{d-1}$ is a consequence of the Sobolev embedding theorem. We argue that y is also a solution to the weak formulation (2.11) by applying integration by parts in (2.8) to obtain

$$\langle u, \chi_{\Omega_c}\varphi\rangle = \langle y, -\partial_t\varphi + A^*\varphi\rangle_I - (y_0, \varphi(0)) = \langle \partial_t y + Ay, \varphi\rangle_I$$

for all $\varphi \in X^{s'}$. Since $X^{s'}$ is dense in $L^2(I, D_{s'}^*)$ the solution y fulfills (2.11), which proves existence for (2.11). Conversely, uniqueness of the solution to the weak formulation (2.11) follows by $X^{s'} \subset L^2(I, D_{s'}^*)$. ∎

Remark 2.6. The weak formulation (2.11) holds also for test functions φ from the subspace $L^2(I, W_\Gamma^{1,s'}(\Omega))$. However, if we restrict the test space in this way, we lose uniqueness of the solution in the general case; cf. the discussion in [29, 11] for the elliptic problem.

We denote the corresponding solution operator for the state equation by $y = S(y_0, u) = S(u)$.

LEMMA 2.7. *The solution operator S is weak-$*$ to weak continuous, i.e., we have*

$$S(u_n) \rightharpoonup S(\hat{u}) \quad \text{in } Y^s$$

for any sequence $u_n \rightharpoonup^* \hat{u}$ in $\mathcal{M}(\Omega_c, L^2(I))$ with $n \to \infty$ (with $s < \frac{d}{d-1}$).

Proof. Since u_n is bounded in $\mathcal{M}(\Omega_c, L^2(I))$, the sequence $y_n = S(u_n)$ is bounded in Y^s with Theorem 2.5. Thus, it contains a weakly converging subsequence (denoted again by y_n) with $y_n \rightharpoonup \hat{y}$ for some $\hat{y} \in Y^s$. By taking the limit in (2.11), we see that $\hat{y} = S(\hat{u})$. The result follows since this argument can be repeated if we start from any subsequence of u_n. ∎

In the following we will implicitly restrict the range of the parameter s to $s \in [\frac{2d}{d+2}, \frac{d}{d-1})$ if we use the spaces Y^s and $X^{s'}$, unless explicitly mentioned otherwise.

2.3. Optimal control problem. With these preparations we can now state the precise problem formulation,

$$(\mathcal{P}) \qquad \min_{u \in \mathcal{M}(\Omega_c, L^2(I))} j(u) = J(S(y_0, u)) + \alpha \|u\|_{\mathcal{M}(\Omega_c, L^2(I))},$$

where j is the reduced cost functional, and

$$J(y) = \tfrac{1}{2} \|\chi_{\Omega_o} y - y_d\|^2_{L^2(I \times \Omega_o)}$$

is a quadratic tracking functional on the observation region $I \times \Omega_o$. We require Ω_o to be an open subset of Ω. By χ_{Ω_o} we denote the embedding

$$\chi_{\Omega_o} \colon Y^s \hookrightarrow L^2(I \times \Omega_o),$$

which is compact due to the compact embedding $W^{1,s}_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$ for $s > \frac{2d}{d+2}$ (with the Aubin–Lions lemma). In the following we sometimes omit χ_{Ω_o} for ease of notation.

LEMMA 2.8. *The functional j is weak-$*$ lower semicontinuous, i.e., we have*

$$\liminf_{n \to \infty} j(u_n) \geq j(\hat{u})$$

for any sequence $u_n \rightharpoonup^* \hat{u}$ for $n \to \infty$ in $\mathcal{M}(\Omega_c, L^2(I))$.

Proof. This is a consequence of Lemma 2.7, the compactness of the embedding χ_{Ω_o}, which yields strong convergence of $S(u_n) \to S(\bar{u})$ in $L^2(I \times \Omega_o)$, continuity of J, and the fact that the norm $\|\cdot\|_{\mathcal{M}(\Omega_c, L^2(I))}$ is weak-$*$ lower semicontinuous. ∎

PROPOSITION 2.9. *There exists an optimal solution $\bar{u} \in \mathcal{M}(\Omega_c, L^2(I))$ to $(\mathcal{P})$ with corresponding optimal state $\bar{y} = S(y_0, \bar{u}) \in Y^s$.*

Proof. We follow classical arguments: Since $J \colon L^2(I \times \Omega) \to \mathbb{R}$ is bounded from below we can construct a minimizing sequence u_n. Since the unit ball in $\mathcal{M}(\Omega_c, L^2(I))$ is weak-$*$ compact and

$$\|u_n\|_{\mathcal{M}(\Omega_c, L^2(I))} \leq \tfrac{1}{\alpha} j(u_n) \leq c,$$

we can extract a weak-$*$ convergent subsequence $u_n \rightharpoonup^* \bar{u}$ (existence of such a subsequence is guaranteed by the sequential version of the Banach–Alaoglu theorem since $\mathcal{C}_0(\Omega_c, L^2(I))$ is separable). Then we apply Lemma 2.8 to complete the proof. ∎

It is clear that the reduced cost functional is convex. Additionally, since the functional J is strictly convex on $L^2(I \times \Omega_o)$, we easily see that the observation $\chi_{\Omega_o}\bar{y}$ is unique. Therefore, we can show that each optimal control will have the same cost and lead to the same observed effect on the state variable. Similarly to the discussion in [28, Section 6], we distinguish two characteristic cases where the first is "full observation," which means $\Omega_c \subset \Omega_o$, and the second is "disjoint control and observation," i.e., $\operatorname{dist}(\Omega_o, \Omega_c) > 0$. Uniqueness can only be generically guaranteed in the former case.

Proposition 2.10. *If $\Omega_c \subset \Omega_o$ the optimal solution $\bar{u}$ to $(\mathcal{P})$ is unique.*

Proof. Under these conditions, the control to observation mapping $u \mapsto \chi_{\Omega_o} y$ is injective, and uniqueness of $\bar{u}$ follows from uniqueness of $\chi_{\Omega_o}\bar{y}$. ∎

2.4. Optimality system. We can characterize optimality of $\bar{u}$ in the following way.

Theorem 2.11. *There exists a unique adjoint state $\bar{p} \in X^{s'}$, which for any optimal solution $\bar{u}$ of $(\mathcal{P})$ and associated state $\bar{y} = S(y_0, \bar{u})$ fulfills the adjoint equation*

$$-\partial_t \bar{p} + A^* \bar{p} = \chi^*_{\Omega_o}(\bar{y} - y_d), \quad \bar{p}(T) = 0, \tag{2.13}$$

and the subgradient condition

$$-\langle u - \bar{u}, \chi_{\Omega_c}\bar{p}\rangle + \alpha \|\bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))} \leq \alpha \|u\|_{\mathcal{M}(\Omega_c, L^2(I))} \tag{2.14}$$

holds for all $u \in \mathcal{M}(\Omega_c, L^2(I))$. The subgradient condition implies

$$\|\bar{p}(x)\|_{L^2(I)} \leq \alpha \quad \text{for all } x \in \Omega_c. \tag{2.15}$$

Proof. The objective function j is the sum of a differentiable and a convex lower semicontinuous function. By the subdifferential calculus of convex optimization (see, e.g., [13, Section 5.3]), we obtain for any optimal solution

$$0 \in \partial j(\bar{u}) = \chi_{\Omega_c}\bar{p} + \alpha\, \partial\|\cdot\|_{\mathcal{M}(\Omega_c, L^2(I))}(\bar{u}) \text{ in } \mathcal{M}(\Omega_c, L^2(I))^*,$$

where $\bar{p} = S^\star(J'(S(y_0, \bar{u})))$ is the adjoint state and $\partial\|\cdot\|_{\mathcal{M}(\Omega_c, L^2(I))}(\bar{u})$ is the convex subdifferential of the norm at the optimal control. By construction $\bar{p}$ fulfills (2.13). Writing out the subgradient condition $-\chi_{\Omega_c}\bar{p} \in \alpha\, \partial\|\cdot\|_{\mathcal{M}(\Omega_c, L^2(I))}(\bar{u})$, we obtain (2.14). Choosing now $u = \bar{u} - v$ in (2.14), we derive

$$\langle v, \chi_{\Omega_c}\bar{p}\rangle \leq \alpha(\|\bar{u} - v\|_{\mathcal{M}(\Omega_c, L^2(I))} - \|\bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))}) \leq \alpha \|v\|_{\mathcal{M}(\Omega_c, L^2(I))}$$

for all $v \in \mathcal{M}(\Omega_c, L^2(I))$. Therefore, by the fact that $\|\varphi\|_{\mathcal{M}(\Omega_c, L^2(I))^*} = \|\varphi\|_{\mathcal{C}_0(\Omega_c, L^2(I))}$ for all $\varphi \in \mathcal{C}_0(\Omega_c, L^2(I))$ we derive

$$\|\chi_{\Omega_c}\bar{p}\|_{\mathcal{C}_0(\Omega_c, L^2(I))} = \sup_{v \in \mathcal{M}(\Omega_c, L^2(I))} \frac{\langle v, \chi_{\Omega_c}\bar{p}\rangle}{\|v\|_{\mathcal{M}(\Omega_c, L^2(I))}} \leq \alpha, \tag{2.16}$$

which proves (2.15). ∎

From the variational inequality (2.14) we can derive additional properties of the optimal control.

THEOREM 2.12. *Let $\bar{u}$ be an optimal solution to* $(\mathcal{P})$ *and* $\bar{p} = S^\star(J'(S(y_0, \bar{u})))$ *the corresponding adjoint state. Then we have*

$$\operatorname{supp}|\bar{u}| \subseteq \left\{ x \in \Omega_c \;\middle|\; \|\bar{p}(x)\|_{L^2(I)} = \alpha \right\}, \tag{2.17}$$

$$\bar{u}' = -\tfrac{1}{\alpha}\chi_{\Omega_c}\bar{p} \qquad \text{in } L^1(\Omega_c, |\bar{u}|, L^2(I)), \tag{2.18}$$

where $\mathrm{d}\bar{u} = \bar{u}'\,\mathrm{d}|\bar{u}|$ *is the polar decomposition.*

Proof. Choose $u = \bar{u} + \bar{u} = 2\bar{u}$ in (2.14). As before this results in

$$-\langle \bar{u}, \chi_{\Omega_c}\bar{p}\rangle \le \alpha\|\bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))}.$$

We obtain the reverse inequality by choosing $u = \bar{u} - \bar{u} = 0$, which implies

$$-\langle \bar{u}, \chi_{\Omega_c}\bar{p}\rangle = \alpha\|\bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))} = \int_{\Omega_c} \alpha\,\mathrm{d}|\bar{u}|.$$

Applying the polar decomposition and reordering we obtain

$$\int_{\Omega_c} \left(\alpha + (\bar{u}'(x), \bar{p}(x))_{L^2(I)}\right) \mathrm{d}|\bar{u}|(x) = 0. \tag{2.19}$$

With the Cauchy–Schwarz inequality and the conditions (2.3) and (2.16) we obtain

$$-(\bar{u}'(x), \bar{p}(x))_{L^2(I)} \le \|\bar{u}'(x)\|_{L^2(I)}\|\bar{p}(x)\|_{L^2(I)} \le \alpha \tag{2.20}$$

for $|\bar{u}|$-almost all $x \in \Omega_c$, which means that the integrand in (2.19) is nonnegative. Therefore it must be zero almost everywhere, i.e.,

$$-(\bar{u}'(x), \bar{p}(x))_{L^2(I)} = \alpha \quad \text{for } |\bar{u}|\text{-almost all } x \in \Omega_c.$$

Considering again (2.20), we see that equality can only hold if the conditions

$$\|\bar{p}(x)\|_{L^2(I)} = \alpha \quad \text{and} \quad \bar{p}(t,x) = -\alpha\bar{u}'(t,x)$$

hold for $|\bar{u}|$-almost all $x \in \Omega_c$ and almost every $t \in I$. This proves (2.18). From the first identity, we can derive (2.17) using basic measure theoretic arguments: Define the function $f\colon \Omega_c \to \mathbb{R}^+$, $f(x) = \alpha - \|\bar{p}(x)\|_{L^2(I)}$, which is positive and continuous due to $\bar{p} \in X^{s'} \hookrightarrow \mathcal{C}_0(\Omega_c, L^2(I))$. Furthermore, it fulfills $\int_{\Omega_c} f(x)\,\mathrm{d}|\bar{u}|(x) = 0$. We can easily argue that $\operatorname{supp}|\bar{u}|$ must be a subset of the zero set $\{ x \in \Omega_c \mid f(x) = 0 \}$, for instance, by a contradiction argument. ∎

If we consider the optimal $\bar{u}$ as a variable of time with (2.5) we can infer from Theorem 2.12 that

$$\operatorname{supp}|\bar{u}(t)| \subseteq \operatorname{supp}|\bar{u}| \subseteq \{ x \in \Omega \mid \|\bar{p}(x)\|_{L^2(I)} = \alpha \}$$

for almost all $t \in I$, where $|u(t)|$ is the total variation measure of the signed measure $u(t) \in \mathcal{M}(\Omega)$, which means that the sparsity pattern is constant over time. Furthermore we can identify a characteristic special case.

