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Measuring Concentration Risk for Regulatory Purposes[+]

by Marc Gürtler[*], Martin Hibbeln[**], and Clemens Vöhringer[***]

Abstract. The measurement of concentration risk in credit portfolios is necessary for the determination of regulatory capital under Pillar 2 of Basel II as well as for managing portfolios and allocating economic capital. Existing multi-factor models that deal with concentration risk are often inconsistent with the Pillar 1 capital requirements. Therefore, we adjust these models to achieve Basel II-compliant results. Within a simulation study we test the impact of sector concentrations on several portfolios and contrast the accuracy of the different models. In this context, we also compare Value at Risk and Expected Shortfall regarding their suitability to assess concentration risk.

Keywords: Concentration Risk; Pillar 2; Multi-Factor Models; Economic Capital; Simulation Study; Value at Risk; Expected Shortfall

JEL classification: G21, G28

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1 Introduction

In recent years there have been significant improvements in understanding and measuring concentration risk in credit portfolios such as undiversified idiosyncratic risk and industry or country risk. The measurement of these risks is important against the background of regulatory capital needs as well as for computing the economic capital. Unfortunately, the existing approaches are mostly not fully consistent with the new capital adequacy framework (Basel II) – sometimes within the derivation and sometimes within the implementation – so that the benefit of these approaches is restricted. Furthermore, comparative analyses on these models are scarce. Against this background we address the following questions:

- How can the existing approaches be modified and adjusted to be consistent with the Basel framework? Is the risk measure Value at Risk problematic when dealing with concentration risk?
- Which methods are capable to measure concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

For answering these questions, we firstly investigate the assumptions underlying the Basel framework. The Basel II formula for measuring the Value at Risk of credit portfolios is based on the so-called asymptotic single risk factor (ASRF) framework as explained in Gordy (2003). In this framework it is assumed that

- the portfolio is infinitely fine grained and thus it consists of a nearly infinite number of credits with small exposures, and
- only one systematic risk factor influences the default risk of all loans in the portfolio.

The first assumption implies that there are no name concentrations within the portfolio, thus all idiosyncratic risk is diversified completely. The second assumption implicates that there are no sector concentrations such as industry- or country-specific risk concentrations. These are idealizations that can be problematic for real world portfolios.

The Basel Committee on Banking Supervision (BCBS) already recognized the high importance of credit risk concentrations in the Basel framework: "Risk concentrations are arguably the single most important cause of major problems in banks."[1] Since it is difficult to incorporate credit risk concentrations in analytic approaches, in Basel II there is no quantitative approach mentioned how to deal with risk concentrations. Instead, it is only qualitatively demanded in Pillar 2 of Basel II that "Banks should have in place effective internal policies, sys-

[1] See BCBS (2005) §770.

tems and controls to identify, measure, monitor, and control their credit risk concentrations."[2] Thus, it is each bank's task how to meet these requirements concretely. But of course the measurement and management of risk concentrations are not only important for the determination of regulatory capital but also for the measurement of the "true" portfolio risk. The capital needs regarding this "true" risk will be denoted as economic capital in the following.

When measuring concentration risk it is important to notice the different interpretation of concentration risk by banks and supervisors. Banks often only look at the one side of concentration risk – the diversification effect. They often argue that the Pillar 1 capital requirement does not measure benefits from diversification. Therefore it is argued that this framework is the non-diversified benchmark and thus an upper barrier for the true capital requirement. Contrary, supervisors interpret concentration risk as "a positive or negative deviation from Pillar 1 minimum capital requirements derived by a framework that does not account explicitly for concentration risk." [3] The latter perception is justified by the fact that the Pillar 1 capital rules were calibrated on well-diversified portfolios with low name and low sector concentration risk.[4] Thus, if a portfolio is low diversified, the risk will be underestimated when using the Basel formula. Therefore, additional capital is required to capture these types of concentration risk. Contrary, if the portfolio is very high diversified, the Basel formula can overestimate the "true" risk. However, in the case of this overestimation of risk it is not allowed – at least at present – to reduce the regulatory capital. For well-diversified portfolios the Basel formula is a good approximation of the "true" risk. This relation is highlighted in Figure 1.

- Figure 1 about here -

Name concentrations as well as sector concentrations are already analyzed in the literature. The theoretical derivation of the so-called granularity adjustment that accounts for name concentrations was done by Wilde (2001) and improved by Pykhtin and Dev (2002) and Gor-

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[2] See BCBS (2005) §773. Furthermore, because of the importance of this topic for the stability of the banking system, the Basel Committee launched the "Research Task Force Concentration Risk" that presented its final report in BCBS (2006). The Task Force collected information about the state of the art in current practice and academic literature, analyzed the impact of departures from the ASRF model and reviewed some methodologies to measure name and sector concentrations. An additional workstream focused on stress testing against the background of risk concentrations.

[3] See BCBS (2006).

[4] See BCBS (2006) and CEBS (2006) §18.

dy (2003). This can be called "portfolio name concentration" because the approach refers to the finite number of credits in the portfolio. The adjustment formulas are derived in a more straightforward approach by Martin and Wilde (2002), Rau-Bredow (2002) and Gordy (2004). Furthermore, the adjustment is extended and numerically analyzed in detail by Gürtler, Heithecker, and Hibbeln (2008). An approach related to Wilde (2001) is the granularity adjustment from Gordy and Lütkebohmert (2007). In contrast, the semi-asymptotic approach from Emmer and Tasche (2005) refers to name concentrations due to a single name while the rest of the portfolio remains infinitely granular. Thus, this type can be called "single name concentration".

There also exist analytic and semi-analytic approaches that account for sector concentrations. One rigorous analytical approach is Pykhtin (2004) that is based on a similar principle as in Martin and Wilde (2002). An alternative is the semi-analytic model from Cespedes *et al.* (2006) that derives an approximation formula through a complex numerical mapping procedure. Another approach from Düllmann (2006) extends the binomial extension technique (BET) model from Moody's. Tasche (2006) suggests an ASRF-extension in an asymptotic multi-factor setting. Some numerical work on the performance of the Pykhtin model is done by Düllmann and Masschelein (2007). Furthermore, Düllmann (2007) presents a first comparison of different approaches on sector concentration risk. The problem is that the derivation and the application of the approaches are often inconsistent with the Basel II framework what is critical for the following reasons:

- Banks are demanded to measure concentration risks and "explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under Pillar 2" of Basel II. Even if a bank uses a high-sophisticated multi-factor model, the results are not comparable with the Pillar 1 capital requirement if the results are not consistent to the Basel framework. Thus, it remains unclear if or how much additional regulatory capital is needed regarding risk concentrations.
- Generally, it is not worthwhile to have a major gap between the regulatory and the "true" economic capital. A homogenization of these values is one goal of the new Capital Accord and would simplify the management of the credit portfolio.

For these reasons we demonstrate how multi-factor models can be used in a way that is consistent with the Basel II framework. This can be seen as expanding the validity of the Basel formula from the inner region of Figure 1 to the whole region. As sector concentrations typically have a significantly higher impact on the capital requirement than name concentrations,[5] we focus on sector concentrations in the following. Furthermore we compare the capability of different multi-factor approaches in approximating the "true" portfolio risk through a simulation study. In this context we also use our framework to test whether the problems mentioned in the literature with the widespread used VaR are relevant in connection with the measurement of concentration risk. The use of the VaR is usually criticized since this risk measure does not fulfill all axioms of coherency.[6] Instead, the application of the coherent risk measure Expected Shortfall (ES) is suggested. Since the non-coherency of the VaR is typically illustrated in contrived portfolio examples the relevance of this issue should be analyzed in more realistic settings.

The rest of the paper is outlined as follows. In section 2 we briefly describe the ASRF framework and the Basel formula. Moreover, we discuss the problems of the non-coherent Value at Risk in the context of concentration risk and present how the coherent ES can be used consistent with Basel II. In section 3 we introduce multi-factor models in general, and the Pykhtin as well as the Cespedes model in particular. In this context we demonstrate how these approaches could be modified and applied to achieve meaningful results. We compare the performance of the models with a simulation study in section 4. Furthermore, we test the accuracy of the VaR in comparison to the ES. The paper concludes with section 5.

2 Coherent Concentration Risk Measurement in the Context of the Basel Framework

2.1 The ASRF Framework and the Basel II Formula

As mentioned before, the Basel II risk quantification formula is based upon the ASRF framework that assumes an infinitely granular portfolio and the existence of only one systematic risk factor $\tilde{x}$. If these two assumptions are fulfilled the relative portfolio loss $\tilde{L}$ in t = T almost surely equals the expected loss (EL) conditional on the realization of the systematic factor $\tilde{x}$[7]

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[5] See BCBS (2006).

[6] See Artzner *et al.* (1999).

[7] To keep track of the model, stochastic variables are marked with a tilde "~". Further, "E" denotes the expectation operator.

$$\tilde{L} - E(\tilde{L} \mid \tilde{x}) \rightarrow 0 \quad \text{a.s.}^{8} \tag{1}$$

If the loss given default (LGD) is assumed to be deterministic, the conditional expectation can be written as

$$E(\tilde{L} \mid \tilde{x}) = \sum_{i=1}^{n} E(w_i \cdot LGD_i \cdot \tilde{I}_{Default,i} \mid \tilde{x}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot E(\tilde{I}_{Default,i} \mid \tilde{x}), \tag{2}$$

where $\tilde{I}_{Default}$ represents the indicator function that is 1 in the event of default and 0 in case of survival of the obligor, n stands for the number of credits, and w_i denotes the weight of credit i in the credit portfolio ($i \in \{1, \ldots, n\}$). For the concrete application of formula (2), the conditional default expectation has to be determined. In the Basel II framework, the well known Vasicek model is used.[9] In this one-period one-factor model the return of each obligor is driven by two components that realize at a future point in time T: a systematic part $\tilde{x}$ that influences all firms and a firm-specific (idiosyncratic) part $\tilde{\varepsilon}_i$. Thus, the "normalized" asset returns[10] $\tilde{a}_i$ of each obligor i in t = T can be represented by the following model

$$\tilde{a}_i = \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i, \tag{3}$$

in which $\tilde{x} \sim N(0,1)$ and $\tilde{\varepsilon}_i \sim N(0,1)$ are independently and identically normally distributed with mean zero and standard deviation one. In this model, the correlation structure of each firm i is represented by the firm-specific correlation $\sqrt{\rho_i}$ to the common factor. Hence, the correlation between two firms i, j can be expressed as $\sqrt{\rho_i} \cdot \sqrt{\rho_j}$ or simply as ρ for the case of a homogeneous correlation structure.

Further, the probability of default of each obligor is exogenously given as PD_i.[11] Corresponding to formula (3), an obligor i defaults at t = T when its "normalized" return falls below a default threshold b_i which can be characterized by

$$\tilde{a}_i < b_i \quad \Leftrightarrow \quad \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i < b_i. \tag{4}$$

Against this background the threshold b_i is determined by the exogenous specification of PD_i:[12]

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[8] See Gordy (2003).

[9] See e.g. Vasicek (1987, 1991, 2002) and Finger (1999, 2001).

[10] The returns are normalized by subtracting the expected return and dividing the resulting term by the standard deviation in order to get standard normally distributed variables.

