



Mechanical analysis of a flat, deformable layer underlain by a rigid foundation and indented by a paraboloidal punch

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ABSTRACT

The deflections are analytically studied in a flat, deformable layer, firmly anchored to a rigid substrate, and frictionlessly indented by a rigid paraboloidal punch, having in mind the design of hip replacements possessing elastomeric layers. An existing perturbation solution of differential character, developed for a layer subject to plane strain conditions and indented by a rigid cylinder, and valid for high contact widths, is here extended to the analogous axisymmetric problem of a layer compressed by a rigid paraboloidal punch approximating a spherical indenter. Perturbed solutions up to the second order are obtained, and the contact pressure profiles activated by a paraboloidal punch are derived for both compressible and incompressible layers. Numerical examples exploring the sensitivity of the stress field to perturbations of the Poisson's ratio are also included.

INTRODUCTION

This paper deals with the deflections of a flat, deformable layer, firmly bonded to a rigid substrate and frictionlessly compressed by a rigid paraboloidal indenter, Fig. 1. This axisymmetric study is relevant in the design of hip replacements whose cup is covered with an elastomeric layer, and compressed by a ball replacing the femoral head, [1]. An analytical perturbation solution of differential character, developed in [2] for a layer subject to plane strain conditions and indented by a rigid cylinder, valid for high contact widths, is here extended to the axisymmetric problem of a layer compressed by a rigid paraboloidal punch approximating a spherical indenter.

This paper is organized as follows. First, the equations describing the layer deflections as functions of an imposed axisymmetric pressure profile are developed up to the second perturbation order. A subsequent section solves the more relevant situation where the



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indenting profile is known, and the contact pressure has to be determined. Numerical examples exploring the sensitivity of the solution to perturbations of the Poisson's ratio end the paper.

THE PERTURBED PRESSURE-DEFLECTION SOLUTION

Following Armstrong [2], the subsequent steps are developed: a) the axisymmetric equilibrium equations are expressed in terms of the displacement field; b) the equilibrium equations are normalized with respect to proper variables, and a small parameter ϵ is identified, which renders this problem amenable to perturbation solution techniques; c) an approximate solution is achieved for parameter ϵ up to the second order, which connects the pressure (as well as its derivatives) acting upon an elastomeric layer to its local deflection. The axisymmetric equilibrium equations in terms of stresses are [3] :

$$\begin{aligned} \mu \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r + \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \quad (1)$$

By introducing into (1) the Hooke law linking stresses to strains, expressed in cylindrical coordinates in terms of radial, u , and axial, w , displacements, [3], the equilibrium equations are formulated in terms of the displacement field as :

$$\mu \frac{\partial^2 u}{\partial z^2} + (\lambda + \mu) \frac{\partial^2 w}{\partial r \partial z} + (\lambda + 2\mu) \frac{\partial^2 u}{\partial r^2} + (\lambda + 2\mu) \frac{1}{r} \frac{\partial u}{\partial r} - (\lambda + 2\mu) \frac{u}{r^2} = 0 \quad (2)$$

$$(\lambda + 2\mu) \frac{\partial^2 w}{\partial z^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial r \partial z} + \mu \frac{\partial^2 w}{\partial r^2} + (\lambda + \mu) \frac{1}{r} \frac{\partial u}{\partial z} + \mu \frac{1}{r} \frac{\partial w}{\partial r} = 0$$

where λ and μ are the so called Lamé constants, [3]. In elastomeric materials, the Poisson's ratio ν approaches the incompressibility value 0.5. In this case, $\lambda = \infty$ and $\mu = E/3$. For a realistic figure of $\nu = 0.499$, [1], then $\lambda = 166.444 E$, $\mu = 0.333 E$, that is, $\lambda \approx 500 \mu$.

The boundary conditions must represent the following aspects : a) the layer is firmly bonded to a rigid substrate; b) due to the smallness of the frictional coefficient in the presence of the synovial fluid, only a normal pressure affects the upper boundary of the layer :

$$\begin{aligned} \text{for } z = 0 \quad & \sigma_z = -p(r); \quad \tau_{rz} = 0 \\ \text{for } z = h \quad & u = 0; \quad w = 0 \end{aligned} \quad (3)$$

where $p(r)$ is the applied pressure and h is the (supposed constant) layer thickness.

