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Measuring Solid Angles Beyond Dimension Three

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Abstract. The dot product formula allows one to measure an angle determined by two vectors, and a formula known to Euler and Lagrange outputs the measure of a solid angle in $\mathbb{R}^3$ given its three spanning vectors. However, there appears to be no closed form expression for the measure of an n-dimensional solid angle for $n > 3$. We derive a multivariable (infinite) Taylor series expansion to measure a simplicial solid angle in terms of the inner products of its spanning vectors. We then analyze the domain of convergence of this hypergeometric series and show that it converges within the natural boundary for solid angles.

1. The Normalized Solid Angle Measure

There are a number of equivalent ways to extend the concept of planar angle measure to higher dimensions. When the solid angle Ω is a simplicial cone in $\mathbb{R}^3$ spanned by unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the *measure of solid angle* Ω is defined to be the area of the spherical triangle $\mathbf{abc}$ on the unit sphere (see Fig. 1). According to Eriksson [1], both Euler and Lagrange knew how to calculate this solid angle measure V_Ω via the triple product:

$$\tan\frac{V_\Omega}{2} = \frac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{1+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{c}}.$$

More generally, if K is a surface in $\mathbb{R}^n$ for which every ray from point C intersects K exactly once, then the *measure of the solid angle* formed by the solid cone of rays that emanate from C and intersect K is the volume of intersection of the solid cone and a unit sphere centered at C (see Fig. 2) [4]. We prefer to normalize the solid angle measure so that the cone of rays emanating from the origin (all of space) has solid angle measure 1. Write vol_n for the usual volume form in $\mathbb{R}^n$, B_n for the unit ball in $\mathbb{R}^n$, and S_n for the

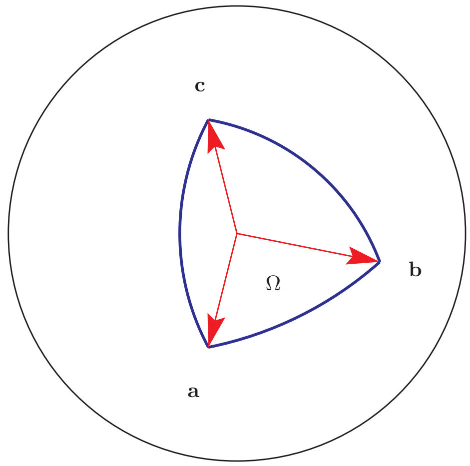

Fig. 1. The solid angle Ω in $\mathbb{R}^3$ spanned by three unit vectors.

unit n-sphere (residing in $\mathbb{R}^{n+1}$). The *normalized* measure of solid angle Ω is defined by

$$\tilde{V}_\Omega = \frac{\mathrm{vol}_n(\Omega \cap B_n)}{\mathrm{vol}_n\, B_n} = \frac{\mathrm{vol}_{n-1}(\Omega \cap S_{n-1})}{\mathrm{vol}_{n-1}\, S_{n-1}}.$$

The normalized measure of a solid angle may be computed using forms other than the volume form vol_n. Any form f invariant under rotations about the origin can be used to compute the normalized solid angle measure of Ω by

$$\tilde{V}_\Omega = \frac{\int_\Omega f\, d\mathbf{x}}{\int_{\mathbb{R}^n} f\, d\mathbf{x}} \tag{1}$$

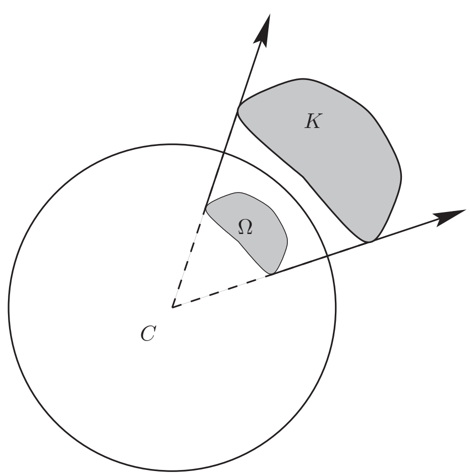

Fig. 2. The solid angle Ω described by a surface K.

as long as the denominator is bounded. We use the form $f = e^{-r^2}$, where $r = |\mathbf{x}|$, because it additionally allows us to calculate certain integrals in the derivation of the series expansion for $\tilde{V}_\Omega$.

