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Measurement in Quantum Mechanics

Hitoshi WAKITA

Research Institute for Theoretical Physics, Hiroshima University
Takehara, Hiroshima-ken

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In quantum mechanics it is well known that, if any two states are superposed, they interfere with each other. It is true, we should not deny such interference in principle, but we may assert what follows. When two states different from each other in a great many degrees of freedom are superposed, the interference effect becomes obscure. If they are different in an infinitely many degrees of freedom, they do not interfere at all, and their superposition is nothing but a mere ***probability function.*** This assertion enables us to understand how the ***probability amplitude*** for a micro-system is converted into a ***probability function*** for a measuring apparatus in the course of measurement.

§ 1. Theory of measurement

Though quantum mechanics has achieved a brilliant success, yet there is lack of a completely unified interpretation of it, and there has been a lively discussion with regard to the subject. Especially the theory of measurement in quantum mechanics has aroused much discussion among the physicists, and it is still in dispute because there is no proper interpretation acceptable for all of them. In the present article this problem will be discussed in connection with the views of H. S. Green.[1)]

At present the following may be considered to be the most "orthodox" interpretation on this subject, that is:

Let R be a Hermitian operator with a pure discrete and simple spectrum, and put

$$R\phi_i = \lambda_i \phi_i, \quad i=1, 2, \cdots. \tag{1}$$

Every wave function ψ can, then, be represented by a series

$$\psi = c_1\phi_1 + c_2\phi_2 + \cdots. \tag{2}$$

When we measure R in the state ψ and get the result λ_i, then the measurement changes ψ into ϕ_i, and the probability of this transition is given by

$$P(\phi_i) = |(\phi_i, \psi)|^2 = |c_i|^2. \tag{3}$$

We cannot accept such an interpretation from a macroscopic point of view unless ψ is only of a statistical nature, because the transition from ψ to ϕ_i is abrupt and non-causal. On the other hand, it is well known that any two states interfere with each other, and so ψ represents something more than a mere probability. In this sense ψ is usually called a *probability amplitude*. Now we are in a dilemma. From a macroscopic point of view ψ should be a mere *probability function*, that is, a mere statistical sum of ϕ's, while from a microscopic point of view it should be a *probability amplitude*. The most typical illustration of this dilemma is afforded by Schrödinger's famous allegory about a cat.[2] It is believed in general that the abrupt and non-causal change of states by measurement is a characteristic feature of quantum mechanics in contrast to classical mechanics. There is, however, no conclusive evidence of this interpretation, and the acceptance of such an interpretation is not necessary to develop the most part of the theory of quantum mechanics. This is the reason why there is no unified theory of measurement as yet.

Some assert that we must treat measuring apparatus by quantum statistics, and that the above dilemma will be solved by such treatment.[1),3)] Let us cite, for instance, Durand III: "the initial state of the object system may be known, but that of the apparatus system is at best delimited only by the results of a few quasi-classical observations. Furthermore, complete knowledge of the quantum mechanical details of the apparatus state is probably impossible even in principle.... A statistical treatment of the measuring process relative to the apparatus state is required." Though there may be some truth in his view, we cannot wholly agree with him. If an apparatus can be treated by quantum statistics, it must be treated by quantum mechanics at least in principle, and so the theory will not be essentially improved by such a statistical treatment alone. Nevertheless, we agree substantially with H. S. Green in his following views:[1)]

"The essential problem to be faced, therefore, concerns the detector. It has to be explained how the *probability amplitude* for the arrival of a micro-system in one of two or more channels gets converted into a *probability* for the transition between the metastable and stable states of a particular detector."†

Now, we interpret the process of "measurement" in the following way and explain it by a very simple example. Let [I] be a micro-system whose physical quantity R is measured by an apparatus [II]. Let, further, R have a pure discrete and simple spectrum and satisfy Eq. (1). At the beginning of "measurement" [II] is in a prepared metastable state $\boldsymbol{\psi}_0$, and in the course of "measurement" [I] interacts on [II] and converts $\boldsymbol{\psi}_0$ into a state $\boldsymbol{\psi}$. If [I] is in the state ϕ_i, then [II] is converted into a corresponding stable state $\boldsymbol{\phi}_i$, and we know that R has the value λ_i. As [II] is a measuring apparatus, these $\boldsymbol{\phi}$'s ought to differ from each other macroscopically. If the state ψ of [I] is given by Eq. (2), then

$$\boldsymbol{\psi}=c_1\boldsymbol{\phi}_1+c_2\boldsymbol{\phi}_2+\cdots. \tag{2'}$$