[11] The probability of default could either be determined by the institution itself or by a rating agency.

$$PD_i = prob\left(\tilde{a}_i < b_i\right) = N(b_i) \quad \Leftrightarrow \quad b_i = N^{-1}(PD_i)\,. \tag{5}$$

Conditional on a realization of the systematic factor the probability of default of each obligor is

$$prob\left(\tilde{a}_i < b_i \mid \tilde{x}\right) = E\left(\tilde{I}_{\tilde{a}_i < b_i} \mid \tilde{x}\right) = N\left(\frac{N^{-1}(PD_i) - \sqrt{\rho_i}\cdot\tilde{x}}{\sqrt{1-\rho_i}}\right) =: p_i(\tilde{x})\,. \tag{6}$$

Applying formula (6) from the Vasicek model to formula (2) from the ASRF framework, the portfolio loss distribution can be computed. For quantification of the credit risk, the Value at Risk (VaR) on confidence level z can be used, that is the z-quantile q_z of the loss variable, in which $z \in (0,1)$ is the target solvency probability. Precisely, like Gordy (2004), we define the VaR as the loss that is only exceeded with the probability of at most 1–z, i.e.

$$VaR_z\left(\tilde{L}\right) := q_z\left(\tilde{L}\right) := \inf\left(l:\; prob\left(\tilde{L} \le l\right) \ge z\right). \tag{7}$$

In the context of the ASRF framework, the VaR can be computed similarly to formula (1) as

$$VaR_z\left(\tilde{L}\right) - E\left(\tilde{L} \mid \tilde{x} = q_{1-z}(\tilde{x})\right) \to 0 \quad \text{a.s.}, \tag{8}$$

where $q_z(\tilde{x})$ stands for the z-quantile of the systematic factor. Recalling formula (2), (6), and the normality of the systematic factor, the VaR of the portfolio equals

$$\begin{aligned} VaR^{Basel}\left(\tilde{L}\right) &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i\left(q_{1-z}(\tilde{x})\right) \\ &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left(\frac{N^{-1}(PD_i) + \sqrt{\rho_i}\cdot N^{-1}(0.999)}{\sqrt{1-\rho_i}}\right), \end{aligned} \tag{9}$$

if we insert the confidence level z = 0.999. This is the (well established) VaR formula used in Basel II. Obviously, the credit risk only relies on the systematic factor since due to the infinite number of exposures the idiosyncratic risks associated with each individual obligor cancel out each other and are diversified completely.

2.2 Concentration Risk and Coherency

In recent years there has been an extensive discussion about reasonable risk measures. Artzner *et al.* (1999) formulated four axioms that a risk measure should satisfy to be coherent: translation invariance, subadditivity, positive homogeneity, and monotonicity. Unfortunately, the commonly used VaR is not coherent because it is not necessarily subadditive. As long as we

[12] The term prob(A) stands for the probability of the occurrence of an uncertain event A. $N(\cdot)$ characterizes the cumulative standard normal distribution and $N^{-1}(\cdot)$ stands for the inverse of $N(\cdot)$.

stay in the ASRF framework this characteristic is not problematic because in this context the VaR is exactly additive.[13] But if we leave the ASRF framework, this behavior is not guaranteed anymore. This is true for non-asymptotic portfolios as well as for multi-factor models. However, many contributions that deal with concentration risk in the context of the Basel II framework use the VaR to quantify credit risk without calling the risk measure into question (possibly to be consistent with the ASRF-framework) even if the subadditivity could get problematic if concentration risk is considered.[14] Thus, it could be beneficial to change the measure of risk, e.g. to use the coherent Expected Shortfall, that is defined as[15]

$$ES_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left(E\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)} + q_z \cdot \left[(1-z) - \text{prob}\left(\tilde{L} \geq q_z\right)\right]\right) \tag{10}$$

with q_z for the VaR on confidence level z (see formula (7)), or simply as

$$ES_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left[E\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)}\right] = E\left(\tilde{L} \mid \tilde{L} \geq q_z\right) \tag{11}$$

for continuous distributions. In addition to the mentioned coherency, the ES is also beneficial from an economic perspective. Instead of focusing on a single quantile which provides no information about tail events, the ES incorporates also information about the degree of losses in the case that the VaR is exceeded. This information is not only relevant for bondholders but also from a regulatory perspective as the shortfall amount could be required to recover the bank.

But despite of the mentioned disadvantages of the VaR, it is still widespread used in practice, so there might be some opposing arguments. One point could be that the mentioned problems of the VaR do not appear in realistic settings and thus both the ES and the VaR lead to plausible results if applied accurately. As this issue is analyzed insufficiently for concentration risk in credit portfolios, we will take up this subject in our simulations later on.[16] A further often stated issue is that the ES is much less robust than the VaR.[17] But as shown in

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[13] This can be seen in formula (8) considering that the expectation operator is additive.

[14] See e.g. Heitfield, Burton, and Chomsisengphet (2006), Cespedes *et al.* (2006), Düllmann (2006), as well as Düllmann and Masschelein (2007).

[15] See Acerbi and Tasche (2002).

[16] Some of our analyses regarding name concentrations show that the corresponding granularity adjustment formulas lead to better results if the ES is used instead of the VaR, particularly if there is a high degree of concentration risk. A numerical study can be requested from the authors. However, it is unclear if this is true for sector concentrations, too.

[17] The standard argument is reproduced by Acerbi (2004) as follows: "VaR does not even try to estimate the leftmost tail events, it simply neglects them altogether, and therefore it is not affected by the statistical uncertainty of rare events. ES on the contrary, being a function of rare events also, has a much larger statistical error."

Acerbi (2004), VaR and ES usually have similar statistical errors, implying this aspect not to be an argument against the use of the ES.[18] An additional problem is that the measured economic capital would be significantly higher if it is determined on the basis of the ES instead of the VaR (by use of the same confidence level). If we exemplary examine a portfolio with PD = 0.5% and ρ = 20% in the ASRF framework, the measured risk on confidence level z = 99.9% is 9.1% for the VaR and 11.81% for the ES, what is not the intended consequence of changing the risk measure. Instead, we would only like to have the appreciated properties when measuring concentration risk without to be bound to increase the required amount of capital. Therefore, we will show how the confidence level can be adjusted to account for this aspect subsequently.

To sum up, from a theoretical perspective it is reasonable to use a coherent risk measure like the ES instead of the VaR when we allow for concentrated credit portfolios. Therefore, we show how the ES can be applied consistently to the Basel II framework in the next section. But as we do not know whether the mentioned disadvantages of the VaR appear in realistic settings of concentrated portfolios, we also apply the VaR during our simulations in chapter 4 and analyze whether and in which degree, respectively, the usage of VaR leads to undesirable results.

2.3 Adjusting for Coherency in Concentrated Portfolios

Against the background of the preceding section we want to implement the ES and compare the outcome with the results by application of the VaR. But if we change the risk measure we have to ensure that the new risk measure (the ES) on the one hand is consistent with the framework presented in Pillar 2 of Basel II to get meaningful results for additional capital requirements stemming from concentration risk. On the other hand the new risk measure should still match the capital requirements of Pillar 1 if the portfolio under consideration fulfills the assumptions of the ASRF framework. I.e. in the context of the ASRF framework, the capital requirements should not differ whether the risk is measured by the VaR or by the ES. Therefore, we examine VaR^{Basel} on the given confidence level z = 99.9% for several (infinitely granular) bank portfolios of different quality. As a next step we determine the confidence level of the ES that is necessary to match the results for both risk measures. We define this ES-confidence level z (= z(ES)) implicitly as

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[18] Actually, due to the lower comparable confidence level – which will be explained in section 2.3 – the statistical error of the ES is smaller compared to the VaR.

$$ES_z^{Basel}(\tilde{L}) = VaR^{Basel}(\tilde{L}), \tag{12}$$

with VaR^{Basel} given by formula (9). ES_z^{Basel} can be calculated using formula (11) and (9), leading to

$$ES_z^{Basel}(\tilde{L}) = \sum_{i=1}^{n} \frac{w_i \cdot LGD_i}{1-z} \cdot N_2\left(-N^{-1}(z), N^{-1}(PD_i), \sqrt{\rho_i}\right), \tag{13}$$

where $N_2(\cdot)$ stands for the bivariate cumulative normal distribution.[19]

Firstly, we investigate the extreme cases that all creditors of a bank have a rating of (I) AAA or (VII) CCC.[20] As can be seen in Table 1, the ES-confidence level must be in a range between 99.67% and 99.74%. Using these confidence levels the required capital is almost identical regardless of whether VaR or ES is used.

- Table 1 about here -

Additionally, we use five portfolios with different credit quality distributions (very high, high, average, low, and very low) that are visualized in Figure 2.[21] All resulting confidence levels are between 99.71% and 99.73% with mean 99.72%. Even if there is some interconnection between the confidence level and the portfolio quality, an ES-confidence level of z = 99.72% seems to be accurate for most real world portfolios.

- Figure 2 about here -

3 Basel II-consistent Credit Risk Modeling in a Multi-Factor Setting

3.1 Multi-Factor Models in Credit Risk Modeling

To obtain a more realistic modeling of correlated defaults in a credit portfolio, we will introduce a typical multi-factor model. In such a model the dependence structure between obligors is not driven by one global systematic risk factor but by sector specific risk factors. Additionally, the group of obligors is divided into S sectors. Hereby a suitable sector assignment is

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[19] Cf. Acerbi and Tasche (2002) and Pykhtin (2004).

[20] We used the idealized default rates from Standard & Poors, see Brand and Bahar (2001), ranging from 0.01% to 18.27%, but the results do not differ widely for different values.

[21] The portfolios with high, average, low, and very low quality are taken from Gordy (2000). We added a portfolio with very high quality.

important,[22] i.e. asset correlations shall be high within a sector and low between different sectors. In contrast to the single factor model in which the correlation structure of each firm i is completely described by ρ, in a multi-factor model we distinguish between an inter-sector correlation ρ_{Inter} and an intra-sector correlation ρ_{Intra}. The inter-sector correlation describes the correlation between the sector factors and the intra-sector correlation characterizes the sensitivity of the asset return to the corresponding sector factor. Thus, the asset return of obligor i in sector s can be represented by

$$\tilde{a}_{s,i} = \sqrt{\rho_{Intra,i}} \cdot \tilde{x}_s + \sqrt{1-\rho_{Intra,i}} \cdot \tilde{\xi}_i, \qquad (14)$$

where $\tilde{x}_s$ is the sector risk factor and $\tilde{\xi}_i$ stands for the idiosyncratic factor. $\tilde{x}_s$ and $\tilde{\xi}_i$ are normally distributed variables with mean zero and standard deviation one that are independent among each other. Since the sector risk factors $\tilde{x}_s$ are potentially dependent random variables that are difficult to deal with[23] we make use of the possibility to present the sector risk factors as a combination of independently and standard normally distributed factors $\tilde{z}_k$ (k = 1, …, K)

$$\tilde{x}_s = \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k \quad \text{with} \sum_{k=1}^{K} \alpha_{s,k}^2 = 1, \qquad (15)$$

in which the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix.[24] Hence the inter-sector correlation is given as

$$\rho_{s,t}^{Inter} = \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k}. \qquad (16)$$

From (14) and (15) the asset correlation between two obligors is given by

$$\text{corr}(\tilde{a}_{s,i}, \tilde{a}_{t,j}) = \begin{cases} \sqrt{\rho_{Intra,i}} \cdot \sqrt{\rho_{Intra,j}}, & \text{if } s = t, \\ \sqrt{\rho_{Intra,i}} \cdot \sqrt{\rho_{Intra,j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k}, & \text{if } s \neq t. \end{cases} \qquad (17)$$

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[22] As shown by Morinaga and Shiina (2005) an assignment of borrowers to the wrong sectors leads to a higher estimation error than a non-optimal sector definition.

