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# Measures of finite (r,p)-energy and potentials on a separable metric space

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**Abstract.** We discuss measures of $(r,p)$-finite energy associated with a Markovian semigroup on a separable metric space. We also investigate a relation with the $(r,p)$-potentials.

## 1. Introduction

In this paper, we discuss measures of finite $(r,p)$-energy on a separable metric space. Our argument is based on the $(r,p)$-capacity. As for the $(r,p)$-capacity, fundamental results were obtained by Fukushima-Kaneko [11]. Further Sugita [22] considered the $(r,p)$-capacity on the Wiener space extensively. He introduced positive generalized functions on the Wiener space and discussed measures of finite $(r,p)$-energy. We will generalize his result in more general setting.

To be precise, let $X$ be a separable metric space. We do not assume that $X$ is complete in general. We denote the Borel $\sigma$-field on $X$ by $\mathcal{B}(X)$. Let $m$ be a *finite* Borel measure on $X$. Suppose a contraction semigroup $\{T_t\}$ on $L^2(X;m)$ is given. We assume that the semigroup is strongly continuous and Markovian but we do not assume that the semigroup is symmetric in general. In addition, we assume that the dual semigroup $\{T_t^*\}$ is also Markovian. Then by the interpolation theorem, $\{T_t\}$ can be defined on $L^p(X;m)$ as a strongly continuous contraction semigroup for $p \geq 1$. For $r > 0$ and $p \geq 1$, set

$$(1.1) \qquad V_r = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t dt,$$

and define the Sobolev space $(\mathcal{F}_{r,p}, \|\cdot\|_{r,p})$ by

$$(1.2) \qquad \mathcal{F}_{r,p} := V_r(L^p(X;m)), \quad \|u\|_{r,p} = \|f\|_p \text{ for } u = V_r f,\ f \in L^p(X;m)$$

where $\|f\|_p$ denotes the $L^p$-norm of $f$. Then, the $(r,p)$-capacity $C_{r,p}$ is defined as follows: for an open set $G \subseteq X$,

$$(1.3) \qquad C_{r,p}(G) := \inf\{\|u\|_{r,p}^p\,;\ u \in \mathcal{F}_{r,p},\ u \geq 1 \quad m\text{-a.e. on } G\}$$

and for an arbitrary set $B \subseteq X$,

$$C_{r,p}(B) := \inf\{C_{r,p}(G)\,;\, G \text{ is open and } G \supseteq B\}. \tag{1.4}$$

We assume the following conditions:

(A.1) $\mathcal{F}_{r,p} \cap C_b(X)$ is dense in $\mathcal{F}_{r,p}$ and $1 \in \mathcal{F}_{r,p}$.

(A.2) There exists an algebra $\mathcal{D} \subseteq \mathcal{F}_{r,p} \cap C_b(X)$ that separates points of $X$.

(A.3) The capacity is *tight*, i.e., for any $\varepsilon > 0$ there exists a compact set $K$ such that $C_{r,p}(X \setminus K) < \varepsilon$.

Here $C_b(X)$ is the set of all bounded continuous functions on $X$. We may and do assume that $1 \in \mathcal{D}$. Note that under the assumption (A.2), $\mathcal{D}$ separates tight measures on $X$, i.e., if two finite tight measures $\mu$ and $\nu$ satisfy

$$\int_X f(x)\mu(dx) = \int_X f(x)\nu(dx), \qquad \forall f \in \mathcal{D},$$

then $\mu = \nu$ (see, e.g., [6, Theorem 4.5]).

Let $(\mathcal{F}_{r,p})^*$ be the dual space of $\mathcal{F}_{r,p}$. We may regard an element of $(\mathcal{F}_{r,p})^*$ as a generalized function. $\varphi \in (\mathcal{F}_{r,p})^*$ is said to be *positive* if for any $f \in \mathcal{F}_{r,p}$ such that $f \geq 0$ $m$-a.e.,

$$\langle f, \varphi \rangle \geq 0. \tag{1.5}$$

We will establish that a positive generalized function defines a measure on $X$. We call it the measure of finite $(r,p)$-energy. We also show that an equilibrium potential is a typical example of non-negative generalized function and give a characterization of a set of capacity zero by using measures of finite $(r,p)$-energy.

On the other hand, Feyel-de La Pradelle [8] discussed the capacity of functions for Gaussian measures. We remark that similar argument can be done in our setting.

The organization of the paper is as follows. We review fundamental properties of Sobolev spaces and $(r,p)$-capacity in the section 2. In the section 3, we define positive generalized functions and give a correspondence with measures. In the section 4, we discuss $(r,p)$-potentials, $(r,p)$-equilibrium potentials and measures of finite $(r,p)$-energy. We also give a characterization of capacity zero set. Lastly, we discuss the capacity of functions and the relation with positive generalized functions.

## 2. $(r,p)$-capacity

We review the Sobolev space $\mathcal{F}_{r,p}$ and fundamental properties of $(r,p)$-capacities. We keep the assumptions (A.1), (A.2) and (A.3) throughout the paper.

For $r > 0$ and $\alpha > 0$ we define an operator $V_r^{(\alpha)}$ on $L^p(X;m)$ by

$$V_r^{(\alpha)} = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-\alpha t} T_t dt. \tag{2.1}$$

For $r = 0$, we set $V_0^{(\alpha)} = I$ for convention where $I$ is the identity operator. Formally, we sometimes write $V_r^{(\alpha)} = (\alpha - A)^{-r/2}$ where $A$ is the generator. It is well-known that for $r, s \geq 0$, $V_{r+s}^{(\alpha)} = V_r^{(\alpha)} V_s^{(\alpha)}$.

**Proposition 2.1.** *$\alpha^{r/2} V_r^{(\alpha)}$ is a Markovian contraction operator. Moreover $\alpha^{r/2} V_r^{(\alpha)} \to I$ strongly as $\alpha \to \infty$.*

*Proof.* The first assertion is easily obtained from the definition. We show the convergence of $\alpha^{r/2} V_r^{(\alpha)}$. By the definition,

$$\alpha^{r/2} V_r^{(\alpha)} = \frac{\alpha^{r/2}}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-\alpha t} T_t dt = \frac{1}{\Gamma(r/2)} \int_0^\infty s^{r/2-1} e^{-s} T_{s/\alpha} ds.$$

Now by noting that $T_t \to I$ strongly as $t \to 0$, we get a desired result. □

If $r = 2$, then $V_r^{(\alpha)} = G_\alpha$ where $G_\alpha$ is the resolvent. The following resolvent equation is well-known:

$$G_\alpha = (I + (\beta - \alpha) G_\alpha) G_\beta.$$

We shall extend this identity. By a formal calculation, we can easily presume

$$(\alpha - A)^{-r/2} = (I + (\beta - \alpha) G_\alpha)^{r/2} (\beta - A)^{-r/2}.$$

Let us justify this identity. First we define $(I + (\beta - \alpha) G_\alpha)^{r/2}$ by

$$(I + (\beta - \alpha) G_\alpha)^{r/2} = \sum_{n=0}^{\infty} c_n \{(\beta - \alpha) G_\alpha\}^n \tag{2.2}$$

where $c_n$, $n = 0, 1, 2, \dots$ are coefficients of Taylor expansion for $(1 + x)^{r/2}$, i.e.,

$$(1 + x)^{r/2} = \sum_{n=0}^{\infty} c_n x^n.$$

If $|\beta - \alpha| < \alpha$, then $\|(\beta - \alpha) G_\alpha\|_{\text{op}} < 1$ and hence (2.2) converges uniformly. Here $\|\cdot\|_{\text{op}}$ denotes the operator norm in $L^p(X; m)$. We have the following:

**Proposition 2.2.** *If $|\beta - \alpha| < \alpha$, then it holds that*

$$V_r^{(\alpha)} = (I + (\beta - \alpha) G_\alpha)^{r/2} V_r^{(\beta)} = V_r^{(\beta)} (I + (\beta - \alpha) G_\alpha)^{r/2}.$$

*Proof.* In general, for any function $h(t)$,

$$\begin{aligned}
&\int_0^\infty e^{-\beta t} h(t) T_t (\beta - \alpha) G_\alpha \, dt \\
&= \int_0^\infty dt \, e^{-\beta t} h(t) T_t \int_0^\infty ds \, e^{-\alpha s} (\beta - \alpha) T_s \\
&= \int_0^\infty dt \int_0^\infty ds \, e^{-\beta t} h(t) e^{-\alpha s} (\beta - \alpha) T_{t+s} \\
&= \int_0^\infty d\sigma \int_0^\sigma d\tau \, e^{-\beta \tau} h(\tau) e^{-\alpha(\sigma - \tau)} (\beta - \alpha) T_\sigma \quad (t + s = \sigma, \ t = \tau) \\
&= \int_0^\infty e^{-\beta \sigma} K h(\sigma) T_\sigma \, d\sigma
\end{aligned}$$

where
$$Kh(\sigma) = (\beta - \alpha) \int_0^\sigma h(\tau) e^{(\beta-\alpha)(\sigma-\tau)} d\tau.$$
Set $g(t) = t^{r/2-1}$. Then repeating above procedure, we have
$$\frac{1}{\Gamma(r/2)} \int_0^\infty e^{-\beta t} t^{r/2-1} T_t \{(\beta - \alpha) G_\alpha\}^n dt = \frac{1}{\Gamma(r/2)} \int_0^\infty e^{-\beta t} K^n g(t) T_t \, dt.$$
Thus we have
$$\begin{aligned} (I + (\beta - \alpha) G_\alpha)^{r/2} V_r^{(\beta)} &= \frac{1}{\Gamma(r/2)} \int_0^\infty e^{-\beta t} t^{r/2-1} T_t \sum_{n=0}^\infty c_n \{(\beta - \alpha) G_\alpha\}^n dt \\ &= \frac{1}{\Gamma(r/2)} \int_0^\infty e^{-\beta t} \sum_{n=0}^\infty c_n K^n g(t) T_t \, dt. \end{aligned}$$
Now it is enough to show
$$e^{-\beta t} \sum_{n=0}^\infty c_n K^n g(t) = e^{-\alpha t} t^{r/2-1}. \tag{2.3}$$
To show this, we use the Laplace transform. By integration by parts, we have
$$\begin{aligned} &\int_0^\infty e^{-\lambda t} e^{-\beta t} K^{n+1} g(t) dt \\ &= \int_0^\infty e^{-\alpha t} e^{-\lambda t} (\beta - \alpha) \int_0^t e^{-(\beta-\alpha)\tau} K^n g(\tau) d\tau \\ &= \int_0^\infty e^{-\lambda t} e^{-\alpha t} (\beta - \alpha)(\alpha + \lambda)^{-1} e^{-(\beta-\alpha)t} K^n g(t) dt \\ &= \int_0^\infty e^{-\lambda t} e^{-\beta t} (\beta - \alpha)(\alpha + \lambda)^{-1} K^n g(t) dt. \end{aligned}$$
Hence, inductively we have
$$\int_0^\infty e^{-\lambda t} e^{-\beta t} t^{r/2-1} (\beta - \alpha)^n (\alpha + \lambda)^{-n} dt = \int_0^\infty e^{-\lambda t} e^{-\beta t} K^n g(t) dt.$$
Therefore
$$\begin{aligned} &\int_0^\infty e^{-\lambda t} e^{-\beta t} \sum_{n=0}^\infty c_n K^n g(t) dt \\ &= \int_0^\infty e^{-\lambda t} e^{-\beta t} t^{r/2-1} \sum_{n=0}^\infty c_n (\beta - \alpha)^n (\alpha + \lambda)^{-n} dt \\ &= \Gamma(r/2)(\beta + \lambda)^{-r/2} (1 + (\beta - \alpha)/(\alpha + \lambda))^{r/2} \\ &= \Gamma(r/2)(\alpha + \lambda)^{-r/2} \\ &= \int_0^\infty e^{-\lambda t} e^{-\alpha t} t^{r/2-1} dt. \end{aligned}$$
By the uniqueness of inverse Laplace transform, we have (2.3). This completes the proof. □

By the above proposition, we have that $\mathrm{Ran}(V_r^{(\alpha)})$ is independent of $\alpha$. We simply denote $V_r^{(1)}$ by $V_r$ and set

$$\mathcal{F}_{r,p} := \mathrm{Ran}(V_r) = V_r(L^p(X;m)). \tag{2.4}$$

Then the following proposition is easily obtained from Proposition 2.2.