COROLLARY 2.13. *Suppose that equality in* (2.15) *is only achieved in a finite collection of points,* $\{ x \in \Omega_c \mid \|\bar{p}(x)\| = \alpha \} = \{ x_i \}_{i=1,\ldots,N}$. *Then $\bar{u}$ is given by a sum of point sources,* $\bar{u} = \sum_{i=1,\ldots,N} u_i\,\delta_{x_i}$, *where* $u_i \in L^2(I)$.

Proof. We infer from (2.17) that $|\bar{u}| = \sum_{i=1,\dots,N} c_i \delta_{x_i}$ for some positive coefficients $c_i \geq 0$. Then the time dependent coefficients $u_i \in L^2(I)$ are given due to (2.18) by the formula $u_i(t) = c_i \bar{u}'(t, x_i) = -\frac{c_i}{\alpha} \bar{p}(t, x_i)$. $\square$

2.5. Comparison. Let us compare conditions (2.18) and (2.17) with the optimality system obtained for the problem

$$\min J(S(y_0, u)) + \alpha \|u\|_{L^2(I,\mathcal{M}(\Omega_c))} \tag{2.21}$$

analyzed in [8], where the order of integration for the control cost term is reversed. We recall the inclusion (2.4), which is strict, i.e.,

$$\mathcal{M}(\Omega_c, L^2(I)) \subsetneq L^2(I, \mathcal{M}(\Omega_c)).$$

For problem (2.21) the optimality condition [8, Theorem 3.3] implies that for almost every $t \in I$

$$\operatorname{supp}|u(t)| \subseteq \left\{\, x \in \Omega_c \;\middle|\; |p(t,x)| = \|p(t)\|_{\mathcal{C}_0(\Omega_c)} \,\right\},$$

which means that the support of $u(t)$ is variable over time. Note, that this implies significantly lower regularity for problem (2.21) in comparison with problem $(\mathcal{P})$ under consideration. For instance, a regularity result such as $\bar{u} \in \mathcal{C}(I, \mathcal{M}(\Omega_c))$ for the solution of $(\mathcal{P})$ (cf. Theorem 3.7 in section 3.1), cannot be expected for (2.21). Indeed, it is false for problem (2.21). We just have to consider as an example the measure $u \in L^2(I, \mathcal{M}(\overline{\Omega}))$, $u \notin \mathcal{M}(\overline{\Omega}, L^2(I))$ with $I = \Omega = (0, 1)$, defined by

$$u(t) = g(t)\delta_t \tag{2.22}$$

with a nontrivial smooth function g with $g(1) = 0$. Here, the Dirac delta function moves in space as time increases. Choosing A as the negative Laplacian with homogeneous Neumann boundary conditions it is possible to construct a desired state y_d such that the optimal solution of (2.21) is given by (2.22). The construction is analogous to the one given in section 5.1.

3. Discretization and numerical analysis. In this section, we consider A to be the negative Laplacian with zero Dirichlet boundary conditions

$$A = -\Delta\colon W_0^{1,s}(\Omega) \to W^{-1,s}(\Omega)$$

on a two-dimensional polygonal and convex domain $\Omega \subset \mathbb{R}^2$. With respect to our previous notation we have now the special case $\Gamma = \emptyset$. Therefore Ω_c is required to be a relatively closed subset of Ω, for instance, $\Omega_c = \Omega$ is valid. We remark that most of the following regularity results can be generalized in a suitable way to the general case and to three dimensions. Some of the finite element estimates employed in section 3.3 are, however, only available for $d = 2$ and the following techniques are in some cases restricted to two dimensions. We require that the desired state fulfills

$$y_d \in L^2(I, L^\infty(\Omega_o)), \tag{3.1}$$

which is only slightly stronger than the natural regularity we can obtain for the state; see Proposition 3.1. Additionally, for the convergence analysis we will suppose that Ω_c is the union of polygons; see (3.8) below for the precise meaning.

3.1. Precise regularity. For a right-hand side $u \in \mathcal{M}(\Omega_c, L^2(I))$, the state $y = S(y_0, u)$ has the regularity

$$y \in L^2(I, W_0^{1,s}(\Omega)) \cap H^1(I, W^{-1,s}(\Omega))$$

for any $s < 2$; see Theorem 2.5. Additionally, we obtain the following regularity.

PROPOSITION 3.1. *The state solution $y = S(y_0, u)$ lies in the space $L^2(I, L^q(\Omega))$ for any $q \in [1, \infty)$ with the a priori estimate*

$$\|y\|_{L^2(I,L^q(\Omega))} \leq c\, q\, (\|u\|_{\mathcal{M}(\Omega_c,L^2(I))} + \|y_0\|_{L^2(\Omega)}) \tag{3.2}$$

with a constant c independent of q.

Proof. We use the Sobolev embedding theorem and argue as in [23, Proposition 2.1] to obtain the dependence of the constant on q in (3.2). ◻

PROPOSITION 3.2. *The state y is continuous in time in the sense that*

$$y \in \mathcal{C}(\bar{I}, (W_0^{1,s}(\Omega), W^{-1,s}(\Omega))_{\frac{1}{2},2}) \hookrightarrow \mathcal{C}(\bar{I}, W^{-\varepsilon,s}(\Omega)) \tag{3.3}$$

for any $s < 2$ and $\varepsilon > 0$, where $(W_0^{1,s}(\Omega), W^{-1,s}(\Omega))_{\frac{1}{2},2}$ is an interpolation space.

Proof. The result follows by an application of the trace theorem [2, Theorem III 4.10.2] and we refer to [32, Theorem 4.6.1] for the embedding of the interpolation space. ◻

Remark 3.3. With methods as in [12, Theorem 2.4], where a single point source is considered, we can show that

$$y \in L^\infty(I, L^s(\Omega))$$

for any $s < 2$. Furthermore the mapping $t \mapsto y(t) \in L^s(\Omega)$ is continuous with respect to the weak topology in $L^s(\Omega)$.

For the adjoint state we can obtain improved regularity as well.

LEMMA 3.4. *Let $f \in L^2(I, L^2(\Omega))$. The solution to the dual equation*

$$-\partial_t p - \Delta p = f, \quad p(T) = 0 \tag{3.4}$$

lies in the space $L^2(I, H^2(\Omega)) \cap H^1(I, L^2(\Omega))$ and $C(\bar{I}, H_0^1(\Omega))$ with the corresponding estimate

$$\|\partial_t p\|_{L^2(I,L^2(\Omega))} + \|p\|_{L^2(I,H^2(\Omega))} + \|p\|_{C(\bar{I},H_0^1(\Omega))} \leq c\, \|f\|_{L^2(I\times\Omega)}.$$

Proof. This can be proved by combining well-known techniques for parabolic equations (see, e.g., [14]), with an elliptic regularity result for convex polygonal domains (see [17]). ◻

With assumption (3.1) and Proposition 3.1 the right-hand side of the adjoint equation (2.13) is even in $L^2(I, L^q(\Omega))$ for any $q < \infty$. If we consider the dual equation (3.4) for an arbitrary $f \in L^2(I, L^q(\Omega))$ we obtain that

$$\partial_t p,\ \Delta p\ \in L^2(I, L^q(\Omega)), \tag{3.5}$$

using maximal parabolic regularity; see [16]. However, from this we cannot in general infer $L^2(I, W^{2,q}(\Omega))$ regularity without further assumptions on $\partial\Omega$. Nevertheless, we can obtain this regularity locally in the interior of the domain.

LEMMA 3.5. *Let $\gamma > 0$ and*

$$\Omega^\gamma = \{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \gamma \,\}.$$

Then we obtain for any solution of (3.4) *with $f \in L^2(I, L^q(\Omega))$ that*

$$p|_{I\times\Omega^\gamma} \in L^2(I, W^{2,q}(\Omega^\gamma)) \cap H^1(I, L^q(\Omega^\gamma)), \tag{3.6}$$

where $q \in [1, \infty)$, with the a priori estimate

$$\|p\|_{L^2(I,W^{2,q}(\Omega^\gamma))} + \|\partial_t p\|_{L^2(I,L^q(\Omega^\gamma))} \leq c\, q\, (\|f\|_{L^2(I,L^q(\Omega^{\frac{\gamma}{2}}))} + \gamma^{-1}\|f\|_{L^2(I,L^2(\Omega))}).$$

Proof. See, for instance, [23, Lemma 2.2], where this is shown for any ball $B \subset \Omega^\gamma$. The result follows since Ω^γ can be covered by finitely many balls $B \subset \Omega^{\frac{\gamma}{2}}$. ∎

By applying this to the optimal adjoint state $\bar p$ and interpolating between both spaces from Lemma 3.5 with $\theta = 1 - \varepsilon$ we obtain

$$\bar p \in H^{1-\varepsilon}(I, \mathcal{C}(\Omega^\gamma)) \tag{3.7}$$

for all $\varepsilon > 0$; see [3, Theorem 5.2]. Here we have used the compact embedding $(W^{2,q}(\Omega^\gamma), L^q(\Omega^\gamma))_{1-\varepsilon,2} \hookrightarrow \mathcal{C}(\Omega^\gamma)$. With the help of the optimality conditions, we can now derive additional regularity for the optimal controls.

PROPOSITION 3.6. *There exists $\gamma > 0$ such that*

$$\operatorname{supp} \bar u \subset \Omega_c \cap \Omega^\gamma = \{\, x \in \Omega_c \mid \operatorname{dist}(x, \partial\Omega) > \gamma \,\},$$

where the constant γ depends only on Ω, Ω_o, Ω_c, the parameter α, and y_d.

Proof. With Lemma 3.4 we have $\bar p \in L^2(I, H^2(\Omega)) \hookrightarrow L^2(I, \mathcal{C}^\delta(\bar\Omega))$ for any $0 < \delta < 1$ and therefore

$$\|\bar p(x) - \bar p(x+h)\|_{L^2(I)} \leq \left[\int_I \big(\sup_{\xi\in\bar\Omega} |\bar p(t,\xi) - \bar p(t,\xi+h)|\big)^2 \,\mathrm{d}t\right]^{\frac{1}{2}} \leq c|h|^\delta$$

for all $x, x+h \in \bar\Omega$, i.e., $\bar p \in \mathcal{C}^\delta(\bar\Omega, L^2(I))$. The result now follows from the sparsity property of the support (2.17) and the zero Dirichlet boundary conditions. We have $\bar p(x) = 0$ for all $x \in \partial\Omega$ and can therefore choose $\gamma < (\frac{\alpha}{2c})^{\frac{1}{\delta}}$ to finish the proof. ∎

THEOREM 3.7. *With assumption* (3.1)*, we obtain the additional regularity*

$$\bar u \in H^{1-\varepsilon}(I, \mathcal{M}(\Omega_c))$$

for all $\varepsilon > 0$.

