MEASURING CONSENSUS IN WEAK ORDERS

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Abstract In this chapter we focus our attention in how to measure consensus in groups of voters when they show their preferences over a fixed set of alternatives or candidates by means of weak orders (complete preorders). We have introduced a new class of consensus measures on weak orders based on distances, and we have analyzed some of their properties paying special attention to seven well-known distances.

1 Introduction

Consensus has different meanings. One of them is related to iterative procedures where voters must change their preferences to improve agreement. Usually, a moderator advise voters to modify some opinions (see, for instance, Eklund, Rusinowska and de Swart [12]). However, in this chapter consensus is related to the degree of agreement in a committee, and voters do not need to change their preferences. For an overview about consensus, see Martínez-Panero [22].

From a technical point of view, it is interesting to note that the problem of measuring the concordance or discordance between two linear orders has been widely explored in the literature. In this way, different *rank correlation indices* have been considered for assigning grades of agreement between two rankings (see Kendall and Gibbons [21]). Some of the most important indices in this context are Spearman's rho [31], Kendall's tau [20], and Gini's cograduation index [16]. On the other hand, some natural extensions of the above mentioned indices have been considered

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for measuring the concordance or discordance among more than two linear orders (see Hays [17] and Alcalde-Unzu and Vorsatz [1, 2]). For details and references, see for instance Borroni and Zenga [6] and Alcalde-Unzu and Vorsatz [1, 2].

In the field of Social Choice, Bosch [7] introduced the notion of *consensus measure* as a mapping that assigns a number between 0 and 1 to every profile of linear orders, satisfying three properties: *unanimity* (in every subgroup of voters, the highest degree of consensus is only reached whenever all individuals have the same ranking), *anonymity* (the degree of consensus is not affected by any permutation of voters) and *neutrality* (the degree of consensus is not affected by any permutation of alternatives).

Recently, Alcalde-Unzu and Vorsatz [1] have introduced some consensus measures in the context of linear orders –related to some of the above mentioned rank correlation indices– and they provide some axiomatic characterizations (see also Alcalde-Unzu and Vorsatz [2]).

In this chapter[1] we extend Bosch's notion of consensus measure to the context of weak orders (indifference among different alternatives is allowed)[2], and we consider some additional properties that such measures could fulfill: *maximum dissension* (in each subset of two voters, the minimum consensus is only reached whenever preferences of voters are linear orders and each one is the inverse of the other), *reciprocity* (if all individual weak orders are reversed, then the consensus does not change) and *homogeneity* (if we replicate a subset of voters, then the consensus in that group does not change). After that, we introduce a class of consensus measures based on the distances among individual weak orders. We pay special attention to seven specific metrics: discrete, Manhattan, Euclidean, Chebyshev, cosine, Hellinger, and Kemeny.

The chapter is organized as follows. Section 2 is devoted to introduce basic terminology and distances used along the chapter. In Section 3 we introduce consensus measures and we analyze their properties. An Appendix contains the most technical proofs.

2 Preliminaries

Consider a set of voters $V = \{v_1, \ldots, v_m\}$ ($m \geq 3$) who show their preferences on a set of alternatives $X = \{x_1, \ldots, x_n\}$ ($n \geq 3$). With $L(X)$ we denote the set of *linear orders* on X, and with $W(X)$ the set of *weak orders* (or *complete preorders*) on X. Given $R \in W(X)$, the *inverse* of R is the weak order R^{-1} defined by $x_i R^{-1} x_j \Leftrightarrow x_j R x_i$, for all $x_i, x_j \in X$.

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[1] A preliminary study can be found in García-Lapresta and Pérez-Román [15].

[2] Recently, García-Lapresta [14] has introduced a class of agreement measures in the context of weak orders when voters classify alternatives within a finite scale defined by linguistic categories with associated scores. These measures are based on distances among individual and collective scores generated by an aggregation operator.

A *profile* is a vector $\mathbf{R} = (R_1,\ldots,R_m)$ of weak or linear orders, where R_i contains the preferences of the voter v_i, with $i = 1,\ldots,m$. Given a profile $\mathbf{R} = (R_1,\ldots,R_m)$, we denote $\mathbf{R}^{-1} = (R_1^{-1},\ldots,R_m^{-1})$.

Given a permutation π on $\{1,\ldots,m\}$ and $\emptyset \neq I \subseteq V$, we denote $\mathbf{R}_\pi = (R_{\pi(1)},\ldots,R_{\pi(m)})$ and $I_\pi = \{v_{\pi^{-1}(i)} \mid v_i \in I\}$, i.e., $v_j \in I_\pi \Leftrightarrow v_{\pi(j)} \in I$.

Given a permutation σ on $\{1,\ldots,n\}$, we denote by $\mathbf{R}^\sigma = (R_1^\sigma,\ldots,R_m^\sigma)$ the profile obtained from $\mathbf{R}$ by relabeling the alternatives according to σ, i.e., $x_i R_k x_j \Leftrightarrow x_{\sigma(i)} R_k^\sigma x_{\sigma(j)}$ for all $i,j \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,m\}$.

The cardinal of any subset I is denoted by $|I|$. With $\mathscr{P}(V)$ we denote the power set of V, i.e., $I \in \mathscr{P}(V) \Leftrightarrow I \subseteq V$; and we also use $\mathscr{P}_2(V) = \{I \in \mathscr{P}(V) \mid |I| \geq 2\}$. Notice that $|\mathscr{P}_2(V)| = |\mathscr{P}(V)| - |V| - 1 = 2^m - m - 1$.

2.1 Codification of weak orders

We now introduce a system for codifying linear and weak orders by means of vectors that represent the relative position of each alternative in the corresponding order.

Given $R \in L(X)$, the position of each alternative in R is defined by the mapping $o_R : X \longrightarrow \{1,\ldots,n\}$. Notice that the vector $\mathbf{o}_R = (o_R(x_1),\ldots,o_R(x_n)) \in \{1,\ldots,n\}^n$ determines R and viceversa (o_R is a bijection).

There does not exist a unique system for codifying weak orders. We propose one based on linearizing the weak order and to assign each alternative the average of the positions of the alternatives within the same equivalence class[3]. As an example, consider $R \in W(\{x_1,\ldots,x_7\})$:

R
$x_2\ x_3\ x_5$
x_1
$x_4\ x_7$
x_6

Then, $o_R(x_2) = o_R(x_3) = o_R(x_5) = \frac{1+2+3}{3} = 2$, $o_R(x_1) = 4$, $o_R(x_4) = o_R(x_7) = \frac{5+6}{2} = 5.5$ and $o_R(x_6) = 7$. Consequently, R is codified by $(4, 2, 2, 5.5, 2, 7, 5.5)$.

Taking into account this idea, given $R \in W(X)$, we may consider the mapping $o_R : X \longrightarrow \mathbb{R}$ that assigns the relative position of each alternative in R. We denote $\mathbf{o}_R = (o_R(x_1),\ldots,o_R(x_n))$ and, depending on the context, $R \equiv \mathbf{o}_R$ or $\mathbf{o}_R \equiv R$.

Remark 1. If $(a_1,\ldots,a_n) \equiv R \in W(X)$, then $R^{-1} \equiv (n+1-a_1,\ldots,n+1-a_n)$.

Remark 2. For every $R \in W(X)$, it holds

1. $o_R(x_j) \in \{1, 1.5, 2, 2.5, \ldots, n-0.5, n\}$ for every $j \in \{1,\ldots,n\}$.