**Proposition 2.3.** *$V_r$ is injective and $\mathcal{F}_{r,p}$ is dense in $L^p(X;m)$.*

Defining a norm $\|\cdot\|_{r,p}$ on $\mathcal{F}_{r,p}$ by

$$\|u\|_{r,p} = \|f\|_p \text{ for } u = V_r f,\ f \in L^p(X;m),$$

$(\mathcal{F}_{r,p}, \|\cdot\|_{r,p})$ forms a Banach space. For negative index $-r$, we define a norm $\|\cdot\|_{-r,p}$ by

$$\|f\|_{-r,p} = \|V_r f\|_p \text{ for } f \in L^p(X;m).$$

We denote the completion of $L^p(X;m)$ under the norm $\|\cdot\|_{-r,p}$ by $\mathcal{F}_{-r,p}$. For $r, s \geq 0$, $V_r\colon \mathcal{F}_{s,p} \to \mathcal{F}_{s+r,p}$ is the isometric isomorphism since $V_{r+s} = V_r V_s$. More generally:

**Proposition 2.4.** *For $s \in \mathrm{R}$ and $r \geq 0$, $V_r\colon \mathcal{F}_{s,p} \to \mathcal{F}_{s+r,p}$ is the isometric isomorphism.*

*Proof.* It is enough to prove this in the case $s + r \leq 0$. First we give a precise definition of $V_r$. For $f \in L^p(X;m) \subseteq \mathcal{F}_{s,p}$, $V_r f$ is already defined and

$$\|V_r f\|_{s+r,p} = \|V_{-s-r} V_r f\|_p = \|V_{-s} f\|_p = \|f\|_{s,p}.$$

Now by noting that $L^p(X;m)$ is dense in $\mathcal{F}_{s,p}$, $V_r$ can be extended uniquely to an isometry on $\mathcal{F}_{s,p}$.

The rest is devoted to prove that $V_r$ is surjective. By noting that $\mathcal{F}_{r,p}$ is dense in $L^p(X;m)$, for any $f \in L^p(X;m) \subseteq \mathcal{F}_{s,p}$, we can choose a sequence $\{f_n\} \subseteq L^p(X;m)$ such that $\lim_{n\to\infty} \|V_r f_n - f\|_p = 0$. Hence

$$\begin{aligned} \|V_r f_n - f\|_{s+r,p} &= \|V_{-s-r}(V_r f_n - f)\|_p \\ &\leq \|V_r f_n - f\|_p \qquad (V_{-s-r} \text{ is the contraction}) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Since $V_r$ is isometric, $V_r(\mathcal{F}_{s,p})$ is a closed subspace of $\mathcal{F}_{s+r,p}$ and hence

$$V_r(\mathcal{F}_{s,p}) \supseteq L^p(X;m).$$

Again using the closeness of $V_r(\mathcal{F}_{s,p})$, we have

$$V_r(\mathcal{F}_{s,p}) \supseteq \overline{L^p(X;m)}^{\|\cdot\|_{s+r,p}} = \mathcal{F}_{s+r,p}$$

which completes the proof. $\square$

Now the following propositions are obvious.

**Proposition 2.5.** *For $r \geq 0$, $s \in \mathbf{R}$, $\mathcal{F}_{r+s,p}$ is a dense subspace in $\mathcal{F}_{s,p}$ and it holds that*

$$\|f\|_{s,p} \leq \|f\|_{s+r,p} \qquad \forall f \in \mathcal{F}_{r+s,p}.$$

*Further, if $p' \geq p$, then $\mathcal{F}_{s,p'}$ is a dense subspace in $\mathcal{F}_{s,p}$ and*

$$\|f\|_{s,p} \leq m(X)^{1/p-1/p'}\|f\|_{s,p'} \qquad \forall f \in \mathcal{F}_{s,p'}.$$

From now on, we restrict ourselves to the case $p > 1$ in order to use the reflexivity of $L^p(X;m)$. Let $q$ be the conjugate exponent of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We denote the dual semigroup of $\{T_t\}_{t\geq 0}$ by $\{\hat{T}_t\}_{t\geq 0}$, i.e., $\hat{T}_t = T_t^*$. Since $L^p(X;m)$ is reflexive, $\{\hat{T}_t\}_{t\geq 0}$ is strongly continuous on $L^q(X;m)$ (see [4, Theorem 1.34]). Moreover $\{\hat{T}_t\}_{t\geq 0}$ is a Markovian contraction semigroup. Hence we can define $\hat{V}_r$, $\hat{\mathcal{F}}_{r,q}$ similarly:

$$\hat{V}_r = \frac{1}{\Gamma(r/2)}\int_0^\infty t^{r/2-1}e^{-t}\hat{T}_t dt \quad \text{and} \quad \hat{\mathcal{F}}_{r,q} = \hat{V}_r(L^q(X;m)).$$

We denote the norm on $\hat{\mathcal{F}}_{r,q}$ by $\|\cdot\|^\wedge_{r,q}$. If $\{T_t\}$ is symmetric, then $\hat{\mathcal{F}}_{r,q} = \mathcal{F}_{r,q}$.

**Proposition 2.6.** *For $r \geq 0$ and $1 < p,q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\hat{\mathcal{F}}_{-r,q}$ is isometrically isomorphic to the dual space $(\mathcal{F}_{r,p})^*$ of $\mathcal{F}_{r,p}$. Moreover under this isomorphism, it holds that for $f \in \mathcal{F}_{r,p} \subseteq L^p(X;m)$ and $g \in L^q(X;m) \subseteq \hat{\mathcal{F}}_{-r,q}$,*

$$_{\mathcal{F}_{r,p}}\langle f, g\rangle_{\hat{\mathcal{F}}_{-r,q}} = {}_{L^p}\langle f, g\rangle_{L^q}. \tag{2.5}$$

*Proof.* By Proposition 2.4, we have isomorphisms

$$\begin{aligned} V_r^* &: (\mathcal{F}_{r,p})^* \longrightarrow (L^p)^* \\ \hat{V}_r &: \hat{\mathcal{F}}_{-r,q} \longrightarrow L^q. \end{aligned}$$

By identifying $(L^p)^*$ and $L^q$ as usual, we get the isomorphism

$$V_r^{*-1}\hat{V}_r : \hat{\mathcal{F}}_{-r,q} \longrightarrow (\mathcal{F}_{r,p})^*.$$

Further for $f \in \mathcal{F}_{r,p}$ and $g \in L^q(X;m)$,

$$\begin{aligned} _{\mathcal{F}_{r,p}}\langle f, g\rangle_{\hat{\mathcal{F}}_{-r,q}} &= {}_{\mathcal{F}_{r,p}}\langle f, V_r^{*-1}\hat{V}_r g\rangle_{(\mathcal{F}_{r,p})^*} \\ &= {}_{L^p}\langle V_r^{-1}f, \hat{V}_r g\rangle_{L^q} \\ &= {}_{L^p}\langle V_r^{-1}f, (V_r)^* g\rangle_{L^q} \\ &= {}_{L^p}\langle f, g\rangle_{L^q}. \end{aligned}$$

This completes the proof. □

We now turn to the $(r,p)$-capacity. The $(r,p)$-capacity has been defined by (1.3), (1.4) and satisfies the following properties: for any subsets $B$, $C$, $B_n, n = 1, 2, \ldots,$

$$m(B) \leq C_{r,p}(B), \tag{2.6}$$

$$B \subseteq C \Rightarrow C_{r,p}(B) \leq C_{r,p}(C), \tag{2.7}$$

$$C_{r,p}(\bigcup_n B_n) \leq \sum_n C_{r,p}(B_n). \tag{2.8}$$

Under the assumption (A.1), any $f \in \mathcal{F}_{r,p}$ has a *quasi-continuous modification.* We denote it by $\tilde{f}$. Then for any subset $B$, set

$$\mathcal{L}_B = \{f \in \mathcal{F}_{r,p}\,;\, \tilde{f} \geq 1 \text{ q.e. on } B\}. \tag{2.9}$$

Here, q.e. is the abbreviation of quasi everywhere, i.e., except for a set of zero capacity.

Fukushima-Kaneko [11] proved that there exists a unique element $e_B \in \mathcal{L}_B$ satisfying

$$C_{r,p}(B) = \|e_B\|_{r,p}^p = \inf\{\|f\|_{r,p}\,;\, f \in \mathcal{L}_B\}. \tag{2.10}$$

$e_B$ is called the $(r,p)$-*equilibrium potential* of $B$. Further Fukushima-Kaneko [11] proved that

$$B_n \uparrow B \Rightarrow C_{r,p}(B_n) \uparrow C_{r,p}(B). \tag{2.11}$$

As is well-known, on a Souslin space, for a capacity satisfying (2.11), every Borel set $B$ is *capacitable*, i.e.,

$$C_{r,p}(B) = \sup\{C_{r,p}(K)\,;\, K \text{ is compact and } K \subseteq B\}. \tag{2.12}$$

See, e.g., Bourbaki [3, Théorèm IX.6.6]. Combining this with (A.3), (2.12) holds on our separable metric space.

## 3. Positive generalized functions

In this section, we introduce positive generalized functions and show that they correspond to finite measures. Using this correspondence we define measures of finite $(r,p)$-energy integral and discuss the relationship with potentials.

Let notations be as before. For $r \geq 0$ and $p > 1$, set

$$(\mathcal{F}_{r,p})_+ := \{f \in \mathcal{F}_{r,p}\,;\, f \geq 0 \quad m\text{-a.e.}\}.$$

For Sobolev space with negative index, we can introduce the notion of positivity. Recall that $\hat{\mathcal{F}}_{-r,q}$ is isomorphic to $(\mathcal{F}_{r,p})^*$ where $1/p + 1/q = 1$. Then we say that $\varphi \in \hat{\mathcal{F}}_{-r,q}$ is *positive* if for any $f \in (\mathcal{F}_{r,p})_+$

$${}_{\mathcal{F}_{r,p}}\langle f, \varphi \rangle_{\hat{\mathcal{F}}_{-r,q}} \geq 0.$$

We denote by $(\hat{\mathcal{F}}_{-r,q})_+$ the set of all positive $\varphi \in \hat{\mathcal{F}}_{-r,q}$. We show that $\varphi \in (\hat{\mathcal{F}}_{-r,q})_+$ defines a measure on $X$.

**Theorem 3.1.** *For $\varphi \in (\hat{\mathcal{F}}_{-r,q})_+$, there exists a unique finite tight measure $\mu$ such that*

$$\langle f, \varphi \rangle = \int_X f(x)\mu(dx), \qquad \forall f \in \mathcal{F}_{r,p} \cap C_b(X). \tag{3.1}$$

*Proof.* We first note that $\alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi \in L_+^q(X;m)$. In fact, for $f \in L_+^p(X;m)$,

$$_{L^p}\langle f, \alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi\rangle_{L^q} = {}_{\mathcal{F}_{r,p}}\langle \alpha^{r/2}V_r^{(\alpha)}f, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}} \geq 0$$

since $V_r^{(\alpha)}f \geq 0$ $m$-a.e. We notice that $\alpha^{r/2}\hat{V}_r^{(\alpha)}$ is the contraction not only on $L^p(X;m)$ but also on $\mathcal{F}_{r,p}$. In fact,

$$\|\alpha^{r/2}V_r^{(\alpha)}f\|_{r,p} = \|V_r^{-1}\alpha^{r/2}V_r^{(\alpha)}f\|_p = \|\alpha^{r/2}V_r^{(\alpha)}V_r^{-1}f\|_p \leq \|V_r^{-1}f\|_p = \|f\|_{r,p}.$$

By using $1 \in \mathcal{F}_{r,p}$, we have

$$\begin{aligned}
\int_X \alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi(x)m(dx) &= \int_X \varphi(x)\alpha^{r/2}V_r^{(\alpha)}1(x)m(dx) \\
&\leq \|\varphi\|_{-r,q}^{\wedge}\|\alpha^{r/2}V_r^{(\alpha)}1\|_{r,p} \\
&\leq \|\varphi\|_{-r,q}^{\wedge}\|1\|_{r,p} < \infty.
\end{aligned}$$

Thus we have that a family of measures $\{\alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi \cdot m\}_{\alpha>0}$ is uniformly bounded.