[23] Concretely, the independence of the risk factors is essential for the derivation of the Pykhtin-model in section 3.3.

[24] This approach is a common mathematical method to generate correlated normal random variables and leads to the identical number of independent risk factors $\tilde{z}_k$ and dependent sector factors $\tilde{x}_s$, that is K equals S. Another common method to determine independent risk factors is the principal component analysis which leads to a reduced number of risk factors.

Obligors in the same sector will be highly correlated with one another when their intra-sector correlation is high. The correlation of obligors in different sectors also depends on the factor weights, which are derived from the inter-sector correlation. Consequently, the dependence structure in the multi-factor model is completely described by the intra- and inter-sector correlations. Taking formula (5) into account, the portfolio loss distribution can be written as

$$\tilde{L} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD_{s,i} \cdot \tilde{I}_{\tilde{a}_{s,i} < N^{-1}(PD_{s,i})}, \tag{18}$$

where n_s is the number of obligors in sector s.

In the next three subsections we will present different approaches to determine the distribution and tail expectations of $\tilde{L}$. Furthermore, we will demonstrate how the models can be parameterized to be Basel II-consistent.

3.2 Monte-Carlo-Simulations and Parameterization through a Correlation Matching Procedure

A common approach to estimate the portfolio loss distribution is the use of Monte-Carlo-Simulations. In each simulation run the sector factors as well as the idiosyncratic factor of each obligor are randomly generated. Herewith the asset return is calculated according to (14). If $\tilde{a}_{s,i}$ is less than a threshold given by $N^{-1}(PD_{s,i})$, obligor i defaults. The portfolio loss is determined from formula (18) by summing up the exposure weights $w_{s,i}$ multiplied by the $LGD_{s,i}$ of each defaulted credit. To get a good approximation of the "true" loss distribution we choose 500,000 runs for our Monte-Carlo-Simulations. After running the simulation and sorting loss outcomes, we get the portfolio loss distribution. To obtain the ES for a given confidence level z, in principle the mean for all loss realizations equal or greater than q_z has to be calculated. The quantile q_z is given by the $z \cdot 500{,}000$th element of the simulated distribution.[25]

To calibrate the multi-factor model, most variables can be chosen identically to the single factor model. The only difference is the correlation structure that generally consists of inter- and intra-sector correlations as described above. The matrix of inter-sector correlations is usually derived from historical default rates or from equity correlations between industry sectors. The intra-sector correlations can be derived from historical default rates, too. The prob-

[25] The exact formulation is given in formula (10).

lem of a derivation based on historical default rates is that there are not always enough observations to get stable results. That is even more problematic if it is assumed (like in Basel II) that the correlation and the PD are interdependent. Furthermore, the results from the multi-factor model would normally not be consistent with Basel II because the correlation structure is completely different. Thus, it would not be possible to identify if there is need for additional regulatory capital under Pillar 2 (measured consistently to Pillar 1) of Basel II.

For both reasons the intra-sector correlations could be chosen analogously to the Basel II formula

$$\rho_{\text{Basel}} = 0,12 \cdot \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}} + 0,24 \cdot \left(1 - \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}}\right) \tag{19}$$

for corporates. This is what Cespedes *et al.* (2006) did in their analyses. But their approach is critical for the following reason: The validity of this formula for the intra-sector correlations is equivalent to the statement that the regulatory capital calculated via the formula of Pillar 1 is an upper barrier of the true risk. This property in turn is only fulfilled if there exists only one sector or if all sectors are perfectly correlated. In all other cases there is an effect of sector diversification that leads to a lower capital requirement compared to the Basel framework. Beyond, the Basel II correlation formula is not intended by the Basel committee to reflect the intra-sector correlation exclusively. Instead, the framework is calibrated on well-diversified portfolios, as demonstrated in Figure 1, implying that the correlation formula is chosen in a way that the single factor model leads to a good approximation of the "true" risk based on the full correlation structure in a multi-factor model. Cespedes *et al.* (2006) already recognized this criticism and mentioned that it should be possible to use some scaling up for the intra-sector correlations and the resulting capital, respectively, but their calculations are based on the formula above.

Alternatively, the intra-sector correlation could be chosen in a way that the regulatory capital RC can be matched with the economic capital EC^{mf} that is simulated for a well-diversified portfolio within a multi-factor model. Therefore, we define the "implicit intra-sector correlation" $\rho_{\text{Intra}}^{(\text{Implied})}$ by

$$\text{EC}^{\text{mf}}\left(\rho_{\text{Inter}}, \rho_{\text{Intra}}^{(\text{Implied})}\right) = \text{RC}\left(\rho_{\text{Basel}}\right). \tag{20}$$

Unfortunately, the portfolios for which the calibration was done by the Basel Committee including the assumed inter-sector correlation structure are not publicly available. Thus, firstly we have to choose a concrete inter-sector correlation and determine the implicit intra-sector correlation for some hypothetical, well-diversified portfolios via Monte-Carlo-Simulations with several parameter trials. This approach is related to Lopez (2004), who empirically determines the single correlation parameter for the ASRF model that leads to the same 99.9% quantile as KMV's multi-factor model for several portfolio types (geographical region, PD, and asset size categories) using a grid search procedure. Thus, in the approach of Lopez (2004) the left-hand side of formula (20) is given and the single correlation parameter of the right-hand side is determined, whereas we are searching for the intra-sector correlation on the left-hand side that leads to a match of both models when the other parameters, especially the single correlation parameter of Basel II, are exogenously given.

As mentioned above, the required inter-sector correlation matrix could be estimated from historical default rates or from time series of stock returns.[26] Düllmann, Küll, and Kunisch (2008) demonstrate on the basis of an extensive simulation study that it is recommendable to use stock prices instead of historical default rates as this involves smaller statistical errors. Against this background we rely on equity correlations, too, and use the correlation matrix of the MSCI EMU industry indices computed by Düllmann and Masschelein (2007) for the inter-sector correlation structure (see Table 2).[27]

- Table 2 about here -

Our definition of a well-diversified portfolio is based on the overall sector concentration of the German banking system.[28] Even if it is theoretically possible to achieve lower capital requirements through different sector decomposition, this can only be done by a restricted number of banks since a deviation from the market structure of all banks immediately leads to a disequilibrium. The composition can be seen in Table 3. In addition, the total number of credits is assumed to be n = 5000 to guarantee a low degree of name concentration.

- Table 3 about here -

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[26] An overview of the literature regarding the measurement of asset correlation parameters can be found in Düllmann, Küll, and Kunisch (2008) and Grundke (2008).

[27] The correlation structure based on the MSCI US is similar, see Düllmann and Masschelein (2007).

[28] Düllmann and Masschelein (2007) notice that the concentration is very similar to other countries like France, Belgium and Spain.

If we assume a constant intra-sector correlation, the best match is achieved by (approximately) $\rho_{\text{Intra}}^{(\text{Implied})} = 25\%$.[29] The concrete results, however, vary with the portfolio quality (see Table 4).[30] Thus, the use of a constant intra-sector correlation can lead to a significant underestimation of economic capital for high-quality portfolios and to an overestimation for low-quality portfolios.

- Table 4 about here -

To reduce the deviation, the intra-sector correlation should be decreasing in PD. We found that the following intra-sector correlation function leads to a good match for portfolios with different quality distributions:

$$\rho_{\text{Intra}}^{(\text{Implied})} = 0.185 \cdot \frac{1-e^{-50 \cdot PD}}{1-e^{-50}} + 0.34 \cdot \left(1 - \frac{1-e^{-50 \cdot PD}}{1-e^{-50}}\right) \tag{21}$$

Thus, we use the correlation function type from Basel II but the correlation range is from 18.5% to 34% instead of 12% to 24%.[31] It has to be noted that this formula is still a substantial simplification as we assume that the intra-sector correlation is PD-dependent only. In addition, empirically there are also inter-sectoral differences of this parameter.[32] In principle it would be possible to capture both effects, e.g. by multiplying a sector-specific factor to formula (21), which covers the relation of the empirically observed correlations.[33] Of course, the absolute level of the resulting correlations would usually be different from the empirical observations to keep Basel II consistent results. But for convenience we rely on the PD-dependent formula (21) in our following analyses.

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[29] This value results on the basis of both measures (VaR and ES) on the respective confidence level as described in section 2.3. The result is consistent with Düllmann and Masschelein (2007) who use a constant intra-sector correlation of 25% in their analysis.

[30] See Figure 4 for the portfolio characteristics.

[31] We tried several different functional forms but the formula above performed best. The multipliers 18.5% and 34% in function (21) were determined with a grid search using a reasonable parameter range, which is similar to the procedure of Lopez (2004) used for the single correlation parameter.

[32] E.g. Heitfield, Burton, and Chomsisengphet (2006) determine the sector loadings, which equal $\sqrt{\rho_{\text{Intra}}}$ for 50 industry sectors using KMV data on asset values. The resulting intra-sector correlation is on average 18.8% and the standard deviation is 8.3%. These inter-sectoral differences are not captured by the formula above.

[33] A correlation structure with one degree of freedom for every PD-/sector-combination is practically unfeasible due to high data requirements.

Thus, all additional input data needed for typical multi-factor models, e.g. using Monte-Carlo-Simulations, are given with Table 2 and formula (21). Using these values, the multi-factor models should be consistent with the Basel framework. Consequently, the measured economic capital is only lower than the regulatory capital if the portfolio is less concentrated than a typical, well-diversified portfolio and the needed economic capital will be above the capital requirement of the regulatory framework if there is more concentration risk in the credit portfolio.

3.3 Implementation for the Pykhtin-Model

In this section we present the multi-factor adjustment of Pykhtin (2004). It is an extension of the granularity adjustment, introduced by Gordy (2003), Wilde (2001) and Martin and Wilde (2002), for multi-factor models and provides an analytical method for calculating the VaR and ES of a credit portfolio.

The basic idea of Pykhtin is to approximate the portfolio loss $\tilde{L}$ in the multi-factor model with the respective portfolio loss $\tilde{\tilde{L}}$ in an accurately adjusted ASRF-model. This is done by mapping the correlation structure of each credit in the multi-factor model into a single correlation factor. This factor is determined by maximizing the correlation between the new single risk factor $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$.

Via this approach it is possible[34] to approximate the z-quantile $q_z(\tilde{L})$ of the portfolio loss by a quadratic Taylor series around the ASRF solution. This leads to

$$q_z(\tilde{L}) \approx q_z(\tilde{\tilde{L}}) + \left.\frac{dq_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2} \cdot \left.\frac{d^2 q_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{22}$$

where ε is the scale of perturbation and $\tilde{U}$ describes the approximation error between $\tilde{L}$ and $\tilde{\tilde{L}}$, i.e. $\tilde{U} = \tilde{L} - \tilde{\tilde{L}}$. The first summand on the right-hand side of (22) is the z-quantile of the loss $\tilde{\tilde{L}}$ within the reasonable adjusted ASRF-model. The corresponding distribution of $\tilde{\tilde{L}}$ can be calculated by

$$\tilde{\tilde{L}} = l(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}}\right], \tag{23}$$

where c_i is the correlation between the systematic risk factor $\tilde{\tilde{x}}$ and the asset return.[35]

[34] See Martin and Wilde (2002).