Proof. Using $\mathrm{d}\bar u = u' \,\mathrm{d}|u| = -\frac{1}{\alpha}\chi_{\Omega_c}\bar p \,\mathrm{d}|\bar u|$ we have that

$$\begin{aligned}\|\bar u(t_1) - \bar u(t_2)\|_{\mathcal{M}(\Omega_c)} &= \sup_{\|\varphi\|_{\mathcal{C}_0(\Omega_c)}=1} \langle \bar u(t_1) - \bar u(t_2), \varphi\rangle \\ &= \sup_{\|\varphi\|_{\mathcal{C}_0(\Omega_c)}=1} \int_{\Omega_c} \tfrac{1}{\alpha}\varphi\,(\bar p(t_2) - \bar p(t_1))\,\mathrm{d}|\bar u| \leq \tfrac{1}{\alpha}\|\bar p(t_2) - \bar p(t_1)\|_{\mathcal{C}_0(\Omega_c\cap\Omega^\gamma)}\, |\bar u|(\Omega_c)\end{aligned}$$

due to Proposition 3.6. Therefore

$$\frac{\|\bar u(t_1) - \bar u(t_2)\|_{\mathcal{M}(\Omega_c)}}{(t_1 - t_2)^{\frac{1}{2}+1-\varepsilon}} \leq \tfrac{1}{\alpha}\|\bar u\|_{\mathcal{M}(\Omega_c, L^2(I))} \frac{\|\bar p(t_2) - \bar p(t_1)\|_{\mathcal{C}_0(\Omega_c\cap\Omega^\gamma)}}{(t_1 - t_2)^{\frac{1}{2}+1-\varepsilon}}.$$

With the regularity (3.7) for $\bar p$, the expression on the right is in $L^2(I \times I)$ which implies the claim. ∎

3.2. Discretization. We discretize the state variable y with (linear) finite elements in space and discontinuous finite elements (of order $r \geq 0$) in time:

$$y_{kh} \in X_k^r(I, V_h) \subset L^2(I, H_0^1(\Omega)).$$

Here $V_h \subset H_0^1(\Omega)$ is the space of linear finite elements on a family of shape regular quasi-uniform triangulations $\{ \mathcal{T}_h \}_h$; see, e.g., [6]. The finite element space associated with $\mathcal{T}_h$ is defined as usual by

$$V_h = \{ v_h \in \mathcal{C}_0(\Omega) \mid v_h|_K \in \mathcal{P}_1(K) \text{ for } K \in \mathcal{T}_h \}.$$

The discretization parameter h denotes the maximal diameter of cells $K \in \mathcal{T}_h$. Furthermore, we suppose that Ω_c can be written as the union of a collection of cells or faces of $\mathcal{T}_h$ for all $\mathcal{T}_h$. In other words, if $\tilde{\mathcal{T}}_h$ is the collection of all cells, line segments, or points, there is a subset $\tilde{\mathcal{T}}_h^c \subseteq \tilde{\mathcal{T}}_h$ such that

$$\Omega_c = \bigcup\nolimits_{K \in \tilde{\mathcal{T}}_h^c} \bar{K} \tag{3.8}$$

for any h; cf. [28, section 6]. For the time discretization we define for any Banach space V the semidiscrete space

$$X_k^r(I, V) = \left\{ v_k \in L^2(I, V) \;\middle|\; v_k|_{I_m} \in \mathcal{P}_r(I_m, V),\ m = 1, 2, \ldots, M \right\}$$

as discontinuous, Banach space valued, piecewise polynomial functions on the disjoint partition of the temporal interval

$$\bar{I} = \{ 0 \} \cup I_1 \cup I_2 \cup \ldots \cup I_M,$$

where $I_m = (t_{m-1}, t_m]$ and $0 = t_0 < t_1 < \cdots < t_M = T$. By $k_m = t_m - t_{m-1}$ we denote the step length and by $k = \max_m k_m$ the maximum thereof. We employ the notation

$$w_m^- = \lim_{\varepsilon \to 0^+} w(t_m - \varepsilon), \quad w_m^+ = \lim_{\varepsilon \to 0^+} w(t_m + \varepsilon), \quad [w]_m = w_m^+ - w_m^-$$

for the left- and right-sided limits and the jump term (for any w where these limits are defined).

The discretized state equation is then given with the bilinear form

$$B(y, \varphi) = \sum_{m=1}^{M} \langle \partial_t y, \varphi \rangle_{I_m} + (\nabla y, \nabla \varphi)_I + \sum_{m=1}^{M-1} ([y]_m, \varphi_m^+) + (y_0^+, \varphi_0^+), \tag{3.9}$$

defined for $y, \varphi \in X_k^r(I, V_h)$. The distributional derivative of a discrete function $\partial_t y_{kh}|_{I_m}$ is defined as the classical derivative of the polynomial (and vanishes for $r = 0$). The duality pairing $\langle \cdot, \cdot \rangle_{I_m}$ denotes the pairing of $L^2(I_m, W_\Gamma^{-1,s}(\Omega))$ with its dual. Therefore this definition can be extended to $y \in X_k^r(I, V_h) + Y^s$ and $\varphi \in X_k^r(I, V_h) + X^{s'}$. Furthermore, by applying integration by parts to (3.9) we obtain the equivalent dual formulation

$$B(y, \varphi) = -\sum_{m=1}^{M} \langle y, \partial_t \varphi \rangle_{I_m} + (\nabla y, \nabla \varphi)_I + \sum_{m=1}^{M-1} (-y_m^-, [\varphi]_m) + (y_M^-, \varphi_M^-). \tag{3.10}$$

Then for any right-hand side $u \in \mathcal{M}(\Omega_c, L^2(I))$ the discrete dG(r)cG(1) formulation of the state equation for the discretized state $y_{kh} \in X_k^r(I, V_h)$ is given as

$$B(y_{kh}, \varphi_{kh}) = \langle u, \chi_{\Omega_c} \varphi_{kh} \rangle + (y_0, \varphi_{kh,0}^+) \tag{3.11}$$

for all $\varphi_{kh} \in X_k^r(I, V_h)$. Since the right-hand side is a linear functional on the discrete solution space, existence of a unique solution can be derived with standard arguments (see [31]). Therefore, we can define a discrete solution operator with $y_{kh} = S_{kh}(u) = S_{kh}(y_0, u)$. This operator and the bilinear form B are compatible with the continuous state solution $y = S(u)$ in the sense that

$$B(y, \varphi) = \langle u, \chi_{\Omega_c} \varphi \rangle + (y_0, \varphi_0^+) \tag{3.12}$$

for any $\varphi \in L^2(I, W_0^{1,s'}(\Omega))$ with $s' > 2$ such that the limits φ_m^+ for $m = 0, \dots, M-1$ are well-defined in $W^{\varepsilon,s'}(\Omega)$ for some $\varepsilon > 0$. This follows from the state equation (2.11) since the jump terms in (3.9) vanish due to Proposition 3.2. With this we can verify the Galerkin orthogonality

$$B(y - y_{kh}, \varphi_{kh}) = 0 \tag{3.13}$$

for all $\varphi_{kh} \in X_k^r(I, V_h)$ and therefore y_{kh} is also referred to as the Galerkin projection of y. We can now formulate a semidiscrete version of $(\mathcal{P})$ by replacing the continuous state equation (2.11) with the discrete equation (3.11), i.e., we formulate the semidiscrete problem as

$$(\mathcal{P}_{kh}^s) \qquad \min_{u \in \mathcal{M}(\Omega_c, L^2(I))} j_{kh}(u) = J(S_{kh}(u)) + \alpha \|u\|_{\mathcal{M}(\Omega_c, L^2(I))}.$$

With the same methods as in the continuous case (cf. Proposition 2.9), we can prove the following results.

PROPOSITION 3.8. *The semidiscrete problem $(\mathcal{P}_{kh}^s)$ possesses an optimal solution $\tilde{u} \in \mathcal{M}(\Omega_c, L^2(I))$.*

PROPOSITION 3.9. *Let $\tilde{u}$ be an optimal solution of $(\mathcal{P}_{kh}^s)$ and $\bar{y}_{kh} = S_{kh}(\tilde{u})$ the corresponding optimal state. There exists a unique discrete adjoint state $\bar{p}_{kh} \in X_k^r(I, V_h)$ solving the adjoint equation*

$$B(\varphi_{kh}, \bar{p}_{kh}) = (\bar{y}_{kh} - y_d, \chi_{\Omega_o} \varphi_{kh}) \tag{3.14}$$

for all $\varphi_{kh} \in X_k^r(I, V_h)$ and fulfilling the subgradient condition

$$-\langle u - \tilde{u}, \chi_{\Omega_c} \bar{p}_{kh} \rangle + \alpha \|\tilde{u}\|_{\mathcal{M}(\Omega_c, L^2(I))} \le \alpha \|u\|_{\mathcal{M}(\Omega_c, L^2(I))} \tag{3.15}$$

for all $u \in \mathcal{M}(\Omega_c, L^2(I))$. We alternatively express the first condition (3.14) *by $\bar{p}_{kh} = S_{kh}^*(J'(\bar{y}_{kh}))$.*

Since S_{kh} has a infinite-dimensional kernel, the solutions to $(\mathcal{P}_{kh}^s)$ cannot be expected to be unique. Therefore, as in [7, 8], we now construct an appropriate subspace of $\mathcal{M}(\Omega_c, L^2(I))$ with the same approximation properties. By $\{x_n\}$, $n = 1, 2 \dots, N_c$ we denote the nodes of the triangulation $\mathcal{T}_h$ contained in Ω_c and by $\{e_n\} \subset V_h$ the corresponding Lagrangian nodal basis functions. We introduce the space $\mathcal{M}_h$ consisting of linear combination of Dirac functionals $\delta_n = \delta_{x_n}$ associated with the nodes x_n

$$\mathcal{M}_h = \operatorname{span} \{ \delta_n \mid n = 1, \dots, N_c \}.$$

A suitable interpolation operator is now defined by the relations

$$\begin{aligned} &\Lambda_{kh}\colon \mathcal{M}(\Omega_c, L^2(I)) \to X_k^r(I, \mathcal{M}_h), \\ &\qquad \langle \Lambda_{kh} u, \varphi\rangle = \langle u,\, \chi_{\Omega_c}\, \pi_k i_h \varphi\rangle \text{ for all } \varphi \in \mathcal{C}_0(\Omega_c, L^2(I)), \end{aligned} \tag{3.16}$$

where $i_h\colon \mathcal{C}(\bar{\Omega}, L^2(I)) \to L^2(I, V_h)$ is the nodal interpolation operator and π_k is the L^2 projection on $X_k^r(I, L^2(\Omega)) \subset L^2(I \times \Omega)$. The interpolation operator i_h is given by

$$(i_h w)(x) = \sum_{n=1}^{N_c} w(x_n)\, e_n(x) \quad \text{for } x \in \Omega_c. \tag{3.17}$$

We can check that for any $w \in \mathcal{C}_0(\Omega_c, L^2(I))$ the projection π_k has the pointwise formula

$$(\pi_k w)(x) = \sum_{i=1}^{M(1+r)} \int_I \psi_i(t) w(t,x)\, \mathrm{d}t\, \psi_i = \tilde{\pi}_k(w(x)) \quad \text{for } x \in \Omega_c, \tag{3.18}$$

where ψ_i is an orthonormal basis of $X_k^r(I, \mathbb{R})$ with respect to the inner product in $L^2(I)$ and $\tilde{\pi}_k$ is the L^2 projection in $L^2(I)$ onto $X_k^r(I, \mathbb{R})$. Therefore π_k and i_h commute and we have for $w \in \mathcal{C}_0(\Omega_c, L^2(I))$

$$i_h(\pi_k(w)) = \pi_k(i_h(w)) = \sum_{n=1}^{N_c} \sum_{i=1}^{M(1+r)} \int_I \psi_i(t) w(t, x_n)\, \mathrm{d}t\, \psi_i\, e_n,$$

which implies

$$\Lambda_{kh} u = \sum_{n=1}^{N_c} \sum_{i=1}^{M(1+r)} \langle u, \psi_i\, e_n\rangle\, \psi_i\, \delta_n.$$

Remark 3.10. In the case $r = 0$ we take the piecewise constant functions $\psi_m = k_m^{-1/2}\, \chi_{I_m}$ as a suitable orthonormal basis for $X_k^r(I, \mathbb{R})$. In this case the operator Λ_{kh} can be written as

$$\Lambda_{kh} u = \sum_{n=1}^{N_c} \sum_{m=1}^{M} \frac{1}{k_m} \int_{I_m} \langle u(t), e_n\rangle\, \mathrm{d}t\, \chi_{I_m}\, \delta_n,$$

which is the same as given in [8, Theorem 4.2].

LEMMA 3.11. *For any $u \in M(\Omega_c, L^2(I))$ we have*

$$\langle \Lambda_{kh} u, \varphi_{kh}\rangle = \langle u, \varphi_{kh}\rangle$$

for all $\varphi_{kh} \in X_k^r(I, V_h)$ and

$$\|\Lambda_{kh} u\|_{\mathcal{M}(\Omega_c, L^2(I))} \le \|u\|_{\mathcal{M}(\Omega_c, L^2(I))}.$$

Proof. The first property is immediately clear from the definition since $\chi_{\Omega_c}(\pi_k i_h \varphi_{kh}) = \chi_{\Omega_c}(i_h \varphi_{kh}) = \chi_{\Omega_c} \varphi_{kh}$ due to (3.8). Furthermore we have

$$(\pi_k(i_h \varphi))(x) = \tilde{\pi}_k((i_h \varphi)(x))$$

for all $x \in \Omega_c$ by (3.18) and since $\tilde{\pi}_k$ is an orthogonal projection we get

$$\|\tilde{\pi}_k((i_h\varphi)(x))\|_{L^2(I)} \le \|(i_h\varphi)(x)\|_{L^2(I)} \le \|\varphi\|_{\mathcal{C}_0(\Omega_c,L^2(I))}$$

for all $x \in \Omega_c$ by (3.17). With this,

$$\|\pi_k(i_h\varphi)\|_{\mathcal{C}_0(\Omega_c,L^2(I))} \le \|\varphi\|_{\mathcal{C}_0(\Omega_c,L^2(I))}$$

is evident and by the duality $\mathcal{M}(\Omega_c, L^2(I)) = \mathcal{C}_0(\Omega_c, L^2(I))^*$,

$$\|\Lambda_{kh}u\|_{\mathcal{M}(\Omega_c,L^2(I))} = \sup_{\varphi\in\mathcal{C}_0(\Omega_c,L^2(I))} \frac{\langle \Lambda_{kh}u, \varphi\rangle}{\|\varphi\|_{\mathcal{C}_0(\Omega_c,L^2(I))}},$$

and the definition of Λ_{kh} (3.16) we obtain the second property. □

By arguments as in [7] it immediately follows that we can restrict the space for the optimal controls to $X_k^r(I, \mathcal{M}_h)$.