According to [2], equations (2) are normalized by introducing the following variables :

$$U = \frac{u}{h}; \quad W = \frac{w}{h}; \quad S = \frac{r}{a}; \quad Z = \frac{z}{h} \quad (4)$$

where a is the contact radius. Equations (2) thus become :

$$\frac{\partial^2 U}{\partial Z^2} + \epsilon \frac{\lambda + \mu}{\mu} \frac{\partial^2 W}{\partial S \partial Z} + \epsilon^2 \frac{\lambda + 2\mu}{\mu} \left\{ \frac{\partial^2 U}{\partial S^2} + \frac{1}{S} \frac{\partial U}{\partial S} - \frac{U}{S^2} \right\} = 0 \quad (5)$$

$$\frac{\partial^2 W}{\partial Z^2} + \epsilon \frac{\lambda + \mu}{\lambda + 2\mu} \left\{ \frac{\partial^2 U}{\partial S \partial Z} + \frac{1}{r} \frac{\partial U}{\partial Z} \right\} + \epsilon^2 \frac{\mu}{\lambda + 2\mu} \left\{ \frac{\partial^2 W}{\partial S^2} + \frac{1}{S} \frac{\partial W}{\partial S} \right\} = 0$$

where $\epsilon = h/a$ is small provided that the contact radius is high. Armstrong, [2], develops an approximate solution to this problem by a perturbation method, [4]. The dimensionless radial, U , and axial, W , displacements are expressed via a series of functions in powers of ϵ :

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots; \quad W = W_0 + \epsilon W_1 + \epsilon^2 W_2 + \dots \quad (6)$$

Expressions (6) are then substituted into (5), and terms in the same power of ϵ are collected, so that the initial problem is split into subproblems. Due to lack of space, only the salient results are reported. The terms of 0 order (i.e., multiplying ϵ^0) supply the following solution :

$$U_0 = 0; \quad W_0 = P(S) (1 - Z); \quad P(S) = \frac{P(r)}{\lambda + 2\mu} \quad (7)$$

The Winkler-type, zero order approximate solution (7) exhibits no radial displacements u of the layer. If the elastomer is ideally incompressible, then $\lambda = \infty$, μ is finite, and w too vanishes. This is an unrealistic result, since an incompressible layer, when indented by a rigid sphere, would still flow radially and, therefore, it would not fully prevent any axial movements of the rigid indenter. Since this behaviour is not simulated by the zero order solution, higher ϵ orders must be accounted for especially in the case of incompressible materials. In particular, the first order ϵ solution is :

$$U_1 = -\frac{dP}{dS} \left\{ \frac{\lambda - \mu}{2\mu} + Z - \frac{\lambda + \mu}{2\mu} Z^2 \right\}; \quad W_1 = 0 \quad (8)$$

With respect to the zero order solution (7), the up to the first order expressions exhibit the same axial displacement function $W_0 + \epsilon W_1$ (it still vanishes for incompressible elastomers), whereas a non vanishing radial displacement formula $U_0 + \epsilon U_1$ is achieved. The ϵ second order solution is :

$$U_2 = 0 \quad (9)$$

$$W_2 = \left\{ \frac{d^2 P}{dS^2} + \frac{1}{S} \frac{dP}{dS} \right\} \left\{ -\frac{\lambda - \mu}{\lambda + 2\mu} \frac{\lambda}{3\mu} + \frac{\lambda - \mu}{\lambda + 2\mu} \frac{\lambda}{2\mu} Z + \frac{\lambda}{2(\lambda + 2\mu)} Z^2 - \frac{\lambda}{6\mu} Z^3 \right\}$$

Finally, for an incompressible material, $\lambda = \infty$, and the dimensional expression for $w(r, z)$ simplifies to :



$$w(r, z) = -\frac{h^3}{6\mu} \left(\frac{d^2 p(r)}{dr^2} + \frac{1}{r} \frac{d p(r)}{dr} \right) \left\{ 2 - 3 \frac{z}{h} + \frac{z^3}{h^3} \right\} \quad (10)$$

which, contrary to the axial displacement of (7), does not necessarily vanish for incompressible materials.