Instead of using ρ as it is done in the ASRF-model, the new correlation parameter c_i is used to match the correlation structure in the multi-factor model. As shown in formula (9), the loss quantile $q_z(\tilde{\tilde{L}})$ is given by $l(N^{-1}(1-z))$ in the ASRF-model. In addition, it can be shown that the first derivative in formula (22) is equal to zero. Hence, the so-called multi-factor adjustment Δq_z is completely described by the second derivative in formula (22). According to Pykhtin (2004) and Wilde (2001) Δq_z can be written as

$$\Delta q_z = q_z(\tilde{L}) - q_z(\tilde{\tilde{L}}) \approx -\frac{1}{2 \cdot l'(\bar{x})} \cdot \left[v'(\bar{x}) - v(\bar{x}) \cdot \left(\frac{l''(\bar{x})}{l'(\bar{x})} + \bar{x} \right) \right] \Bigg|_{\bar{x}=N^{-1}(1-z)}, \tag{24}$$

in which $l'(\bar{x})$ and $l''(\bar{x})$ are the first and second derivative of l according to formula (23) and $v(\bar{x})$ is the conditional variance of U. Further, $v(\bar{x})$ can be decomposed into two terms, $v_\infty(\bar{x})$ and $v_{GA}(\bar{x})$. The first term $v_\infty(\bar{x})$ describes the systematic risk adjustment, which is given by the difference between the multi-factor and single-factor loss distribution in infinitely granular portfolios. The second term $v_{GA}(\bar{x})$ is the granularity adjustment, which measures the influence of single-name concentration.[36] Using these terms the multi-factor adjustment can be presented as

$$\Delta q_z = \Delta q_z^\infty + \Delta q_z^{GA}, \tag{25}$$

i.e. the multi-factor adjustment can be split into a systematic risk adjustment component and a granularity adjustment component. Finally, the approximation of a loss quantile $q_z(\tilde{L})$ in (22) is given by (23) and the multi-factor adjustment:

$$q_z(\tilde{L}) = q_z(\tilde{\tilde{L}}) + \Delta q_z^\infty + \Delta q_z^{GA}. \tag{26}$$

After dealing with the VaR we now present the ES in a multi-factor model. In this context formula (11) can be rewritten as

$$\begin{aligned} ES_z(\tilde{L}) &= E\left(\tilde{L} \mid \tilde{L} \geq q_z(\tilde{L})\right) = \frac{1}{1-z} \cdot \int_z^1 (q_s(\tilde{\tilde{L}}) + \Delta q_s) ds \\ &= ES_z(\tilde{\tilde{L}}) + \frac{1}{1-z} \cdot \int_z^1 \Delta q_s ds =: ES_z(\tilde{\tilde{L}}) + \Delta ES_z \end{aligned}. \tag{27}$$

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[35] The derivation of c_i to obtain the maximum correlation between $\tilde{\tilde{x}}$ and $\{\tilde{x}_s\}$ can be found in Appendix A.1. From Appendix A.1 we also know that for determination of c_i both (the intra- and inter-sector) correlations are needed, which can be taken from section 3.2.

[36] The derivatives and the conditional variances can be found in Appendix A.2.

To get this result the quantile $q_z(\tilde{L})$ is substituted by approximation (26). The first summand of the right-hand side describes the ES for the single factor portfolio and the second summand is the multi-factor adjustment.

As shown by Pykhtin (2004) $ES_z(\tilde{\tilde{L}})$ and $\Delta ES_z(\tilde{L})$ can be calculated as

$$ES_z(\tilde{\tilde{L}}) = \frac{1}{1-z}\sum_{i=1}^{n} w_i \cdot LGD_i \cdot N_2\left[N^{-1}(PD_i), N^{-1}(1-z), c_i\right], \tag{28}$$

and

$$\Delta ES_z \approx -\frac{1}{2\cdot(1-z)} \cdot n\left[N^{-1}(1-z)\right]\frac{v\left[N^{-1}(1-z)\right]}{l'\left[N^{-1}(1-z)\right]}, \tag{29}$$

with $n(\cdot)$ denoting the density function of the standard normal distribution. Again, the multi-factor adjustment can be decomposed into a systematic and an idiosyncratic part by decomposing the conditional variance. Hence the ES for a portfolio in a multi-factor model is given by

$$ES_z(\tilde{L}) = ES_z(\tilde{\tilde{L}}) + \Delta ES_z^{\infty} + \Delta ES_z^{GA}. \tag{30}$$

In principle it is straightforward to implement the Pykhtin model. For calculating the ES we have to compute formula (29).[37] If applied to large portfolios, its computation can be extremely time-consuming since the calculation procedure inter alia requires n^2-times the computation of the conditional asset correlation,[38] with n being the number of credits. An alternative procedure performed by Düllmann and Masschelein (2007) is to neglect the multi-factor adjustment and to use (23) only to aggregate all credits for each sector and thus using the formulas on sector and not on borrower level. To consider the multi-factor adjustment and thus to increase the accuracy, we propose to built PD-classes for each of the sectors and aggregate the credits to these buckets for the calculation of the multi-factor adjustment. With this approach, the computation time is basically controlled by

$$Loops = (N_{PD} \cdot S_{Sectors})^2, \tag{31}$$

where N_{PD} and S denote the number of PD-classes and sectors.[39] If the number of PD-classes is sufficient high, the approximation error resulting from aggregating individual PDs to PD-classes is negligible. As the number of loops will not grow with bigger portfolios, it is possi-

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[37] For the implementation we need the derivatives and conditional variances given in Appendix A.2.

[38] The quadratic computation effort is due to the determination of a double sum (see Appendix A.2, (A.10)).

[39] The results of the multi-factor adjustment do not differ whether different exposures with the same PD are aggregated or handled separately on borrower level. For details see Appendix A.2.

ble to perform the adjustment on bucket level within reasonable time. Only the granularity adjustment should be calculated on borrower level but this is no computational burden.[40]

3.4 Implementation for the Cespedes-Model

Cespedes *et al.* (2006) present a method to relate the economic capital in the multi-factor model to the regulatory capital via a diversification factor DF( ), which depends on two parameters:[41]

- the average sector concentration CDI and
- the average weighted inter-sector correlation $\overline{\beta}$.

Herewith the economic capital of a portfolio can be approximated by:

$$EC^{mf} \approx DF \cdot RC. \tag{32}$$

Thus, the economic capital in the multi-factor model EC^{mf} can be approximated by a well-defined diversification factor DF multiplied by the regulatory capital RC of the ASRF-model. As mentioned before, Cespedes *et al.* assume that the regulatory capital of Pillar 1 is an upper barrier of the true risk because no diversification effects between the sectors are considered, which in turn implies the parameter DF to be always less than or equal to one. In contrast, if we use our definition of the intra-sector correlation ρ_{Intra} from section 3.2, it is possible to obtain $EC^{mf} > RC$ as well as $EC^{mf} < RC$ depending on the degree of diversification in comparison to the well-diversified portfolio defined in section 3.2. Hence, our later on calculated DF-function can be greater than one, i.e. the DF-function measures not only the benefit from sector diversification but also the risk resulting from high sector concentration. As the regulatory capital is additive in the ASRF-model (32) can be substituted by

$$EC_z^{mf} = DF \cdot \sum_{s=1}^{S} RC^s , \tag{33}$$

in which EC_z^{mf} is the economic capital in the multi-factor model and RC^s is the regulatory capital of sector s. In principle, the approach can be characterized as follows: Firstly, EC_z^{mf} is calculated for a multitude of portfolios via Monte-Carlo-Simulations. For each simulated port-

[40] The computation time when calculating the multi-factor adjustment on bucket- instead on borrower-level can be reduced from 67 minutes to 5 seconds for a portfolio with 11 sectors, 7 PD-classes, and 5000 creditors.

[41] In the strict sense Cespedes *et al.* relate the multi-factor model to the *economic* capital in a single-factor model. But since they apply the regulatory capital formula and we require a relation to this formula, too, we use the term regulatory capital instead.

folio the diversification factor can be calculated according to formula (33). Finally, a regression is performed to get an approximation for DF as a function of the two parameters CDI and $\overline{\beta}$. If DF can capture the industry diversification effects, we are able to approximate EC_z^{mf} with formula (33) without additional Monte-Carlo-Simulations.

To derive the parameters which explain the effect of diversification and concentration in a multi-factor model, Cespedes *et al.* suggest to use the average inter-sector correlation $\overline{\beta}$. This can be interpreted as a scale of the dependence between the sectors. The formula for $\overline{\beta}$ is given as

$$\overline{\beta} = \frac{\sum_{s=1}^{S}\sum_{j\neq s}\rho_{sj}^{inter}\cdot RC^s\cdot RC^j}{\sum_{s=1}^{S}\sum_{j\neq s}RC^s\cdot RC^j}. \quad (34)$$

The correlation is weighted by the expected shortfall in order to account for the contribution of each sector. The second suggested parameter is the capital diversification index denoted by CDI. It describes the sector concentration measured by the relative weight of each RC^s:[42]

$$CDI = \frac{\sum_{s=1}^{S}\left(RC^s\right)^2}{\left(\sum_{s=1}^{S}RC^s\right)^2}. \quad (35)$$

The parameter CDI lies between the two extreme values:

- $CDI = \frac{1}{n}$, i.e. perfect sector diversification,
- $CDI = 1$, i.e. perfect sector concentration.

To avoid a too complex model Cespedes *et al.* neglect further potential input parameters to determine the DF-function. To approximate the multi-factor model, formula (33) can be rewritten as

$$EC_z^{mf}\left(CDI,\overline{\beta}\right) = DF\left(CDI,\overline{\beta}\right)\cdot\sum_{s=1}^{S}RC^s. \quad (36)$$

In the following, we present the procedure to estimate the DF-function. To get a universally valid DF-factor as many portfolios as possible have to be generated and simulated. To reduce the necessary number of trials, the portfolios should be restricted to those with reasonable characteristics. Our portfolios are randomly generated using the following parameter setting.

[42] This concentration measure is also known as the Herfindahl-Hirschmann-Index.

When we state several parameter values or a parameter range, the parameter is randomly drawn from this set.

For the intra-sector correlations we use the functional form of formula (21). The inter-sector correlation structure is taken from Table 2, so that all simulated portfolios are stemming from this sector definition. Each portfolio consists of {2, …, 11} sectors that are randomly drawn from the different industries. The sector weights are in [0, 1]. The total number of credits is 5000, equally divided for each sector. Each sector in turn consists of credits from the PD classes {AAA, AA, A, BBB, BB, B, CCC}. Instead of using equally distributed PD classes we draw the quality distribution from our predefined credit portfolio qualities {very high, high, average, low, very low} for every sector.[43] We draw 25,000 and 50,000 portfolios, respectively, and compute the economic capital in the multi-factor model for each portfolio.