On the other hand, by the assumption (A.3), for any $\varepsilon > 0$ there exists a compact set $K$ such that

$$C_{r,p}(X \setminus K) < \varepsilon^p.$$

Set $G = X \setminus K$ and let $e_G$ be the $(r,p)$-equilibrium potential of $G$. Then

$$\|e_G\|_{r,p} = C_{r,p}(X \setminus K)^{1/p} < \varepsilon$$

and

$$\begin{aligned}
\int_G \alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi\, m(dx) &\leq \int_X e_G\alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi\, m(dx) \\
&= {}_{\mathcal{F}_{r,p}}\langle \alpha^{r/2}V_r^{(\alpha)}e_G, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}} \\
&\leq \|e_G\|_{r,p}\|\varphi\|_{-r,q}^{\wedge} \\
&\leq \varepsilon\|\varphi\|_{-r,q}^{\wedge}
\end{aligned}$$

which implies $\{\alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi \cdot m\}_{\alpha>0}$ is tight. Hence there exists a sequence $\{\alpha_j\}$ and a finite tight measure $\mu$ such that $\lim_{j\to\infty}\alpha_j = \infty$ and for $f \in \mathcal{F}_{r,p} \cap C_b(X)$,

$$\begin{aligned}
\int_X f(x)\,\mu(dx) &= \lim_{j\to\infty}\int_X \alpha_j^{r/2}\hat{V}_r^{(\alpha_j)}\varphi(x)f(x)\,m(dx) \\
&= \lim_{j\to\infty} {}_{\mathcal{F}_{r,p}}\langle \alpha_j^{r/2}V_r^{(\alpha_j)}f, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}} \\
&= {}_{\mathcal{F}_{r,p}}\langle f, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}}
\end{aligned}$$

which proves (3.1). Here we used that $\alpha^{r/2}\hat{V}_r^{(\alpha)} \to I$ strongly in $\mathcal{F}_{r,p}$. Uniqueness follows from the assumption that $\mathcal{D}$ is separating tight measures. □

In the above proof, it is easy to see that $\mu$ does not depend on the choice of $\{\alpha_j\}$ and hence $\{\alpha^{r/2}\hat{V}_r^{(\alpha)}\varphi \cdot m\}_{\alpha>0}$ itself converges weakly to $\mu$.

From now on, we regard an element of $(\hat{\mathcal{F}}_{-r,q})_+$ as a measure on $X$ by this correspondence.

**Proposition 3.2.** *Take any $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$. Then for any open set $G$, it holds that*

$$\mu(G) \leq \|\mu\|^{\wedge}_{-r,q}\, C_{r,p}(G)^{1/p}. \tag{3.2}$$

*In particular, $\mu(B) = 0$ for any Borel set $B$ with $C_{r,p}(B) = 0$.*

*Proof.* Set $\mu_n = n^{r/2}\hat{V}_r^{(n)}\mu \cdot m$. Since $\mu_n \to \mu$ weakly as $n \to 0$, we have

$$\begin{aligned}
\mu(G) &\leq \varliminf_{n\to\infty} \mu_n(G) \\
&\leq \varliminf_{n\to\infty} \int_X e_G(x) n^{r/2}\hat{V}_r^{(n)}\mu\, m(dx) \\
&= \lim_{n\to\infty} \langle e_G, n^{r/2}\hat{V}_r^{(n)}\mu\rangle \\
&= \langle e_G, \mu\rangle \\
&\leq \|e_G\|_{r,p}\|\mu\|^{\wedge}_{-r,q} \\
&= C_{r,p}(G)^{1/p}\|\mu\|^{\wedge}_{-r,q}
\end{aligned}$$

which completes the proof. □

By the above proposition, $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$ charges no set of zero capacity. We also have the following proposition.

**Proposition 3.3.** *Take any $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$. Let $\tilde{f}$ be the quasi-continuous modification of $f \in \mathcal{F}_{r,p}$. Then $\tilde{f} \in L^1(X;\mu)$ and*

$$\langle f, \mu\rangle = \int_X \tilde{f}(x)\mu(dx). \tag{3.3}$$

*Further, it holds that*

$$\|\tilde{f}\|_{L^1(\mu)} \leq \|f\|_{r,p}\|\mu\|^{\wedge}_{-r,q}. \tag{3.4}$$

*Proof.* We first prove (3.3) when $f$ is bounded and non-negative. We may assume that $\tilde{f}$ is bounded. By the assumption (A.3) and the proof of Theorem 3.1, there exists a sequence of compact sets $\{K_n\}$ such that $\tilde{f}|_{K_n}$ is continuous and

$$\lim_{n\to\infty} C_{r,p}(X \setminus K_n) = 0, \tag{3.5}$$

$$\lim_{n\to\infty} \sup_{\alpha}\{(\alpha^{r/2}\hat{V}_r^{(\alpha)}\mu \cdot m)(X \setminus K_n)\} = 0. \tag{3.6}$$

Since $\alpha^{r/2}\hat{V}_r^{(\alpha)}\mu \in L^q(X;m)$, it holds that

$$\langle f, \alpha^{r/2}\hat{V}_r^{(\alpha)}\mu\rangle = \int_X \tilde{f}(x)\alpha^{r/2}\hat{V}_r^{(\alpha)}\mu(x)m(dx). \tag{3.7}$$

We easily see that L.H.S. of (3.7) converges to $\langle f,\mu\rangle$ as $\alpha\to\infty$. We show that R.H.S. of (3.7) converges to $\int_X \tilde f(x)\mu(dx)$. To do this, for any $n\in\mathbf{N}$, we take $f_n\in C_b(X)$ such that $\tilde f = f_n$ on $K_n$ and $\|f_n\|_\infty \le \|\tilde f\|_\infty$ by using Urysohn's extension theorem. Now we have

$$\begin{aligned}
&\left|\int_X \tilde f(x)\alpha^{r/2}\hat V_r^{(\alpha)}\mu(x)m(dx) - \int_X \tilde f(x)\mu(dx)\right| \\
&\le \left|\int_X f_n(x)\alpha^{r/2}\hat V_r^{(\alpha)}\mu(x)m(dx) - \int_X f_n(x)\mu(dx)\right| \\
&\quad + \int_{X\setminus K_n} |\tilde f(x) - f_n(x)|\alpha^{r/2}\hat V_r^{(\alpha)}\mu(x)m(dx) + \int_{X\setminus K_n} |\tilde f(x) - f_n(x)|\mu(dx) \\
&\le \left|\int_X f_n(x)\alpha^{r/2}\hat V_r^{(\alpha)}\mu(x)m(dx) - \int_X f_n(x)\mu(dx)\right| \\
&\quad + 2\|\tilde f\|_\infty\{(\alpha^{r/2}\hat V_r^{(\alpha)}\mu\cdot m)(X\setminus K_n) + \mu(X\setminus K_n)\}.
\end{aligned}$$

Letting $\alpha\to\infty$ and then $n\to\infty$, we have

$$\overline{\lim_{\alpha\to\infty}}\left|\int_X \tilde f(x)\alpha^{r/2}\hat V_r^{(\alpha)}\mu(x)m(dx) - \int_X \tilde f(x)\mu(dx)\right| = 0.$$

Thus we have (3.3) for bounded $f$.

For general $f\in\mathcal{F}_{r,p}$, write $f = V_r g$, $g\in L^p(X;m)$. Let $g_+$, $g_-$ be positive part and negative part of $g$, respectively. Then $f = V_r g_+ - V_r g_-$. Further considering sequences $\{V_r(g_\pm\wedge k)\}_{k\in\mathbf{N}}$, we can reduce it to the first case.

Lastly, we show the estimate (3.4). By the above notations, $\tilde f = \widetilde{V_r g_+} - \widetilde{V_r g_-}$ q.e. and $|\tilde f| \le \widetilde{V_r|g|}$ q.e. Hence

$$\begin{aligned}
\|\tilde f\|_{L^1(\mu)} &\le \int_X \widetilde{V_r|g|}\mu(dx) \\
&= \langle V_r|g|,\mu\rangle \\
&\le \|V_r|g|\|_{r,p}\|\mu\|^\wedge_{-r,q} \\
&= \||g|\|_p\|\mu\|^\wedge_{-r,q} \\
&= \|g\|_p\|\mu\|^\wedge_{-r,q} \\
&= \|f\|_{r,p}\|\mu\|^\wedge_{-r,q}
\end{aligned}$$

which completes the proof. □

**Lemma 3.4.** *For $p'\ge p>1$, $r'\ge r\ge 1$ and $f\in(\mathcal{F}_{r,p})_+$, there exists a sequence $\{f_n\}\subseteq(\mathcal{F}_{r',p'})_+$ such that $\|f-f_n\|_{r,p}\to 0$ as $n\to 0$.*

*Proof.* Set

$$f_M = V_r^{-1}\{(-M)\vee V_r f\wedge M\}$$

Then $f_m\to f$ in $\mathcal{F}_{r,p}$ and hence in $L^p(X;m)$. Since $f\ge 0$, we have $f_M\vee 0\to f$ in $L^p(X;m)$.

On the other hand, by noting $V_r^{-1}f \in L^p(X;m)$, we have

$$\lim_{\alpha\to\infty} \|\alpha^{r'/2}V_{r'}^{(\alpha)}f - f\|_{r,p} = \lim_{\alpha\to\infty} \|\alpha^{r'/2}V_{r'}^{(\alpha)}V_r^{-1}f - V_r^{-1}f\|_p = 0.$$

Further, for any fixed $\alpha$,

$$\begin{aligned} &\lim_{M\to\infty} \|\alpha^{r'/2}V_{r'}^{(\alpha)}(f_M \vee 0) - \alpha^{r'/2}V_{r'}^{(\alpha)}f\|_{r,p} \\ &\quad\leq \lim_{M\to\infty} \|\alpha^{r'/2}V_{r'}^{(\alpha)}(f_M \vee 0) - \alpha^{r'/2}V_{r'}^{(\alpha)}f\|_{r',p} \\ &\quad= \lim_{M\to\infty} \|V_{r'}^{-1}\alpha^{r'/2}V_{r'}^{(\alpha)}(f_M \vee 0 - f)\|_p \\ &\quad\leq \lim_{M\to\infty} \alpha^{r'/2}\|V_{r'}^{-1}V_{r'}^{(\alpha)}\|_{\mathrm{op}}\|f_M \vee 0 - f\|_p \to 0. \end{aligned}$$

Here $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm in $L^p(X;m)$. Therefore for any $\varepsilon > 0$, we can choose $\alpha$ and $M$ so that

$$\|\alpha^{r'/2}V_{r'}^{(\alpha)}(f_M \vee 0) - f\|_{r,p} \leq \varepsilon.$$

Since $f_M$ is bounded, we can easily see that $\alpha^{r'/2}V_{r'}^{(\alpha)}(f_M \vee 0) \in (\mathcal{F}_{r',p'})_+$. The proof is complete. □

Now the following proposition is easily obtained.

**Proposition 3.5.** *For $q \geq q' > 1$, $r' \geq r \geq 0$*

$$(\hat{\mathcal{F}}_{-r',q'})_+ \cap \hat{\mathcal{F}}_{-r,q} = (\hat{\mathcal{F}}_{-r,q})_+.$$

We give an example of positive generalized function. Recall that the following Kato's inequality: for $u \in \mathcal{F}_{2,q}$ and $f \in (\mathcal{F}_{2,p})_+$, $1/p + 1/q = 1$,

$$\langle Af, |u|\rangle \geq \langle f, (\mathrm{sgn}u)Au\rangle. \tag{3.8}$$

Here

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{3.9}$$

and we denote the generator by $A$. (3.8) means that $A|u| - (\mathrm{sgn}u)Au \in (\hat{\mathcal{F}}_{-2,q})_+$ where $q$ is the conjugate exponent of $p$.