In principle, this Eq. (2′) is of the same nature as Eq. (2), that is, $\boldsymbol{\psi}$ is a super-

† Even if we accept this view of Green's completely, we cannot accept his explanation. At the end of § 2 we shall deal with this problem.

position of ϕ's, and it is a *probability amplitude.* We shall show, however, in the following section that, *if a state is a superposition of two or more states different from each other macroscopically, it can be considered as a mere probability function.* This implies that Eq. (2′) does not mean more than the following equation:

$$P(\phi_i)=|(\phi_i,\ \psi)|^2=|c_i|^2. \tag{3′}$$

Thus we can consider that, when we "observe" the state ψ, it changes into one of ϕ's abruptly and non-causally. Even if the relation between a micro-system and a measuring apparatus be more complicated, the above interpretation will remain essentially unchanged.

In conclusion we assert what follows: *In the course of "measurement" the micro-system interacts on a measuring apparatus, and the prepared metastable state of the apparatus changes into a new state, which is a superposition of several stable states. As these stable states should differ macroscopically, we can treat the new state as a mere probability function, and so, when we "observe" it, it changes into one of these stable states abruptly and non-causally.* In this sense we can agree with H. S. Green, and the above statement enables us to accept the "orthodox" view on measurement in a slightly modified sense. Care must be taken of the fact that we have distinguished the word "observation" from the word "measurement" in the preceding assertion. "Observation" implies the final process of the usual measuring process, that is, the cognizance by our own organs of perception. On the contrary, "measurement" implies the usual measuring process except for the final process, "observation." In our terminology there may be an automatic "measuring" apparatus, but not an automatic "observing" apparatus.

§ 2. Systems with a great many degrees of freedom

Let S_N be a system with N degrees of freedom, where N is a very large number, and $\mathfrak{h}_i$, $i=1, 2, \cdots, N$, be a Hilbert space corresponding to the i-th degree of freedom. Put

$$\mathfrak{H}_N=\otimes_{i=1}^{N}\mathfrak{h}_i, \tag{4}$$

and then it is a Hilbert space corresponding to S_N. For example, let S_N be made up of N particles with the same properties, and $\varphi_i(x_i)$ be the wave-function of the i-th particle. Then every state $\psi\in\mathfrak{H}_N$ can be represented by one of direct products

$$\varphi_1(x_1)\times\varphi_2(x_2)\times\cdots\times\varphi_N(x_N),\quad \varphi_i\in\mathfrak{h}_i,$$

or by their linear combination. In principle, S_N should obey quantum mechanics, but, if we apply the usual interpretation of quantum mechanics indiscriminately to S_N, we cannot but arrive at somewhat unphysical conclusions.

Let $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ be given by

$$\left.\begin{aligned}\boldsymbol{\psi}' &= \psi_1'(x_1)\times\psi_2'(x_2)\times\cdots\times\psi_N'(x_N),\\ \text{and}\quad & \\ \boldsymbol{\psi}'' &= \psi_1''(x_1)\times\psi_2''(x_2)\times\cdots\times\psi_N''(x_N).\end{aligned}\right\}\tag{5}$$

At first, we assume that $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ are the same except for the N-th degree of freedom, that is,

$$\left.\begin{aligned}&\psi_i'(x_i)=\psi_i''(x_i),\quad i=1,2,\cdots,N-1,\\ &\text{and that}\\ &(\psi_N',\ \psi_N'')=\int\overline{\psi_N'(x_N)}\psi_N''(x_N)\,dx_N=0.\end{aligned}\right\}\tag{6}$$

(5) and (6) show that

$$(\boldsymbol{\psi}',\ \boldsymbol{\psi}'')=\Pi_{i=1}^N(\psi_i',\ \psi_i'')=0,\tag{7}$$

and so these two states are "completely" different from each other in the usual quantum mechanical sense. $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$, however, should be very near to each other from a macroscopic point of view, because N is very large. In fact, it is not an operator q_i in $\mathfrak{h}_i$ but a mean of q_1, q_2, $\cdots$, q_N, that has a macroscopically important meaning, and $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ give almost equal expectation values to the mean. This implies that, in order that two states $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ in (5) can be said to be different macroscopically, they should differ in many degrees of freedom; namely, they should satisfy the following equations:

$$\Pi'^N_{i=1}(\psi_i',\ \psi_i'')=0,\tag{7$'$}$$

where each Π' is a product of (ψ_i',ψ_i''), $i=1, 2, \cdots, N$, with the exception of a few arbitrary members. In other words, a great many of $|(\psi_i',\ \psi_i'')|$, $i=1, 2, \cdots$, should be zero.