[3] Similar procedures have been considered in the generalization of scoring rules from linear orders to weak orders (see Smith [30], Black [5] and Cook and Seiford [9], among others).

2. $\sum_{j=1}^{n} o_R(x_j) = 1+2+\cdots+n = \frac{n(n+1)}{2}$.

In Proposition 1, we provide a complete characterization of the vectors that codify weak orders.

Given $(a_1,\ldots,a_n) \equiv R \in W(X)$, we denote $M_i(R) = \{m \in \{1,\ldots,n\} \mid a_m = a_i\}$, for $i = 1,\ldots,n$. Given a permutation σ on $\{1,\ldots,n\}$, we denote $R^\sigma \equiv (a_1^\sigma,\ldots,a_n^\sigma)$, with $a_i^\sigma = a_{\sigma(i)}$.

Proposition 1. *Given $(a_1,\ldots,a_n) \in \mathbb{R}^n$, $(a_1,\ldots,a_n) \equiv R \in W(X)$ if and only if there exists a permutation σ on $\{1,\ldots,n\}$ such that R^σ satisfies the following conditions:*

1. $a_1^\sigma \leq \cdots \leq a_n^\sigma$.
2. $a_1^\sigma + \cdots + a_n^\sigma = \frac{n(n+1)}{2}$.
3. For all $i \in \{1,\ldots,n\}$ and $m \in M_i(R^\sigma)$ it holds

$$a_m^\sigma = \sum_{l=0}^{k-1} \frac{j+l}{k} = j + \frac{k-1}{2}\ ,$$

where $j = \min M_i(R^\sigma)$ and $k = |M_i(R^\sigma)|$.

Proof. See the Appendix. □

Remark 3. Notice that if $|M_i(R)| = 1$ for every $i \in \{1,\ldots,n\}$, then $R \in L(X)$. Moreover, if $R \in L(X)$, then there exists a unique permutation σ such that $a_i^\sigma = i$ for every $i \in \{1,\ldots,n\}$.

Definition 1. We denote by A_W the set of vectors that codify weak orders, i.e.,

$$A_W = \{(a_1,\ldots,a_n) \in \mathbb{R}^n \mid (a_1,\ldots,a_n) \equiv R \text{ for some } R \in W(X)\}.$$

Remark 4. For every $R \in W(X)$, the mapping $o_R : X \longrightarrow A_W$ that assigns the relative position of each alternative in R is a bijection. Thus, we can identify $W(X)$ and A_W.

Remark 5. A_W is stable under permutations, i.e., for every permutation σ on $\{1,\ldots,n\}$, if $(a_1,\ldots,a_n) \in A_W$, then $(a_1^\sigma,\ldots,a_n^\sigma) \in A_W$.

Example 1. Consider $R \in W(\{x_1,\ldots,x_8\})$:

R
x_3
$x_1\ x_6$
x_4
$x_5\ x_7\ x_8$
x_2

Then, $R \equiv (2.5, 8, 1, 4, 6, 2.5, 6, 6)$.

Let σ be the permutation on $\{1,\ldots,8\}$ represented by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 6 & 4 & 5 & 7 & 8 & 2 \end{pmatrix}$, i.e., $\sigma(1)=3,\ \sigma(2)=1,\ \ldots,\sigma(8)=2$. Then,

R^σ
x_1^σ
$x_2^\sigma\ x_3^\sigma$
x_4^σ
$x_5^\sigma\ x_6^\sigma\ x_7^\sigma$
x_8^σ

and, consequently, $R^\sigma \equiv (1,2.5,2.5,4,6,6,6,8)$.

1. $a_i^\sigma \leq a_j^\sigma$, for $1 \leq i < j \leq 8$.
2. $a_1^\sigma + \cdots + a_8^\sigma = \frac{8(8+1)}{2} = 36$.
3. For $i=7$, we have $M_7(R^\sigma) = \left\{m \in \{1,\ldots,8\} \mid a_m^\sigma = a_7^\sigma\right\} = \{5,6,7\}$, $j = \min\{M_7(R^\sigma)\} = 5$, $k = |M_7(R^\sigma)| = 3$ and

$$a_5^\sigma = a_6^\sigma = a_7^\sigma = \sum_{l=0}^{3-1} \frac{5+l}{2} = 6 = 5 + \frac{3-1}{2} = j + \frac{k-1}{2}\ .$$

Definition 2. $W_{\leq}(X) = \{R \in W(X) \mid R \equiv (a_1,\ldots,a_n) \text{ and } a_1 \leq \cdots \leq a_n\}$.

Remark 6. By Proposition 1, for every $R \in W(X)$ there exists some permutation σ on $\{1,\ldots,n\}$ such that $R^\sigma \in W_{\leq}(X)$. Notice that if $R \in L(X)$, then σ is unique, but if $R \in W(X) \setminus L(X)$, then there exist more than one σ satisfying $R^\sigma \in W_{\leq}(X)$.

Lemma 1. *For all $(a_1,\ldots,a_n) \equiv R_1 \in W(X)$ and $(b_1,\ldots,b_n) \equiv R_2 \in W(X)$, there exists a permutation σ on $\{1,\ldots,n\}$ such that:*

1. $a_1^\sigma \leq \cdots \leq a_n^\sigma$.
2. For every $i \in \{1,\ldots,n\}$ such that $|M_i(R_1^\sigma)| > 1$, if $j = \min M_i(R_1^\sigma)$ and $k = |M_i(R_1^\sigma)|$, then $b_j^\sigma \geq \cdots \geq b_{j+k-1}^\sigma$.

Proof. See the Appendix. □

Example 2. In order to illustrate Lemma 1, consider $R_1, R_2 \in W(\{x_1,\ldots,x_8\})$:

R_1	R_2
x_3	x_4
$x_1\ x_4\ x_6$	$x_5\ x_8$
x_8	x_7
$x_5\ x_7$	x_1
x_2	x_3
	$x_2\ x_6$

Then, $R_1 \equiv (3,8,1,3,6.5,3,6.5,5)$ and $R_2 \equiv (5,7.5,6,1,2.5,7.5,4,2.5)$. Let σ' be the permutation on $\{1,\ldots,n\}$ represented by $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 3\,1\,6\,4\,8\,5\,7\,2 \end{pmatrix}$. Then,

$$\begin{aligned} R_1^{\sigma'} &\equiv (1,\ 3,\ 3,\ \ 3,\ \ \ 5,\ \ 6.5,\ 6.5,\ 8), \\ R_2^{\sigma'} &\equiv (6,\ 5,\ 1,\ 7.5,\ 2.5,\ 2.5,\ \ 4,\ \ 6). \end{aligned}$$

Let σ'' be the permutation on $\{1,\ldots,n\}$ represented by $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,4\,2\,3\,5\,7\,6\,8 \end{pmatrix}$. It is clear that $R_1^{\sigma'} = (R_1^{\sigma'})^{\sigma''}$. Thus, if $\sigma = \sigma' \cdot \sigma''$, i.e., $\sigma(i) = \sigma'(\sigma''(i))$, then σ is represented by $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 3\,4\,1\,6\,8\,7\,5\,2 \end{pmatrix}$. Therefore,

$$\begin{aligned} R_1^{\sigma} &\equiv (1,\ \ 3,\ \ 3,\ 3,\ \ 5,\ \ 6.5,\ 6.5,\ 8)\ , \\ R_2^{\sigma} &\equiv (6,\ 7.5,\ 5,\ 1,\ 2.5,\ \ 4,\ \ 2.5,\ 6)\ . \end{aligned}$$

2.2 Distances

The use of distances for designing and analyzing voting system has been widely considered in the literature. On this, see Kemeny [18], Slater [29], Nitzan [26], Baigent [3, 4], Nurmi [27, 28], Meskanen and Nurmi [23, 24], Monjardet [25], Gaertner [13, 6.3] and Eckert and Klamler [11], among others. A general and complete survey on distances can be found in Deza and Deza [10].