To determine the economic capital we tried both Monte-Carlo-Simulations with 100,000 trials[44] for every portfolio and the Pykhtin formula from section 3.3. Because the computation time for Monte-Carlo-Simulations is materially longer, the corresponding results are based on 25,000 random portfolios whereas we computed the economic capital for 50,000 portfolios when using the Pykhtin formula instead. Furthermore, since Cespedes *et al.* (2006) use the VaR as the relevant risk measure and thus define economic capital as $\mathrm{EC}^{\mathrm{mf}} = \mathrm{VaR}^{\mathrm{mf}} - \mathrm{EL}$ we redefine the economic capital of the multi-factor model with respect to ES as argued in section 2.3: $\mathrm{EC}^{\mathrm{mf}} = \mathrm{ES}^{\mathrm{mf}} - \mathrm{EL}$.[45] In contrast, for the regulatory capital we use $\mathrm{RC} = \mathrm{VaR}^{\mathrm{Basel}} - \mathrm{EL}$. The result could also be related to the Expected Shortfall in the ASRF-model but we detected that the results differ only marginally and the VaR is easier to implement in typical spreadsheet applications.[46] The results for the diversification factor DF are very similar whether they are based on Monte-Carlo-Simulations or on the Pykhtin formula. Figure 3 presents characteristics of the diversification factor when using the Pykhtin formula.

[43] The setting is similar to Cespedes *et al.* Until this point, the main difference is the definition of the intra- and inter-sector correlations.

[44] For the determination of the economic capital for one specific portfolio the number of trials is slightly low but as we perform 25,000 simulations and the simulation noise of each simulation is unsystematic, the error terms should cancel out each other to a large extent.

[45] We also tested the results when using the ES instead of the unexpected loss but the coefficient of determination is higher when subtracting the EL in the corresponding formulas when performing the simulations.

[46] To determine the Expected Shortfall with formula (13), a bivariate cumulative normal distribution has to be computed whereas the Value at Risk only makes use of univariate distributions.

- Figure 3 about here -

For determination of the functional form of DF we use a regression of the type[47]

$$\mathrm{DF} = a_0 + a_1 \cdot (1-\mathrm{CDI}) \cdot (1-\bar{\beta}) + a_2 \cdot (1-\mathrm{CDI})^2 \cdot (1-\bar{\beta}) + a_3 \cdot (1-\mathrm{CDI}) \cdot (1-\bar{\beta})^2 \tag{37}$$

in both cases. The resulting function when using Monte-Carlo-Simulations is

$$\begin{aligned} \mathrm{DF}_{\mathrm{MC}} = {} & 1.4626 - 1.4475 \times (1-\mathrm{CDI}) \times (1-\bar{\beta}) \\ & -0.0382 \times (1-\mathrm{CDI})^2 \times (1-\bar{\beta}) + 0.3289 \times (1-\mathrm{CDI}) \times (1-\bar{\beta})^2 \end{aligned} \tag{38}$$

with a coefficient of determination of R^2 = 95.5%. Analogously, we determined the DF-function when using the Pykhtin formula

$$\begin{aligned} \mathrm{DF}_{\mathrm{Pykhtin}} = {} & 1.4598 - 1.4168 \times (1-\mathrm{CDI}) \times (1-\bar{\beta}) \\ & -0.0213 \times (1-\mathrm{CDI})^2 \times (1-\bar{\beta}) + 0.2421 \times (1-\mathrm{CDI}) \times (1-\bar{\beta})^2 \end{aligned} \tag{39}$$

with R^2 = 97.9%. The latter function is plotted in Figure 4.[48] In order to get the approximation for the multi-factor model, formula (36) has to be computed using either function (38) or (39).

- Figure 4 about here -

It can be seen that the maximum diversification factor is about 1.46. Thus, in the case of (almost) no diversification effects the measured capital requirement is 46% above the regulatory capital under Pillar 1. This will appear in the case of being concentrated to a single sector, leading to $\mathrm{CDI} = 1$, as well as in the theoretical case of perfect correlations between the relevant sectors, leading to $\bar{\beta} = 1$. Furthermore, the diversification factor is strongly increasing in CDI and in $\bar{\beta}$ which is consistent with the intuition.

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[47] We tried several different regressions but similar to Cespedes *et al.* this function worked best. In contrast to Cespedes *et al.* we do not set the first parameter a_0 to one because our DF-factor is not bound by the single-factor-model.

[48] The shape of the function is similar to Cespedes *et al.* but their range is from 0.1 to 1.0 whereas our function ranges from 0.2 to 1.5. In addition, they received a little higher R^2 (99.4% instead of 95.5% and 97.9%, respectively) but this is mainly due to the different simulation setting. Cespedes *et al.* directly draw the parameter $\bar{\beta}$ as an input parameter for each simulation, implying $\bar{\beta}$ to fully define their correlation structure. We use a heterogeneous correlation structure instead and compute $\bar{\beta}$ for the portfolios. Thus, in our setting $\bar{\beta}$ does not reflect the complete correlation structure which results in a lower R^2 but does not imply a worse approximation.

4 Performance of the Concentration Risk Models

4.1 Analysis for Deterministic Portfolios

To determine the quality of the presented models, we start our analysis with calculating the risk for five deterministic portfolios of different quality.[49] We generate well-diversified portfolios consisting of 5,000 credits. Consequently, we have neither high name nor high sector concentration risk. Concretely, we choose the sectors and their weights as given in Table 3. The inter-sector correlation is given in Table 2 whereas the intra-sector correlation is calculated on the basis of formula (21). The five portfolios differ in their PD distribution which is presented in Figure 4. Portfolio 1 is the portfolio with the highest and Portfolio 5 is the one with the lowest credit quality distribution.

In Table 5 we compare the results from the Monte-Carlo-Simulation (MC-Sim.), the Basel II formula (Basel II), the Pykhtin model (Pykhtin), the Cespedes model calibrated with Monte-Carlo-Simulations (Cespedes I) and the Cespedes model calibrated with the Pykhtin formula (Cespedes II). As can be seen in the table, the benchmark portfolio is constructed in a way that the Basel II formula represents a very good approximation[50] of the "real" ES in a multi-factor model given by Monte Carlo Simulations.[51] Besides, the simulated VaR^{mf} matches the simulated ES^{mf}, our benchmark, almost exactly. The calculated values of the Pykhtin model are very good approximations of the ES in almost all cases, too. The outcomes of the Cespedes model are somewhat more imprecise in both cases. With better credit quality the estimation error is increasing, which leads to an underestimation of risk in high quality portfolios.

- Table 5 about here -

As a next step, we change the portfolio structure towards high sector concentration. Therefore, we increase the sector weights of two sectors. We assume that 45% of the creditors – in terms of their exposure – belong to the Information Technology sector and an equal amount

[49] The results refer to the total gross loss of a portfolio in terms of ES and VaR, respectively. To relate this to the unexpected net loss, the results have to be multiplied by the LGD and the EL has to be subtracted.

[50] The small mismatch is mainly due to keeping the ES-confidence level constant and not a result of the chosen intra-sector correlation function. If we directly compare the results from Monte-Carlo-Simulations with the ES in the ASRF-framework, the relative root mean squared error is reduced from 0.97% to 0.28%.

[51] In our analyses the number of simulation runs is 500,000.

belongs to the Telecommunication Services sector. The remaining 10% of exposure are equally assigned to the miscellaneous sectors. As shown in Table 6 the risk materially increases for all types of portfolio quality. Again, the simulated values for ES^{mf} and VaR^{mf} are very close to each other. However, the Basel formula underestimates the risk by 14% to 20% depending on the portfolio quality. This is the (relative) amount that should be considered in the assessment of capital adequacy under Pillar 2. The approximation formula of Pykhtin can capture this concentration risk with a negligible error in all cases. Cespedes I leads to an underestimation of risk in high quality portfolios and to an overestimation of risk in low quality portfolios with a maximum deviation of nearly 4%. Contrary, Cespedes II underestimates the risk in most cases with up to 6%. Thus, the sector concentration risk is not fully captured for high quality portfolios.

- Table 6 about here -

Furthermore, we build credit portfolios with low sector concentration. For this purpose, we use the concept of naïve diversification implying each sector to have an equal weight of 1/11. As can be seen in Table 7, the economic capital is significantly lower than the regulatory capital. Moreover, this shows that it is easy to construct portfolios that are better diversified than the overall credit market.[52] Apart from insignificant deviations both simulated risk measures lead to the same solutions. Again the Pykhtin model approximates the "real" risk very good for all types of credit quality. The Cespedes model I underestimates the risk for high quality portfolios with up to 3%. The Cespedes model II underestimates the risk, too, but the approximation error is negligible.

- Table 7 about here -

4.2 Simulation Study for Homogeneous and Heterogeneous Portfolios

To achieve more general results we test the models for different, randomly generated portfolios. For this reason, we implement four simulation studies. In these studies we analyze the

[52] If we consider all 25,000 simulated portfolios from section 3.4, the lowest measured economic capital requirement was even 26% lower than the regulatory capital. This result underlines the prospects of actively managing credit portfolios, e.g. with credit derivatives, but this is not in the scope of this paper.

accuracy for homogeneous as well as for heterogeneous portfolios with respect to PD and EAD. In each simulation run we generate a portfolio and determine its ES by the three models. After 100 runs we calculate the root mean squared error for the outcomes of the Pykhtin model and of the Cespedes models I and II[53] in absolute and relative terms to quantify its performance in comparison to Monte-Carlo-Simulations using 500,000 trials. Furthermore, we calculate the VaR with the Basel II formula and with Monte-Carlo-Simulation to measure its accuracy compared to ES^{mf}. In the following we describe the four simulation settings.

Simulation I: In this scenario we generate portfolios with homogenous exposure sizes and homogenous PDs, that is, $w_i = 1/5000$ and $PD_i = PD = const$ for each credit. To test the accuracy for different portfolio qualities a PD is drawn from a uniformly distribution between 0% and 10% before each new run. The sector structure and correlation is the same as in section 4.1.

Simulation II: We generate portfolios with homogenous exposure sizes but heterogeneous PDs. For each sector we determine randomly one of the quality distributions from section 2.3. After that we draw the PD for each credit of the sector according to this quality distribution. The exposure size remains as in Simulation I. Again, the sector structure and correlation is taken from section 4.1.

Simulation III: We generate portfolios with homogenous PDs as in Simulation I but with heterogeneous exposure sizes. Firstly, we choose the number of sectors randomly between 2 and 11. Then we apply a uniform distribution between 0 and 1 for the weight of every sector and scale this in order to sum up the weights to one. The weights for the credits in each sector are determined in the same manner. The correlations remain unchanged.

Simulation IV: In this setting the PDs as well as the exposure sizes of the generated portfolios are heterogeneous. The PDs are determined as in Simulation II and the exposure sizes as in Simulation III.

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[53] Cespedes I still corresponds to the DF-function based on Monte-Carlo-Simulation and Cespedes II complies with the DF-function based on the Pykhtin formula.

In each simulation we calculate the intra-sector correlations with formula (21) and choose 5,000 credits. These portfolios contain a relatively low amount of name concentration. Instead we focus on sector concentration. The reason is that the identical methodology for measuring name concentrations, the granularity adjustment, can be used within both approaches. Thus, we prefer to avoid name concentrations to be able to separately analyze the effect of sector concentrations. The degree of sector concentration differs between the simulations. In Simulations I and II the portfolios consist of homogenous exposures leading to a CDI of 9.1% in each case. This equals the CDI for a naïve diversified portfolio. On the contrary in Simulation III and IV exposures are chosen randomly and the CDI of the generated portfolios can take values between 9.1%. (naïve diversification) and 1 (perfect concentration). The mean of these CDIs is around 30% in each simulation, which is only slightly higher than the CDIs of the bank portfolios analyzed by Acharya, Hasan, and Saunders (2006), which shows that the setting leads to a realistic degree of diversification.[54] The results of our simulation study can be found in Table 8.