PROPOSITION 3.12. *The semidiscrete solution operator $S_{kh} \colon \mathcal{M}(\Omega_c, L^2(I)) \to X_k^r(I, V_h)$ fulfills $S_{kh} = S_{kh} \circ \Lambda_{kh}$ and for each optimal solution $\tilde{u} \in \mathcal{M}(\Omega_c, L^2(I))$ of $(\mathcal{P}_{kh}^s)$ the discrete control $\bar{u}_{kh} = \Lambda_{kh}\tilde{u} \in X_k^r(I, \mathcal{M}_h)$ fulfills*

$$j_{kh}(\tilde{u}) = j_{kh}(\bar{u}_{kh}).$$

Thus, $\bar{u}_{kh} = \Lambda_{kh}\tilde{u}$ is also an optimal solution of $(\mathcal{P}_{kh}^s)$.

Therefore, in the following, it suffices to consider the fully discrete problem

$$(\mathcal{P}_{kh}) \qquad \min_{u_{kh}\in X_k^r(I,\mathcal{M}_h)} j_{kh}(u_{kh}) = J(S_{kh}(u_{kh})) + \alpha\|u_{kh}\|_{\mathcal{M}(\Omega_c,L^2(I))},$$

which can be solved in practice. Note, that for this problem the same optimality system holds as in Proposition 3.9, where we are allowed to insert any control from $\mathcal{M}(\Omega_c, L^2(I))$ in the subgradient condition (3.15), instead of only discrete controls. This is a direct consequence of Proposition 3.12 and will be important for the following error analysis.

3.3. Error analysis for the state. For the error analysis, we restrict attention to dG(0), which is a variant of the implicit Euler method. This restriction arises since we employ optimal estimates for the dG(r)cG(1) method in the $L^\infty(\Omega, L^2(I))$ norm, which are not considered in the standard finite element literature. These estimates were obtained recently for two dimensions in [23] in the case $r = 0$.

Define $i_k \colon \mathcal{C}(\bar{I}, V) \to X_k^0(I, V)$ as the pointwise interpolation at the right time point in each interval

$$i_k w = \sum_{m=1}^{M} w(t_m)\chi_{I_m}, \tag{3.19}$$

where χ_{I_m} is the indicator function of the interval I_m. We can obtain the following interpolation estimates for i_k.

LEMMA 3.13. *For any $w = S^\star(f)$ with $f \in L^2(I, L^2(\Omega))$ we have*

$$\|w - i_k w\|_{L^2(I,L^2(\Omega))} \le c\,k\|f\|_{L^2(I,L^2(\Omega))}, \tag{3.20}$$

$$\|w - i_k w\|_{L^2(I,H_0^1(\Omega))} \le c\,k^{\frac{1}{2}}\|f\|_{L^2(I,L^2(\Omega))}. \tag{3.21}$$

Proof. First, we note that $i_k w$ is in $L^2(I, H_0^1(\Omega))$ since $w \in \mathcal{C}(\bar{I}, H_0^1(\Omega))$ by Lemma 3.4. The interpolation estimates can be obtained with standard techniques and are given in Appendix A. ◻

In the following estimates we are going to apply the best approximation properties obtained in [23, Theorems 3.1, 3.5].

THEOREM 3.14 (best approximation). *Let $w = S^\star(f)$ be an adjoint solution and $w_{kh} = S_{kh}^*(f)$ its Galerkin projection for some $f \in L^2(I, L^2(\Omega))$ and $1 \le q \le \infty$. Then we have for every $x \in \Omega$ that*

$$\begin{aligned}
&\|w(x) - w_{kh}(x)\|_{L^2(I)}^2 \\
&\qquad \le c|\ln h|^2 \inf_{\chi \in X_k^0(I,V_h)} \int_I \|w(t) - \chi(t)\|_{L^\infty(\Omega)}^2 + h^{-\frac{4}{q}} \|i_k w(t) - \chi(t)\|_{L^q(\Omega)}^2 \,\mathrm{d}t.
\end{aligned}$$

Furthermore, for $x \in \Omega^\gamma$ with $\gamma > 4h > 0$ we have the local estimate

$$\begin{aligned}
&\|w(x) - w_{kh}(x)\|_{L^2(I)}^2 \\
&\quad \le c|\ln h|^3 \inf_{\chi \in X_k^0(I,V_h)} \int_I \|w(t) - \chi(t)\|_{L^\infty(B_\gamma(x))}^2 + h^{-\frac{4}{q}} \|i_k w(t) - \chi(t)\|_{L^q(B_\gamma(x))}^2 \,\mathrm{d}t \\
&\qquad + c\,\gamma^{-2} |\ln h| \int_I \|w(t) - w_{kh}(t)\|_{L^2(\Omega)}^2 \,\mathrm{d}t.
\end{aligned}$$

With this we can prove the following a priori error estimates.

THEOREM 3.15. *Let $y = S(y_0, u)$ and its Galerkin projection $y_{kh} = S_{kh}(y_0, u)$ for arbitrary $u \in \mathcal{M}(\Omega_c, L^2(I))$ and $y_0 \in H_0^1(\Omega)$. Then we have the a priori estimate*

$$\|y - y_{kh}\|_{L^2(I\times\Omega)} \le c|\ln h|^2 (k^{\frac{1}{2}} + h) \left(\|u\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{H^1(\Omega)}\right). \tag{3.22}$$

If additionally the measure is supported in the interior of the domain, i.e., $\operatorname{supp} u \subset \Omega^\gamma$ *for some $\gamma > 0$, we obtain the improved estimate in a weaker norm*

$$\|y - y_{kh}\|_{L^2(I, L^1(\Omega))} \le c\,\gamma^{-1} |\ln h|^{\frac{5}{2}} (k + h^2) \left(\|u\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{H^1(\Omega)}\right). \tag{3.23}$$

Proof. Consider that $y = S(y_0, 0) + S(0, u)$ and $y_{kh} = S_{kh}(y_0, 0) + S_{kh}(0, u)$. We have $S(y_0, 0) \in L^2(I, H^2(\Omega)) \cap H^1(I, L^2(\Omega))$ and the corresponding error estimate

$$\|S(y_0, 0) - S_{kh}(y_0, 0)\|_{L^2(I, L^2(\Omega))} \le c\,(k + h^2) \|y_0\|_{H^1(\Omega)}$$

can be found, e.g., in [26]. Without restriction, we suppose $y_0 = 0$ in the following and employ a duality argument. Define the error $e = y - y_{kh}$ and introduce

$$\begin{aligned}
g_2 &= e \in L^2(I, L^2(\Omega)), \\
g_1 &= \|e(t)\|_{L^1(\Omega)} \operatorname{sgn} e(t,x) \in L^2(I, L^\infty(\Omega))
\end{aligned}$$

for the first and second estimate, respectively. For $l \in \{1, 2\}$ we define the auxiliary dual variable $w = S^\star(g_l)$ and its Galerkin projection $w_{kh} = S_{kh}^*(g_l)$. We can verify that $B(\varphi, w) = (\varphi, g_l)_I$ holds for any $\varphi \in L^2(I, W^{1,s}(\Omega))$ with $\varphi_m^- \in H^{-1}(\Omega)$ for $m = 1, \ldots, M$, since the jump terms in the dual description of the bilinear form (3.9) vanish due to Lemma 3.4. We rewrite the error using this identity for w, Galerkin orthogonality for y (see (3.13)), Galerkin orthogonality for w, and (3.12) to obtain

$$\begin{aligned}(3.24)\quad \|y-y_{kh}\|^2_{L^2(I,L^l(\Omega))} &= (y-y_{kh},g_l)_I = B(y-y_{kh},w)\\ &= B(y-y_{kh},w-w_{kh}) = B(y,w-w_{kh})\\ &= \langle u,\chi_{\Omega_c}(w-w_{kh})\rangle \le \|u\|_{\mathcal{M}(\Omega_c,L^2(I))}\|w-w_{kh}\|_{\mathcal{C}_0(\Omega_c,L^2(I))}.\end{aligned}$$

In the following, we estimate the last term.

For the first estimate, where $l=2$, we apply the global best approximation property from Theorem 3.14 with the choice $\chi=\pi_h i_k w$, where i_k is the pointwise interpolation defined in (3.19) and $\pi_h\colon L^1(\Omega)\to V_h$ is the Clément interpolation; see, e.g., [4]. This results in

$$\begin{aligned}(3.25)\quad &\|w-w_{kh}\|_{\mathcal{C}_0(\Omega_c,L^2(I))}\\ &\le c|\ln h|\left(\|w-\pi_h i_k w\|_{L^2(I,L^\infty(\Omega))}+h^{-\frac{2}{q}}\|i_k(w-\pi_h w)\|_{L^2(I,L^q(\Omega))}\right),\end{aligned}$$

where we choose any $q<\infty$. The first term is further estimated by

$$\begin{aligned}\|w-\pi_h i_k w\|_{L^2(I,L^\infty(\Omega))} &\le \|w-\pi_h w\|_{L^2(I,L^\infty(\Omega))}+\|\pi_h(w-i_k w)\|_{L^2(I,L^\infty(\Omega))}\\ &\le c\,h\|w\|_{L^2(I,H^2(\Omega))}+c\,h^{-\frac{2}{q}}\|\pi_h(w-i_k w)\|_{L^2(I,L^q(\Omega))}\end{aligned}$$

with an interpolation estimate for the Clément interpolation and an inverse estimate with the same q as above. With the stability of the Clément interpolation in $L^q(\Omega)$ and the Sobolev embedding we obtain

$$\|\pi_h(w-i_k w)\|_{L^2(I,L^q(\Omega))}\le c\,\|w-i_k w\|_{L^2(I,L^q(\Omega))}\le c\,q\,\|w-i_k w\|_{L^2(I,H^1_0(\Omega))};$$

see, e.g., [1, Theorem 8.8] for the dependence of the embedding constant on $q<\infty$. With Lemma 3.13 we then get the estimate

$$\|\pi_h(w-i_k w)\|_{L^2(I,L^q(\Omega))}\le c\,q\,k^{\frac{1}{2}}\|g_2\|_{L^2(I,L^2(\Omega))}.$$

The second term in (3.25) is estimated by the triangle inequality

$$\begin{aligned}&\|i_k(w-\pi_h w)\|_{L^2(I,L^q(\Omega))}\\ &\quad\le \|i_k w-w\|_{L^2(I,L^q(\Omega))}+\|w-\pi_h w\|_{L^2(I,L^q(\Omega))}+\|\pi_h(w-i_k w)\|_{L^2(I,L^q(\Omega))}.\end{aligned}$$

The single terms are treated as before and we arrive at

$$\begin{aligned}&\|w-w_{kh}\|_{\mathcal{C}_0(\Omega_c,L^2(I))}\\ &\le c|\ln h|\left(h\|w\|_{L^2(I,H^2(\Omega))}+h^{-\frac{2}{q}}\left(q\,k^{\frac{1}{2}}\|g_2\|_{L^2(I,L^2(\Omega))}+h^{1+\frac{2}{q}}\|w\|_{L^2(I,H^2(\Omega))}\right)\right).\end{aligned}$$

Finally, with the choice $q=|\ln h|$ and Lemma 3.4 this implies

$$\begin{aligned}(3.26)\quad \|w-w_{kh}\|_{\mathcal{C}_0(\Omega_c,L^2(I))} &\le c|\ln h|\left(h+q\,h^{-\frac{2}{q}}k^{\frac{1}{2}}\right)\|g_2\|_{L^2(I,L^2(\Omega))}\\ &\le c|\ln h|^2\left(h+k^{\frac{1}{2}}\right)\|y-y_{kh}\|_{L^2(I,L^2(\Omega))}.\end{aligned}$$

Combining (3.24) and (3.26) we obtain the result (3.22).