Let us discuss the essential self-adjointness of the operator $-A + V$ where $V$ is a potential function. We suppose that the semigroup is symmetric. Then $A$ is a self-adjoint operator in $L^2(X;m)$. We can give an sufficient condition as follows.

**Theorem 3.6.** *Let $p$, $p' > 2$ be exponents such that $\frac{1}{2} + \frac{1}{p} + \frac{1}{p'} = 1$. Suppose that $V \in L^p(X;m)_+$ and $C$ is a dense subspace of $\mathcal{F}_{2,p'}$. Then $-A + V$ is essentially self-adjoint on $C$.*

*Proof.* It is easy to see that $-A+V$ is well-defined on $\mathcal{C}$ and symmetric in $L^2(X;m)$. To show the essential self-adjointness, we shall prove

$$\operatorname{Ker}(I-A+V\uparrow\mathcal{C})^*=\{0\}.$$

Take $g\in\operatorname{Ker}(I-A+V\uparrow\mathcal{C})^*$. Then, for $f\in\mathcal{C}$,

$$\langle (I-A+V)f,g\rangle=0. \tag{3.10}$$

By the denseness of $\mathcal{C}$, (3.10) holds for $f\in\mathcal{F}_{2,p'}$ which means

$$(A-I)g=Vg \quad \text{in } \mathcal{F}_{-2,q'},$$

where $q'$ is the conjugate exponent of $p'$. Hence, by Kato's inequality, for any $u\in(\mathcal{F}_{2,p'})_+$

$$\langle (A-I)|g|,u\rangle\geq\langle(\operatorname{sgn} g)(A-I)g,u\rangle\geq\langle(\operatorname{sgn} g)Vg,u\rangle\geq 0.$$

which implies $(A-I)|g|\in(\mathcal{F}_{-2,q'})_+$.

On the other hand, $(A-I)|g|\in\mathcal{F}_{-2,2}$. Hence by Proposition 3.5, we obtain $(A-I)|g|\in(\mathcal{F}_{-2,2})_+$. Now by noting $T_{2t}|g|\in(\mathcal{F}_{2,2})_+$

$$0\leq\langle (A-I)|g|,T_{2t}|g|\rangle=\langle (A-I)T_t|g|,T_t|g|\rangle\leq 0.$$

Thus we have $T_t|g|=0$ and hence $g=0$. This completes the proof. □

## 4. Measures of finite $(r,p)$-energy and potentials

In this section, we define the $(r,p)$-energy and discuss the potentials. First we define the mapping $U\colon\hat{\mathcal{F}}_{-r,q}\to\mathcal{F}_{r,p}$. This operator is introduced by Maz'ya-Khavin [14] in the connection with the Riesz potential on the finite dimensional Euclidean space. For $\varphi\in\hat{\mathcal{F}}_{-r,q}$, $U\varphi$ is defined by

$$U\varphi=V_r\{|\hat{V}_r\varphi|^{q-1}\operatorname{sgn}(\hat{V}_r\varphi)\} \tag{4.1}$$

where sgn is defined by (3.9). By noting that $\hat{V}_r\varphi\in L^q$ and $|\hat{V}_r\varphi|^{q-1}\in L^p$ for $\varphi\in\hat{\mathcal{F}}_{-r,q}$, $U$ is well-defined. Moreover $U$ is bijective and the inverse mapping $U^{-1}$ is given by

$$U^{-1}u=\hat{V}_r^{-1}\{|V_r^{-1}u|^{p-1}\operatorname{sgn}(V_r^{-1}u)\}. \tag{4.2}$$

To see the continuity of $U$, we need the following estimates: for $q\geq 2$,

$$\|U\varphi-U\psi\|_{r,p}\leq(q-1)\{(\|\varphi\|^{\wedge}_{-r,q})^q+(\|\psi\|^{\wedge}_{-r,q})^q\}^{(q-2)/q}\|\varphi-\psi\|^{\wedge}_{-r,q}, \quad \forall\varphi,\psi\in\hat{\mathcal{F}}_{-r,q} \tag{4.3}$$

and for $q\in(1,2)$,

$$\|U\varphi-U\psi\|_{r,p}\leq(\|\varphi-\psi\|^{\wedge}_{-r,q})^{q-1}, \qquad \forall\varphi,\psi\in(\hat{\mathcal{F}}_{-r,q})_+. \tag{4.4}$$

Hence, $U$ is continuous if $q \geq 2$, and $U$ is continuous at least on $(\hat{\mathcal{F}}_{-r,q})_+$ if $q \in (1,2)$. For the proof, see Maz'ya-Khavin [14, Lemma 3.5].

Similarly, we have the following estimates: for $p \geq 2$,

$$\|\varphi - \psi\|^{\wedge}_{-r,q} \leq (p-1)\{\|U\varphi\|^p_{r,p} + \|U\psi\|^p_{r,p}\}^{(p-2)/p}\|U\varphi - U\psi\|_{r,p}, \qquad \forall \varphi, \psi \in \hat{\mathcal{F}}_{-r,q} \tag{4.5}$$

and for $p \in (1,2)$,

$$\|\varphi - \psi\|^{\wedge}_{-r,q} \leq \|U\varphi - U\psi\|^{p-1}_{r,p}, \qquad \forall \varphi, \psi \in (\hat{\mathcal{F}}_{-r,q})_+. \tag{4.6}$$

Define a function $\mathcal{E}_{r,p}: \mathcal{F}_{r,p} \times \mathcal{F}_{r,p} \to \mathbf{R}$ by

$$\begin{aligned} \mathcal{E}_{r,p}(u,v) &= {}_{\mathcal{F}_{r,p}}\langle u, U^{-1}v\rangle_{\hat{\mathcal{F}}_{-r,q}} \\ &= \int_X V_r^{-1}u(x)|V_r^{-1}v(x)|^{p-1}\mathrm{sgn}(V_r^{-1}v(x))m(dx). \end{aligned} \tag{4.7}$$

Note that $\mathcal{E}_{r,p}$ is linear in the first variable but *non-linear* in the second variable if $p \neq 2$. We can easily see the following: for $u, v \in \mathcal{F}_{r,p}$,

$$\mathcal{E}_{r,p}(u,u) = \|u\|^p_{r,p} \tag{4.8}$$

$$|\mathcal{E}_{r,p}(u,v)| \leq \|u\|_{r,p}\|v\|^{p-1}_{r,p}. \tag{4.9}$$

$\mathcal{E}_{r,p}$ is a natural extension of $\mathcal{E}_1$, in fact if $r = 1$, $p = 2$, then $U = G_1 = (I - A)^{-1}$ and $\mathcal{E}_{1,2}(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(m)}$. An element $\varphi \in \hat{\mathcal{F}}_{-r,q}$ is said to be of *finite* $(r,p)$*-energy* and the $(r,p)$-energy of $\varphi$ is given by

$$(\|\varphi\|^{\wedge}_{-r,q})^q = \|U\varphi\|^p_{r,p} = \mathcal{E}_{r,p}(U\varphi, U\varphi) = {}_{\mathcal{F}_{r,p}}\langle U\varphi, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}}.$$

**Definition 4.1.** $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$ is called the *measure of finite* $(r,p)$*-energy*. Further $u = U\mu \in \mathcal{F}_{r,p}$ is called the $(r,p)$*-potential* of $\mu$. We set $S_0 = (\hat{\mathcal{F}}_{-r,q})_+$.

Note that the $(r,p)$-energy of $\mu$ is equal to $\int_X \tilde{u}(x)\mu(dx)$. The following theorems are fundamental.

**Theorem 4.1.** *For $u \in \mathcal{F}_{r,p}$, the following are equivalent:*

(i) *$u$ is an $(r,p)$-potential, i.e., there exists $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$ such that $u = U\mu$.*
(ii) *For any $v \in \mathcal{F}_{r,p}$ with $v \geq 0$ a.e., it holds that $\mathcal{E}_{r,p}(u,v) \geq 0$.*
(iii) *For any $v \in \mathcal{F}_{r,p}$ with $v \geq u$ a.e., it holds that $\|v\|_{r,p} \geq \|u\|_{r,p}$.*

*Further, under the above conditions, it holds that $u \geq 0$ a.e.*

*Proof.* Noticing the identity $\mathcal{E}_{r,p}(v, Uu) = {}_{\mathcal{F}_{r,p}}\langle v, \mu\rangle_{\hat{\mathcal{F}}_{-r,q}}$, the equivalence of (i) and (ii) can be easily seen. We postpone a proof of the equivalence of (i) and (iii) until the section 5, since we need the notion of capacity of functions.

Lastly, we show $u \geq 0$ a.e. In the proof of Theorem 3.1, we have shown that $\hat{V}_r\mu \geq 0$ a.e. So it is easy to see that $U\mu \geq 0$ from the definition of $U$. □

Next we shall show that the $(r,p)$-equilibrium potential is a potential.

**Theorem 4.2.** *Let $e_B$ be the $(r,p)$-equilibrium potential of a set $B$. Then it holds that for $v \in \mathcal{F}_{r,p}$ such that $\tilde{v} \geq 0$ q.e. on $B$,*

$$\mathcal{E}_{r,p}(v, e_B) \geq 0. \tag{4.10}$$

*In particular, if $\tilde{v} = 0$ q.e. on $B$, then $\mathcal{E}_{r,p}(v, e_B) = 0$.*

*Proof.* Set

$$\mathcal{L}_B = \{u \in \mathcal{F}_{r,p}\,;\, \tilde{u} \geq 1 \text{ q.e. on } B\}.$$

$e_B$ is the unique element of $\mathcal{L}_B$ which minimizes $\|u\|_{r,p}^p$. Take $v \in \mathcal{F}_{r,p}$ such that $\tilde{v} \geq 0$ q.e. on $B$. Then for any $\varepsilon \geq 0$, $e_B + \varepsilon\tilde{v} \in \mathcal{L}_B$ and hence

$$\|e_B + \varepsilon v\|_{r,p}^p \geq \|e_B\|_{r,p}^p.$$

Therefore

$$\frac{d}{d\varepsilon}\|e_B + \varepsilon v\|_{r,p}^p\Big|_{\varepsilon=0} \geq 0.$$

Following Maz'ya-Khavin [14], we calculate L.H.S.

$$\begin{aligned}
&\frac{d}{d\varepsilon}\|e_B + \varepsilon v\|_{r,p}^p\Big|_{\varepsilon=0} \\
&\quad= \int_X \frac{d}{d\varepsilon}|V_r^{-1}e_B + \varepsilon V_r^{-1}v|^p\, m(dx) \\
&\quad= \int_X p|V_r^{-1}e_B + \varepsilon V_r^{-1}v|^{p-1}\mathrm{sgn}(V_r^{-1}e_B + \varepsilon V_r^{-1}v)V_r^{-1}v\, m(dx).
\end{aligned}$$

Thus

$$0 \leq \frac{d}{d\varepsilon}\|e_B + \varepsilon v\|_{r,p}^p\Big|_{\varepsilon=0} = \int_X p|V_r^{-1}e_B|^{p-1}\mathrm{sgn}(V_r^{-1}e_B)V_r^{-1}v\, m(dx) = p\mathcal{E}_{r,p}(v, e_B)$$

which proves (4.10). □

We can introduce the notion of smooth measure as follows;

**Definition 4.2.** Borel measure $\mu$ (not necessarily finite) on $X$ is said to be $(r,p)$-smooth if the following conditions are satisfied:

(i) $\mu$ charges no set of zero capacity,
(ii) There exists an increasing sequence of compact sets $\{K_n\}$ such that

$$\mu(K_n) < \infty \quad \text{for } n = 1, 2, \ldots, \tag{4.11}$$

$$\lim_{n\to\infty} C_{r,p}(X \setminus K_n) = 0. \tag{4.12}$$

We denote the set of all $(r,p)$-smooth measures by $S$.

We remark that $\mu(X \setminus \bigcup_n K_n) = 0$ follows from (i) and (4.12).