- Table 8 about here -

Again, the outcomes of the Pykhtin model are a good approximation of the "true" result from the Monte-Carlo Simulations. Especially, when EADs are homogeneous the results are very good. Both types of the Cespedes model lead to very stable results in all simulation settings. Interestingly, the Cespedes model performs even better when PDs are heterogeneous, probably because the portfolios used for calculation of the functional form have heterogeneous PDs, too, and thus the resulting portfolios are more similar. Somewhat surprising, in Simulation III the Cespedes model shows a better performance than the Pykhtin model even if the Pykhtin formula is used for determination of the diversification factor. Probably the approximation errors of the Pykhtin model are partially smoothed by the regression from formula (37).

The comparison of the risk measures with different confidence levels shows an almost perfect match between ES^{mf} and VaR^{mf}. The relative error is smaller than 1% in each case. Thus, our simulation study clarifies that the above-mentioned theoretical problems of the non-

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[54] Acharya, Hasan, and Saunders (2006) examined credit portfolios of 105 Italian banks during the period 1993-1999. In this study, most bank portfolios had a CDI between 20% and 30%. However, it has to be considered that the number of different industry sectors was 23 whereas we used 11 different sectors. Thus, for a comparable degree of diversification their calculated CDIs have to be slightly smaller than our CDIs.

coherent VaR are not practically relevant for a very broad range of credit portfolios. Hence, the use of the VaR for determining the credit risk seems to be unproblematic from a practical point of view even if the portfolio incorporates sector concentration risk. The Basel formula, however, shows the largest inaccuracy of all tested models for any simulation. Since in Simulation I and II a naïve diversified portfolio is taken as a basis, the Basel formula overestimates the risk in every case due to the diversification effect. A plot of the relative errors of the Basel formula and of VaR^{mf} in Simulation III, sorted in ascending order, can be found in Figure 5. Apart from slightly higher deviations, a plot with a similar characteristics results for Simulation IV.

- Figure 5 about here -

It can be seen that for more than 50% of the simulated portfolios the Basel VaR is too low. That means the risk measured under Pillar 1 is underestimated compared to the "real" risk. In general this happens when the sector concentration of the generated portfolio increases, as already demonstrated for deterministic portfolios. Consequently, the simulation study accentuates the need for considering sector concentration when calculating the risk of a credit portfolio. Otherwise the risk can be massively underestimated. This conclusion coincides with that of BCBS (2006), which points out that sector concentration can increase the capital requirement up to 40%. The maximal deviation of VaR^{mf} is around 3%, which is negligible for practical implementation. Actually, for most of the generated portfolios the error is almost zero. In order to verify if there is a systematic pattern, which may help to explain the occurrence of these deviations in the multi-factor setting, we tried to find portfolio variables such as CDI, average correlation or average PD that can explain these deviations. Since our analyses did not show a link between the deviations and any of the mentioned variables, it seems that the occurrence is unsystematic

As the purpose of deriving (semi-)analytical approximation formulas for the VaR or the ES is an acceleration of the computation time, we compare the runtime of the demonstrated methods in Table 9.[55] The main advantage of the Pykhtin model is that it can be applied without an excessive calibration procedure and it is considerably faster than Monte-Carlo-Simulations without leading to major approximation errors. When comparing both alternative implementations of the Cespedes model, we strongly propose to use the Pykhtin model for ca-

[55] The runtimes refer to a quad-core PC with 2.66 GHz CPUs (calculated on one core).

libration (Cespedes II) instead of Monte-Carlo-Simulations (Cespedes I) as the approximation accuracy is almost identically but the computation time for determination of the DF-function is significantly lower. As this calibration procedure only has to be computed once for a specified correlation structure and the application of the formula is very fast, in most situations the Cespedes type model should be a very good choice.

- Table 9 about here -

5 Conclusion

In this paper we proposed a methodology to perform multi-factor models that are able to measure concentration risk in credit portfolios in terms of economic capital and still deliver results that are consistent with Basel II. Furthermore, we applied this to different multi-factor approaches and compared their performance. It could be shown that it is possible to achieve good approximations in reasonable time when the approaches are adjusted in the proposed way.

We also discussed the shortcomings of the Value at Risk, which can arise when leaving the ASRF-framework. From a theoretical point of view, it is advisable to use a coherent risk measure like the ES. Since the ES, by definition, is higher than the VaR if we use the same confidence level, we performed a mapping procedure that determines the confidence level ($z = 99.72\%$) of the ES to get reasonable results. Despite the mentioned shortcomings, however, the accuracy of the VaR turned out to be almost perfect compared to the ES for a multitude of generated portfolios. Thus, in our opinion, it is unproblematic to use the VaR for measuring concentration risk of credit portfolios.

Furthermore, we chose input parameters, especially the inter- and intra-sector correlations, in a way that the results are comparable with the regulatory Pillar 1 capital. Consequently, we do not follow some approaches that assume a pure diversification effect compared with the Basel II formula. Instead, we relate the results to a well-diversified portfolio as assumed when calibrating the Basel II formula and determine a function for the implied intra-sector correlation. Hence, it is possible to directly consider the extent of credit risk concentrations in the assessment of capital adequacy under Pillar 2. Using these modifications, we performed an extensive numerical study similar to Cespedes *et al.* (2006) to get a closed form approximation formula. In addition, we suggest computing the multi-factor adjustment on bucket instead of borrower level. This allows to compute the Pykhtin formula much faster than Monte-Carlo-Simulations even for a high number of credits.

Having assured a Basel II consistent capital requirement, we analyzed the impact of credit concentration risk and carried out a simulation study to compare the performance of the (modified) models from Cespedes *et al.* (2006) and Pykhtin (2004). We detect that the Pykhtin model leads to very good results for homogeneous as well as heterogeneous PDs when EADs are homogeneous. The performance is slightly lower for heterogeneous EADs. The results of the Cespedes model have a throughout high accuracy. Interestingly, the approach works better for heterogeneous portfolios. In general, both models can be used for approximating the economic capital in a multi-factor setting when adjusted in the proposed way. The main advantage of the Pykhtin model is that it can be directly applied to an arbitrary portfolio type, whereas the approach of Cespedes *et al.* (2006) should not be used without initially performing the demonstrated extensive numerical work when the portfolio structure is very different. On the contrary, the results of the Cespedes model were slightly better for heterogeneous portfolios and it allows for ad-hoc analyses including sensitivity analyses when the non-recurring extensive numerical work is progressed.

In further analyses it would be interesting to analyze the approach of Cespedes *et al.* (2006) when adjusted to a specific bank portfolio. Under the (plausible) assumption that a bank's portfolio will only be faced to minor changes for a finite period, it should be possible to get a higher accuracy for this bandwidth of scenarios. Moreover, it would be helpful to know how much numerical work is necessary when the parameters are highly restricted to these realistic cases to achieve stable results because the extensive computation time is still a challenge.

Appendix A.1

To relate $\tilde{\tilde{L}}$ to $\tilde{L}$ the systematic factor $\tilde{\tilde{x}}$ is defined as

$$\tilde{\tilde{x}} = \sum_{k=1}^{K} b_k \cdot \tilde{z}_k, \quad \text{with} \quad \sum_{k=1}^{K} b_k^2 = 1. \tag{A.1}$$

On condition that $\tilde{\tilde{L}} = E\left[\tilde{L} \mid \tilde{\tilde{x}}\right]$ Pykhtin (2004) shows that c_i can be calculated as

$$c_i = \rho_{\text{Intra},i} \cdot \bar{\rho}_i^2 = \rho_{\text{Intra},i} \cdot \left(\sum_{k=1}^{K} \alpha_{i,k} \cdot b_k \right)^2, \text{ in which } \bar{\rho}_i = \text{cor}(\tilde{\tilde{x}}, \tilde{x}_i) \tag{A.2}$$

with $\alpha_{i,k} = \alpha_{s,k}$ and $\tilde{x}_i = \tilde{x}_s$ for obligor i in sector s.

Since there is no unique method to determine the coefficients $\{b_k\}$, we use the approach presented by Pykhtin (2004). Thus, the coefficients are chosen in a way that the correlation between $\tilde{\tilde{x}}$ and $\{\tilde{x}\}$ will be maximized, in order to minimize the difference given by (24) between the quantiles $q_z(\tilde{\tilde{L}})$ and $q_z(\tilde{L})$. This leads to the following maximization problem:

$$\max_{\{b_k\}} \left(\sum_{i=1}^{n} d_i \cdot \bar{\rho}_i \right) \quad \text{subject to} \sum_{k=1}^{K} b_k^2 = 1. \tag{A.3}$$

The solutions of $\{b_k\}$ are given as

$$b_k = \sum_{i=1}^{n} \frac{d_i \cdot \alpha_{ik}}{\lambda}, \tag{A.4}$$

where the Lagrange multiplier λ is chosen so that $\{b_k\}$ satisfy the constraint. There is no obvious choice of the weighting factors d_i but

$$d_i = w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) + \sqrt{\rho_{\text{Intra},i}} \cdot N^{-1}(z)}{\sqrt{1-\rho_{\text{Intra},i}}} \right] \tag{A.5}$$

leads to good results, which is the VaR formula in a single factor model. The intuition behind this choice is that obligors with a high exposure in terms of VaR should get a high weight in the maximization problem.

Appendix A.2

The derivatives of (23) are calculated as follows:

$$l'(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i'(\tilde{\tilde{x}}), \quad l''(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i''(\tilde{\tilde{x}}). \tag{A.6}$$

The derivatives $p_i'(\tilde{\tilde{x}})$ and $p_i''(\tilde{\tilde{x}})$ of the conditional default probability are calculated by differentiation of equation (6) as

$$p_i'(\tilde{\tilde{x}}) = -\frac{\sqrt{c_i}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i}\cdot\tilde{\tilde{x}}}{\sqrt{1-c_i}}\right], \qquad \text{(A.7)}$$

$$p_i''(\tilde{\tilde{x}}) = -\frac{c_i}{1-c_i} \cdot \frac{N^{-1}(PD_i) - \sqrt{c_i}\cdot\tilde{\tilde{x}}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i}\cdot\tilde{\tilde{x}}}{\sqrt{1-c_i}}\right]. \qquad \text{(A.8)}$$

Since $\tilde{\tilde{L}}$ is deterministic for given $\tilde{\tilde{x}}$, $v(\tilde{\tilde{x}})$ equals the conditional variance of $\tilde{L}$, this means $v(\tilde{\tilde{x}}) = var(\tilde{L} - \tilde{\tilde{L}} \mid \tilde{\tilde{x}}) = var(\tilde{L} \mid \tilde{\tilde{x}})$. To calculate $v(\tilde{\tilde{x}})$ the conditional variance can be decomposed as the sum of systematic and idiosyncratic parts:

$$v(\tilde{\tilde{x}}) = \underbrace{var[E(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_\infty(\tilde{\tilde{x}})} + \underbrace{E[var(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_{GA}(\tilde{\tilde{x}})}. \qquad \text{(A.9)}$$

The first summand $v_\infty(\tilde{\tilde{x}})$ of (A.9) can be calculated as

$$\begin{aligned} v_\infty(\tilde{\tilde{x}}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot \\ \left[N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \sqrt{\rho_{ij}^{\tilde{\tilde{x}}}}\right) - p_i(\tilde{\tilde{x}})\cdot p_j(\tilde{\tilde{x}})\right], \end{aligned} \qquad \text{(A.10)}$$

where $\rho_{ij}^{\tilde{\tilde{x}}}$ describes the conditional asset correlation

$$\sqrt{\rho_{ij}^{\tilde{\tilde{x}}}} = \frac{\sqrt{\rho_{Intra,i}\cdot\rho_{Intra,j}} \cdot \sum_{k=1}^{K}\alpha_{ik}\cdot\alpha_{jk} - \sqrt{c_i \cdot c_j}}{\sqrt{(1-c_i)\cdot(1-c_j)}}. \qquad \text{(A.11)}$$