The second estimate, where $l = 1$, can be obtained in a similar fashion using the local estimate from Theorem 3.14 and choosing again $\chi = \pi_h i_k w$. Then we can use the approximation properties of the Clément interpolation, Lemma 3.13, and the regularity estimate from Lemma 3.5 for the first two terms and an L^2 estimate from [26] for the term $\|w - w_{kh}\|_{L^2(I\times\Omega)}$. We get

$$\begin{aligned}\|w - w_{kh}\|_{\mathcal{C}_0(\Omega_c, L^2(I))} &\le c\,\gamma^{-1}|\ln h|^{\frac{1}{2}}(1 + qh^{-\frac{2}{q}})(k + h^2)\|g_1\|_{L^2(I,L^q(\Omega))}\\ &\le c\,\gamma^{-1}|\ln h|^{\frac{3}{2}}(k + h^2)\|y - y_{kh}\|_{L^2(I,L^1(\Omega))}\end{aligned} \tag{3.27}$$

with $q = |\ln h|$ as above and we obtain result (3.23). We omit a more detailed argument since it is analogous to the one in [23, Theorem 4.1], where an estimate for the special case $u(t) = \hat{u}(t)\delta_{x_0}$ for some $x_0 \in \Omega$ and $\hat{u} \in L^2(I)$ is proved. $\square$

Remark 3.16. It is possible to derive a sharpened version of (3.22) without any $|\ln h|$ term if we require a coupling of k and h of the form

$$k = ch^2$$

for a constant independent of k and h; see [8, Theorem 4.6]. Whether we can improve (3.22) without such a coupling is an open question to the best of our knowledge. However, such an improvement alone would yield no improvement for the estimates in section 3.4. Furthermore, such a strong coupling seems overly restrictive for an implicit discretization scheme.

For the error analysis in the following section we need an additional stability property of the space-time discretization.

LEMMA 3.17. *We have for every $y_0 \in L^2(\Omega)$ and $u \in \mathcal{M}(\Omega_c, L^2(I))$ that*

$$\|y_{kh}\|_{L^2(I,L^\infty(\Omega))} \le c|\ln h|(\|y_0\|_{L^2(\Omega)} + \|u\|_{\mathcal{M}(\Omega_c,L^2(I))}).$$

Proof. We start by applying the discrete Sobolev inequality

$$\|y_{kh}\|^2_{L^2(I,L^\infty(\Omega))} = \int_I \|y_{kh}(t)\|^2_{L^\infty(\Omega)}\,\mathrm{d}t \le c|\ln h|\|\nabla y_{kh}\|^2_{L^2(I\times\Omega)}; \tag{3.28}$$

see [6, Lemma 4.9.1]. Now, we can add the primal and dual representation of the bilinear form (3.9) and (3.10) with $y = \varphi = y_{kh}$ and divide by two to obtain

$$B(y_{kh}, y_{kh}) = (\nabla y_{kh}, \nabla y_{kh})_I + \tfrac{1}{2}\sum_{m=1}^{M} \|[y_{kh}]_m\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|y^+_{kh,0}\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|y^-_{kh,M}\|^2_{L^2(\Omega)}.$$

This allows us to estimate the $L^2(I, H^1_0(\Omega))$ seminorm in terms of the bilinear form and then apply the definition of the discrete state equation (3.11) to obtain

$$\begin{aligned}\|\nabla y_{kh}\|^2_{L^2(I\times\Omega)} &\le B(y_{kh}, y_{kh}) - \tfrac{1}{2}\|y^+_{kh,0}\|^2_{L^2(\Omega)}\\ &= \langle u, \chi_{\Omega_c} y_{kh}\rangle + (y_0, y^+_{kh,0}) - \tfrac{1}{2}\|y^+_{kh,0}\|^2_{L^2(\Omega)}\\ &= \langle u, \chi_{\Omega_c} y_{kh}\rangle - \tfrac{1}{2}\|y^+_{kh,0} - y_0\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|y_0\|^2_{L^2(\Omega)}\\ &\le \|u\|_{\mathcal{M}(\Omega_c,L^2(I))}\|y_{kh}\|_{L^2(I,L^\infty(\Omega))} + \tfrac{1}{2}\|y_0\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.29}$$

Finally, we combine this with (3.28), use Young's inequality to produce the term $\frac{1}{2}\|\nabla y_{kh}\|^2_{L^2(I\times\Omega)}$, bring it on the other side, and take the square root to finish the proof. $\square$

3.4. Error analysis for the optimal control problem. First we will consider convergence of the functional values.

LEMMA 3.18. *For every optimal control $\bar{u}$ or $\bar{u}_{kh}$ we have*

$$\max\left\{\|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))}, \|\bar{u}_{kh}\|_{\mathcal{M}(\Omega_c,L^2(I))}\right\} \leq c\,(\|y_0\|_{L^2(\Omega)} + \|y_d\|_{L^2(I\times\Omega_o)}). \tag{3.30}$$

Proof. For $\bar{u}_{kh}$ this is a consequence of the minimality, since

$$\|\bar{u}_{kh}\|_{\mathcal{M}(\Omega_c,L^2(I))} \leq \tfrac{1}{\alpha} J(S_{kh}(y_0,0)) \leq \tfrac{1}{2\alpha}(\|S_{kh}(y_0,0)\|_{L^2(I\times\Omega_o)} + \|y_d\|_{L^2(I\times\Omega_o)}).$$

The result follows by the stability estimate $\|S_{kh}(y_0,0)\|_{L^2(I\times\Omega)} \leq c\|y_0\|_{L^2(\Omega)}$ for the dG(r)cG(1) method; see (3.29) for $u=0$. The proof for $\bar{u}$ is similar. ∎

THEOREM 3.19. *Let $\bar{u} \in \mathcal{M}(\Omega_c, L^2(I))$ be an optimal solution to $(\mathcal{P})$ and $\bar{u}_{kh} \in X_k^0(I,\mathcal{M}_h)$ be a discrete optimal solution to $(\mathcal{P}_{kh})$. We have for the associated optimal functional values*

$$|j(\bar{u}) - j_{kh}(\bar{u}_{kh})| \leq c\,\gamma^{-1}|\ln h|^4(k+h^2) \tag{3.31}$$

with a constant c independent of k and h, where γ is the constant from Proposition 3.6.

Proof. Since we have

$$j(\bar{u}) - j_{kh}(\bar{u}) \leq j(\bar{u}) - j_{kh}(\bar{u}_{kh}) \leq j(\bar{u}_{kh}) - j_{kh}(\bar{u}_{kh})$$

by minimality of $\bar{u}$ and $\bar{u}_{kh}$, and Proposition 3.12 we obtain

$$|j(\bar{u}) - j_{kh}(\bar{u}_{kh})| \leq \max\{|j(\bar{u}) - j_{kh}(\bar{u})|,\, |j(\bar{u}_{kh}) - j_{kh}(\bar{u}_{kh})|\}.$$

Therefore we estimate the functional error $j(u) - j_{kh}(u) = J(S(u)) - J(S_{kh}(u))$ for a fixed $u \in \mathcal{M}(\Omega_c, L^2(I))$. We define $y = S(u)$ and $y_{kh} = S_{kh}(u)$ and by reordering terms and applying Hölders inequality we get

$$\begin{aligned}|j(u) - j_{kh}(u)| &= \tfrac{1}{2}|(\chi_{\Omega_o}(y-y_{kh}), y - y_{kh} + 2y_{kh} - 2y_d)_I| \\ &\leq \tfrac{1}{2}\|y-y_{kh}\|^2_{L^2(I\times\Omega)} + \|y-y_{kh}\|_{L^2(I,L^1(\Omega))}\|y_{kh}-y_d\|_{L^2(I,L^\infty(\Omega))}.\end{aligned} \tag{3.32}$$

The terms which contain $y - y_{kh}$ are treated with estimates (3.22) and (3.23) from Theorem 3.15, respectively. Furthermore we have

$$\|y_{kh} - y_d\|_{L^2(I,L^\infty(\Omega))} \leq \|y_d\|_{L^2(I,L^\infty(\Omega))} + c|\ln h|\left(\|y_0\|_{L^2(\Omega)} + \|u\|_{\mathcal{M}(\Omega_c,L^2(I))}\right)$$

by Lemma 3.17 and (3.1). Together with Lemma 3.18 we have shown (3.31). ∎

We also provide an error estimate for the optimal state solutions on the observation domain.

THEOREM 3.20. *Let $\bar{u} \in \mathcal{M}(\Omega_c, L^2(I))$ be an optimal solution to $(\mathcal{P})$ with associate state $\bar{y} = S(y_0,\bar{u})$ and $\bar{u}_{kh} \in X_k^0(I,\mathcal{M}_h)$ be a discrete optimal solution to $(\mathcal{P}_{kh})$ with $\bar{y}_{kh} = S_{kh}(y_0,\bar{u}_{kh})$. With assumption* (3.1) *we have the estimate*

$$\|\bar{y} - \bar{y}_{kh}\|_{L^2(I\times\Omega_o)} \leq c\,\gamma^{-\frac{1}{2}}|\ln h|^2(k^{\frac{1}{2}} + h),$$

where $\gamma > 0$ is the constant from Proposition 3.6.

Proof. We test the continuous subgradient condition (2.14) with the discrete solution, and the discrete one (3.15) with the continuous solution (which is possible due to Proposition 3.12), to obtain

$$\begin{aligned}-\langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}\bar{p}\rangle + \alpha\|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))} &\leq \alpha\|\bar{u}_{kh}\|_{\mathcal{M}(\Omega_c,L^2(I))}, \\ -\langle \bar{u} - \bar{u}_{kh}, \chi_{\Omega_c}\bar{p}_{kh}\rangle + \alpha\|\bar{u}_{kh}\|_{\mathcal{M}(\Omega_c,L^2(I))} &\leq \alpha\|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))}.\end{aligned}$$

Adding both implies

$$-\langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \bar{p}_{kh})\rangle \leq 0.$$

We introduce as auxiliary variables the Galerkin projections of $\bar{y}$ and $\bar{p}$ as $\hat{y}_{kh} = S_{kh}(\bar{u})$ and $\hat{p}_{kh} = S^*_{kh}(\chi^*_{\Omega_o}(\bar{y} - y_d))$. With this, we can reformulate the inequality above to

$$\begin{aligned}
0 &\leq \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \bar{p}_{kh})\rangle \\
&= \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \hat{p}_{kh})\rangle + \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\hat{p}_{kh} - \bar{p}_{kh})\rangle \\
&= \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \hat{p}_{kh})\rangle + (\bar{y}_{kh} - \hat{y}_{kh}, \chi_{\Omega_o}(\bar{y} - \bar{y}_{kh})) \\
&= \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \hat{p}_{kh})\rangle + (\bar{y} - \hat{y}_{kh}, \chi_{\Omega_o}(\bar{y} - \bar{y}_{kh})) - \|\bar{y} - \bar{y}_{kh}\|^2_{L^2(I\times\Omega_o)}.
\end{aligned}$$

We bring the last term above on the other side and treat the second with Young's inequality to obtain

$$\begin{aligned}
\tfrac{1}{2}\|\bar{y} - \bar{y}_{kh}\|^2_{L^2(I\times\Omega_o)} &\leq \langle \bar{u}_{kh} - \bar{u}, \chi_{\Omega_c}(\bar{p} - \hat{p}_{kh})\rangle + \tfrac{1}{2}\|\bar{y} - \hat{y}_{kh}\|^2_{L^2(I\times\Omega_o)} \\
&\leq \|\bar{u}_{kh} - \bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))}\|\bar{p} - \hat{p}_{kh}\|_{\mathcal{C}_0(\Omega_c, L^2(I))} + \tfrac{1}{2}\|\bar{y} - \hat{y}_{kh}\|^2_{L^2(I\times\Omega_o)}.
\end{aligned}$$

Since $\|\bar{u}_{kh} - \bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))}$ can be bounded independently of k and h with Lemma 3.18 and the triangle inequality we obtain an estimate of the optimal state in terms of two Galerkin projection errors:

$$\|\bar{y} - \bar{y}_{kh}\|^2_{L^2(I\times\Omega_o)} \leq c\,(\|\bar{p} - \hat{p}_{kh}\|_{\mathcal{C}_0(\Omega_c, L^2(I))} + \|\bar{y} - \hat{y}_{kh}\|^2_{L^2(I\times\Omega_o)}).$$

For the second term on the right-hand side we apply Theorem 3.15 to obtain

$$\|\bar{y} - \hat{y}_{kh}\|^2_{L^2(I\times\Omega_o)} \leq c\,|\ln h|^4(k + h^2)\big(\|\bar{u}\|^2_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|^2_{H^1(\Omega)}\big).$$

For the first term we argue as in Theorem 3.15 for estimate (3.27) to obtain

$$\|\bar{p} - \hat{p}_{kh}\|_{\mathcal{C}_0(\Omega_c, L^2(I))} \leq c\,\gamma^{-1}|\ln h|^{\frac{1}{2}}(1 + q\,h^{\frac{2}{q}})(k + h^2)\|\bar{y} - y_d\|_{L^2(I, L^q(\Omega_o))}.$$