The consensus measures introduced in this chapter are based on distances on weak orders. After presenting the general notion, we show the distances on $\mathbb{R}^n$ used for inducing the distances on weak orders. We pay special attention to the Kemeny distance.

Definition 3. A *distance* (or *metric*) on a set $A \neq \emptyset$ is a mapping $d : A \times A \longrightarrow \mathbb{R}$ satisfying the following conditions for all $a,b,c \in A$:

1. $d(a,b) \geq 0$.
2. $d(a,b) = 0 \Leftrightarrow a = b$.
3. $d(a,b) = d(b,a)$.
4. $d(a,b) \leq d(a,c) + d(c,b)$.

2.2.1 Distances on $\mathbb{R}^n$

Example 3. Typical examples of distances on $\mathbb{R}^n$ or $[0,\infty)^n$ are the following:

1. The *discrete* distance $d' : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d'((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \begin{cases} 1, & \text{if } (a_1,\ldots,a_n) \neq (b_1,\ldots,b_n)\ , \\ 0, & \text{if } (a_1,\ldots,a_n) = (b_1,\ldots,b_n)\ . \end{cases}$$

2. The *Minkowski* distance $d_p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, with $p \geq 1$,

$$d_p((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \left(\sum_{i=1}^{n} |a_i - b_i|^p\right)^{\frac{1}{p}} .$$

   For $p = 1$ and $p = 2$ we obtain the *Manhattan* and *Euclidean* distances, respectively.
3. The *Chebyshev* distance $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d_\infty((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \max\big\{|a_1 - b_1|,\ldots,|a_n - b_n|\big\} .$$

4. The *cosine* distance $d_c : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d_c((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = 1 - \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\,\sqrt{\sum_{i=1}^{n} b_i^2}} .$$

5. The *Hellinger* distance $d_H : [0,\infty)^n \times [0,\infty)^n \longrightarrow \mathbb{R}$,

$$d_H((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \left(\sum_{i=1}^{n} \left(\sqrt{a_i} - \sqrt{b_i}\right)^2\right)^{\frac{1}{2}} .$$

Notice that all the previous distances may be defined on $[0,\infty)^n$. In fact, we only need distances on $[0,\infty)^n$ to introduce a simple procedure for constructing distances on $W(X)$.

Definition 4. Given $D \subseteq \mathbb{R}^n$ stable under permutations, a distance $d : D \times D \longrightarrow \mathbb{R}$ is *neutral* if for every permutation σ on $\{1,\ldots,n\}$, it holds

$$d\left((a_1^\sigma,\ldots,a_n^\sigma),(b_1^\sigma,\ldots,b_n^\sigma)\right) = d\left((a_1,\ldots,a_n),(b_1,\ldots,b_n)\right) ,$$

for all $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in D$.

Remark 7. All the distances introduced in Example 3 are neutral.

2.2.2 Distances on weak orders

We now introduce a direct way of defining distances on weak orders. They are induced by distances on $\mathbb{R}^n$ by considering the position vectors.

Definition 5. Given a distance $d : [0,\infty)^n \times [0,\infty)^n \longrightarrow \mathbb{R}$, the *distance on* $W(X)$ *induced by* d is the mapping $\bar{d} : W(X) \times W(X) \longrightarrow \mathbb{R}$ defined by

$$\bar{d}(R_1,R_2) = d\big((o_{R_1}(x_1),\ldots,o_{R_1}(x_n)),(o_{R_2}(x_1),\ldots,o_{R_2}(x_n))\big) ,$$

for all $R_1, R_2 \in W(X)$.

Given a distance d_- on $[0,\infty)^n$, we use $\bar{d}_-$ to denote the distance on $W(X)$ induced by d_-.

2.2.3 The Kemeny distance

The Kemeny distance was initially defined on linear orders by Kemeny [18], as the sum of pairs where the orders' preferences disagree. However, it has been generalized to the framework of weak orders (see and Eckert and Klamler [11], among others).

The *Kemeny* distance on weak orders $d^K : W(X) \times W(X) \longrightarrow \mathbb{R}$ is usually defined as one half[4] of the cardinal of the symmetric difference between the weak orders, i.e.,

$$d^K(R_1,R_2) = \frac{|(R_1 \cup R_2) \setminus (R_1 \cap R_2)|}{2} .$$

We now consider $d_K : A_W \times A_W \longrightarrow \mathbb{R}$, given by

$$d_K\big((a_1,\dots,a_n),(b_1,\dots,b_n)\big) = \sum_{\substack{i,j=1 \\ i<j}}^{n} |\,\mathrm{sgn}\,(a_i - a_j) - \mathrm{sgn}\,(b_i - b_j)| \,,$$

where sgn is the *sign function*:

$$\mathrm{sgn}\,(a) = \begin{cases} 1, & \text{if } a > 0 \,, \\ 0, & \text{if } a = 0 \,, \\ -1, & \text{if } a < 0 \,. \end{cases}$$

Notice that d_K is a neutral distance on A_W.

Taking into account Kemeny and Snell [19, p. 18], it is easy to see that d^K coincides with the distance on $W(X)$ induced by d_K, i.e.,

$$\begin{aligned} d^K(R_1,R_2) &= \bar{d}_K(R_1,R_2) = d_K\big((a_1,\dots,a_n),(b_1,\dots,b_n)\big) = \\ &= \frac{1}{2} \sum_{i,j=1}^{n} |\,\mathrm{sgn}\,(a_i - a_j) - \mathrm{sgn}\,(b_i - b_j)| = \\ &= \sum_{\substack{i,j=1 \\ i<j}}^{n} |\,\mathrm{sgn}\,(a_i - a_j) - \mathrm{sgn}\,(b_i - b_j)| \,, \end{aligned}$$

where $R_1 \equiv (a_1,\dots,a_n)$ and $R_2 \equiv (b_1,\dots,b_n)$.

[4] Sometimes "one half" is removed.

3 Consensus measures

Consensus measures have been introduced and analyzed by Bosch [7] in the context of linear orders. We now extend this concept to the framework of weak orders.

Definition 6. A *consensus measure* on $W(X)^m$ is a mapping

$$\mathscr{M} : W(X)^m \times \mathscr{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

1. *Unanimity*. For all $\mathbf{R} \in W(X)^m$ and $I \in \mathscr{P}_2(V)$, it holds
$$\mathscr{M}(\mathbf{R},I) = 1 \Leftrightarrow R_i = R_j \text{ for all } v_i, v_j \in I.$$
2. *Anonymity*. For all permutation π on $\{1,\dots,m\}$, $\mathbf{R} \in W(X)^m$ and $I \in \mathscr{P}_2(V)$, it holds
$$\mathscr{M}(\mathbf{R}_\pi, I_\pi) = \mathscr{M}(\mathbf{R},I) .$$
3. *Neutrality*. For all permutation σ on $\{1,\dots,n\}$, $\mathbf{R} \in W(X)^m$ and $I \in \mathscr{P}_2(V)$, it holds
$$\mathscr{M}(\mathbf{R}^\sigma, I) = \mathscr{M}(\mathbf{R},I) .$$

Unanimity means that the maximum consensus in every subset of decision makers is only achieved when all opinions are the same. Anonymity requires symmetry with respect to decision makers, and neutrality means symmetry with respect to alternatives.