The first derivative of $v_\infty(\tilde{\tilde{x}})$ is given by:

$$\begin{aligned} v_\infty'(\tilde{\tilde{x}}) = 2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot p_i'(\tilde{\tilde{x}}) \\ \cdot\left[N\left(\frac{N^{-1}\left[p_j(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ij}^{\tilde{\tilde{x}}}}\cdot N^{-1}\cdot\left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ij}^{\tilde{\tilde{x}}}}}\right) - p_j(\tilde{\tilde{x}})\right]. \end{aligned} \qquad \text{(A.12)}$$

The second summand $v_{GA}(\tilde{\tilde{x}})$ of (A.9) and its derivative $v_{GA}'(\tilde{\tilde{x}})$ are

$$v_{GA}(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot \left(LGD_i^2\left[p_i(\tilde{\tilde{x}}) - N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}}\right)\right]\right), \qquad \text{(A.13)}$$

$$v_{GA}'(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot p_i'(\tilde{\tilde{x}}) \cdot \left(LGD_i^2\left[1 - 2\cdot N\left(\frac{N^{-1}\left[p_i(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}}\cdot N^{-1}\left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ii}^{\tilde{\tilde{x}}}}}\right)\right]\right). \qquad \text{(A.14)}$$

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FIGURE 1 Accuracy of the Pillar 1 capital requirements considering risk concentrations