**Lemma 4.3.** *Let $\nu$ be a bounded Borel measure on $X$. Suppose that there exists a constant $\kappa > 0$ such that*

$$\nu(B) \leq \kappa C_{r,p}(B), \qquad \forall B \in \mathcal{B}(X).$$

*Then $\nu \in S_0$.*

*Proof.* For $v \in \mathcal{F}_{r,p}$ with $\|v\|_{r,p} = 1$,

$$\begin{aligned} \int_X |\tilde{v}(x)|\nu(dx) &\leq \nu(X) + \sum_{k=0}^{\infty} 2^{k+1}\nu\{2^k < \tilde{v} \leq 2^{k+1}\} \\ &\leq \nu(X) + \kappa \sum_{k=0}^{\infty} 2^{k+1} C_{r,p}\{2^k < \tilde{v} \leq 2^{k+1}\} \\ &\leq \nu(X) + \kappa \sum_{k=0}^{\infty} 2^{k+1} C_{r,p}\{\tilde{v} > 2^k\} \\ &\leq \nu(X) + \kappa \sum_{k=0}^{\infty} 2^{k+1}(2^{-k})^p \|v\|_{r,p}^p < \infty. \end{aligned}$$

Hence a function $v \mapsto \int_X \tilde{v}(x)\nu(dx)$ belongs to $\hat{\mathcal{F}}_{-r,q}$ and satisfies the positivity. Thus we have $\nu \in (\hat{\mathcal{F}}_{-r,q})_+ = S_0$. $\square$

Now the following lemma and theorem can be obtained by the same proof as in Fukushima [10, Lemma 3.2.5 and Theorem 3.2.3] .

**Lemma 4.4.** *Let $\nu$ be a bounded Borel measure on $X$ charging no set of zero capacity. Then there exists a decreasing sequence of open sets $\{G_n\}$ such that*

$$\lim_{n\to\infty} C_{r,p}(G_n) = \lim_{n\to\infty} \nu(G_n) = 0, \tag{4.13}$$

$$\nu(B) \leq 2^n C_{r,p}(B) \quad \text{for } B \in \mathcal{B}(X),\ B \subseteq X \setminus G_n. \tag{4.14}$$

**Theorem 4.5.** *Borel measure $\nu$ on $X$ is $(r,p)$-smooth if and only if there exists an increasing sequence of closed set $\{F_n\}$ such that $\nu(X \setminus \bigcup_{n=1}^{\infty} F_n) = 0$, $\lim_{n\to\infty} C_{r,p}(X \setminus F_n) = 0$ and $1_{F_n} \cdot \nu \in S_0$.*

Now we can give a characterization of capacity zero set by using $S_0$. Let $e_B$ be the $(r,p)$-equilibrium potential of a set $B$. As was shown in Theorem 4.2, $(r,p)$-equilibrium potentials are potentials. Hence there exists a measure $\nu_B \in (\hat{\mathcal{F}}_{-r,q})_+$ such that

$$\int_X \tilde{u}(x)\nu_B(dx) = \mathcal{E}_{r,p}(u, e_B).$$

We call $\nu_B$ the $(r,p)$-*equilibrium measure* of $B$.

**Lemma 4.6.** *Let $K$ be a compact set and $\nu_K$ be the $(r,p)$-equilibrium measure of $K$. Then* $\operatorname{supp}[\nu_K] \subseteq K$.

*Proof.* It is enough to show that for $g \in C_b(X)$, $g \geq 0$ on $K$,

$$\int_X g \, d\nu_K \geq 0.$$

We show this by following three steps.

*Step 1.* For $g \in \mathcal{D}$, $g \geq 0$ on $K$, $\int_X g e^{-\varepsilon g^2} \, d\nu_K \geq 0$.

First we note that for $x \geq 0$,

$$e^{-x} = \sum_{k=0}^{n} \frac{(-x)^k}{k!} + \frac{(-1)^{n+1}}{(n+1)!} e^{-\theta x}, \qquad 0 \leq \theta \leq 1.$$

In particular, for even $n$,

$$\sum_{k=0}^{n} \frac{(-x)^k}{k!} = e^{-x} + \frac{1}{(n+1)!} e^{-\theta x} \geq 0.$$

Now set

$$g_m = \sum_{k=0}^{2m} g \frac{(-\varepsilon g^2)^k}{k!}.$$

Then $g_m \in \mathcal{D}$ and $g_m \geq 0$ on $K$. Here we used that $\mathcal{D}$ is an algebra. Noticing that $g_m$ converges to $g e^{-\varepsilon g^2}$ uniformly on $X$, we have

$$\int_X g e^{-\varepsilon g^2} \, d\nu_K = \lim_{m \to \infty} \int_X g_m \, d\nu_K = \lim_{m \to \infty} {}_{\mathcal{F}_{r,p}}\langle g_m, \nu_K \rangle_{\hat{\mathcal{F}}_{-r,q}} \geq 0.$$

*Step 2.* For $g \in C_b(X)$, $g \geq 0$ on $K$, $\int_X g e^{-\varepsilon g^2} \, d\nu_K \geq 0$.

Take a sequence of increasing compact sets $K = K_0 \subset K_1 \subset K_2 \subset \cdots$ such that $C_{r,p}(X \setminus K_n) \to 0$. By the Stone-Weierstrass theorem, we can take $g_n \in \mathcal{D}$ satisfying $|g_n - g| \leq \frac{1}{n}$ on $K_n$. Setting $f_n = g_n + \frac{1}{n}$, we have

$$f_n \geq 0 \quad \text{on } K \quad \text{and} \quad |f_n - g| \leq \frac{2}{n} \quad \text{on } K_n.$$

Hence $f_n$ converges to $g$ q.e. and $\nu_K$-a.e. Note that the function $x e^{-\varepsilon x^2}$ is bounded. By the dominated convergence theorem and the Step 1, we have

$$\int_X g e^{-\varepsilon g^2} \, d\nu_K = \lim_{n \to \infty} \int_X f_n e^{-\varepsilon f_n^2} \, d\nu_K \geq 0.$$

*Step 3.* For $g \in C_b(X)$, $g \geq 0$ on $K$, $\int_X g \, d\nu_K \geq 0$.

In fact, by using Step 2,

$$\int_X g \, d\nu_K = \lim_{\varepsilon \to 0} \int_X g e^{-\varepsilon g^2} \, d\nu_K \geq 0.$$

□

**Theorem 4.7.** *For $B \in \mathcal{B}(X)$, the following conditions are equivalent:*

(i) $C_{r,p}(B) = 0$.
(ii) $\mu(B) = 0$ *for any* $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$.

*Proof.* The implication (i) $\Rightarrow$ (ii) was proven in Proposition 3.2.

Conversely, if $C_{r,p}(B) > 0$, then there exists a compact set $K \subseteq B$ with $C_{r,p}(K) > 0$ since $B$ is capacitable. Let $e_K$ be the $(r,p)$-equilibrium potential of $K$ and $\nu_K$ be the $(r,p)$-equilibrium measure of $K$, respectively. Then, by Lemma 4.6

$$0 < C_{r,p}(K) = \mathcal{E}_{r,p}(e_K, e_K) = \int_X \widetilde{e_K}(x)\nu_K(dx) = \int_K \widetilde{e_K}(x)\nu_K(dx).$$

Noting that $\widetilde{e_K}(x) \geq 1$ q.e. on $K$, we obtain $\nu_K(K) > 0$. Thus we have $\nu_K(B) \geq \nu_K(K) > 0$. This shows (ii) $\Rightarrow$ (i). $\square$

## 5. Capacity of functions

Following Feyel-de La Pradelle [8], we introduce the $(r,p)$-capacity of functions. For $[0,\infty]$-valued lower semicontinuous (l.s.c. in abbreviation) function $h$, define $C_{r,p}(h)$ by

(5.1) $$C_{r,p}(h) := \inf\{\|u\|_{r,p}^p\,;\ u \in \mathcal{F}_{r,p},\ u \geq h,\ m\text{-a.e.}\}$$

and for an arbitrary $[-\infty,\infty]$-valued function $f$ (not assumed to be measurable),

(5.2) $$C_{r,p}(f) := \inf\{C_{r,p}(h)\,;\ h \text{ is l.s.c. and } h(x) \geq |f(x)|,\ \forall x \in X\}.$$

Here and sequel we use the convention $\inf \phi = \infty$.

Then the following properties hold as well as for the capacity of sets. For any functions $f$, $f_1, f_2, \ldots$, and $\lambda \geq 0$,

(5.3) $$C_{r,p}(|f|) = C_{r,p}(f),$$

(5.4) $$C_{r,p}(\lambda f) = \lambda^p C_{r,p}(f),$$

(5.5) $$|f_1(x)| \leq |f_2(x)| \quad \forall x \in X \Rightarrow C_{r,p}(f_1) \leq C_{r,p}(f_2),$$

(5.6) $$C_{r,p}(\sup_n |f_n|) \leq \sum_n C_{r,p}(f_n),$$

(5.7) $$C_{r,p}(\sum_n f_n)^{1/p} \leq \sum_n C_{r,p}(f_n)^{1/p},$$

(5.8) $$C_{r,p}(\{x \in X; f(x) \geq \lambda\}) \leq \frac{1}{\lambda^p} C_{r,p}(f).$$

Moreover this capacity is consistent with the capacity of sets. In fact, for any set $B$,

(5.9) $$C_{r,p}(1_B) = C_{r,p}(B).$$

Here $1_B$ denotes the indicator function of $B$.

To show (5.6), we need the following fact as in Fukushima-Kaneko [11]: for any non-negative l.s.c. function $h$ with $C_{r,p}(h) < \infty$, there exists a unique (up to a.e. equivalence) $u \in \mathcal{F}_{r,p}$ satisfying

$$C_{r,p}(h) = \|u\|_{r,p}^p \text{ and } h \leq u, \ m\text{-a.e.} \tag{5.10}$$

We will extend this to all functions. First, the following lemma is easily obtained.

**Lemma 5.1.** (i) *Let $h$ be non-negative l.s.c. and $u$ be quasi-continuous. If $h \leq u$, $m$-a.e., then $h \leq u$ q.e.*
(ii) *For any function $f$, $C_{r,p}(f) = 0$ if and only if $f = 0$ q.e.*
(iii) *Let $f_1$, $f_2$ be any functions. If $|f_1| \leq |f_2|$ q.e., then $C_{r,p}(f_1) \leq C_{r,p}(f_2)$ and if $|f_1| = |f_2|$ q.e., then $C_{r,p}(f_1) = C_{r,p}(f_2)$.*

*Proof.* We first prove (i). Since $h$ is l.s.c., there exists a sequence of continuous functions $\{c_n\}$ such that

$$h(x) = \sup_{n\geq 1} c_n(x).$$

Noticing that $c_n \leq u$ $m$-a.e., we obtain $c_n \leq u$ q.e. (see e.g., [11, §3]). Now we easily have $h \leq u$ q.e.

To show (ii), assume $C_{r,p}(f) = 0$. By (5.8), for any $\lambda > 0$,

$$C_{r,p}(|f| \geq \lambda) \leq \frac{1}{\lambda^p} C_{r,p}(f) = 0.$$

Hence we have $C_{r,p}(|f| > 0) = 0$, i.e., $f = 0$ q.e.

Conversely, assume that $f = 0$ q.e. Then there exists a decreasing sequence of open sets $\{O_n\}$ such that

$$C_{r,p}(O_n) \leq \frac{1}{2^n}, \quad f = 0 \text{ on } (\bigcap_{n=1}^{\infty} O_n)^c.$$

Set $f_n = \sup_{m\geq n} m 1_{O_m}$. Clearly, $f_n$ is l.s.c. and $|f(x)| \leq f_n(x)$, for $x \in X$. Hence by (5.5), (5.6) and (5.9), we have

$$C_{r,p}(f) \leq C_{r,p}(f_n) \leq \sum_{m=n}^{\infty} C_{r,p}(m 1_{O_m}) \leq \sum_{m=n}^{\infty} m^p C_{r,p}(O_m) \leq \sum_{m=n}^{\infty} \frac{m^p}{2^m}.$$

Letting $n \to \infty$, we get $C_{r,p}(f) = 0$.