We now introduce other properties that a consensus measure may satisfy.

Definition 7. Let $\mathscr{M} : W(X)^m \times \mathscr{P}_2(V) \longrightarrow [0,1]$ be a consensus measure.

1. $\mathscr{M}$ satisfies *maximum dissension* if for all $\mathbf{R} \in W(X)^m$ and $v_i, v_j \in V$ such that $i \neq j$, it holds
$$\mathscr{M}(\mathbf{R}, \{v_i, v_j\}) = 0 \Leftrightarrow R_i, R_j \in L(X) \text{ and } R_j = R_i^{-1} .$$
2. $\mathscr{M}$ is *reciprocal* if for all $\mathbf{R} \in W(X)^m$ and $I \in \mathscr{P}_2(V)$, it holds
$$\mathscr{M}(\mathbf{R}^{-1}, I) = \mathscr{M}(\mathbf{R},I) .$$
3. $\mathscr{M}$ is *homogeneous* if for all $\mathbf{R} \in W(X)^m$, $I \in \mathscr{P}_2(V)$ and $t \in \mathbb{N}$, it holds
$$\mathscr{M}^t(t\,\mathbf{R}, t\,I) = \mathscr{M}(\mathbf{R},I) ,$$
where $\mathscr{M}^t : W(X)^{tm} \times \mathscr{P}_2(t\,V) \longrightarrow [0,1]$, $t\mathbf{R} = (\mathbf{R}, \overset{t}{\dots}, \mathbf{R}) \in W(X)^{tm}$ is the profile defined by t copies of $\mathbf{R}$ and $t\,I = I \uplus \overset{t}{\cdots} \uplus I$ is the multiset of voters[5] defined by t copies of I.

[5] List of voters where each voter occurs as many times as the multiplicity. For instance, $2\{v_1, v_2\} = \{v_1, v_2\} \uplus \{v_1, v_2\} = \{v_1, v_2, v_1, v_2\}$.

Maximum dissension means that in each subset of two voters[6], the minimum consensus is only reached whenever preferences of voters are linear orders and each one is the inverse of the other. Reciprocity means that if all individual weak orders are reversed, then the consensus does not change. And homogeneity means that if we replicate a subset of voters, then the consensus in that group does not change.

We now introduce our proposal for measuring consensus in sets of weak orders.

Definition 8. Given a distance $\bar{d}: W(X) \times W(X) \longrightarrow \mathbb{R}$, the mapping

$$\mathscr{M}_{\bar{d}}: W(X)^m \times \mathscr{P}_2(V) \longrightarrow [0,1]$$

is defined by

$$\mathscr{M}_{\bar{d}}(\mathbf{R}, I) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i<j}} \bar{d}(R_i, R_j)}{\displaystyle\binom{|I|}{2} \cdot \Delta_n},$$

where

$$\Delta_n = \max \left\{ \bar{d}(R_i, R_j) \mid R_i, R_j \in W(X) \right\}.$$

Notice that the numerator of the quotient appearing in the above expression is the sum of all the distances between the weak orders of the profile, and the denominator is the number of terms in the numerator's sum multiplied by the maximum distance between weak orders. Consequently, that quotient belongs to the unit interval and it measures the disagreement in the profile.

Proposition 2. *For every distance $\bar{d}: W(X) \times W(X) \longrightarrow \mathbb{R}$, $\mathscr{M}_{\bar{d}}$ satisfies unanimity and anonymity.*

Proof. Let $\mathbf{R} \in W(X)^m$ and $I \in \mathscr{P}_2(V)$.

1. Unanimity.

$$\mathscr{M}_{\bar{d}}(\mathbf{R}, I) = 1 \quad \Leftrightarrow \quad \sum_{\substack{v_i, v_j \in I \\ i<j}} \bar{d}(R_i, R_j) = 0 \quad \Leftrightarrow$$

$$\forall v_i, v_j \in I \quad \bar{d}(R_i, R_j) = 0 \quad \Leftrightarrow \quad \forall v_i, v_j \in I \quad R_i = R_j.$$

2. Anonymity. Let π be a permutation on $\{1, \ldots, m\}$.

$$\sum_{\substack{v_i, v_j \in I_\pi \\ i<j}} \bar{d}(R_{\pi(i)}, R_{\pi(j)}) = \sum_{\substack{v_{\pi(i)}, v_{\pi(j)} \in I \\ \pi(i)<\pi(j)}} \bar{d}(R_{\pi(i)}, R_{\pi(j)}) = \sum_{\substack{v_i, v_j \in I \\ i<j}} \bar{d}(R_i, R_j).$$

Thus, $\mathscr{M}_{\bar{d}}(\mathbf{R}_\pi, I_\pi) = \mathscr{M}_{\bar{d}}(\mathbf{R}, I)$. □

[6] It is clear that a society reach maximum consensus when all the opinions are the same. However, in a society with more than two members it is not an obvious issue to determine when there is minimum consensus (maximum disagreement).

If $\mathcal{M}_{\bar{d}}$ is neutral, then we say that $\mathcal{M}_{\bar{d}}$ is the *consensus measure associated with* $\bar{d}$.

Proposition 3. *If* $d : [0,\infty)^n \times [0,\infty)^n \longrightarrow \mathbb{R}$ *is a neutral distance, then* $\mathcal{M}_{\bar{d}}$ *is a consensus measure.*

Proof. By Proposition 2, $\mathcal{M}_{\bar{d}}$ satisfies unanimity and anonymity. Obviously, if $\bar{d}$ is neutral, then $\mathcal{M}_{\bar{d}}$ is neutral and thus $\mathcal{M}_{\bar{d}}$ is a consensus measure. □

Proposition 4. *If* $\bar{d}$ *is the distance on* $W(X)$ *induced by* d', d_1, d_2, d_∞, d_c *or* d_K, *then* $\mathcal{M}_{\bar{d}}$ *is a reciprocal consensus measure.*

Proof. See the Appendix. □

Remark 8. $\mathcal{M}_{\bar{d}_H}$ is not a reciprocal consensus measure.

Let us consider $R_1, R_2 \in W(\{x_1, x_2, x_3\})$:

R_1	R_2	R_1^{-1}	R_2^{-1}
x_1	$x_1\ x_2\ x_3$	$x_2\ x_3$	$x_1\ x_2\ x_3$
$x_2\ x_3$		x_1	

The above weak orders are codified by $R_1 \equiv (1, 2.5, 2.5)$, $R_1^{-1} \equiv (3, 1.5, 1.5)$ and $R_2 = R_2^{-1} \equiv (2,2,2)$. We have

$$\bar{d}_H(R_1,R_2) =$$
$$= \left(\left(\sqrt{1}-\sqrt{2}\right)^2 + \left(\sqrt{2.5}-\sqrt{2}\right)^2 + \left(\sqrt{2.5}-\sqrt{2}\right)^2\right)^{\frac{1}{2}} = 0.476761 \neq$$
$$\neq 0.415713 = \left(\left(\sqrt{3}-\sqrt{2}\right)^2 + \left(\sqrt{1.5}-\sqrt{2}\right)^2 + \left(\sqrt{1.5}-\sqrt{2}\right)^2\right)^{\frac{1}{2}} =$$
$$= \bar{d}_H(R_1^{-1}, R_2^{-1}) \, .$$

In order to prove that the maximum dissension property is satisfied for some of the consensus measures introduced above, we need two lemmas.