2. Derivation of Solid Angle Formula

We think of the simplicial solid angle $\Omega \subseteq \mathbb{R}^n$ as the unbounded cone spanned by the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$. For solid angles bounded by more than n extreme rays, compute the solid angle measure by first dissecting into simplicial cones. Without loss of generality, take each $\mathbf{v}_i$ to be a unit vector, and let

$$V = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

The matrix $V^T V$ has as its (i, j)-entry the inner product between $\mathbf{v}_i$ and $\mathbf{v}_j$, that is, the cosine of the angle between vectors $\mathbf{v}_i$ and $\mathbf{v}_j$. We consider the entries of $V^T V = (\alpha_{ij})$ as a multivariable $\boldsymbol{\alpha} = (\alpha_{12}, \ldots, \alpha_{1n}, \alpha_{23}, \ldots, \alpha_{n-1,n})$ in $\binom{n}{2}$ variables α_{ij}, $1 \le i < j \le n$. For a multiexponent $\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}$, we define $\boldsymbol{\alpha}^{\mathbf{a}} = \prod \alpha_{ij}^{a_{ij}}$. The notation α_{ji} for $j > i$ means the variable α_{ij}, similarly for the exponents a_{ji}. Hence, the term $\sum_{m \neq l} a_{lm}$ appearing in the expression of Lemma 2.1 means the sum over all terms in $\mathbf{a}$ in which l appears as either the first or second index.

Lemma 2.1. *Given a symmetric $n \times n$ matrix $A = (\alpha_{ij})$ with 1's on the diagonal and a column vector $\mathbf{u} = (u_1, \ldots, u_n)^T$, the coefficient of $\boldsymbol{\alpha}^{\mathbf{a}}$ in the multinomial series expansion of $e^{-\mathbf{u}^T A \mathbf{u}}$ is*

$$\left(\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{l=1}^{n} u_l^{\sum_{m \neq l} a_{lm}} \right) \cdot \prod_i e^{-u_i^2}.$$

Proof. For a symmetric $n \times n$ matrix $A = (\alpha_{ij})$ with 1's on the diagonal, we can separate the quadratic form

$$\mathbf{u}^T A \mathbf{u} = \sum_{1 \le i, j \le n} \alpha_{ij} u_i u_j$$

into the diagonal and off-diagonal terms to obtain

$$e^{-\mathbf{u}^T A \mathbf{u}} = e^{-\sum_{i<j} 2\alpha_{ij} u_i u_j - \sum_i u_i^2} = \prod_{i<j} \left(\sum_{k_{ij}=0}^{\infty} \frac{(-2\alpha_{ij} u_i u_j)^{k_{ij}}}{k_{ij}!} \right) \cdot \prod_i e^{-u_i^2}.$$

Extracting the coefficient of $\boldsymbol{\alpha}^{\mathbf{a}}$ in the previous expression and organizing this product by the variables u_i achieves the desired result:

$$\begin{aligned} e^{-\mathbf{u}^T A \mathbf{u}}|_{\boldsymbol{\alpha}^{\mathbf{a}}} &= \prod_{i<j}\left(\frac{(-2u_i u_j)^{a_{ij}}}{a_{ij}!}\right)\cdot\prod_i e^{-u_i^2} \\ &= \left(\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!}\prod_{i<j}(u_i u_j)^{a_{ij}}\right)\cdot\prod_i e^{-u_i^2} \\ &= \left(\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!}\prod_{l=1}^{n} u_l^{\sum_{m\neq l} a_{lm}}\right)\cdot\prod_i e^{-u_i^2}. \end{aligned}$$ □