In quantum mechanics it is usually assumed that for any two states, ψ' and ψ'', there is at least one such physically meaningful operator q that

$$(\psi',\ q\psi'')\neq 0,\tag{8}$$

and this assumption seems to have some grounds. The validity of the above assumption, however, becomes doubtful for the system S_N. In S_N every physically meaningful operator is given by

$$\boldsymbol{q}_n=q_{i_1}\times q_{i_2}\times\cdots\times q_{i_n},\tag{9}$$

or by their linear combination, where q_{i_j} is an operator in $\mathfrak{h}_{i_j}$. Let $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$ satisfy Eqs. (5) and (6), and put

$$\boldsymbol{q}_N=q_1\times q_2\times\cdots\times q_N.$$

As

$$(\boldsymbol{\psi},\ \boldsymbol{q}_N\boldsymbol{\psi})=(\psi_1,\ q_1\psi_1)\times(\psi_2,\ q_2\psi_2)\times\cdots\times(\psi_N,\ q_N\psi_N),$$

there may be such q_N that $|(\psi', q_N\psi') - (\psi'', q_N\psi'')| \gg 1$. This is unphysical at least from a macroscopic point of view, and so such q_N will be physically meaningless. Furthermore, it is obvious that any q_n corresponding to a proper physical quantity, such as energy and momentum, is a product of a few q's. From these facts, we may conclude that, the larger n becomes, the less meaningful becomes q_n physically.† If ψ' and ψ'' are macroscopically different, every operator q satisfying

$$(\psi', q\psi'') \fallingdotseq 0 \tag{8'}$$

cannot be q_n with small n, and has little physical meaning. Therefore, we may say that

$$(\psi', q\psi'') = 0 \tag{10}$$

for almost every q which is physically meaningful.

Put

$$\psi^\theta = \psi' + e^{i\theta}\psi'' \tag{11}$$

for any macroscopically different states ψ' and ψ'', where θ is any real number. From (10) it is easy to show that

$$(\psi^\theta, q\psi^\theta) = (\psi^{\theta'}, q\psi^{\theta'})$$

for any θ and θ' and for almost every physically meaningful operator q. Thus every ψ^θ can be considered as representing the same state, and this means that ψ' and ψ'' do not interfere at all. Accordingly,

$$\psi = \psi' + \psi'' \tag{11'}$$

can be considered as a mere statistical sum of ψ' and ψ'', that is, a mere *probability function*, though it is a *probability amplitude* as a matter of principle. This is what we desired to show in the preceding section.

We shall compare the foregoing assertion with that of H. S. Green.[1)] Let $\rho(x, y)$ be a statistical matrix

$$\rho(x, y) = \{\psi'(x) + \psi''(x)\} \cdot \{\psi'(y) + \psi''(y)\}^*. \tag{12}$$

The interference of ψ' and ψ'' can be represented by

$$\begin{aligned}\rho'(x, y) &= \psi'(x) \cdot \psi''(y)^* \\ &= [\phi_1'(x_1) \cdot \phi_1''(y_1)^*] \cdot [\phi_2'(x_2) \cdot \phi_2''(y_2)^*] \cdots [\phi_N'(x_N) \cdot \phi_N''(y_N)^*].\end{aligned} \tag{13}$$

When ψ' and ψ'' are macroscopically different,

$$\begin{aligned}&\int \rho'(x, x)\, dx \\ &= \int \phi_1'(x_1)\overline{\phi_1''(x_1)}\, dx_1 \cdot \int \phi_2'(x_2)\overline{\phi_2''(x_2)}\, dx_2 \cdots \int \phi_N'(x_N)\overline{\phi_N''(x_N)}\, dx_N \\ &= \overline{(\psi', \psi'')} = 0.\end{aligned} \tag{14}$$

† The validity of this assertion will be clear in the quantum theory of fields. (See also § 3.)