THE PRESSURE PROFILE FOR A PARABOLOIDAL INDENTER

Once the axisymmetric pressure distribution, $p(r)$, is known, equations (7), (9) or (10) permit the axial deflections of the layer upper boundary ($z=0$) to be computed with various precisions. The practically more relevant situation is now treated of an imposed displacement function, where the pressure profile is the unknown. In particular, three axisymmetric cases are treated, all referring to a rigid paraboloidal indenter : a) a compressible layer studied via the zero order equation (7) ; b) an incompressible layer analyzed through the second order expression (10); c) a compressible layer described via (9). As usual, [4], the imposed displacement is approximated by a second degree (paraboloidal) expression :

$$w(r, 0) = \delta - \frac{r^2}{2R} \quad ; \quad \frac{1}{R} = \frac{1}{R_s} - \frac{1}{R_t} \quad (11)$$

where δ denotes the rigid body axial displacement (that is, the indentation depth) and R is the equivalent radius of curvature of the two contacting surfaces, namely of the indenting sphere of radius R_s , and of the upper surface ($z=0$) of the elastomeric layer, of radius R_t . A note of caution is here introduced on the accuracy of (11) in describing very high contact radii. Anyway, the influence of the indenter profile (paraboloidal or spherical) does not appear to be generally very relevant, as the numerical forecasts of [1] indicate.

The zero order expression of the pressure profile as a function of the peak pressure, p_0 , and of the normalized S coordinate is :

$$p(r) = p_0 (1 - S^2) \quad ; \quad p_0 = \frac{(\lambda + 2\mu)\delta}{h} \quad ; \quad a = \sqrt{2R\delta} \quad (12)$$

This Winkler solution coincides with formula (5.73) of Johnson, [5], valid for plane contacts.

The incompressible case according to (10) furnishes :

$$p(r) = \frac{3\mu\alpha^2\delta}{8h^3} \left(\frac{r^4}{a^4} - 2\frac{r^2}{a^2} + 1 \right) = p_0 (1 - S^2)^2$$

$$p_0 = \frac{3\mu\alpha^2\delta}{8h^3} \quad ; \quad a = 2\sqrt{R\delta} \quad (13)$$

Formula (13) differs from (5.75) of [5], valid for plane contacts.

Finally, the second order complete solution in dimensional form

for the case of a paraboloidal indenter is expressible in terms of the Bessel function I_0 :

$$\frac{p(r)}{\lambda + 2\mu} = \frac{I_0\left(\frac{r}{h} \sqrt{\frac{3\mu}{\lambda} \frac{\lambda + 2\mu}{\lambda - \mu}}\right)}{I_0\left(\frac{a}{h} \sqrt{\frac{3\mu}{\lambda} \frac{\lambda + 2\mu}{\lambda - \mu}}\right)} \left(\frac{a^2}{2hR} - \frac{\delta}{h} + \frac{2}{3} \frac{\lambda}{\mu} \frac{\lambda - \mu}{\lambda + 2\mu} \frac{h}{R} \right) + \frac{\delta}{h} - \frac{r^2}{2hR} - \frac{2}{3} \frac{\lambda}{\mu} \frac{\lambda - \mu}{\lambda + 2\mu} \frac{h}{R} \quad (14)$$

where the following equation involving I_0 and I_1 determines the value of a :

$$\frac{I_1(\alpha)}{I_0(\alpha)} \sqrt{\frac{3\mu}{\lambda} \frac{\lambda + 2\mu}{\lambda - \mu}} \left(\frac{4}{3} \frac{\lambda}{\mu} \frac{\lambda - \mu}{\lambda + 2\mu} \frac{h}{a} + \frac{a}{h} - 2 \frac{\delta R}{a h} \right) - 2 = 0 \quad (15)$$

The maximum contact pressure is obtained from (15) for $r = 0$. Due to lack of space, a critical discussion on the applied boundary conditions is omitted.

NUMERICAL RESULTS

Two different kinds of diagrams comprise this section. First, the validity fields of compressible and incompressible solutions are numerically explored for a realistic configuration. Secondly, various wide spectrum diagrams reporting the normalized peak contact pressure, the maximum shear stress at the layer-backing interface, and the contact radius, versus the non-dimensional indentation depth are presented for a selection of Poisson's ratios.