Our main result gives an expression for measuring a simplicial solid angle in $\mathbb{R}^n$. It is the only formula we know for measuring solid angles beyond dimension three.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a solid angle spanned by unit vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, let V be the matrix whose ith column is $\mathbf{v}_i$, and let $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$, as above. Let T_{α} be the following infinite multivariable Taylor series*:

$$T_{\alpha} = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a}\in\mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!}\prod_i \Gamma\left(\frac{1+\sum_{m\neq i} a_{im}}{2}\right)\right]\boldsymbol{\alpha}^{\mathbf{a}}.$$

The series T_{α} agrees with $\tilde{V}_{\Omega}$, the normalized measure of solid angle Ω, wherever T_{α} converges.

Proof. We evaluate (1) with the form $f = e^{-r^2}$ to obtain

$$\tilde{V}_{\Omega} = \frac{\int_{\Omega} e^{-r^2} d\mathbf{x}}{\int_{\mathbb{R}^n} e^{-r^2} d\mathbf{x}} = \frac{\int_{\Omega} e^{-\mathbf{x}^T\mathbf{x}}\, d\mathbf{x}}{\pi^{n/2}}.$$

The numerator can be evaluated by means of a change of coordinates that takes each vector $\mathbf{v}_i$ to the unit vector $\mathbf{e}_i$. This transformation maps the region Ω to the positive orthant of $\mathbb{R}^n$ so that limits of integration are manageable:

$$\tilde{V}_{\Omega} = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n_{\geq 0}} e^{-\mathbf{u}^T V^T V \mathbf{u}} |\det V|\, d\mathbf{u}.$$

Bringing the Jacobian outside the integral sign and applying Lemma 2.1 gives

$$\begin{aligned} \tilde{V}_{\Omega} &= \frac{|\det V|}{\pi^{n/2}} \int_{\mathbb{R}^n_{\geq 0}} \prod_{i<j}\left(\sum_{k_{ij}=0}^{\infty} \frac{(-2\alpha_{ij} u_i u_j)^{k_{ij}}}{k_{ij}!}\right)\cdot\prod_i e^{-u_i^2}\, d\mathbf{u} \\ &= \frac{|\det V|}{\pi^{n/2}} \int_{\mathbb{R}^n_{\geq 0}} \sum_{\mathbf{a}\in\mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!}\prod_i u_i^{\sum_{m\neq i}^{n} a_{im}}\right]\prod_i e^{-u_i^2}\boldsymbol{\alpha}^{\mathbf{a}}\, d\mathbf{u}. \end{aligned}$$

Finally, moving the integral sign inside the summation and product and substituting the expression

$$\int_0^\infty x^k e^{-x^2}\,dx = \frac{\Gamma\left((k+1)/2\right)}{2}$$

for the Euler gamma function evaluated at half-integers yields the series T_α:

$$\begin{aligned}\tilde{V}_\Omega &= \frac{|\det V|}{\pi^{n/2}} \sum_{\mathbf{a}\in\mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod\limits_{i<j} a_{ij}!} \prod_i \int_0^\infty u_i^{\sum_{m\neq i} a_{im}} e^{-u_i^2}\,du_i \right] \boldsymbol{\alpha}^{\mathbf{a}} \\ &= \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a}\in\mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod\limits_{i<j} a_{ij}!} \prod_i \Gamma\left(\frac{1+\sum_{m\neq i} a_{im}}{2}\right)\right] \boldsymbol{\alpha}^{\mathbf{a}}.\end{aligned}$$

□

3. The Domain of Convergence of the Solid Angle Formula

The multivariable Taylor series T_α appearing in Theorem 2.2, divided by $|\det V|$, is an example of a *complete multivariable hypergeometric series with step size* 2. In this section we determine the domain of absolute convergence of $T_\alpha/|\det V|$ using what is known of the general theory of convergence for multivariable hypergeometric series. Namely, we employ a generalized version of Horn's theorem for double (two-variable) hypergeometric series.