Then we use the regularity assumption on the desired state (3.1) and estimate (3.2) from Proposition 3.1 for

$$\|\bar{y} - y_d\|_{L^2(I, L^q(\Omega_o))} \leq \|y_d\|_{L^2(I, L^\infty(\Omega_o))} + c\,q\,(\|\bar{u}\|_{\mathcal{M}(\Omega_c, L^2(I))} + \|y_0\|_{L^2(\Omega)}).$$

Setting $q = |\ln h|$ and combining the above estimates we complete the proof. ☐

4. Algorithmic treatment. To implement $(\mathcal{P})$ numerically, we can choose to develop a first or second order iterative scheme. Here we shall utilize a Newton method. In a function space setting this necessitates an additional regularization, which is introduced next. For $\epsilon > 0$, which is chosen small relative to α, we search for optimal controls in the Hilbert space $L^2(I \times \Omega_c)$; cf. [21]:

$$(\mathcal{P}_\varepsilon) \qquad \min_{u \in L^2(I\times\Omega_c)} j_\varepsilon(u) = J(S(y_0, u)) + \alpha\|u\|_{L^1(\Omega_c, L^2(I))} + \tfrac{\varepsilon}{2}\|u\|^2_{L^2(I\times\Omega_c)}.$$

For simplicity we exclude Ω_c with complicated topology and only consider Ω_c which is the relative closure of an open set or a Lipschitz manifold (intersected with $\Omega \cup \Gamma$). In the first case $L^q(\Omega_c)$ for $q \in \{1, 2\}$ is to be understood with respect to the Lebesgue measure, in the second with respect to a Hausdorff measure. It is then clear that the canonical embedding $L^1(\Omega_c, L^2(I)) \hookrightarrow \mathcal{M}(\Omega_c, L^2(I))$ is isometric and therefore

$$\|u\|_{\mathcal{M}(\Omega_c,L^2(I))} = \|u\|_{L^1(\Omega_c,L^2(I))} = \int_{\Omega_c} \|u(x)\|_{L^2(I)}\,\mathrm{d}x$$

for $u \in L^1(\Omega_c, L^2(I))$. We abbreviate the inner product in $L^2(I \times \Omega_c)$ by $(\cdot,\cdot)$ and recall that $L^1(\Omega_c, L^2(I))^* = L^\infty(\Omega_c, L^2(I))$. The problem $(\mathcal{P}_\varepsilon)$ is investigated in [21], where the following optimality system is obtained.

THEOREM 4.1. *Let $\varepsilon > 0$. Problem $(\mathcal{P}_\varepsilon)$ possesses a unique optimal solution $u_\varepsilon \in L^2(I \times \Omega_c)$ with corresponding state $y_\varepsilon = S(y_0, u_\varepsilon)$ and adjoint state $p_\varepsilon = S^\star(J'(y_\varepsilon))$. The optimality is characterized by the subgradient condition*

$$-(u - u_\varepsilon,\, \varepsilon u_\varepsilon + \chi_{\Omega_c} p_\varepsilon) + \alpha\|u_\varepsilon\|_{L^1(\Omega_c,L^2(I))} \le \alpha\|u\|_{L^1(\Omega_c,L^2(I))} \tag{4.1}$$

for all $u \in L^1(\Omega_c, L^2(I))$, which is equivalent to the "stripewise" projection formula

$$u_\varepsilon(t,x) = -\frac{1}{\varepsilon}\max\left(0, 1 - \frac{\alpha}{\|p_\varepsilon(x)\|_{L^2(I)}}\right) p_\varepsilon(t,x) \tag{4.2}$$

for almost all $(t,x) \in I \times \Omega_c$. This implies that $\operatorname{supp}|u_\varepsilon|$ *is contained in the closure of $\{\, x \in \Omega_c \mid \|p_\varepsilon(x)\|_{L^2(I)} > \alpha \,\}$.*

The regularized problem $(\mathcal{P}_\varepsilon)$ can be solved efficiently with a semismooth Newton method, which admits a Banach space analysis; see [21, Theorem 3.7, Example 1.2]. Moreover, we obtain the original problem $(\mathcal{P})$ in the limiting case for $\varepsilon \to 0^+$.

PROPOSITION 4.2. *For $\varepsilon \to 0^+$ we have $j(\bar u) \le j_\varepsilon(u_\varepsilon) \to j(\bar u)$, where $\bar u$ is an (arbitrary) optimal solution of $(\mathcal{P})$. Moreover, the sequence of solutions of $(\mathcal{P}_\varepsilon)$ contains an accumulation point in the sense of weak-$*$ convergence and any such accumulation point is an optimal solution of $(\mathcal{P})$.*

Proof. We observe that

$$\alpha\|u_\varepsilon\|_{\mathcal{M}(\Omega_c,L^2(I))} + \tfrac{\varepsilon}{2}\|u_\varepsilon\|^2_{L^2(I\times\Omega_c)} \le j_\varepsilon(u_\varepsilon) \le J(S(y_0, 0)) \tag{4.3}$$

for all $\varepsilon > 0$, which implies that u_ε is bounded in $\mathcal{M}(\Omega_c, L^2(I))$. Take any subsequence converging in a weak-$*$ sense (again denoted by u_ε) and denote the limit point by $\hat u$. With Lemma 2.7 and the compact embedding we have that $y_\varepsilon = S(u_\varepsilon) \rightharpoonup \hat y = S(\hat u)$ in Y^s with strong convergence in $L^2(I \times \Omega)$ and $p_\varepsilon = S^*(J'(y_\varepsilon)) \to \hat p = S^*(J'(\hat y))$ strongly in $\mathcal{C}_0(\Omega \cup \Gamma, L^2(I))$. As in the proofs of Theorems 2.11 and 2.12 we can obtain from the variational inequality (4.1) that

$$\|\varepsilon u_\varepsilon + p_\varepsilon\|_{L^\infty(\Omega_c,L^2(I))} \le \alpha, \tag{4.4}$$

$$\alpha\|u_\varepsilon\|_{L^1(\Omega_c,L^2(I))} = -(u_\varepsilon, \varepsilon u_\varepsilon + p_\varepsilon). \tag{4.5}$$

With (4.3) we have $\|\varepsilon u_\varepsilon\|_{L^2(I\times\Omega_c)} \to 0$ for $\varepsilon \to 0$ and therefore $\varepsilon u_\varepsilon + p_\varepsilon \to \hat p$ in $L^2(I \times \Omega_c)$. Since $\|\cdot\|_{L^\infty(\Omega_c,L^2(I))}$ is weakly lower semicontinuous this implies $\|\hat p\|_{L^\infty(\Omega_c,L^2(I))} \le \alpha$. Taking the limit in (4.5) shows

$$\begin{aligned}\alpha\|\hat u\|_{\mathcal{M}(\Omega_c,L^2(I))} &\le \liminf_{\varepsilon\to 0}\, -(u_\varepsilon, \varepsilon u_\varepsilon + p_\varepsilon) \le \limsup_{\varepsilon\to 0}\, -(u_\varepsilon, \varepsilon u_\varepsilon + p_\varepsilon)\\ &\le \limsup_{\varepsilon\to 0} -\varepsilon\|u_\varepsilon\|^2_{L^2(I\times\Omega_c)} - \lim_{\varepsilon\to 0}\langle u_\varepsilon, p_\varepsilon\rangle \le -\langle \hat u, \hat p\rangle,\end{aligned}$$

using weak-$*$ convergence of u_ε and strong convergence of p_ε. The bound on $\hat{p}$ and this inequality imply the variational inequality

$$-\langle u-\hat{u},\hat{p}\rangle + \alpha\|\hat{u}\|_{\mathcal{M}(\Omega_c,L^2(I))} \le \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}$$

for all $u \in \mathcal{M}(\Omega_c, L^2(I))$. Therefore $-\hat{p} \in \alpha\partial\|\cdot\|_{\mathcal{M}(\Omega_c,L^2(I))}(\hat{u})$ and $\hat{u}$ is optimal (the first order necessary conditions are sufficient since j is convex). It remains to consider convergence of the functional values. We have

$$\begin{aligned} j(\bar{u}) \le j_\varepsilon(u_\varepsilon) &= J(y_\varepsilon) + \alpha\|u_\varepsilon\|_{L^1(\Omega_c,L^2(I))} + \tfrac{\varepsilon}{2}\|u_\varepsilon\|^2_{L^2(I\times\Omega_c)} \\ &= J(y_\varepsilon) - (u_\varepsilon,p_\varepsilon) - \tfrac{\varepsilon}{2}\|u_\varepsilon\|^2_{L^2(I\times\Omega_c)} \end{aligned}$$

by (4.5) and with similar arguments as before we obtain

$$j(\bar{u}) \le \limsup_{\varepsilon\to 0} j_\varepsilon(u_\varepsilon) \le j(\hat{u}) = j(\bar{u}).$$

We have shown convergence of the functional values for a specific subsequence with $u_\varepsilon \to \hat{u}$. The result for an arbitrary sequence follows by repeating the argument for any given subsequence and noting that $j(\bar{u})$ is unique. $\square$

Motivated by this we will use the following procedure to compute $\bar{u}$ in practice. In an inner loop, we use the semismooth Newton method from [21] to compute the minimizer u_ε for a small value of ε. Then we decrease ε by a constant factor, e.g., tenfold, and use the previous solution as an initial guess for the new iteration. In our experiments the Newton method exhibited robust convergence in each iteration and a globalization strategy was not needed.

5. Numerical examples. In this section we consider two examples, where one is geared towards verification of the convergence results in section 3.4 and the other is motivated by an application to inverse problems.

5.1. Order of convergence. We design an example for the specific setting in section 3 with an explicit solution on the interval $I = (0,T)$ and the two-dimensional domain $\Omega = \Omega_c = \Omega_o = (-1,1)\times(-1,1)$. For the construction of the example the optimal control is chosen as

$$\bar{u}(t) = T^{-2}\,(T-t)\,\delta_0$$

with a Dirac delta function in the origin. We can give the analytical solution $\bar{y}$ of $\partial_t y - \Delta y = \bar{u}$ with zero Dirichlet boundary conditions; see Figure 1. It can be represented by the series

$$\bar{y}(t,x) = \sum_{k\in\mathbb{Z},l\in\mathbb{Z}} (-1)^{k+l}\, G(t,\, x_1+2k,\, x_2+2l), \tag{5.1}$$

where $x = (x_1,x_2)^t$ and G is the free space solution given by

$$G(t,x_1,x_2) = \frac{1}{4\pi T^2}\left(\left(\tfrac{r^2}{4} - T + t\right)\mathrm{Ei}\left(-\tfrac{r^2}{4t}\right) + te^{-\frac{r^2}{4t}}\right)$$

and $r = \sqrt{x_1^2+x_2^2}$ is the distance to the origin. The function $\mathrm{Ei}(s) = \int_{-s}^{\infty} \frac{e^{-h}}{h}\,\mathrm{d}h$ is the exponential integral. The polar decomposition for $\bar{u} = \bar{u}'|\bar{u}|$ is given by

$$\bar{u}'(t) = \sqrt{3}\,T^{-\frac{3}{2}}\,(T-t)\,, \quad |\bar{u}| = \tfrac{1}{\sqrt{3}}T^{-\frac{1}{2}}\delta_0,$$

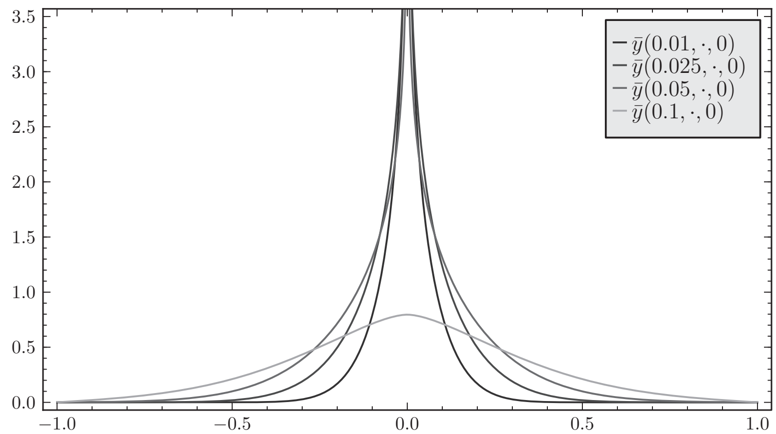


FIG. 1. *Snapshots of the exact state solution $\bar{y}$ at $x_2 = 0$ (for $T = 0.1$).*

and a matching adjoint state $\bar{p}$ which fulfills (2.14) and $\bar{p}(T) = 0$ can be chosen as

$$\bar{p}(t,x) = -\alpha\sqrt{3}\,T^{-\frac{3}{2}}\,(T-t)\,\cos\left(\tfrac{\pi}{2}x_1\right)\,\cos\left(\tfrac{\pi}{2}x_2\right).$$

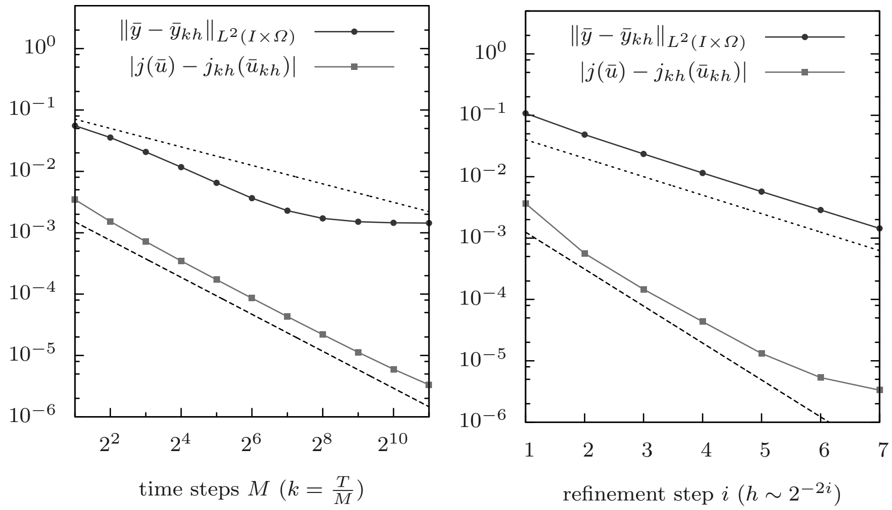
The reader may verify that this $\bar{p}$ fulfills (2.15), (2.17), and (2.18). By inspection of the adjoint equation (2.13) this determines the desired state y_d to be

$$y_d = \bar{y} + \partial_t\bar{p} + \Delta\bar{p}$$

for which we can now derive an explicit formula by differentiating $\bar{p}$.