(iii) is obtained easily from (ii). □

**Proposition 5.2.** *For $u \in \mathcal{F}_{r,p}$, it holds that $C_{r,p}(\tilde{u}) \leq \|u\|_{r,p}^p$ where $\tilde{u}$ is the quasi-continuous modification of $u$.*

*Proof.* For $u \in \mathcal{F}_{r,p} \cap C_b(X)$, there exists $f \in L^p(X;m)$ such that $u = V_r f$. Hence $|u| \leq V_r|f|$ $m$-a.e. Since $|u|$ is l.s.c.,

$$C_{r,p}(u) \leq \|V_r|f|\|_{r,p} = \||f|\|_p = \|f\|_p = \|u\|_{r,p}.$$

For general $u$, we take a sequence $\{u_n\} \subseteq \mathcal{F}_{r,p} \cap C_b(X)$ such that $u_n \to u$ in $\mathcal{F}_{r,p}$. By taking a subsequence if necessary, we may assume that

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_{r,p} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |u_{n+1}(x) - u_n(x)| < \infty \quad \text{q.e.-}x.$$

Then we have

$$|\tilde{u}(x)| \leq |u_n(x)| + \sum_{k=n}^{\infty} |u_{k+1}(x) - u_k(x)| \quad \text{q.e.-}x.$$

Now by (5.7), we have

$$\begin{aligned} C_{r,p}(\tilde{u})^{1/p} &\leq C_{r,p}(u_n)^{1/p} + \sum_{k=n}^{\infty} C_{r,p}(u_{k+1} - u_k)^{1/p} \\ &\leq \|u_n\|_{r,p} + \sum_{k=n}^{\infty} \|u_{k+1} - u_k\|_{r,p} \end{aligned}$$

Letting $n \to \infty$, we have $C_{r,p}(\tilde{u})^{1/p} \leq \|u\|_{r,p}$ as desired. $\square$

Now we shall extend (5.10). For any function $f$, set

$$\mathcal{L}_f = \{u \in \mathcal{F}_{r,p};\ \tilde{u} \geq |f| \text{ q.e.}\}.$$

**Proposition 5.3.** *For any function $f$, it holds that*

$$C_{r,p}(f) = \inf\{\|u\|_{r,p}^p;\ u \in \mathcal{L}_f\}. \tag{5.11}$$

*Moreover, there exists a unique element $u \in \mathcal{F}_{r,p}$ which attains the infimum of the right hand side of (5.11).*

*Proof.* Since $\mathcal{F}_{r,p}$ is uniformly convex, we can show the existence and the uniqueness of the element which attains the infimum of the right hand side of (5.11).

Take $u \in \mathcal{L}_f$. Then by Lemma 5.1 (iii) and Proposition 5.2,

$$C_{r,p}(f) \leq C_{r,p}(\tilde{u}) \leq \|u\|_{r,p}^p.$$

Hence we have

$$C_{r,p}(f) \leq \inf\{\|u\|_{r,p}^p;\ u \in \mathcal{L}_f\}.$$

Next we show the converse. We may assume that $C_{r,p}(f) < \infty$. For any $\varepsilon > 0$, there exists a non-negative l.s.c. function $h$ such that

$$C_{r,p}(f) + \varepsilon > C_{r,p}(h) \text{ and } |f(x)| \leq h(x) \quad \forall x \in X.$$

Further we can take $u \in \mathcal{F}_{r,p}$ such that

$$C_{r,p}(h) + \varepsilon > \|u\|_{r,p}^p \text{ and } h(x) \leq u(x) \quad m\text{-a.e. } x.$$

By Lemma 5.1 (i), $\tilde{u} \geq h$ q.e. and hence $\tilde{u} \geq |f|$ q.e. Thus we have

$$C_{r,p}(f) + 2\varepsilon \geq \inf\{\|u\|_{r,p}^p; u \in \mathcal{L}_f\}.$$

Letting $\varepsilon \to 0$, we get a desired result. □

Now the following proposition can be proven in the same way as in Fukushima-Kaneko [11, Theorem 2].

**Proposition 5.4.** (i) *Let $\{f_n\}$ be a sequence of functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \quad \text{q.e.-}x.$$

*Then it holds that*

$$C_{r,p}(\sup_n f_n) = \sup_n C_{r,p}(f_n). \tag{5.12}$$

(ii) *Let $\{f_n\}$ be a sequence of functions with $f_n \geq 0$ q.e. Then it holds that*

$$C_{r,p}(\varliminf_{n\to\infty} f_n) \leq \varliminf_{n\to\infty} C_{r,p}(f_n). \tag{5.13}$$

Following Feyel-de La Pradelle [8], let us define the Banach space $L^1(X; C_{r,p})$ as follows. First note that $C_{r,p}(\cdot)^{1/p}$ is a norm on $\mathcal{F}_{r,p} \cap C_b(X)$ by (5.7). So we define $L^1(X; C_{r,p})$ to be the completion of $\mathcal{F}_{r,p} \cap C_b(X)$ under the norm $C_{r,p}(\cdot)^{1/p}$. We can give another characterization. Let $\mathcal{L}^1(X; C_{r,p})$ be the set of all functions $f$ satisfying the following condition: there exists a sequence $\{u_n\}_n \subset \mathcal{F}_{r,p} \cap C_b(X)$ such that $\lim_{n\to\infty} C_{r,p}(f - u_n) = 0$ and $f = \lim_{n\to\infty} u_n$ q.e. Then $L^1(X; C_{r,p})$ is the quotient space of $\mathcal{L}^1(X; C_{r,p})$ under the equivalence relation $\sim$: $f_1 \sim f_2$ if and only if $f_1 = f_2$ q.e. To avoid the complexity, we identify $L^1(X; C_{r,p})$ and $\mathcal{L}^1(X; C_{r,p})$.

**Proposition 5.5.** *$\mathcal{F}_{r,p}$ is the dense subspace of $L^1(X; C_{r,p})$ and the inclusion is continuous.*

*Proof.* First we give a precise meaning. For any $u \in \mathcal{F}_{r,p}$, there exists a quasi continuous modification $\tilde{u}$ and $\tilde{u}$ is unique up to q.e. equivalence. The assertion says that $\tilde{u} \in L^1(X; C_{r,p})$.

To show this, we take a sequence $\{u_n\}_n \subset \mathcal{F}_{r,p} \cap C_b(X)$ such that $\lim_{n\to\infty} \|u - u_n\|_{r,p} = 0$. We may assume that $\lim_{n\to\infty} u_n = u$ q.e. by taking a subsequence if necessary. Further by Proposition 5.2,

$$\lim_{n,m\to\infty} C_{r,p}(u_n - u_m) \le \lim_{n,m\to\infty} \|u_n - u_m\|_{r,p}^p = 0.$$

Hence we have $u \in L^1(X; C_{r,p})$.

Continuity of the inclusion follows from Proposition 5.2. Further it is easy to see that $\mathcal{F}_{r,p}$ is dense subspace in $L^1(X; C_{r,p})$ by the definition of $L^1(X; C_{r,p})$. □

**Proposition 5.6.** *The following holds:*

$$C_b(X) \subseteq L^1(X; C_{r,p}), \tag{5.14}$$

$$f \in L^1(X; C_{r,p}) \Rightarrow |f| \in L^1(X; C_{r,p}). \tag{5.15}$$

*Proof.* We divide a proof of (5.14) into three steps. We take arbitrary $\varepsilon > 0$.

*Step 1.* If $g \in \mathcal{D}$, then $ge^{-\varepsilon g^2} \in L^1(X; C_{r,p})$.

Set

$$g_n = \sum_{k=0}^{n} g\frac{(-\varepsilon g^2)^k}{k!}.$$

Then $g_n \in \mathcal{D}$ and $g_n$ converges to $ge^{-\varepsilon g^2}$ uniformly on $X$. Hence we have

$$C_{r,p}(g_n - ge^{-\varepsilon g^2}) \to 0$$

which implies $ge^{-\varepsilon g^2} \in L^1(X; C_{r,p})$.

*Step 2.* If $g \in C_b(X)$, then $ge^{-\varepsilon g^2} \in L^1(X; C_{r,p})$.

Take a sequence of increasing compact sets $\{K_n\}$ such that $C_{r,p}(X \setminus K_n) \to 0$. By the Stone-Weierstrass theorem, we can take $g_n \in \mathcal{D}$ satisfying $|g_n - g| \le \frac{1}{n}$ on $K_n$. Set

$$M = \sup_{x\ge 0} xe^{-\varepsilon x^2} < \infty.$$

Then we have

$$\begin{aligned}
&C_{r,p}(g_n e^{-\varepsilon g_n^2} - ge^{-\varepsilon g^2})^{1/p} \\
&= C_{r,p}((g_n e^{-\varepsilon g_n^2} - ge^{-\varepsilon g^2})1_{K_n} + (g_n e^{-\varepsilon g_n^2} - ge^{-\varepsilon g^2})1_{X\setminus K_n})^{1/p} \\
&\le \frac{1}{n} C_{r,p}(K_n)^{1/p} + MC_{r,p}(X \setminus K_n)^{1/p} \to 0
\end{aligned}$$

and hence $ge^{-\varepsilon g^2} \in L^1(X; C_{r,p})$.

*Step 3.* If $g \in C_b(X)$, then $g \in L^1(X; C_{r,p})$.

Noting that $ge^{-\varepsilon g^2}$ converges uniformly to $g$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} C_{r,p}(ge^{-\varepsilon g^2} - g) = 0.$$

This implies $g \in L^1(X; C_{r,p})$.

Next we show (ii). By definition, we can take a sequence $\{u_n\} \subseteq \mathcal{F}_{r,p} \cap C_b(X)$ such that $f = \lim_{n\to\infty} u_n$ q.e. and $\lim_{n\to\infty} C_{r,p}(u_n - u) = 0$. Then, we easily obtain $|f| = \lim_{n\to\infty} |u_n|$ q.e. and $\lim_{n\to\infty} C_{r,p}(|u_n| - |f|) \leq \lim_{n\to\infty} C_{r,p}(u_n - f) = 0$ which shows $|f| \in L^1(X; C_{r,p})$. □

Let us discuss positive generalized function in this context.

**Definition 5.1.** An element $\Phi \in L^1(X; C_{r,p})^*$ is said to be positive if

$${}_{L^1(X;C_{r,p})}\langle f, \Phi \rangle_{L^1(X;C_{r,p})^*} \geq 0 \quad \text{for } f \in L^1(X; C_{r,p}) \text{ with } f \geq 0 \text{ q.e.}$$

We denote the set of all positive elements by $L^1(X; C_{r,p})^*_+$.

By using Proposition 5.6, we can give an alternative proof of Theorem 3.1. In fact, Feyel-de La Pradelle proved it for Gaussian measures in this manner.

*Proof of Theorem 3.1.* We first note that $\mathcal{F}_{r,p}$ is contained in $L^1(X; C_{r,p})$ by Proposition 5.5. Take any $\varphi \in (\mathcal{F}_{r,p})^*_+$. Then by Proposition 5.3, for any $f \in L^1(X; C_{r,p})$, there exists $u \in \mathcal{F}_{r,p}$ such that $|f| \leq \tilde{u}$ q.e. Now by using the extension theorem of the positive linear functional (e.g., [16, XI, T3]), $\varphi$ can be extended to a positive linear functional on $L^1(X; C_{r,p})$ which we denote by $\Phi$. Since $C_b(X) \subseteq L^1(X; C_{r,p})$ by Proposition 5.6, we can define the functional $I$ by

$$I(u) = {}_{L^1(X;C_{r,p})}\langle u, \Phi \rangle_{L^1(X;C_{r,p})^*}, \quad u \in C_b(X).$$

$I$ is clearly a positive linear functional on $C_b(X)$. We show the continuity of $I$ in the following sense: for any decreasing sequence $\{f_n\} \subseteq C_b(X)$, such that $f_n(x) \downarrow 0$, $\forall x \in X$, it holds that $\lim_{n\to\infty} I(f_n) = 0$.

To see this, take any compact set $K$. Then

$$\begin{aligned} C_{r,p}(f_n)^{1/p} &\leq C_{r,p}(f_n 1_K)^{1/p} + C_{r,p}(f_n 1_{K^c})^{1/p} \\ &\leq \|f_n\|_{\infty;K} C_{r,p}(K)^{1/p} + \|f_1\|_\infty C_{r,p}(K^c)^{1/p}. \end{aligned}$$

Here $\|\cdot\|_{\infty;K}$ denote the supremum norm on $K$. Since $\lim_{n\to\infty} \|f_n\|_{\infty;K} = 0$ by Dini's theorem, we have

$$\varlimsup_{n\to\infty} C_{r,p}(f_n)^{1/p} \leq \|f_1\|_\infty C_{r,p}(K^c)^{1/p}.$$

We can make the right hand side as small as we want, and hence $\lim_{n\to\infty} I(f_n) = 0$.