Gauss gave the first systematic treatment of hypergeometric series in his 1812 thesis [6]. Gauss considered the single (complex) variable hypergeometric series

$$ {}_2\mathrm{F}_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\quad z\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \tag{2}$$

where $(\lambda)_n$ denotes the Pochammer symbol for rising factorial:

$$(\lambda)_n = \begin{cases} 1 & \text{if} \quad n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if} \quad n = 1, 2, \ldots. \end{cases}$$

By d'Alembert's ratio test the series (2) converges absolutely inside the unit circle where $|z| < 1$. Moreover, for the particular boundary value $z = 1$, Gauss's summation theorem gives

$$ {}_2\mathrm{F}_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\quad 1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

where $\mathrm{Re}(\gamma-\alpha-\beta) > 0$ and $\gamma \neq 0, -1, -2, \ldots$.

In his 1889 paper "Über die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen," Horn defined double hypergeometric series and described their domains of convergence [3]. Horn calls the two-variable power series

$$\sum_{a,b=0}^{\infty} A_{a,b} x^a y^b \tag{3}$$

hypergeometric if the two quotients

$$f_1(a,b) = \frac{A_{a+1,b}}{A_{a,b}} \quad \text{and} \quad f_2(a,b) = \frac{A_{a,b+1}}{A_{a,b}}$$

are rational functions of a and b. If $f_1(a,b)$ and $f_2(a,b)$ are ratios of polynomials of the same degree, then the double hypergeometric series is said to be *complete*. More generally we call the series

$$\sum_{a_1,\ldots,a_n=0}^{\infty} A_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n} \tag{4}$$

hypergeometric if the coefficients have the form

$$A_{a_1,\ldots,a_n} = \frac{\Gamma(l_1)\cdots\Gamma(l_s)}{\Gamma(m_1)\cdots\Gamma(m_t)},$$

where the l_i's and m_i's are rational linear combinations of the indices $a_1, \ldots, a_n$. If the l_i's and m_i's are integer linear combinations of $a_1, \ldots, a_n$, then the series (4) is hypergeometric in Horn's sense. If the l_i's and m_i's are rational with common denominator d, then we say the series (4) is *hypergeometric with step size d*. In this case the series can be written as a finite sum of series, each of which is a monomial times a hypergeometric series in Horn's sense in the variables x_i^d.

Given the double hypergeometric geometric series (3) and associated rational functions f_i, $i = 1, 2$, let

$$\Psi_i(a,b) = \lim_{\varepsilon \to 0} f_i(\varepsilon a, \varepsilon b).$$

For the complete hypergeometric series (3), the f_i are again rational functions of a and b. Let r_i denote the radii of convergence, meaning the series converges absolutely for $|x| < r_1$ and $|y| < r_2$ and diverges for $|x| > r_1$ and $|y| > r_2$. Then Horn's theorem describes the domain of absolute convergence.

Theorem 3.1 (Horn). *The complete double hypergeometric series*

$$\sum_{a,b=0}^{\infty} A_{a,b} x^a y^b$$

converges absolutely within the domain bounded by the coordinate axes, the lines $x = 1$ *and* $y = 1$, *and the curve defined parametrically by*

$$r_i = |\Psi_i(a,b)|^{-1}.$$

Horn's theorem generalizes to multivariable hypergeometric series [2], and we now apply his theorem to analyze the domain of convergence of the series $T_\alpha/|\det V|$. Since T_α is a series in the multivariable $\boldsymbol{\alpha} = (\alpha_{12}, \ldots, \alpha_{n-1,n})$ with $\binom{n}{2}$ terms, it is natural to employ the two term subscript ij, $1 \le i < j \le n$, to index all variables, parameters, exponents, coordinate functions, and radii of convergence in this context, and we take