We choose the final time as $T = 0.1$ and a relatively small parameter $\alpha = 0.01$. For the practical verification of the convergence results we compute the optimal solutions $\bar{u}_{kh}, \bar{y}_{kh}$ on an equidistant time grid with M steps and with a uniform triangulation of the square of different refinement levels. The series in (5.1) is approximated by the first nine terms, which yield a pointwise accuracy of about 10^{-12}. We use an adapted iterated quadrature formula in space to evaluate the integrals containing the singularity near $x = 0$ with sufficient accuracy. For the temporal integration, we use the box rule. The convergence plots are given in Figure 2. We also plot the corresponding rates of convergence as predicted in Theorems 3.19 and 3.20 without the logarithmic factor. As we can see, the rates for the functional match the predicted order of almost $\mathcal{O}(k)$ and $\mathcal{O}(h^2)$, which are plotted for visual comparison. For the state error we make this observation only in the case of refinement in space: Figure 2(b) clearly shows a rate of $\mathcal{O}(h)$ in this case. For the case of time refinement, we seem to observe in Figure 2(a) a slightly better rate than the predicted $\mathcal{O}(\sqrt{k})$ (until the spatial error starts to dominate from 128 time steps on). For this reason we give the experimental orders of convergence in Table 1, which seem to indicate a possible rate close to $\mathcal{O}(k^{0.8})$.

5.2. Reconstruction of a point source. In this section we discuss a practical application of the abstract problem formulation to an inverse source problem. The example falls slightly outside of the theoretical framework given in section 2 due to an additional transport term in the parabolic equation, Robin boundary conditions, and an objective functional involving boundary observation. However, the necessary



(a) Time refinement on grid level 7. (b) Space refinement with 2048 time steps.

FIG. 2. *Error plots of the optimal solutions.*

TABLE 1
Time refinement on grid level 7 *(as in Figure* 2(a)*).*

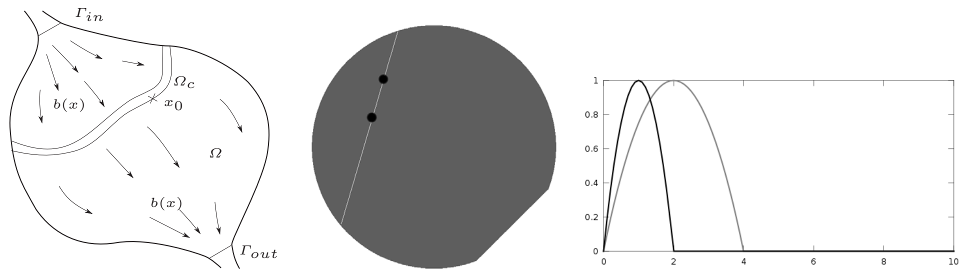
Time steps	$\lvert j(\bar{u}) - j_{kh}(\bar{u}_{kh})\rvert$	Rate	$\|\bar{y} - \bar{y}_{kh}\|_{L^2(I\times\Omega)}$	Rate
2	$3.458 \cdot 10^{-3}$	–	$5.543 \cdot 10^{-2}$	–
4	$1.527 \cdot 10^{-3}$	1.17924	$3.553 \cdot 10^{-2}$	0.641629
8	$7.160 \cdot 10^{-3}$	1.09267	$2.072 \cdot 10^{-2}$	0.778014
16	$3.470 \cdot 10^{-4}$	1.04502	$1.172 \cdot 10^{-2}$	0.822051
32	$1.714 \cdot 10^{-4}$	1.01757	$6.509 \cdot 10^{-3}$	0.848465
64	$8.569 \cdot 10^{-5}$	1.00013	$3.658 \cdot 10^{-3}$	0.831381
128	$4.316 \cdot 10^{-5}$	0.98937	$2.291 \cdot 10^{-3}$	0.675078
256	$2.193 \cdot 10^{-5}$	0.97669	$1.716 \cdot 10^{-3}$	0.416928
512	$1.131 \cdot 10^{-5}$	0.95550	$1.512 \cdot 10^{-3}$	0.182591

modifications would be mainly of a technical nature. The state equation for the example is a simplified model for the transport and diffusion of a pollutant y in a lake, given as

$$\left.\begin{aligned} \partial_t y - \nu\Delta y + b\cdot\nabla y = u \quad & \text{in } I\times\Omega, \\ \nu\partial_n y = 0 \quad & \text{on } I\times\partial\Omega\setminus\Gamma_{\text{in}}, \\ \nu\partial_n y - n\cdot b\, y = 0 \quad & \text{on } I\times\Gamma_{\text{in}}, \end{aligned}\right\} \tag{5.2}$$

with initial condition $y(0) = 0$. The domain Ω describes the surface of the lake, the inflow boundary Γ_{in} is a subset of $\partial\Omega$, $\nu > 0$ is a diffusion parameter, and b is assumed to be a static, smooth, and divergence-free vector field (i.e., we assume the influence of y on the flow b to be negligible). We additionally define an outflow boundary Γ_{out} such that b has the property

$$n\cdot b \quad \begin{cases} \le 0 & \text{on } \Gamma_{\text{in}}, \\ \ge 0 & \text{on } \Gamma_{\text{out}}, \\ = 0 & \text{on } \partial\Omega\setminus(\Gamma_{\text{in}}\cup\Gamma_{\text{out}}), \end{cases}$$



(a) Setup for (5.2). (b) Ω, Ω_c, $\hat{x}_1$, and $\hat{x}_2$. (c) Coefficients $\hat{u}_1$ and $\hat{u}_2$.

FIG. 3. *Inverse problem setup.*

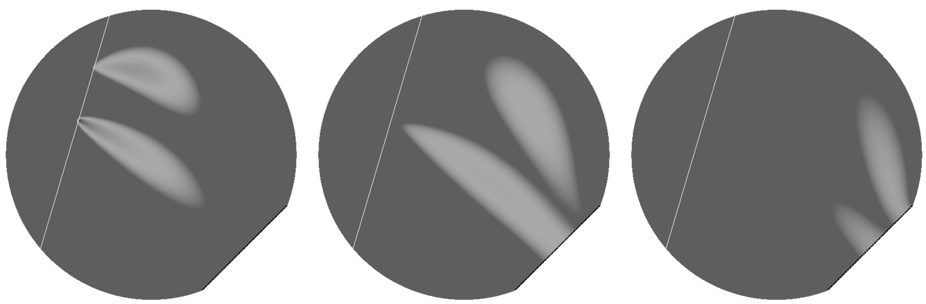


FIG. 4. *Snapshots of the exact state $\hat{y}$ at $t = 2, 4, 6$.*

where $n \colon \partial\Omega \to \mathbb{R}^d$ is the outer normal. The source term u is assumed to consist of a finite number of pointwise inflows

$$\hat{u} = \sum_{i=1}^{N} \hat{u}_i(t)\delta_{\hat{x}_i} \tag{5.3}$$

where $\hat{x}_i \in \Omega_c$ are unknown locations and $\hat{u}_i(t)$ describes the unknown amount of substance leaking into the lake at $\hat{x}_i$ and time t. Furthermore we assume it is known that $\hat{x}_i \in \Omega_c$, where Ω_c is a line (e.g., a pipeline) intersecting Ω.

A schematic depiction of the setup and exemplary exact data is given in Figure 3. Furthermore, the diffusion coefficient is chosen as $\nu = 0.002$ and the vector field b is given by the negative gradient of a potential v on Ω, which fulfills an elliptic equation to guarantee the condition $\nabla \cdot b = 0$ in the domain and inhomogeneous Neumann boundary conditions according to the conditions on $n \cdot b$ on the boundary. Corresponding snapshots for some $t \in I = (0, 10)$ of the state solution corresponding to the exact data are given in Figure 4.

For the inverse problem we have available only the concentration of y on the outflow boundary in the form $y_{\text{obs}} = \hat{y}|_{I\times\Gamma_{\text{out}}} + \delta$, where $\hat{y}$ is the solution of (5.2) corresponding to the true source (5.3) and the noise term $\delta \in L^2(I \times \Gamma_{\text{out}})$ stands for an additional measurement error (which we will set to a deterministic function in our numerical experiments). For the concrete example from Figure 4 the corresponding observations are depicted in Figure 5.

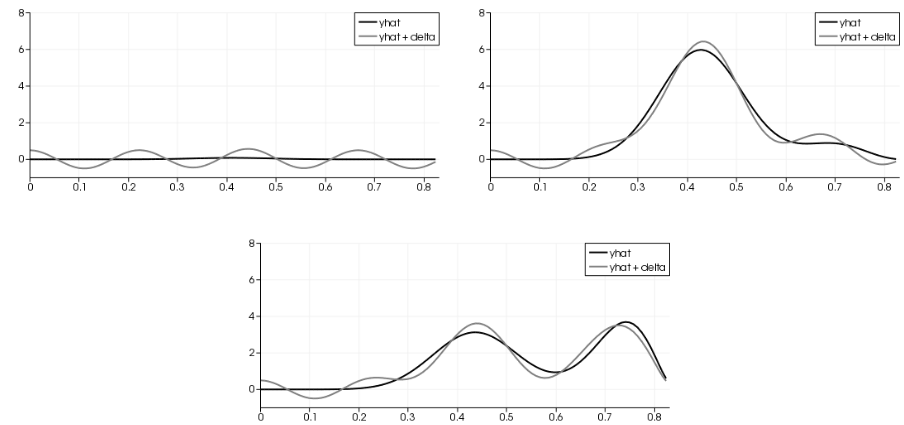


FIG. 5. *Snapshots of the observation* y_{obs} *on the observation boundary* Γ_{out} *(with and without noise) at* $t = 2, 4, 6$.

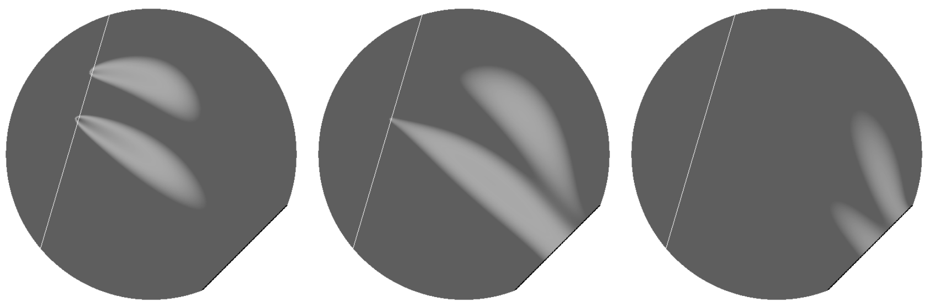
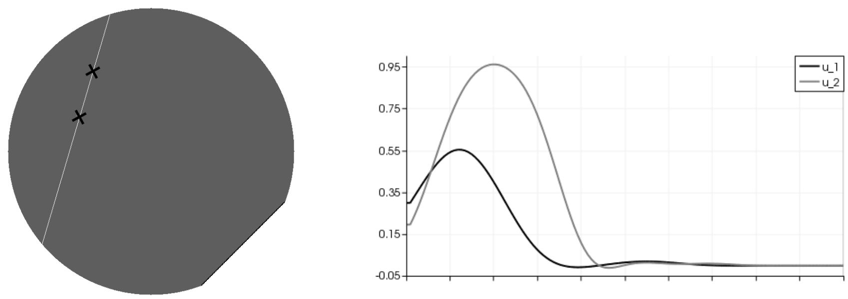


FIG. 6. *Snapshots of the reconstructed* $\bar{y}$ *at* $t = 2, 4, 6$ *for* $\alpha = 0.5$.