It seems to us that Green's assertion is, in essence, nothing but to assert what follows:

Eq. (14) *shows that* $\rho'(x, y)$ *is a very small operator, accordingly it has no physical effect.*

This seems to us incorrect, however, because the norm of the operator $\rho'(x, y)$ is 1. On the contrary, our assertion is as follows:

For any physically meaningful operator $q(x, y)$,

$$\int \rho'(x, y) q(x, y) dx dy = 0, \tag{15}$$

therefore $\rho'(x, y)$ *has no physical effect.*

Lastly, it will be necessary to add a few words about the relations between ϕ's in (2'). At first sight, it seems paradoxical to apply (10) to these ϕ's, because all these states result from the same state ψ_0. For example, if

$$\phi_i = U_i \psi_0, \quad i = 1, 2, \cdots, N,$$

then

$$\phi_i = U_i U_j^{-1} \phi_j,$$

and

$$(\phi_i, U_i U_j^{-1} \phi_j) = (\phi_i, \phi_i) = 1$$

in contradiction to (10). In order to solve this dilemma, it is necessary to remember the fact that ψ_0 is the prepared *metastable* state; therefore, even if the operator U_i be physically meaningful, U_i^{-1} has no physical meaning. *Maxwell's demon* will be responsible for taking these operators into consideration. Only in this respect, it is necessary to take account of the statistical nature of the apparatus system. It is noteworthy that the statistical consideration is necessary not to show how ψ in (2') can be a mere probability function but to explain how various macroscopically different states can result from the same metastable state.

§ 3. Systems with an infinitely many degrees of freedom

In the preceding section we have shown that *a system with a great many degrees of freedom* is fairly different from a system with a few degrees of freedom in its physical import. The difference is, however, only quantitative and not qualitative. In order to clarify the distinction between these systems, we shall consider a system S_∞ with an infinitely many degrees of freedom, because it can be considered as an extreme case of S_N. Comparing S_∞ with S_N, we can show how they are qualitatively different, and this serves as an extreme illustration of the aforementioned difference.

In place of Eq. (4), we put

$$\mathfrak{H}_\infty = \otimes_{i=1}^{\infty} \mathfrak{h}_i, \tag{16}$$

where the right-hand side is the infinite direct product,[4] and it is a Hilbert space corresponding to S_∞.† The set of all physically meaningful operators of this system forms an algebra $\mathfrak{A}$, which is generated by operators in all $\mathfrak{h}$'s.[5] For example, in the quantum theory of fields every physical quantity can be represented by creation and annihilation operators. This means that they belong to $\mathfrak{A}$ or to its closure $\mathfrak{A}^c$ in an appropriate topology. Even a unitary operator $\exp(iHt)$ belongs to $\mathfrak{A}^c$ where H is the Hamiltonian. On the other hand, every state of this system is nothing but a linear functional f on $\mathfrak{A}$ or $\mathfrak{A}^c$,[5] and can be represented by a vector $\Psi_f \in \mathfrak{H}_\infty$ as

$$f(\boldsymbol{q}) = (\Psi_f,\ \boldsymbol{q}\Psi_f), \quad \boldsymbol{q} \in \mathfrak{A}^c. \tag{17}$$

Now we consider the superposition of two states Ψ' and Ψ'', each of which belongs to an incomplete direct product.†† There are two cases to be distinguished. First, we consider the case in which Ψ' and Ψ'' belong to the same incomplete direct product. This case is similar to that of quantum mechanics, and there is such an operator $\boldsymbol{q} \in \mathfrak{A}^c$ that

$$(\Psi',\ \boldsymbol{q}\Psi'') \neq 0. \tag{18}$$

Put

$$\Psi^\theta = \Psi' + e^{i\theta}\Psi''; \tag{19}$$

then Ψ^θ represent different states for different θ, and we may say that Ψ' and Ψ'' interfere. It is obvious that there are two operators $\boldsymbol{q}'$ and $\boldsymbol{q}''$ and a vector Ψ^0 such that

$$\Psi' = \boldsymbol{q}'\Psi^0, \quad \Psi'' = \boldsymbol{q}''\Psi^0, \quad \boldsymbol{q}',\ \boldsymbol{q}'' \in \mathfrak{A}^c. \tag{20}$$

As

$$\Psi' + e^{i\theta}\Psi'' = (\boldsymbol{q}' + e^{i\theta}\boldsymbol{q}'')\Psi^0, \quad \boldsymbol{q}' + e^{i\theta}\boldsymbol{q}'' \in \mathfrak{A}^c, \tag{21}$$

we can represent the superposition as the sum of operators, but we cannot represent it by using f. In fact,

$$(f' + e^{i\theta}f'')(\boldsymbol{q}) = (\Psi',\ \boldsymbol{q}\Psi') + e^{i\theta}(\Psi'',\ \boldsymbol{q}\Psi''),$$

and it is not equal to

$$f^\theta(\boldsymbol{q}) = (\Psi^\theta,\ \boldsymbol{q}\Psi^\theta).$$

The second is the case in which Ψ' and Ψ'' belong to different incomplete direct products. In this case there is no operator satisfying Eq. (18), and Ψ^θ