Now by Daniel's extension theorem (see e.g., [15, p. 29]), there exists a Borel measure $\mu$ on $X$ such that

$$I(f) = \int_X f(x)\mu(dx), \quad \forall f \in C_b(X)$$

which completes the proof. □

Next let us discuss the relation between $L^1(X;C_{r,p})^*_+$ and $(\hat{\mathcal{F}}_{-r,q})_+$. We have the following:

**Theorem 5.7.** (i) *$L^1(X;C_{r,p})^*_+ = (\hat{\mathcal{F}}_{-r,q})_+$ and the norm is preserved.*
(ii) *For any $\mu \in L^1(X;C_{r,p})^*_+$, which is regarded as a measure on $X$, it holds that $L^1(X;C_{r,p}) \subseteq L^1(X;\mu)$. Moreover it holds that*

$$\|f\|_{L^1(X;\mu)} \leq \|\mu\|_{L^1(X;C_{r,p})^*} C_{r,p}(f)^{1/p}, \quad \forall f \in L^1(X;C_{r,p}). \tag{5.16}$$

*Proof.* We denote the inclusion by $i : \mathcal{F}_{r,p} \longrightarrow L^1(X;C_{r,p})$. Let $i^*$ be the dual operator, i.e., $i^* : L^1(X;C_{r,p})^* \longrightarrow (\mathcal{F}_{r,p})^* = \hat{\mathcal{F}}_{-r,q}$. Take any $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$. We regard $\mu$ as a Borel measure on $X$ as in the section 3. For any $f \in L^1(X;C_{r,p})$, there exists $u \in \mathcal{F}_{r,p}$ such that $|f| \leq \tilde{u}$ q.e. and $C_{r,p}(f) = \|u\|^p_{r,p}$ by Proposition 5.3. Then, by Proposition 3.3 and

$$\int_X |f|\, d\mu \leq \int_X \tilde{u}\, d\mu \leq \|u\|_{r,p}\|\mu\|^\wedge_{-r,q} = C_{r,p}(f)^{1/p}\|\mu\|^\wedge_{-r,q}.$$

Thus we have $\mu \in L^1(X,C_{r,p})^*_+$ and $\|\mu\|_{L^1(X,C_{r,p})^*} \leq \|\mu\|^\wedge_{-r,q}$. Conversely, we have $\|\mu\|_{L^1(X,C_{r,p})^*_+} \geq \|\mu\|^\wedge_{-r,q}$ since

$$\|\mu\|^\wedge_{-r,q} \leq \|i^*\|_{\mathrm{op}}\|\mu\|_{L^1(X;C_{r,p})^*} \leq \|i\|_{\mathrm{op}}\|\mu\|_{L^1(X;C_{r,p})^*} \leq \|\mu\|_{L^1(X;C_{r,p})^*}.$$

Hence we have $\|\mu\|_{L^1(X,C_{r,p})^*} = \|\mu\|^\wedge_{-r,q}$. This completes the proof. □

(5.15) shows that $L^1(X;C_{r,p})$ is a Banach lattice (as for Banach lattice, see e.g., [17, Chapter V]). Here the order in this space is given as follows: $f \geq g$ if and only if $f \geq g$ q.e. Hence its dual space $L^1(X;C_{r,p})^*$ is a Banach lattice as well. $\Phi \in L^1(X;C_{r,p})^*$, thereby can be written as $\Phi = \Phi_+ - \Phi_-$ where $\Phi_+ = \Phi \vee 0$ and $\Phi_- = (-\Phi) \vee 0$. This means that $\Phi$ defines a signed measure and the above decomposition corresponds to the Hahn decomposition. Further, combining this with Theorem 5.7, we have that the range $i^*(L^1(X;C_{r,p})^*)$ is the set of all $\varphi \in \hat{\mathcal{F}}_{-r,q}$ which can be written as $\varphi = \varphi_+ - \varphi_-$, $\varphi_+, \varphi_- \in (\hat{\mathcal{F}}_{-r,q})_+$.

Lastly, we shall give a proof of Theorem 4.1 which was put off. We prove it in the following theorem.

**Theorem 5.8.** *For $u \in \mathcal{F}_{r,p}$, the following conditions are equivalent each other:*

(i) *$u$ is an $(r,p)$-potential.*
(ii) *For any $v \in \mathcal{F}_{r,p}$ with $v \geq u$ a.e., it holds that $\|v\|_{r,p} \geq \|u\|_{r,p}$.*
(iii) *$C_{r,p}(\tilde{u}) = \|u\|^p_{r,p}$ and $u \geq 0$ a.e.*

*Proof.* The equivalence of (ii) and (iii) is clear. We can prove the implication (ii)⇒(i) in the same manner as Theorem 4.2. In fact for any $w \in (\mathcal{F}_{r,p})_+$

$$0 \leq \frac{d}{d\varepsilon}\|u + \varepsilon w\|^p_{r,p}\bigg|_{\varepsilon=0} = p\mathcal{E}_{r,p}(u,w).$$

Next we shall show (i)⇒(iii). Noting that $C_{r,p}(\tilde{u}) \leq \|u\|_{r,p}^p$, it is enough to show $C_{r,p}(\tilde{u}) \geq \|u\|_{r,p}^p$. Without loss of generality, we may assume $u \neq 0$. Take $\varphi \in (\hat{\mathcal{F}}_{-r,q})_+ = L^1(X; C_{r,p})_+^*$ so that $u = U\varphi$. Then noting

$$\|u\|_{r,p}^p = (\|\varphi\|_{-r,q}^\wedge)^q = {}_{\mathcal{F}_{r,p}}\langle u, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}}$$

and

$$\|\varphi\|_{-r,q}^\wedge = \|\varphi\|_{L^1(X;C_{r,p})^*},$$

we obtain

$$\begin{aligned}
\|u\|_{r,p}^p &= {}_{\mathcal{F}_{r,p}}\langle u, \varphi\rangle_{\hat{\mathcal{F}}_{-r,q}} \\
&= {}_{L^1(X;C_{r,p})}\langle \tilde{u}, \varphi\rangle_{L^1(X;C_{r,p})^*} \\
&\leq C_{r,p}(\tilde{u})^{1/p}\|\varphi\|_{L^1(X;C_{r,p})^*} \\
&= C_{r,p}(\tilde{u})^{1/p}\|\varphi\|_{-r,q}^\wedge \\
&= C_{r,p}(\tilde{u})^{1/p}\|u\|_{r,p}^{p/q}.
\end{aligned}$$

Now dividing both hands by $\|u\|_{r,p}^{q/p}$ we get a desired result. □

Using the fact that $L^1(X; C_{r,p})^*$ is the Banach lattice, we can give an another expression of the capacity.

**Lemma 5.9.** *For any $f \in L^1(X; C_{r,p})$,*

(5.17) $C_{r,p}(f)^{1/p} = \sup\{{}_{L^1(X;C_{r,p})}\langle |f|, \mu\rangle_{L^1(X;C_{r,p})^*};\ \mu \in (\hat{\mathcal{F}}_{-r,q})_+, \|\mu\|_{-r,q}^\wedge \leq 1\}.$

*Moreover, the infimum of the right hand side is obtained by a unique element $\nu \in (\hat{\mathcal{F}}_{-r,q})_+$ with $\|\nu\|_{-r,q}^\wedge = 1$.*

*In particular, if $f = e_B$, i.e., the $(r,p)$-equilibrium potential for a set $B$, then $\nu_B/\|\nu_B\|_{-r,p}^\wedge$ is the unique minimizing element where $\nu_B = U^{-1}e_B$.*

*Proof.* By the general theory of Banach lattice, we have

$$C_{r,p}(f)^{1/p} = \sup\{{}_{L^1(X;C_{r,p})}\langle |f|, \Phi\rangle_{L^1(X;C_{r,p})^*};\ \Phi \in L^1(X; C_{r,p})_+^*, \|\Phi\|_{L^1(X;C_{r,p})^*} \leq 1\}.$$

Now, (5.17) is easily obtained by Theorem 5.7 (i). On the other hand, $\{\mu \in (\hat{\mathcal{F}}_{-r,q})_+;\ \|\mu\|_{-r,q}^\wedge \leq 1\}$ is bounded closed set in $\hat{\mathcal{F}}_{-r,q}$ and hence weakly compact. The existence of a maximizing element easily follows from this.

To show the uniqueness, we recall that $\hat{\mathcal{F}}_{-r,q}$ is uniformly convex. Suppose that $\mu$ and $\nu$ maximize the left hand side of (5.17). If $\mu \neq \nu$, then $\|\mu + \nu\| < 2$ by the uniform convexity. Therefore

$$\begin{aligned}
&{}_{L^1(X;C_{r,p})}\langle |f|, (\mu+\nu)/\|\mu+\nu\|_{\hat{\mathcal{F}}_{-r,q}}\rangle_{L^1(X;C_{r,p})^*} \\
&> \frac{1}{2}\{{}_{L^1(X;C_{r,p})}\langle |f|, \mu\rangle_{L^1(X;C_{r,p})^*} + {}_{L^1(X;C_{r,p})}\langle |f|, \nu\rangle_{L^1(X;C_{r,p})^*}\} = C_{r,p}(f)^{1/p}
\end{aligned}$$

which is the contradiction.

If $f = \widetilde{e_B}$, then

$$C_{r,p}(\widetilde{e_B}) = \|e_B\|_{r,p}^p = \int_X \widetilde{e_B}(x)\nu_B(dx) = (\|\nu_B\|_{-r,q}^\wedge)^q$$

where $\nu_B = U^{-1}e_B$. Hence

$$\begin{aligned} C_{r,p}(\widetilde{e_B})^{1/p} &= C_{r,p}(\widetilde{e_B})C_{r,p}(\widetilde{e_B})^{-1/q} \\ &= C_{r,p}(\widetilde{e_B})(\|\nu_B\|_{-r,q}^\wedge)^{-1} \\ &= \int_X \widetilde{e_B}(x)\nu_B(dx)/\|\nu_B\|_{-r,q}^\wedge \end{aligned}$$

which shows that $\nu_B/\|\nu_B\|_{-r,q}^\wedge$ is the maximizing element. □

**Lemma 5.10.** *Suppose that a sequence $\{\nu_n\} \subseteq (\hat{\mathcal{F}}_{-r,q})_+$ converges to $\nu$ *-weakly in $\hat{\mathcal{F}}_{-r,q}$. Then $\nu \in L^1(X;C_{r,p})_+^*$ and $\{\nu_n\}$ converges to $\nu$ *-weakly in $L^1(X;C_{r,p})^*$.*

*Proof.* $\{\nu_n\}$ is bounded in $\hat{\mathcal{F}}_{-r,q}$, since $\{\nu_n\}$ converges. But $\|\nu_n\|_{-r,q}^\wedge = \|\nu_n\|_{L^1(X;C_{r,p})^*}$ by Theorem 5.7, $\{\nu_n\}$ is bounded in $L^1(X;C_{r,p})^*$ as well. Moreover,

$$\lim_{n\to\infty}\langle f,\nu_n\rangle = \langle f,\nu\rangle \quad \text{for } f \in \mathcal{F}_{r,p}.$$

Noticing that $\mathcal{F}_{r,p}$ is dense in $L^1(X;C_{r,p})$, we can see that $\nu \in L^1(X;C_{r,p})_+^*$ and $\{\nu_n\}$ converges to $\nu$ *-weakly in $L^1(X;C_{r,p})^*$. □

**Proposition 5.11.** *Let $e_B$, $e_C$ be $(r,p)$-equilibrium potentials for sets $B$ and $C$. Set $\nu_B = U^{-1}e_B$. Then we have*

(i) $\widetilde{e_B} \le 1 \quad \nu_B$-a.e.
(ii) $B \subseteq C \Rightarrow \widetilde{e_B} \le \widetilde{e_C} \quad \nu_B$-a.e.