$ji = ij$ for $j > i$. Let $M(d, x_{ij})$ denote the symmetric matrix with diagonal entries d and off-diagonal entry x_{ij}, $i \neq j$, in the ith row and jth column:

$$M(d, x_{ij}) = \begin{bmatrix} d & & x_{ij} \\ & \ddots & \\ & & d \end{bmatrix}.$$

We also adopt the shorthand notation $s_i = \sum_{j \neq i} a_{ij}$. It may help to think of s_i as the ith row (or column) sum of the symmetric exponent matrix $M(0, a_{ij})$.

Consider the $n \times n$ matrix $M(1, -x_{ij})$, where $\{x_{ij}\colon\ 1 \leq i < j \leq n\}$ is the set of coordinate functions on $\mathbb{R}^{\binom{n}{2}}$. The polynomial equation $\det M(1, -x_{ij}) = 0$ describes an affine hypersurface in $\mathbb{R}^{\binom{n}{2}}$, and we claim this hypersurface bounds the domain of convergence for the series $T_{\boldsymbol{\alpha}}/|\det V|$.

Theorem 3.2. *The series $T_{\boldsymbol{\alpha}}$ converges absolutely to $\tilde{V}_\Omega$ in the domain $\{\boldsymbol{\alpha}^{\mathbf{a}}\colon\ |\alpha_{ij}| < r_{ij}\}$, where the $r_{ij} \in \mathbb{R}_{\geq 0}$ satisfy* $\det M(1, -r_{ij}) = 0$.

Proof. For the purposes of analyzing convergence, we ignore the constant term appearing before the summation sign of $T_{\boldsymbol{\alpha}}/|\det V|$ and consider only the series part

$$\left(\sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} A_{\mathbf{a}} \boldsymbol{\alpha}^{\mathbf{a}}\right) / |\det V| = \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_i \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2}\right) \right] \boldsymbol{\alpha}^{\mathbf{a}}.$$

Step 1: *In the manner of* [3], *we compute the parameters r_{ij}*. To ease the strain of notation, we compute r_{12} and use symmetry to obtain r_{ij} in general. Stirling's formula implies that for any real parameter $t \to \infty$,

$$\frac{\Gamma(1 + t/2)}{\Gamma((1 + t)/2)} \sim \sqrt{\frac{t}{2}},$$

so we have

$$f_{12}(\mathbf{a}) = \frac{A_{a_{12}+1, a_{13}, \ldots, a_{n-1,n}}}{A_{a_{12}, a_{13}, \ldots, a_{n-1,n}}} = \frac{-2}{(a_{12} + 1)} \frac{\Gamma\left(1 + s_1/2\right) \Gamma\left(1 + s_2/2\right)}{\Gamma\left((1 + s_1)/2\right) \Gamma\left((1 + s_2)/2\right)} \sim -\frac{\sqrt{s_1 s_2}}{a_{12} + 1}.$$

The function

$$\Psi_{ij}(\mathbf{a}) = \lim_{\varepsilon \to \infty} f_{ij}(\varepsilon \mathbf{a}) = -\frac{\sqrt{s_i s_j}}{a_{ij}}$$

keeps only the highest multidegree terms of f_{ij}. Hence,

$$r_{ij} = |\Psi_{ij}(\mathbf{a})|^{-1} = \frac{a_{ij}}{\sqrt{s_i s_j}}.$$

Step 2: *Eliminate the parameters $\{a_{ij}\}$ by finding an equation satisfied by $\{r_{ij}\}$*. We show that $\det M(1, -r_{ij}) = 0$ is the equation we seek. By multilinearity of the determinant function, we can pull a factor of $1/\sqrt{s_i}$ out of the ith row for each i, $1 \le i \le n$; likewise for the columns. Therefore,

$$\det M(1, -r_{ij}) = \det M\left(\frac{s_i}{\sqrt{s_i s_i}}, \frac{-a_{ij}}{\sqrt{s_i s_j}}\right) = \frac{\det M(s_i, -a_{ij})}{s_1 s_2 \cdots s_n}.$$