Lemma 2. *Let* $R_1 \in W(X) \setminus L(X)$ *and* $R_2 \in W(X)$. *If* $\bar{d}_-$ *is the distance induced by* d_2, d_c, d_H *or* d_K, *then there exists* $R_3 \in W(X)$ *such that* $\bar{d}_-(R_1,R_2) < \bar{d}_-(R_1,R_3)$.

Proof. See the Appendix. □

Lemma 3. *Let* $R_1, R_2 \in L(X)$ *such that* $R_2 \neq R_1^{-1}$. *If* $\bar{d}_-$ *is the distance induced by* d_2, d_c, d_H *or* d_K, *then there exists* $R_3 \in W(X)$ *such that* $\bar{d}_-(R_1,R_2) < \bar{d}_-(R_1,R_3)$.

Proof. See the Appendix. □

Proposition 5. *If* $\bar{d}_-$ *is the distance induced by* d_2, d_c, d_H *or* d_K, *then* $\mathcal{M}_{\bar{d}_-}$ *satisfies the maximum dissension property.*

Proof. First of all, notice that $\mathscr{M}_{\bar{d}_-}(\mathbf{R},\{v_i,v_j\})=0$ if and only if $\bar{d}_-(R_i,R_j)=\Delta_n$. By Lemma 2 and Lemma 3, $d_-(R_1,R_2)=\Delta_n$ if and only if $R_2=R_1^{-1}$. □

Remark 9. If $\bar{d}_-$ is the distance induced by d', d_1 or d_∞, then $\mathscr{M}_{\bar{d}_-}$ does not satisfy the maximum dissension property.

Let us consider the following profile $\mathbf{R}=(R_1,R_2)\in L(X)^2$:

R_1	R_2	R_1^{-1}
x_1	x_3	x_3
x_2	x_1	x_2
x_3	x_2	x_1

Notice that $R_2\neq R_1^{-1}$. Since the above linear orders are codified by $R_1\equiv(1,2,3)$, $R_2\equiv(2,3,1)$ and $R_1^{-1}\equiv(3,2,1)$, we have

1. $\bar{d}'(R_1,R_2)=\bar{d}'(R_1,R_1^{-1})=1$ and $\mathscr{M}_{\bar{d}'}(\mathbf{R},\{v_1,v_2\})=0$.
2. $\bar{d}_1(R_1,R_2)=\bar{d}_1(R_1,R_1^{-1})=4$ and $\mathscr{M}_{\bar{d}_1}(\mathbf{R},\{v_1,v_2\})=0$.
3. $\bar{d}_\infty(R_1,R_2)=\bar{d}_\infty(R_1,R_1^{-1})=2$ and $\mathscr{M}_{\bar{d}_\infty}(\mathbf{R},\{v_1,v_2\})=0$.

As presented in the following result, none of the introduced consensus measures is homogeneous. In García-Lapresta and Pérez-Román [15] we introduced the Borda consensus measures, based on the Euclidean distance. We note that they are homogeneous and reciprocal, but they do not satisfy the maximum dissension property.

Proposition 6. *The consensus measure $\mathscr{M}_{\bar{d}}$ is not homogeneous for any distance $\bar{d}$ on $W(X)$.*

Proof. Let $\mathbf{R}\in W(X)^m$ such that $R_1\neq R_2$ and $I=\{v_1,v_2\}\in\mathscr{P}_2(V)$. We now consider $2\mathbf{R}=(R'_1,R'_2,R'_3,R'_4)$, where $R'_1=R'_3=R_1$ and $R'_2=R'_4=R_2$, and $2I=\{v'_1,v'_2,v'_3,v'_4\}$, with $v'_1=v'_3=v_1$ and $v'_2=v'_4=v_2$. Then, we have

$$\mathscr{M}^2_{\bar{d}}(2\mathbf{R},2I)=$$

$$=1-\frac{\bar{d}(R'_1,R'_2)+d(R'_1,R'_3)+d(R'_1,R'_4)+d(R'_2,R'_3)+d(R'_2,R'_4)+d(R'_3,R'_4)}{\binom{|2I|}{2}\cdot\Delta_n}=$$

$$=1-\frac{4\cdot\bar{d}(R_1,R_2)}{6\cdot\Delta_n}=1-\frac{2\cdot\bar{d}(R_1,R_2)}{3\cdot\Delta_n}\neq 1-\frac{\bar{d}(R_1,R_2)}{\Delta_n}=\mathscr{M}_{\bar{d}}(\mathbf{R},I)\,. \qquad \square$$

Proposition 7. *If $\mathbf{R}\in L(X)^2$ and $R_2=R_1^{-1}$, then for any distance $\bar{d}$ on $W(X)$ it holds:*

$$\lim_{t\to\infty}\mathscr{M}^t_{\bar{d}}(t\,\mathbf{R},t\,I)=\frac{1}{2}\,.$$

Proof. Let $I=\{v_1,v_2\}$ and $\mathbf{R}=(R_1,R_2)$ with $R_2=R_1^{-1}$. For every $t\in\mathbb{N}$, we have $t\,\mathbf{R}=(R_1,R_2,\dots,R_{2t})$, where $R_{2k-1}=R_1$ and $R_{2k}=R_2$ for every $k\in\{1,2,\dots,t\}$.

We should calculate the limit of the following expression:

$$\mathscr{M}^t_{\bar{d}}(t\,\mathbf{R}, t\,I) = 1 - \frac{\sum\limits_{\substack{v_i,v_j \in tI \\ i<j}} \bar{d}(R_i,R_j)}{\binom{|tI|}{2} \cdot \Delta_n} \, .$$

Since

$$\bar{d}(R_i,R_j) = \begin{cases} 0, & \text{if } i,j \text{ are both even} , \\ 0, & \text{if } i,j \text{ are both odd} , \\ \Delta_n, & \text{otherwise} , \end{cases}$$

we obtain

$$\sum_{\substack{v_i,v_j \in tI \\ i<j}} \bar{d}(R_i,R_j) = \sum_{i=1}^{2t-1} \sum_{j=i+1}^{2t} \bar{d}(R_i,R_j) = \left(\sum_{i=1}^{t} i + \sum_{j=1}^{t-1} j \right) \cdot \Delta_n = t^2 \cdot \Delta_n \, .$$

On the other hand, we have

$$\binom{|tI|}{2} = \binom{2t}{2} = 2t^2 - t \, .$$

Consequently,

$$\lim_{t\to\infty} \mathscr{M}^t_{\bar{d}}(t\,\mathbf{R}, t\,I) = 1 - \lim_{t\to\infty} \frac{t^2 \cdot \Delta_n}{(2t^2 - t) \cdot \Delta_n} = \frac{1}{2} \, . \quad \square$$

Remark 10. Homogeneity ensures that a society has no consensus at all when it is divided into two groups each one ranks order the alternatives just in the opposite way to the other group. According to Proposition 6, our consensus measures are not homogeneous. Thus, they perceive some consensus in polarized societies and, by Proposition 7, consensus tends to 0.5 when the number of voters tends to infinity, regardless of the distance used. Notice that this result holds even when the distance used does not verify the maximum dissension property, which guaranteed that the consensus between two profiles is zero if and only if they are opposites.