† $\mathfrak{H}_\infty$ is not separable. On the contrary, an incomplete direct product $\mathfrak{H}_\sigma$ is separable, but we cannot tell which of them is more appropriate for S_∞ from a physical point of view. $\mathfrak{H}_\sigma$, however, resembles $\mathfrak{H}_N$ in its character, so we use not $\mathfrak{H}_\sigma$ but $\mathfrak{H}_\infty$ in the following.

†† Essentially this means that Ψ' and Ψ'' are direct products

$$\Psi' = \times_{i=1}^{\infty}\phi_i', \quad \Psi'' = \times_{i=1}^{\infty}\phi_i'', \quad \phi_i',\ \phi_i'' \in \mathfrak{h}_i.$$

In the first case of the following, $(\phi_i',\ \phi_i'') = 0$ only for a finite number of i's, and in the second case, $(\phi_i',\ \phi_i'') = 0$ for an infinite number of i's.

represent the same state for all θ; namely, Ψ' and Ψ'' do not interfere. Ψ^{θ} can be given by

$$f^{\theta}(\boldsymbol{q})=(f'+f'')(\boldsymbol{q})=(\Psi',\ \boldsymbol{q}\Psi')+(\Psi'',\ \boldsymbol{q}\Psi''). \tag{22}$$

As there are no operators satisfying (20), Ψ^{θ} cannot be represented by a sum of operators. *To sum up, in the first case* $(\Psi'+\Psi'')$ *is a probability amplitude and can be represented by a sum of operators as in Eq.* (21). *On the contrary, in the second case* $(\Psi'+\Psi'')$ *is a mere probability function and can be represented by a sum of states as in Eq.* (22). In each case the representation by operators or by f's will be more essential than that by Ψ's.

As an illustration of the above argument, we shall consider the difference between the classical field theory and the quantum field theory. Divide the whole space into many small domains V_i, $i=1, 2, \cdots$. In the quantum theory there is a Hilbert space $\mathfrak{H}_{V_i}$ corresponding to each V_i, and the Hilbert space $\mathfrak{H}_{V_\infty}$ which corresponds to V_∞ is given by[5]

$$\mathfrak{H}_{V_\infty}=\bigotimes_{i=1}^{\infty}\mathfrak{H}_{V_i}. \tag{23}$$

That is, any state $\Psi\in\mathfrak{H}_{V_\infty}$ is given by a direct product

$$\Psi=\times_{i=1}^{\infty}\Psi_i,\quad \Psi_i\in\mathfrak{H}_{V_i}, \tag{24}$$

or by their linear combination

$$\Psi=\sum_{j}\{\times_{i=1}^{\infty}\Psi_i^j\}. \tag{25}$$

Every state Ψ can be represented as in (25), but it is not necessarily represented as in (24). All these states, however, are equivalent. In fact, whether a state Ψ can be written as in (24) or not is relative to the division $\{V_i\}$. If a state is written as in (24) for any division $\{V_i\}$, it is nothing but the vacuum. On the contrary, in the classical theory a state is completely determined when we know one in each V_i. Therefore, we may say that a state is always given by (24) in the classical theory.

In principle, the classical theory should be an approximation to quantum theory. Hence, it is necessary to explain how to settle the difference above mentioned. A domain V_i sufficiently small from the view-point of the classical theory may be large enough to contain a great many degrees of freedom from the view-point of quantum theory. In the following we take only such divisions. Two states different with each other from the view-point of the classical theory should be different macroscopically at least in one V_i, and the sum of two such states is a mere probability function. This implies that a state able to be written as in (24) plays a principal role, and that a state unable to be in the form of (24) plays only a subordinate role. Of course, there is no state which can be written as in (24) for any division $\{V_i\}$ except for the vacuum, but there are states which can be written as in (24) for any division $\{V_i\}$ in an approximate sense, and it is only such states that are fundamental in the classical theory.† We may say, therefore, that in the classical theory a "state" means only such a state as in (24), and this explains the aforementioned difference.

References

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† It is obvious that every ϕ_i in Eq. (2′) should be such a fundamental state. On the contrary, a state ϕ which is a superposition of these ϕ's cannot be such a fundamental state.