*Proof.* We first prove (ii). To show this, we note

$$\begin{aligned} C_{r,p}(\widetilde{e_B}) &= \|e_B\|_{r,p}^p && \text{(Theorem 5.8 (iii))} \\ &= C_{r,p}(B) \\ &= C_{r,p}(1_B) \\ &\le C_{r,p}(\widetilde{e_B}\wedge\widetilde{e_C}) && (\widetilde{e_B}\wedge\widetilde{e_C} \ge 1_B \quad \text{q.e.}) \\ &\le C_{r,p}(\widetilde{e_B}). \end{aligned}$$

Therefore we have

(5.18) $$C_{r,p}(\widetilde{e_B}) = C_{r,p}(\widetilde{e_B}\wedge\widetilde{e_C}).$$

Since $L^1(X; C_{r,p})$ is a Banach lattice, we have $\widetilde{e_B} \wedge \widetilde{e_C} \in L^1(X; C_{r,p})$. Now from Lemma 5.10, there exist $\nu$, $\mu \in (\hat{\mathcal{F}}_{-r,q})_+$ with $\|\nu\|^\wedge_{-r,q} = \|\mu\|^\wedge_{-r,q} = 1$ such that

$$\int_X \widetilde{e_B} d\nu = C_{r,p}(\widetilde{e_B})^{1/p},$$

$$\int_X \widetilde{e_B} \wedge \widetilde{e_C} d\mu = C_{r,p}(\widetilde{e_B} \wedge \widetilde{e_C})^{1/p}.$$

Since $\nu$ is the maximizing element for $\widetilde{e_B}$, we have

$$\int_X \widetilde{e_B} d\mu \leq \int_X \widetilde{e_B} d\nu = C_{r,p}(\widetilde{e_B})^{1/p}.$$

Therefore

$$C_{r,p}(\widetilde{e_B} \wedge \widetilde{e_C})^{1/p} = \int_X \widetilde{e_B} \wedge \widetilde{e_C} d\mu \leq \int_X \widetilde{e_B} d\mu \leq \int_X \widetilde{e_B} d\nu = C_{r,p}(\widetilde{e_B})^{1/p}.$$

Combining this with (5.18), we have

$$\int_X \widetilde{e_B} \wedge \widetilde{e_C} d\mu = \int_X \widetilde{e_B} d\mu$$

which implies $\widetilde{e_B} \wedge \widetilde{e_C} = \widetilde{e_B}$ $\mu$-a.e. and further we have

$$\int_X \widetilde{e_B} d\mu = \int_X \widetilde{e_B} d\nu$$

which implies $\mu = \nu$ by the uniqueness of the maximizing element.

To show (i), it is enough to replace $e_B$ by 1 in the above proof. □

Now we can discuss the support of $\nu_B$ for a general set $B$.

**Theorem 5.12.** *For any set $B$, let $e_B$ be the $(r,p)$-equilibrium potential of $B$ and $\nu_B$ be the $(r,p)$-equilibrium measure, i.e., $\nu_B = U^{-1}e_B$. Then we have*

(i) $\mathrm{supp}[\nu_B] \subseteq \overline{B}$
(ii) $C_{r,p}(B) = \nu_B(\overline{B})$.

*Proof.* We first prove (i) and (ii) for a compact set $K$. (i) was proven in Lemma 4.4. To see (ii), note that $\widetilde{e_K} \geq 1$ q.e. on $K$ by the definition and $\widetilde{e_K} \leq 1$ $\nu_K$-a.e. Accordingly, we have $\widetilde{e_K} = 1$ $\nu_K$-a.e. and hence

$$C_{r,p}(K) = \int_X \widetilde{e_K} d\nu_K = \int_K 1 d\nu_K = \nu_K(K)$$

which shows (ii) for $K$.

Next we prove the assertion for an open set $G$. Since $G$ is capacitable, there exists an increasing sequence of compact sets $\{K_n\}$ such that $K_n \subseteq G$ and

$$\lim_{n\to\infty} C_{r,p}(K_n) = C_{r,p}(G), \quad m(G \setminus \bigcup_{n=1}^{\infty} K_n) = 0.$$

Define $\mathcal{L}_G$ and $\mathcal{L}_{K_n}$, $n = 1, 2, \dots$ by (2.9). We claim that $\mathcal{L}_G = \bigcap_{n=1}^{\infty} \mathcal{L}_{K_n}$. In fact, $\mathcal{L}_G \subseteq \mathcal{L}_{K_n}$ is evident. To see the converse, take $u \in \bigcap_{n=1}^{\infty} \mathcal{L}_{K_n}$. Then $\tilde{u} \geq 1$ q.e. on $K_n$ for all $n$. Therefore $\tilde{u} \geq 1$ q.e. on $\bigcup_{n=1}^{\infty} K_n$. Since we have chosen $\{K_n\}$ so that $m(G \setminus \bigcup_{n=1}^{\infty} K_n) = 0$, we have $\tilde{u} \geq 1$ $m$-a.e. on $G$. But $\tilde{u}$ is quasi-continuous, we eventually obtain $\tilde{u} \geq 1$ q.e. on $G$, i.e., $u \in \mathcal{L}_G$. Thus we get $\mathcal{L}_G = \bigcap_{n=1}^{\infty} \mathcal{L}_{K_n}$.

Set $e_n = e_{K_n}$. Then

$$\lim_{n\to\infty} \|e_n\|_{r,p}^p = \lim_{n\to\infty} C_{r,p}(K_n) = C_{r,p}(G) < \infty.$$

Hence we can take a subsequence $\{e_{n_j}\}$ such that

$$e_{n_j} \longrightarrow e \quad \text{weakly in } \mathcal{F}_{r,p}.$$

Here we used that $\mathcal{F}_{r,p}$ is reflexive. Note that $\bigcap_{n=1}^{\infty} \mathcal{L}_{K_n}$ is convex closed set in $\mathcal{F}_{r,p}$. Therefore $\bigcap_{n=1}^{\infty} \mathcal{L}_{K_n}$ is weakly closed. Moreover it is easy to see that $e \in \bigcap_{n=1}^{\infty} \mathcal{L}_{K_n} = \mathcal{L}_G$. Hence $\|e\|_{r,p}^p \geq C_{r,p}(G)$.

On the other hand,

$$\|e\|_{r,p}^p \leq \varliminf_{j\to\infty} \|e_{n_j}\|_{r,p}^p \leq \varliminf_{j\to\infty} C_{r,p}(K_{n_j}) = C_{r,p}(G).$$

Thus we have $\|e\|_{r,p}^p = C_{r,p}(G)$ and hence $e = e_G$ by the uniqueness of minimizing element. The limit $e_G$ does not depend on a choice of subsequence, we eventually obtain that

(5.19) $$e_n \longrightarrow e_G \quad \text{weakly in } \mathcal{F}_{r,p}.$$

Moreover

(5.20) $$\lim_{j\to\infty} \|e_n\|_{r,p}^p = \lim_{j\to\infty} C_{r,p}(K_n) = C_{r,p}(G) = \|e_G\|_{r,p}^p.$$

Since $\mathcal{F}_{r,p}$ is uniformly convex, (5.19) and (5.20) implies

$$\lim_{n\to\infty} \|e_n - e_G\|_{r,p} = 0,$$

(see, e.g., [5, II.4.28]). Now, using inequalities (4.5) or (4.6), we have

$$\lim_{n\to\infty} \|\nu_{K_n} - \nu_G\|_{-r,q}^{\wedge} = 0.$$

Further, by Lemma 5.10,

$$\nu_{K_n} \longrightarrow \nu_G \quad \text{*-weakly in } L^1(X; C_{r,p})^*.$$

Take $u \in C_b(X)$ with $\operatorname{supp}[u] \cap \overline{G} = \phi$. Noting that $C_b(X) \subseteq L^1(X; C_{r,p})$, we get

$$\int_X u \, d\nu_G = \lim_{n\to\infty} \int_X u \, d\nu_{K_n} = 0$$

which asserts that $\operatorname{supp}[\nu_G] \subseteq \overline{G}$.

(ii) can be obtained as follows:

$$\begin{aligned} C_{r,p}(G) &= \lim_{n\to\infty} C_{r,p}(K_n) = \lim_{n\to\infty} \nu_{K_n}(K_n) = \int_X 1_{K_n}\, d\nu_{K_n} \\ &= \lim_{n\to\infty} \int_X 1\, d\nu_{K_n} = \int_X 1\, d\nu_G = \nu_G(\overline{G}). \end{aligned}$$

Lastly we prove the assertions for a general set $B$. For any $u \in C_b(X)$ with $\mathrm{supp}[u] \cap \overline{B} = \phi$, take a decreasing sequence of open sets $\{G_n\}$ such that $G_n \supseteq B$, $\mathrm{supp}[u] \cap \overline{G}_n = \phi$ and $C_{r,p}(G_n) \downarrow C_{r,p}(B)$. Noting that $e_{G_n} \in \mathcal{L}_B$ and $\mathcal{L}_B$ is weakly closed, we can obtain

$$e_{G_n} \to e_B \quad \text{strongly in } \mathcal{F}_{r,p}$$

by the same argument as above. Moreover, we have similarly

$$\nu_{G_n} \to \nu_B \quad \text{*-weakly in } L^1(X; C_{r,p})^*.$$

Hence

$$\int_X u\, d\nu_B = \lim_{n\to\infty} \int_X u\, d\nu_{G_n} = 0$$

which implies $\mathrm{supp}[\nu_B] \subseteq \overline{B}$.

(ii) can be shown by the same way. □

Lastly, we shall give an example satisfying the conditions (A.1), (A.2) and (A.3).

Let $(B, H, \mu)$ be an abstract Wiener space: $B$ is a separable real Banach space, $H$ is a separable real Hilbert space which is embedded densely and continuously in $B$ and $\mu$ is the Gaussian measure satisfying

$$\hat{\mu}(l) = \int_B \exp\left\{\sqrt{-1}\, {}_B\langle x, l\rangle_{B^*}\right\} \mu(dx) = \exp\left\{-\frac{1}{2}|l|_{H^*}^2\right\}, \qquad l \in B^* \hookrightarrow H^*.$$

We consider the following Ornstein-Uhlenbeck semigroup:

$$T_t f(x) = \int_B f(e^{-tA}x + \sqrt{1 - e^{-2tA}}y)\mu(dy) \quad \text{for } f \in L^2(\mu). \tag{5.21}$$

Here $A$ is a strictly positive definite self-adjoint operator in $H$.

We assume that $C^\infty(A^*) \bigcap B^*$ is dense in $\mathrm{Dom}(A^{*k})$ under the graph norm of $A^{*k}$ for any $k \in \mathbf{Z}_+$. Here $A^*: H^* \to H^*$ is the dual operator of $A$ and $C^\infty(A^*) = \bigcap_{k=1}^\infty \mathrm{Dom}(A^{*k})$.

We define $\mathcal{F}C_b^\infty$ to be the set of all functions of the form

$$f(x) = F({}_{B^*}\langle l_1, x\rangle_B, \ldots, {}_{B^*}\langle l_n, x\rangle_B), \quad l_1, \cdots, l_n \in C^\infty(A^*) \cap B^* \tag{5.22}$$

where $n \in \mathbf{N}$, $F \in C_b^\infty(\mathbf{R}^n)$. The associated Dirichlet form is given by

$$\mathcal{E}(f, g) = \int_B (\sqrt{A^*}Df(x), \sqrt{A^*}Dg(x))_{H^*}\, \mu(dx), \qquad f, g \in \mathcal{F}C_b^\infty.$$

Here $Df(x)$ is an $H$-derivative of $f$ at $x$:

$$Df(x)[h] = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} \quad \text{for } h \in H.$$

Let us see that the above Ornstein-Uhlenbeck semigroup satisfies the conditions. In [19, Proposition 4.2] it is shown that $\mathcal{F}C_b^\infty$ is dense in $\mathcal{F}_{r,p}$. Hence (A.1) is satisfied. It is clear that $\mathcal{F}C_b^\infty$ satisfies (A.2). As for (A.3), the tightness of $(r,p)$-capacity is proven by Feyel-de La Pradelle [9]. Hence all conditions are satisfied in this case.

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