We summarize the properties of the analyzed consensus measures in Table 1.

Appendix

Proof of Proposition 1. Consider $(a_1, \ldots\, a_n) \equiv R \in W(X)$.

1. Obvious.
2. By Remark 2.

Table 1 Summary

	Max. diss.	Reciproc.	Homogen.
$\mathscr{M}_{\bar{d}'}$	No	Yes	No
$\mathscr{M}_{\bar{d}_1}$	No	Yes	No
$\mathscr{M}_{\bar{d}_2}$	Yes	Yes	No
$\mathscr{M}_{\bar{d}_\infty}$	No	Yes	No
$\mathscr{M}_{\bar{d}_c}$	Yes	Yes	No
$\mathscr{M}_{\bar{d}_H}$	Yes	No	No
$\mathscr{M}_{\bar{d}_K}$	Yes	Yes	No

3. Consider $i \in \{1,\ldots,n\}$ with $|M_i(R^\sigma)| = k > 1$, and $j = \min M_i(R^\sigma)$. Then, $M_i(R^\sigma) = \{j, j+1, \ldots, j+(k-1)\}$ and $a^\sigma_j = a^\sigma_{j+1} = \cdots = a^\sigma_{j+(k-1)} = a^\sigma_i$. Since we assign each alternative the average of the positions of the alternatives within the same equivalence class, then we have

$$a^\sigma_i = \frac{\sum\limits_{m \in M_i(R^\sigma)} m}{k} = \frac{j+(j+1)+\cdots+(j+(k-1))}{k} = j + \frac{k-1}{2}.$$

Reciprocally, given $(a_1,\ldots,a_n) \in \mathbb{R}^n$ and a permutation σ verifying conditions 1, 2 and 3, then we can consider that a^σ_i is the relative position of the alternative x_i in $R \in W(X)$. Then, $R \equiv (a_1,\ldots,a_n)$. $\square$

Proof of Lemma 1. By Proposition 1, there exists a permutation σ' on $\{1,\ldots,n\}$ such that $R^{\sigma'}_1$ satisfies 1. We consider $R^{\sigma'}_2$. Let σ'' be a permutation on $\{1,\ldots,n\}$ such that:

1. If $\left|M_i\left(R^{\sigma'}_1\right)\right| = 1$, then $\sigma''(i) = i$.
2. If $\left|M_i\left(R^{\sigma'}_1\right)\right| = k > 1$ and $j = \min M_i\left(R^{\sigma'}_1\right)$, then let σ'' be a permutation on $\{1,\ldots,n\}$ such that $b^{\sigma'}_{\sigma''(j)} \geq b^{\sigma'}_{\sigma''(j+1)} \geq \cdots \geq b^{\sigma'}_{\sigma''(j+k-1)}$. Obviously, $a^{\sigma'}_{\sigma''(j)} = \cdots = a^{\sigma'}_{\sigma''(j+k-1)} = a^{\sigma'}_i$.

Therefore, $\sigma = \sigma' \cdot \sigma''$. $\square$

Proof of Proposition 4. By Remark 7 and Proposition 3, we only need to prove that the corresponding distances are reciprocal.

1. <u>Case $\bar{d}'$.</u>

   Since $R_i = R_j \Leftrightarrow R^{-1}_i = R^{-1}_j$, we have $\bar{d}'(R_i,R_j) = \bar{d}'(R^{-1}_i,R^{-1}_j)$. Consequently, $\mathscr{M}_{\bar{d}'}(\mathbf{R}^{-1},I) = \mathscr{M}_{\bar{d}'}(\mathbf{R},I)$.

2. <u>Cases $\bar{d}_p$ $(p \in \{1,2\})$.</u>

   By Remark 1, we have

$$\bar{d}_p(R_i^{-1},R_j^{-1}) = \left(\sum_{k=1}^{n} |(n+1-o_{R_i}(x_k)) - (n+1-o_{R_j}(x_k))|^p \right)^{\frac{1}{p}} =$$

$$= \left(\sum_{k=1}^{n} |o_{R_i}(x_k) - o_{R_j}(x_k)|^p \right)^{\frac{1}{p}} = \bar{d}_p(R_i,R_j) \ .$$

   Thus, $\mathscr{M}_{\bar{d}_p}(\mathbf{R}^{-1},I) = \mathscr{M}_{\bar{d}_p}(\mathbf{R},I)$.

3. <u>Case $\bar{d}_\infty$.</u>

   By Remark 1, we have

$$\bar{d}_\infty(R_i^{-1},R_j^{-1}) =$$
$$= \max \left\{ |(n+1-o_{R_i}(x_k)) - (n+1-o_{R_j}(x_k))| \ \mid k \in \{1,\ldots,n\} \right\} =$$
$$= \max \left\{ |o_{R_i}(x_k) - o_{R_j}(x_k)| \ \mid k \in \{1,\ldots,n\} \right\} = \bar{d}_\infty(R_i,R_j) \ .$$

   Thus, $\mathscr{M}_{\bar{d}_\infty}(\mathbf{R}^{-1},I) = \mathscr{M}_{\bar{d}_\infty}(\mathbf{R},I)$.

4. <u>Case $\bar{d}_c$.</u>

$$\bar{d}_c(R_i^{-1},R_j^{-1}) = 1 - \frac{\sum_{k=1}^{n} (n+1-o_{R_i}(x_k))\,(n+1-o_{R_j}(x_k))}{\sqrt{\sum_{k=1}^{n} (n+1-o_{R_i}(x_k))^2}\sqrt{\sum_{k=1}^{n} (n+1-o_{R_j}(x_k))^2}} \ .$$

   By Remark 2, we have $\sum_{k=1}^{n} \big((n+1) - (o_{R_i}(x_k) + o_{R_j}(x_k))\big) = 0$. Thus,

$$\sum_{k=1}^{n} (n+1-o_{R_i}(x_k))\,(n+1-o_{R_j}(x_k)) =$$
$$= (n+1) \left[\sum_{k=1}^{n} \big((n+1) - (o_{R_i}(x_k) + o_{R_j}(x_k))\big) \right] + \sum_{k=1}^{n} o_{R_i}(x_k)\, o_{R_j}(x_k) =$$
$$= \sum_{k=1}^{n} o_{R_i}(x_k)\, o_{R_j}(x_k) \ .$$

   By Remark 2, we also have $\sum_{k=1}^{n} ((n+1) - 2o_{R_i}(x_k)) = \sum_{k=1}^{n} ((n+1) - 2o_{R_j}(x_k)) = 0$. Thus,

$$\sum_{k=1}^{n}(n+1-o_{R_i}(x_k))^2 = (n+1)\sum_{k=1}^{n}((n+1)-2o_{R_i}(x_k)) + \sum_{k=1}^{n} o_{R_i}(x_k)^2 = \sum_{k=1}^{n} o_{R_i}(x_k)^2 \ ,$$

and

$$\sum_{k=1}^{n}(n+1-o_{R_j}(x_k))^2 = (n+1)\sum_{k=1}^{n}((n+1)-2o_{R_j}(x_k)) + \sum_{k=1}^{n} o_{R_j}(x_k)^2 = \sum_{k=1}^{n} o_{R_j}(x_k)^2 \ .$$

Consequently,

$$\bar{d}_c(R_i^{-1},R_j^{-1}) = 1 - \frac{\sum_{k=1}^{n} o_{R_i}(x_k)\, o_{R_j}(x_k)}{\sqrt{\sum_{k=1}^{n}(o_{R_i}(x_k))^2}\sqrt{\sum_{k=1}^{n}(o_{R_j}(x_k))^2}} = \bar{d}_c(R_i,R_j) \ .$$

Thus, $\mathscr{M}_{\bar{d}_c}(\mathbf{R}^{-1},I) = \mathscr{M}_{\bar{d}_c}(\mathbf{R},I)$.