To give a reconstruction of the source $\hat{u}$, we propose to solve the deterministic inverse problem

$$\min_{u \in \mathcal{M}(\Omega_c, L^2(I))} \tfrac{1}{2}\|S(u) - y_{\text{obs}}\|^2_{L^2(I\times\Gamma_{\text{out}})} + \alpha\|u\|_{\mathcal{M}(\Omega_c, L^2(I))},$$

where $S(u)$ is the solution of (5.2) corresponding to u. This inverse problem formulation is similar to the approach described in [5], if we would somehow replace the Hilbert space $L^2(I)$ with $\mathbb{R}^M$ for some $M \in \mathbb{N}$. For the concrete example with the depicted data we empirically determine $\alpha = 0.5$ to be an appropriate regularization parameter. The optimal state solution $\bar{y}$ is visualized in Figure 6. For the numerical realization we added a small L^2 regularization term as described in section 4, with a value of $\varepsilon = 10^{-6}$ in the depicted simulation. Due to discretization and the additional L^2 regularization, the discrete $\bar{u}_{kh}$ does have not the structure as in (5.3) (for $N = 2$) since it is the linear combination of more than two Dirac delta functions. As a post-processing strategy, to obtain the visualization in Figure 7, we group all the connected components of the grid points in the support of $\bar{u}_{kh}$ and identify each of them with a central point $\tilde{x}_i$ of the component. In the concrete case we have exactly two components. Then we identify the spatial part of the of $\bar{u}_{kh}$ with $|\bar{u}_{kh}| \approx \sum_{i=1,2} c_i \delta_{\tilde{x}_i}$, where the c_i is the sum over all coefficients of $|\bar{u}_{kh}|$ in each component. From the optimality condition (2.18) we derive a reconstruction of the coefficients of the form $\tilde{u}_i(t) = -\frac{c_i}{\alpha}\bar{p}_{kh}(t, \tilde{x}_i)$; cf. Corollary 2.13.



(a) supp $\bar{u}_{kh}$ and reconstructed $\tilde{x}_1$ and $\tilde{x}_2$.

(b) Reconstructed $\tilde{u}_1$ and $\tilde{u}_2$.

FIG. 7. *Postprocessing: visualization of the reconstructed $\bar{u}$.*

We see that the outlined reconstruction procedure gives the main structural features of the exact source $\hat{u}$, such as the number and location of the points x_i, and a quantitatively adequate estimate of the coefficients u_i (which is in contrast to the results we would obtain with a regularization approach based on the L^2-norm). Certainly, there is a qualitative error between $\hat{u}$ and $\bar{u}$ which stems from the noise δ and the nonzero regularization parameter α. However, a detailed study of the reconstruction error for a systematic choice of α depending on the magnitude of δ (as in [5]) is beyond the scope of this paper.

Appendix A. Interpolation estimate. To prove the second part of the interpolation estimate from Lemma 3.13 we need an auxiliary lemma.

LEMMA A.1. *For any $w \in L^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I, L^2(\Omega))$ we have the estimate*

$$\sup_{t\in I} \|\nabla(w(t) - w(T))\|^2_{L^2(\Omega)} \le c\, \|\partial_t w\|_{L^2(I,L^2(\Omega))} \|\Delta w\|_{L^2(I,L^2(\Omega))},$$

where the constant c is independent of T.

Proof. Since $w \in C(\bar{I}, H_0^1(\Omega))$ with the trace theorem [2, Theorem III 4.10.2] we have a unique, continuous representation $[0,T] \ni t \mapsto w(t) \in H_0^1(\Omega)$ and hence for $\Delta w(t) \in H^{-1}(\Omega)$. Since $\|\Delta w(\cdot)\|_{L^2(\Omega)}$ is square integrable, it is finite almost everywhere and we can choose a point $t_0 \in [0,T]$, such that

$$\|\Delta w(t_0)\|^2_{L^2(\Omega)} \le \frac{1}{T} \int_0^T \|\Delta w(t)\|^2_{L^2(\Omega)} \,\mathrm{d}t.$$

We can estimate with the triangle inequality that

$$\sup_{t\in I} \|\nabla(w(t) - w(T))\|_{L^2(\Omega)} \le 2 \sup_{t\in I} \|\nabla(w(t) - w(t_0))\|_{L^2(\Omega)}. \tag{A.1}$$

To estimate the term on the right we define the function $v = w - w(t_0)$, which is an element of $L^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I, L^2(\Omega))$. By construction, v fulfills $v(t_0) = 0$ and $\partial_t v = \partial_t w$ and we can estimate

$$\|\Delta v\|_{L^2(I,L^2(\Omega))} \le 2\|\Delta w\|_{L^2(I,L^2(\Omega))}$$

by the choice of t_0. Now, we can apply a well-known identity, integration by parts, and Hölder's inequality to obtain for any $t \in I$ that

$$\begin{aligned}\|\nabla v(t)\|^2_{L^2(\Omega))} &= \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\|\nabla v(s)\|^2_{L^2(\Omega))}\,\mathrm{d}s = \int_{t_0}^{t} 2\,(\partial_t v(s), -\Delta v(s))\,\mathrm{d}s \\ &\leq 2\,\|\partial_t v\|_{L^2(I,L^2(\Omega))}\|\Delta v\|_{L^2(I,L^2(\Omega))} \leq 4\,\|\partial_t w\|_{L^2(I,L^2(\Omega))}\|\Delta w\|_{L^2(I,L^2(\Omega))},\end{aligned}$$

and we finish the proof by combining this with (A.1). ∎

Proof. (proof of Lemma 3.13). We first consider the second estimate

$$\|w - i_k w\|_{L^2(I,H^1_0(\Omega))} \leq c\,k^{\frac{1}{2}}\|f\|_{L^2(I,L^2(\Omega))}$$

on a reference interval $I' = (0,1)$ for an arbitrary $\hat{w} \in L^2(I', H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(I', L^2(\Omega))$. With Lemma A.1 it holds on the reference interval that

$$\begin{aligned}\|\nabla(\hat{w} - \hat{w}(1))\|^2_{L^2(I',L^2(\Omega))} &\leq \sup_{t\in I'}\|\nabla(\hat{w}(t) - \hat{w}(1))\|^2_{L^2(\Omega)} \\ &\leq c\,\|\partial_t \hat{w}\|_{L^2(I',L^2(\Omega))}\|\Delta\hat{w}\|_{L^2(I',L^2(\Omega))}.\end{aligned}$$

By linear transformation this implies for w, restricted to an arbitrary time interval I_m, that

$$\begin{aligned}\|w - w(t_m)\|^2_{L^2(I_m,H^1_0(\Omega))} &\leq c\,k_m\|\partial_t w\|_{L^2(I_m,L^2(\Omega))}\|\Delta w\|_{L^2(I_m,L^2(\Omega))} \\ &\leq c\,k_m\left(\|\partial_t w\|^2_{L^2(I_m,L^2(\Omega))} + \|\Delta w\|^2_{L^2(I_m,L^2(\Omega))}\right).\end{aligned}$$

The final result is obtained by summing these estimates over all intervals I_m for $m = 1\ldots M$ and using the parabolic regularity from Lemma 3.4. The proof for the first estimate, which is standard,

$$\|w - i_k w\|_{L^2(I,L^2(\Omega))} \leq c\,k\|f\|_{L^2(I,L^2(\Omega))},$$

can be based in a similar way on the estimate

$$\|w - w(t_m)\|^2_{L^2(I_m,L^2(\Omega))} \leq k_m^2\|\partial_t w\|^2_{L^2(I_m,L^2(\Omega))}$$

on each interval. ∎

REFERENCES

[1] H. W. Alt, *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung, 6th ed.*, Springer, Berlin, 2011.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems: Volume I: Abstract Linear Theory*, Birkhäuser Basel, 1995.

[3] H. Amann, *Compact embeddings of vector valued Sobolev and Besov spaces*, Glas. Mat., 35 (2000), pp. 161–177.

[4] C. Bernardi and V. Girault, *A local regularization operator for triangular and quadrilateral finite elements*, SIAM J. Numer. Anal., 35 (1998), pp. 1893–1916.

[5] K. Bredies and H. K. Pikkarainen, *Inverse problems in spaces of measures*, ESAIM: Control Optim. Calc. Var., 19 (2013), pp. 190–218.

[6] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts Appl. Math. 15, Springer, New York, 2008.

[7] E. Casas, C. Clason, and K. Kunisch, *Approximation of elliptic control problems in measure spaces with sparse solutions*, SIAM J. Control Optim., 50 (2012), pp. 1735–1752.

[8] E. Casas, C. Clason, and K. Kunisch, *Parabolic control problems in measure spaces with sparse solutions*, SIAM J. Control Optim., 51 (2013), pp. 28–63.

[9] E. Casas and E. Zuazua, *Spike controls for elliptic and parabolic PDEs*, Systems Control Lett., 62 (2013), pp. 311–318.

[10] J. Diestel and J. Uhl, *Vector Measures*, Math. Surveys, AM Society, Providence, RI, 1977.

[11] J. Droniou, *Solving convection-diffusion equations with mixed, Neumann and Fourier boundary conditions and measures as data, by a duality method*, Adv. Differential Equations, 5 (2000), pp. 1341–1396.

[12] J. Droniou and J.-P. Raymond, *Optimal pointwise control of semilinear parabolic equations*, Nonlinear Anal., 39 (2000), pp. 135–156.

[13] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Classics Appl. Math. 28, SIAM, Philadelphia, 1999.

[14] L. C. Evans, *Partial Differential Equations*, 2nd ed., American Mathematical Society, Providence, RI, 2010.

[15] W. Gong, M. Hinze, and Z. Zhou, *A priori error analysis for finite element approximation of parabolic optimal control problems with pointwise control*, SIAM J. Control Optim., 52 (2014), pp. 97–119.

[16] J. Griepentrog, H. Kaiser, and J. Rehberg, *Resolvent and heat kernel properties for second order elliptic differential operators with general boundary conditions on L^p*, Adv. Math. Sci. Appl., 11(2001), pp. 87–112.

[17] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing, Boston, 1985.

[18] K. Gröger, *A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann., 283 (1989), pp. 679–687.

[19] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela, *Hölder continuity and optimal control for nonsmooth elliptic problems*, Appl. Math. Optim., 60 (2009), pp. 397–428.

[20] R. Haller-Dintelmann and J. Rehberg, *Maximal parabolic regularity for divergence operators including mixed boundary conditions*, J. Differential Equations, 247 (2009), pp. 1354–1396.

[21] R. Herzog, G. Stadler, and G. Wachsmuth, *Directional sparsity in optimal control of partial differential equations*, SIAM J. Control Optim., 50 (2012), pp. 943–963.

[22] S. Lang, *Real Analysis, 2nd ed.*, Addison-Wesley, Reading, MA, 1983.

[23] D. Leykekhman and B. Vexler, *Optimal a priori error estimates of parabolic optimal control problems with pointwise control*, SIAM J. Numer. Anal., 51 (2013), pp. 2797–2821.

[24] A. Martínez, C. Rodríguez, and M. E. Vázquez-Méndez, *Theoretical and numerical analysis of an optimal control problem related to wastewater treatment*, SIAM J. Control Optim., 38 (2000), pp. 1534–1553.

[25] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.

[26] D. Meidner and B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems Part* I*: Problems without control constraints*, SIAM J. Control Optim., 47 (2008), pp. 1150–1177.

[27] L. Meziani, *On the dual space $C_0^*(S,X)$.*, Acta Math. Univ. Comenian (N.S.), 78 (2009), pp. 153–160.

[28] K. Pieper and B. Vexler, *A priori error analysis for discretization of sparse elliptic optimal control problems in measure space*, SIAM J. Control Optim., 51 (2013), pp. 2788–2808.

[29] A. Prignet, *Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures*, Rend. Mat. Appl. (7), 15 (1995), pp. 321–337.

[30] G. Stampacchia, *Le probléme de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Annales Inst. Fourier (Grenoble), 15 (1965), pp. 189–257.

[31] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, 2nd ed., Springer Ser. Comput. Math. 25, Springer-Verlag, Berlin, 2006.

[32] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, J. A. Barth Verlag, Heidelberg, 1978.