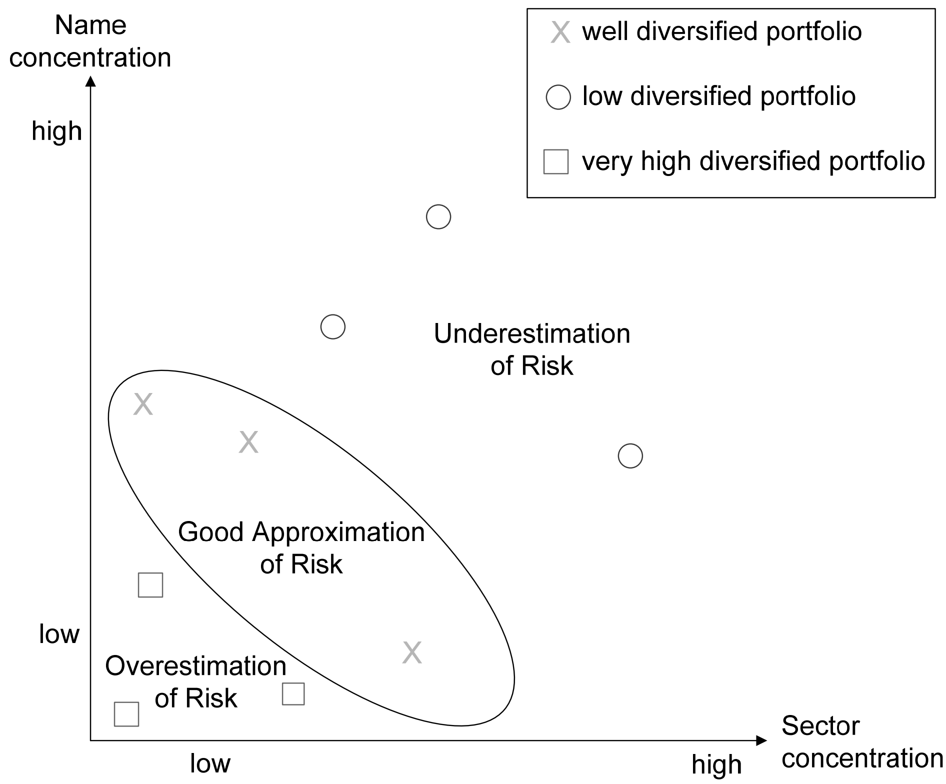

FIGURE 2 Portfolio quality distributions

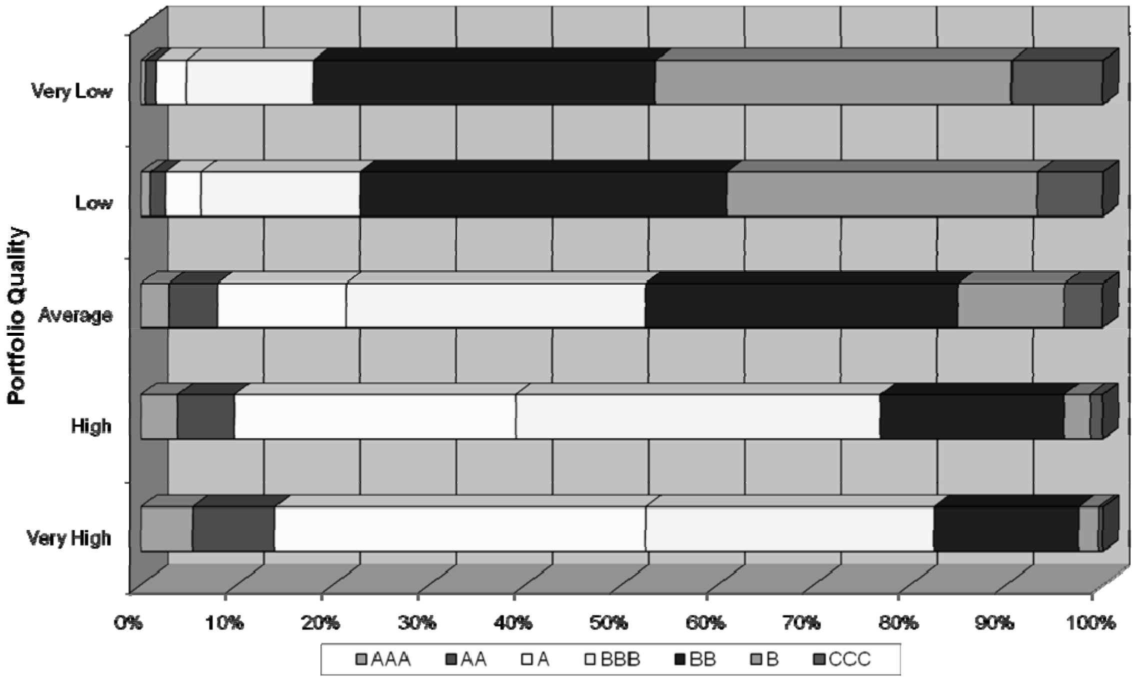

FIGURE 3 Diversification Factor realizations on the basis of 50,000 simulations

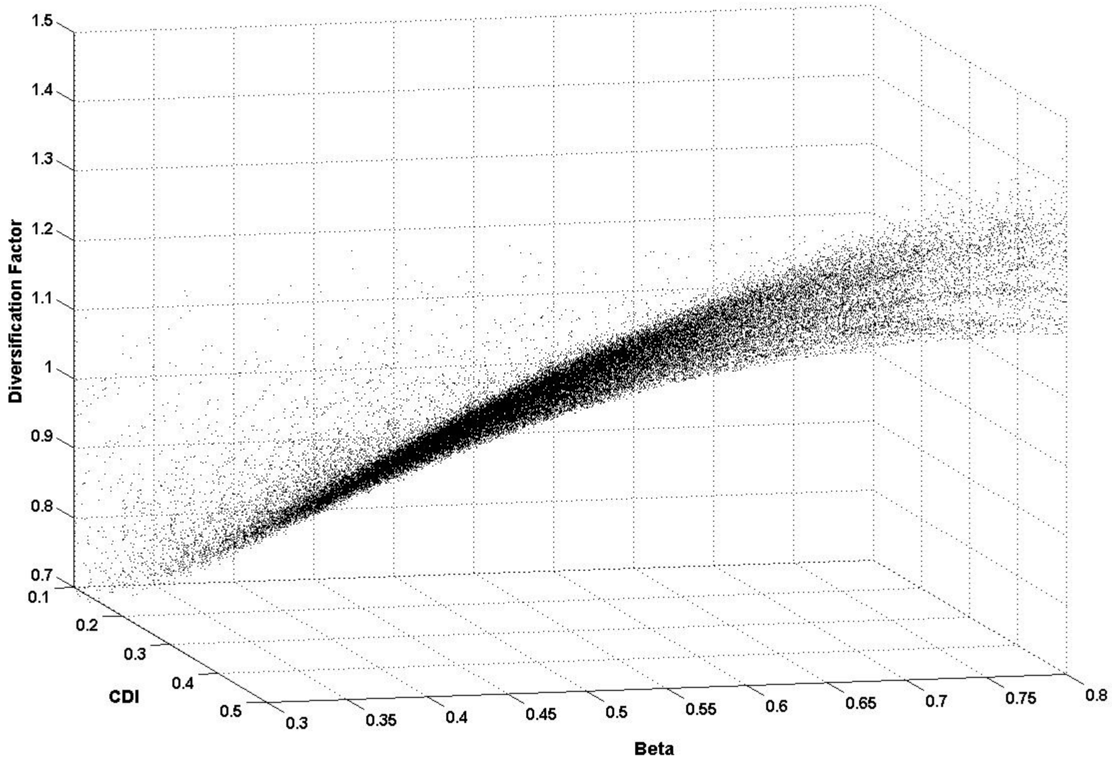

FIGURE 4 Surface plot of the DF-function

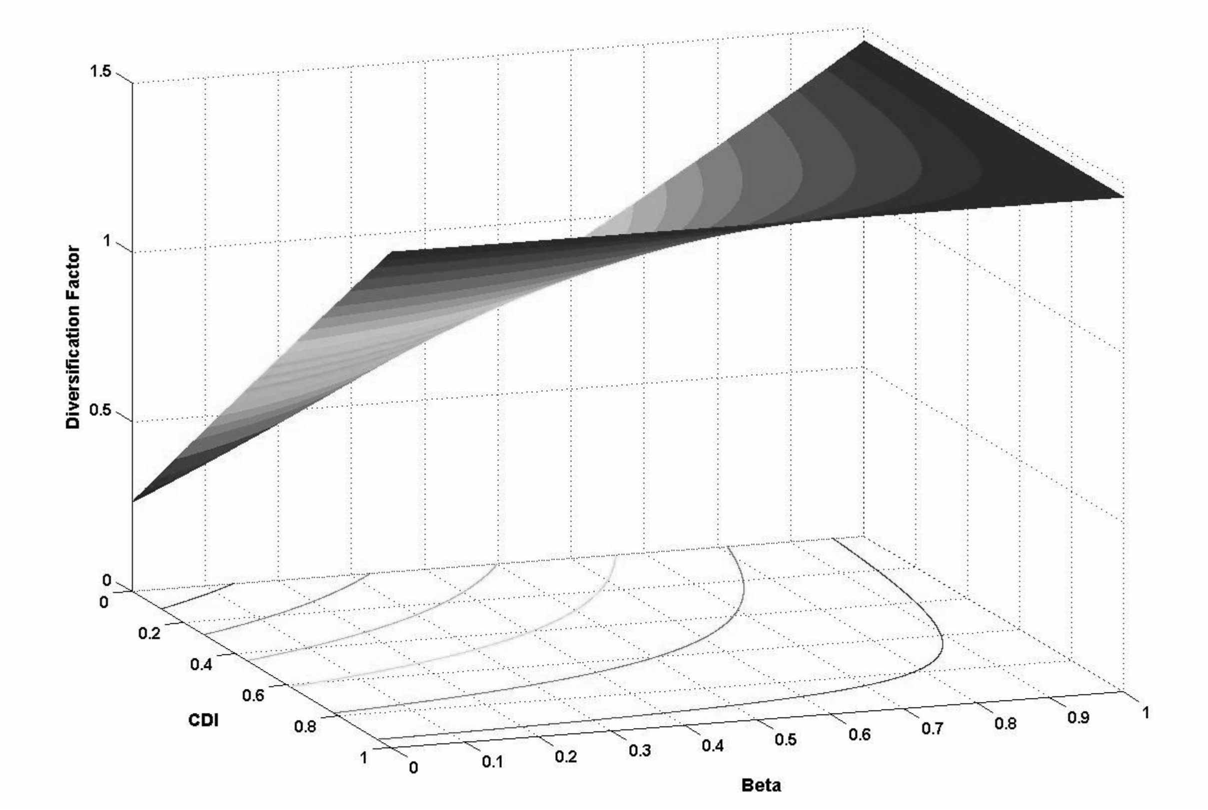

FIGURE 5 Deviations of VaR^{Basel} and VaR^{mf} from ES^{mf}

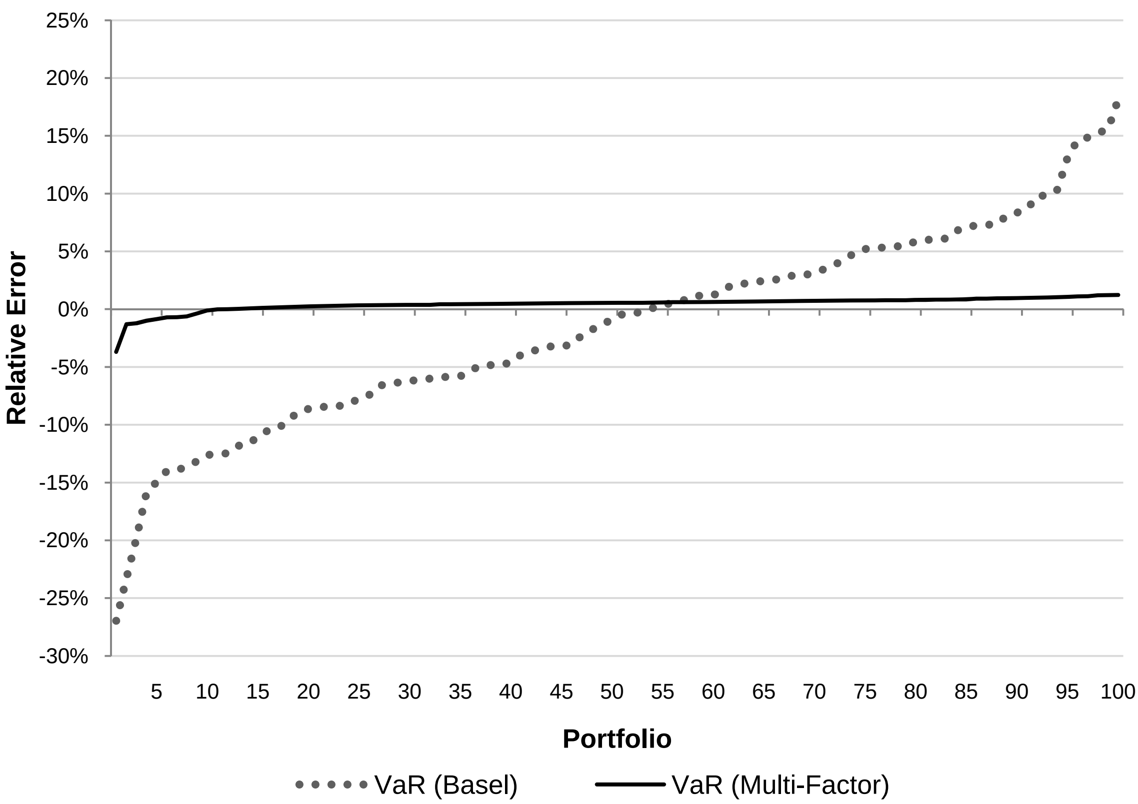

TABLE 1 Confidence levels for the ES implying ES_z^{Basel} to be matched with the VaR^{Basel} for portfolios of different quality

Portfolio Type / Quality	VaR^{Basel} & ES_z^{Basel}	Confidence Level z(ES)
(I) AAA only	0.57%	99.672%
(II) Very High	6,12%	99.709%
(III) High	7.59%	99.711%
(IV) Average	12.94%	99.719%
(V) Low	20.89%	99.726%
(VI) Very Low	23.30%	99.727%
(VII) CCC only	57.00%	99.741%

TABLE 2 Inter-sector correlation structure based on MSCI industry indices (in %)

Sector	A	B	C1	C2	C3	D	E	F	H	I	J
A: Energy	100	50	42	34	45	46	57	34	10	31	69
B: Materials		100	87	61	75	84	62	30	56	73	66
C1: Capital Goods			100	67	83	92	65	32	69	82	66
C2: Comm. Svs. & Supplies				100	58	68	40	8	50	60	37
C3: Transportation					100	83	68	27	58	77	67
D: Consumer Discretionary						100	76	21	69	81	66
E: Consumer Staples							100	33	46	56	66
F: Health Care								100	15	24	46
H: Information Technology									100	75	42
I: Telecommunication Services										100	62
J: Utilities											100

TABLE 3 Overall sector composition of the German banking system

Sector	Exposure Weight
A: Energy	0.18%
B: Materials	6.01%
C1: Capital Goods	11.53%
C2: Comm. Svs. & Supplies	33.69%
C3: Transportation	7.14%
D: Consumer Discretionary	14.97%
E: Consumer Staples	6.48%
F: Health Care	9.09%
H: Information Technology	3.20%
I: Telecommunication Services	1.04%
J: Utilities	6.67%

TABLE 4 Implicit intra-sector correlations for different portfolio quality

Portfolio Type / Quality	Implicit Intra-Sector Correlation
(I) Very High	30%
(II) High	28%
(III) Average	25%
(IV) Low	23%
(V) Very Low	21%

TABLE 5 Comparison of the models for the 5 benchmark portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
	ES	6.23%	7.68%	12.95%	20.88%	23.15%
MC-Sim.	VaR	6.18%	7.62%	12.94%	20.93%	23.3%
	Absolute Error	-5 bp	-6 bp	-1 bp	5 bp	15 bp
	Relative Error	-0.80%	-0.78%	0.08%	0.24%	0.65%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	-11 bp	-9 bp	0 bp	1 bp	11 bp
	Relative Error	-1.77%	-1.17%	0.00%	0.05%	0.48%
Pykhtin	ES	6.21%	7.66%	12.91%	20.80%	23.20%
	Absolute Error	-2 bp	-2 bp	-4 bp	-8 bp	5 bp
	Relative Error	-0.32%	-0.26%	-0.31%	-0.38%	0.22%
Cespedes I	ES	6.07%	7.51%	12.70%	20.43%	22.79%
	Absolute Error	-16 bp	-17 bp	-25 bp	-45 bp	-36 bp
	Relative Error	-2.57%	-2.21%	-1.93%	-2.16%	-1.56%
Cespedes II	ES	6.00%	7.45%	12.68%	20.48%	22.87%
	Absolute Error	-23 bp	-23 bp	-27 bp	-40 bp	-28 bp
	Relative Error	-3.69%	-2.99%	-2.08%	-1.92%	-1.21%

TABLE 6 Comparison of the models for 5 high concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	7.69%	9.22%	15.41%	24.41%	27.10%
	VaR	7.48%	9.17%	15.36%	24.51%	27.06%
	Absolute Error	-21 bp	-5 bp	-5 bp	10 bp	-6 bp
	Relative Error	-2.73%	-0.54%	-0.32%	0.41%	0.15%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	-157 bp	-163 bp	-246 bp	-352 bp	-384 bp
	Relative Error	-20.42%	-17.68%	-15.96%	-14.42%	-14.17%
Pykhtin	ES	7.66%	9.29%	15.46%	24.39%	27.03%
	Absolute Error	-3 bp	7 bp	5 bp	-2 bp	-7 bp
	Relative Error	-0.35%	0.76%	0.31%	-0.08%	-0.24%
Cespedes I	ES	7.40%	9.08%	15.59%	25.07%	27.95%
	Absolute Error	-29 bp	-14 bp	18 bp	66 bp	85 bp
	Relative Error	-3.77%	1.52%	1.17%	2.70%	3.14%
Cespedes II	ES	7.22%	8.86%	15.19%	24.38%	27.14%
	Absolute Error	-47 bp	-36 bp	-22 bp	-3 bp	4 bp
	Relative Error	-6.11%	-3.90%	-1.43%	-0.12%	0.15%

TABLE 7 Comparison of the models for 5 low concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	5.66%	6.98%	12.16%	19.78%	22.06%
	VaR	5.64%	6.94%	12.17%	19.81%	22.10%
	Absolute Error	-2 bp	-4 bp	1 bp	3 bp	4 bp
	Relative Error	-0.35%	-0.57%	0.08%	0.15%	0.18%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	46 bp	61 bp	79 bp	111 bp	120 bp
	Relative Error	8.13%	8.74%	6.50%	5.61%	5.44%
Pykhtin	ES	5.67%	6.98%	12.14%	19.74%	22.08%
	Absolute Error	1 bp	0 bp	-2 bp	-4 bp	2 bp
	Relative Error	0.26%	-0.07%	-0.16%	-0.21%	0.09%
Cespedes I	ES	5.66%	6.94%	11.92%	19.17%	21.38%
	Absolute Error	0 bp	-4 bp	-24 bp	-61 bp	-68 bp
	Relative Error	0.0%	-0.57%	-1.97%	-3.08%	-3.08%
Cespedes II	ES	5.64%	6.94%	12.06%	19.52%	21.81%
	Absolute Error	-2 bp	-4 bp	-10 bp	-26 bp	-25 bp
	Relative Error	-0.35%	-0.57%	-0.82%	-1.31%	-1.13%

TABLE 8 Comparison of the models resulting from simulation studies with different parameter settings

		Simulation I	Simulation II	Simulation III	Simulation IV
MC-Sim. VaR	Ø Absolute Error	18 bp	6 bp	22 bp	8 bp
	Ø Relative Error	0.67%	0.43%	0.77%	0.60%
Basel II	Ø Absolute Error	259 bp	186 bp	264 bp	379 bp
	Ø Relative Error	11.66%	13.70%	8.81%	25.76%
Pykhtin	Ø Absolute Error	14 bp	11 bp	54 bp	18 bp
	Ø Relative Error	0.64%	0.81%	3.40%	1.26%
Cespedes I	Ø Absolute Error	54 bp	11 bp	47 bp	20 bp
	Ø Relative Error	1.73%	0.79%	1.65%	1.53%
Cespedes II	Ø Absolute Error	54 bp	12 bp	46 bp	21 bp
	Ø Relative Error	1.72%	0.84%	1.56%	1.59%

TABLE 9 Comparison of the runtime

	Runtime: Calibration	**Runtime: Application**
MC-Simulation		20 min
Pykhtin		~ 10 sec - 2 min
Cespedes I	30 days	0.01 sec
Cespedes II	150 min	0.01 sec