5. Case $\bar{d}_K$.

$$\begin{aligned}
&\bar{d}_K(R_1^{-1},R_2^{-1}) = \\
&= \sum_{\substack{i,j=1\\ i<j}}^{n} \big|\mathrm{sgn}\big(n+1-o_{R_1}(x_i)-(n+1-o_{R_1}(x_j))\big) - \\
&\quad - \mathrm{sgn}\big(n+1-o_{R_2}(x_i)-(n+1-o_{R_2}(x_j))\big)\big| = \\
&= \sum_{\substack{i,j=1\\ i<j}}^{n} \big|\,\mathrm{sgn}\ \big(o_{R_1}(x_j)-o_{R_1}(x_i)\big) - \mathrm{sgn}\ \big(o_{R_2}(x_j)-o_{R_2}(x_i)\big)\big| = \\
&= \sum_{\substack{i,j=1\\ i<j}}^{n} \big|\,\mathrm{sgn}\ \big(o_{R_1}(x_i)-o_{R_1}(x_j)\big) - \mathrm{sgn}\ \big(o_{R_2}(x_i)-o_{R_2}(x_j)\big)\big| = \bar{d}_K(R_1,R_2) \ .
\end{aligned}$$

Thus, $\mathscr{M}_{\bar{d}_K}(\mathbf{R}^{-1},I) = \mathscr{M}_{\bar{d}_K}(\mathbf{R},I)$. □

Proof of Lemma 2. Let $(a_1,\dots,a_n) \equiv R_1 \in W(X) \setminus L(X)$, $(b_1,\dots,b_n) \equiv R_2 \in W(X)$ and $(a'_1,\dots,a'_n) \equiv R' \in W(X)$. By Proposition 1 and Lemma 1, and taking into account that all the considered distances are neutral (Remark 7), we can assume without loss of generality:

- $R_1 \equiv (a_1,\dots,a_j,\dots,a_{j+k-1},\dots,a_n) \in W_{\leq}(X)$ with $j = \min\{i \mid |M_i(R_1)| > 0\}$, $|M_j(R_1)| = k$ and $a_j = \cdots = a_{j+k-1} = j + \frac{k-1}{2}$.
- $R_2 \equiv (b_1,\dots,b_j,\dots,b_{j+k-1},\dots,b_n) \in W(X)$ with $b_j \geq \cdots \geq b_{j+k-1}$.

Let now $(a'_1,\dots,a'_n) = (a_1,\dots,a_{j-1},j,\dots,j+k-1,a_{j+k},\dots,a_n) \equiv R_3 \in W(X)$ and

$$m = \begin{cases} \frac{k-1}{2} - 1, & \text{if } k \text{ is odd,} \\ \frac{k-1}{2} - \frac{1}{2}, & \text{if } k \text{ is even.} \end{cases}$$

1. Case $\bar{d}_2$.

$$\bar{d}_2(R_1,R_2) < \bar{d}_2(R_3,R_2) \Leftrightarrow 0 < \left(\bar{d}_2(R_3,R_2)\right)^2 - \left(\bar{d}_2(R_1,R_2)\right)^2 \Leftrightarrow$$

$$\Leftrightarrow 0 < \sum_{i=1}^{n} (a_i' - b_i)^2 - (a_i - b_i)^2 = \sum_{l=0}^{k-1} (j+l-b_{j+l})^2 - \left(j + \frac{k-1}{2} - b_{j+l}\right)^2 =$$

$$= \sum_{l=0}^{m} (j+l-b_{j+l})^2 - \left(j + \frac{k-1}{2} - b_{j+l}\right)^2 + ((j+k-1)-l-b_{j+k-1-l})^2 -$$

$$- \left(j + \frac{k-1}{2} - b_{j+k-1-l}\right)^2 =$$

$$= \sum_{l=0}^{m} 2\left(l - \frac{k-1}{2}\right)^2 + \left((j-b_{j+k-1-l}) - (j-b_{j+l})\right)\left((k-1)-2l)\right).$$

Since $0 < l < \frac{k-1}{2}$ and $0 < b_{j+k-1-l} \leq b_{j+l}$, we have $\bar{d}_2(R_1,R_2) < \bar{d}_2(R_3,R_2)$.

2. Case $\bar{d}_c$. Consider $\|R\| = \sqrt{a_1^2 + \cdots + a_n^2}$ whenever $R \equiv (a_1, \ldots, a_n)$.

$$\|R_3\|^2 - \|R_1\|^2 = \sum_{i=1}^{n} \left((a_i')^2 - (a_i)^2\right) =$$

$$= \sum_{l=0}^{m} (j+l)^2 - \left(j + \frac{k-1}{2}\right)^2 + (j+(k-1)-l)^2 - \left(j - \frac{k-1}{2}\right)^2 =$$

$$= 2\left(\frac{k-1}{2} - l\right)^2 > 0. \tag{1}$$

Thus, $\|R_3\| > \|R_1\|$.

$$\bar{d}_c(R_3,R_2) - \bar{d}_c(R_1,R_2) = \frac{\sum_{i=1}^n (a_i\, b_i)}{\|R_1\|\, \|R_2\|} - \frac{\sum_{i=1}^n (a_i'\, b_i)}{\|R_3\|\, \|R_2\|} \overset{\text{by }(1)}{>}$$

$$> \frac{1}{\|R_1\|\, \|R_2\|} \left(\sum_{i=0}^{n} \left(a_i\, b_i - a_i'\, b_i\right) \right) =$$

$$= \frac{1}{\|R_1\|\, \|R_2\|} \left(\sum_{l=0}^{m} \left(\left(j + \frac{k-1}{2} \right) b_{j+l} - (j+l) b_{j+l} + \right.\right.$$

$$\left.\left. + \left(j + \frac{k-1}{2} \right) b_{j+k-1-l} - (j+k-1+l) b_{j+k-1-l} \right) \right) =$$

$$= \frac{1}{\|R_1\|\, \|R_2\|} \left(\sum_{l=0}^{m} \left(b_{j+l} \left(\frac{k-1}{2} + l \right) + b_{j+k-1-l} \left(l - \frac{k-1}{2} \right) \right) \right) \geq$$

$$\geq \frac{1}{\|R_1\|\, \|R_2\|} \left(\sum_{l=0}^{m} b_{j+k-1-l}\, 2l \right) \geq 0.$$

Thus, $\bar{d}_c(R_1,R_2) < \bar{d}_c(R_3,R_2)$.