As all the row sums of $M(s_i, -a_{ij})$ equal zero, $M(1, -r_{ij})$ has determinant zero. □

In principle, Horn's theorem may involve a bounding surface described by the equations $\{r_{ij} = |\Psi(\mathbf{a})|^{-1}\}$ for every subset of the multivariable indices. These extra conditions arise from setting some subset of the variables $\{\alpha_{ij}\}$ equal to zero since, if some $\alpha_{pq} = 0$, then only terms where $a_{pq} = 0$ contribute a nonzero summand. However, in our application the radii of convergence in (3) only grow larger when some of the a_{pq}'s other than a_{ij} are zero, so the extra boundary conditions are redundant.

The boundary for the domain of convergence of the solid angle formula, as described in Theorem 3.2, coincides with the natural boundary of the solid angle $\Omega \subseteq \mathbb{R}^n$ in the following sense. When the affine span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ determines a linear hyperplane but not all of $\mathbb{R}^n$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ describes a degenerate solid angle Ω that is a half-space of $\mathbb{R}^n$. When this occurs, $|\det V| = 0$ so that $T_\alpha/|\det V|$ is undefined. For obtuse solid angles where $\alpha_{ij} < 0$ and $|\det V| > 0$, the series T_α converges to a well-defined normalized solid angle measure $\tilde{V}_\Omega$. As each of the angles between $\mathbf{v}_i$ and $\mathbf{v}_j$ are simultaneously increased, $|\det V| \to 0$, $T_\alpha/|\det V| \to \infty$, and $\boldsymbol{\alpha}$ moves toward the boundary of the domain of convergence of $T_\alpha/|\det V|$. For an arbitrary solid angle the condition for convergence of the series T_α depends only on the absolute values of the α_{ij}'s. The preceding remarks and Theorem 3.2 imply the following corollary for testing convergence.

Corollary 3.3. *The series T_α converges to $\tilde{V}_\Omega$ if and only if the matrix $M(1, -|\alpha_{ij}|)$ is positive definite.*

4. Computing with the Solid Angle Formula

We have enjoyed successes and met obstacles in trying to apply the multivariable Taylor series expansion to solid angles beyond dimension three. The original motivation for having such a formula was to be able to compute the expected number of simplices occurring in a random regular triangulation of the n-cube. Our technique requires computing solid angle measures in the Gale transform space of the n-cube. In the case of the 3-cube, the solid angle formula can be used to compute the measures of the solid angles occurring in its four-dimensional Gale transform space. Moreover, the symmetry of the distinguished Gale transform vectors in this space can be used to verify the calculation and assess the rate of convergence. See [5] for a more detailed discussion of applying the solid angle formula to triangulate the n-cube efficiently.

While the solid angle formula converges for solid angles in the Gale transform space of the 3-cube, two issues hinder computing solid angles in the Gale transform spaces of larger polytopes. One quickly recognizes the issue of computational feasibility. The input angle $\boldsymbol{\alpha}$ for an n-dimensional solid angle requires $\binom{n}{2}$ coordinates. In the case of the

Gale transform space of the 4-cube, $\binom{11}{2} = 55$ coordinates are required. While the series expansion can be simplified since the inner products take on only a few different cosine values, accurate series approximations will require theorems allowing us to reduce the number of terms that need computing.

The issue of convergence seems to be a more significant obstacle. The inner products that arise between distinguished vectors in the Gale transform space of a polytope often lie outside the domain of convergence of the solid angle formula. Perhaps some notion of analytic continuation might rescue the solid angle formula in such cases. Another idea worth exploring may be whether solid angles lying outside the domain of convergence can be systematically subdivided into solid angles lying within the domain of convergence.

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