3. <u>Case $\bar{d}_H$.</u>

$$\bar{d}_H(R_1,R_2) < \bar{d}_H(R_3,R_2) \ \Leftrightarrow\ 0 < \left(\bar{d}_H(R_3,R_2)\right)^2 - \left(\bar{d}_H(R_1,R_2)\right)^2 \ \Leftrightarrow$$

$$\Leftrightarrow\ 0 < \sum_{i=1}^{n} \left(\sqrt{a_i'} - \sqrt{b_i} \right)^2 - \left(\sqrt{a_i} - \sqrt{b_i} \right)^2 = \sum_{l=0}^{m} \left(\sqrt{j+l} - \sqrt{b_{j+l}} \right)^2 -$$

$$- \left(\sqrt{j + \frac{k-1}{2}} - \sqrt{b_{j+l}} \right)^2 + \left(\sqrt{(j+k-1)-l} - \sqrt{b_{j+k-1-l}} \right)^2 -$$

$$- \left(\sqrt{j + \frac{k-1}{2}} - \sqrt{b_{j+k-1-l}} \right)^2 =$$

$$= 2 \sum_{l=0}^{m} \sqrt{b_{j+l}} \left(\sqrt{j + \frac{k-1}{2}} - \sqrt{j+l} \right) +$$

$$+ \sqrt{b_{j+k-1-l}} \left(\sqrt{j+k-1} - \sqrt{j+k-1-l} \right).$$

Since $0 < l < \frac{k-1}{2}$, we have $\bar{d}_H(R_1,R_2) < \bar{d}_H(R_3,R_2)$.

4. <u>Case $\bar{d}_K$.</u>

$$\bar{d}_K(R_1,R_2) = \sum_{\substack{i,h=1 \\ i<h}}^{n} \left| \operatorname{sgn}\,(a_i - a_h) - \operatorname{sgn}\,(b_i - b_h) \right|.$$

$$\bar{d}_K(R_3,R_2) = \sum_{\substack{i,h=1\\ i<h}}^{n} |\operatorname{sgn}(a'_i - a'_h) - \operatorname{sgn}(b_i - b_h)| \, .$$

$$|\operatorname{sgn}(a_i - a_h) - \operatorname{sgn}(b_i - b_h)| =$$

$$= \begin{cases} |\operatorname{sgn}(a'_i - a'_h) - \operatorname{sgn}(b_i - b_h)| \, , & \text{if } \{i,h\} \not\subseteq a_j(R), \\ |-\operatorname{sgn}(b_i - b_h)| < |\operatorname{sgn}(a'_i - a'_h) - \operatorname{sgn}(b_i - b_h)| \, , & \text{if } \{i,h\} \subseteq a_j(R). \end{cases}$$

Thus, $\bar{d}_K(R_1,R_2) < \bar{d}_K(R_3,R_2)$. $\square$

Proof of Lemma 3. Consider $R_1,R_2,R_3 \in L(X)$, $R_1 \equiv (a_1,\dots a_n)$, $R_2 \equiv (b_1,\dots b_n)$ and $R' \equiv (a'_1,\dots a'_n)$. By Proposition 1 and Lemma 1, and taking into account that all the considered distances are neutral (Remark 7), we can assume without loss of generality that $R_1 \equiv (1,2,\dots,n)$.

If $R_2 \neq R_1^{-1}$, then we consider $j = \min\{i \mid b_i \neq n-i+1\}$ and $k = j+l$ such that $b_k = n-j+1$. Let now $R_3 \equiv (b'_1,\dots,b'_n)$ such that $b'_i = b_j$ for every $i \notin \{j,k\}$, $b'_j = b_k = n-j+1$ and $b'_k = b_j < n-j+1$.

1. Case $\bar{d}_2$.

$$\bar{d}_2(R_1,R_3)^2 - \bar{d}_2(R_1,R_2)^2 = |j-b'_j|^2 + |k-b'_k|^2 - \left(|j-b_j|^2 + |k-b_k|^2\right).$$

$$\begin{aligned} |j-b'_j|^2 + |k-b'_k|^2 &= (j+l-b_k-l)^2 + (j-b_j+l)^2 = \\ &= (k-b_k)^2 + (j-b_j)^2 + 2l(l+(j-b_j)-(k-b_k)) = \\ &= (j-b_j)^2 + (k-b_k)^2 + 2l(b_k-b_j) > |j-b_j|^2 + |k-b_k|^2. \end{aligned}$$

Thus, $\bar{d}_2(R_1,R_2) < \bar{d}_2(R_1,R_3)$.

2. Case $\bar{d}_c$.

It is clear that $\|R_2\| = \|R_3\|$.

$$\begin{aligned} \bar{d}_c(R_1,R_3) - \bar{d}_c(R_1,R_2) &= \frac{\sum_{i=1}^n i\,b_i}{\|R_1\|\,\|R_2\|} - \frac{\sum_{i=1}^n i\,b'_i}{\|R_1\|\,\|R_3\|} = \\ &= \frac{j\,b_j + k\,b_k - (j\,b'_j + k\,b'_k)}{\|R_1\|\,\|R_2\|} = \frac{j\,b_j + k\,b_k - (j\,b_k + k\,b_j)}{\|R_1\|\,\|R_2\|} = \\ &= \frac{(b_k - b_j)(k-j)}{\|R_1\|\,\|R_2\|} > 0. \end{aligned}$$

Thus, $\bar{d}_c(R_1,R_2) < \bar{d}_c(R_1,R_3)$.

3. Case $\bar{d}_H$.

$$\begin{aligned}
&\bar{d}_H(R_1,R_3)^2 - \bar{d}_H(R_1,R_2)^2 = \\
&= \left(\sqrt{j}-\sqrt{b'_j}\right)^2 + \left(\sqrt{k}-\sqrt{b'_k}\right)^2 - \left(\sqrt{j}-\sqrt{b_j}\right)^2 + \left(\sqrt{k}-\sqrt{b_k}\right)^2 = \\
&= \left(\sqrt{j}-\sqrt{b_k}\right)^2 + \left(\sqrt{k}-\sqrt{b_j}\right)^2 - \left(\sqrt{j}-\sqrt{b_j}\right)^2 + \left(\sqrt{k}-\sqrt{b_k}\right)^2 = \\
&= 2\left(\sqrt{jb_j}+\sqrt{kb_k}-\sqrt{jb_k}-\sqrt{kb_j}\right) = \\
&= 2\left(\left(\sqrt{k}-\sqrt{j}\right)-\left(\sqrt{b_k}-\sqrt{b_j}\right)\right) > 0.
\end{aligned}$$

Thus, $\bar{d}_H(R_1,R_2) < \bar{d}_H(R_1,R_3)$.

4. Case $\bar{d}_K$.

$$\begin{aligned}
&\bar{d}_K(R_1,R_3) - \bar{d}_K(R_1,R_2) = \\
&= |\,\mathrm{sgn}\,(j-k) - \mathrm{sgn}\,(b'_j - b'_k)| - |\,\mathrm{sgn}\,(j-k) - \mathrm{sgn}\,(b_j - b_k)| = \\
&= |\,\mathrm{sgn}\,(j-k) - \mathrm{sgn}\,(b_k - b_j)| - |\,\mathrm{sgn}\,(j-k) - \mathrm{sgn}\,(b_j - b_k)| = \\
&= |-1-1| - |-1-(-1)| = 2 > 0.
\end{aligned}$$

Thus, $\bar{d}_K(R_1,R_2) < \bar{d}_K(R_1,R_3)$. □

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