Turning to the first kind of diagrams, expressions (12) (Winkler solution) and (13) (incompressible solution) are compared to (14) (complete second order solution) for various Poisson's ratios, in order to define their validity fields. The following realistic values have been selected, [1]: $E = 3.52 \text{ MPa}$, $h = 3 \text{ mm}$, $R = 4000 \text{ mm}$, $\delta = 0.3 \text{ mm}$. Fig. 2 shows that the maximum normalized contact pressure according to a Winkler model approaches the second order complete solution (here assumed as a benchmark) for Poisson's ratios lower than 0.48 . The incompressible idealization holds true when $\nu \geq 0.4999$. Unfortunately, there is a sizeable ν interval ($0.48 \leq \nu \leq 0.4999$) including most of the physical values of ν for elastomers, [1], for which neither the Winkler modelling nor the incompressible idealization supply results sufficiently close to the second order complete solution, so that they are not valid for the above ν interval. There is also a need to assess the accuracy of the complete second order solution by accounting for more perturbation terms, an aspect which is deferred to a future paper. Anyway, the accuracy of the second order complete perturbed solution for intermediate Poisson's ratios has been proved in [1] for a plane situation.



The second kind of diagrams includes results retrieved from the complete second order perturbed solution (14), namely the normalized variables $pR/(Ea)$ (where p denotes the peak contact pressure), $\tau R/(Ea)$ (where τ indicates the maximum shear stress at the interface between layer and backing, a relevant parameter in forecasting debonding phenomena), and a/h , versus the normalized indentation depth, $\delta R/h^2$, up to 200 (a realistic top value for hip joints, [1]), and for $\nu = 0.4999$; 0.499 ; 0.48 . The asymptotic incompressible results (13) as well as the Winkler findings (12) are included where pertinent.

By comparing Fig. 3 to 5, it appears that, when $\nu = 0.48$, the Winkler approach produces acceptable results in terms of peak contact pressure, whereas for $\nu = 0.4999$ the incompressible solution becomes accurate. It clearly emerges that the contact pressure for an imposed indentation depth is particularly sensitive to perturbations of the Poisson's ratio, especially when ν approaches the incompressibility figure 0.5 and for high contact widths. Figs 6-8 show that the peak contact pressure normally ranges between $1/6$ and $1/20$ times the maximum interface shear stress. Fig. 9 details this ratio as a function of the indentation depth, for $\nu = 0.4999$ and 0.49 . From Figs 10-12 it appears that the contact width is less sensitive to perturbations of ν .

For small contacts, the precision of these perturbed solutions is expected to decline. In [1] it was found that, for plane cases, these asymptotic solutions are applicable for a/h ratios beyond, say, 10, a figure which is comparable to the maximum a/h values encountered in a hip replacement with elastomeric layer subject to normal gait loads.

CONCLUSIONS

A perturbation solution valid for high contact widths has been developed up to the second order for the case of a deformable layer firmly bonded to a rigid substrate and indented by a paraboloidal punch. Winkler-type compressible, as well as incompressible solutions have been derived as particular cases. They have been found to be valid for $\nu < 0.48$ and $\nu > 0.4999$, respectively. More generally, the contact pressure for an imposed penetration depth has been shown to be very sensitive to perturbations of the Poisson's ratio especially when the following situations occur simultaneously: a) the Poisson's ratio is close to its incompressibility figure 0.5; b) the contact width is considerably larger than the layer thickness.

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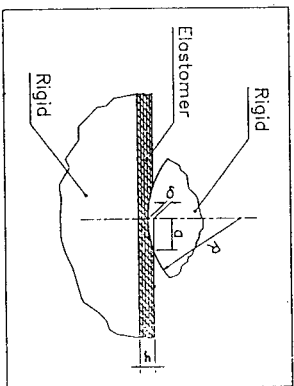


Fig. 1

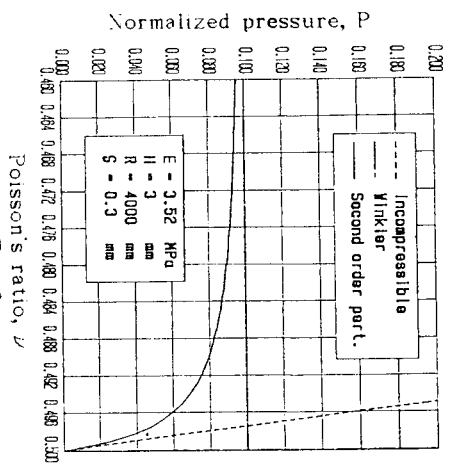


Fig. 2

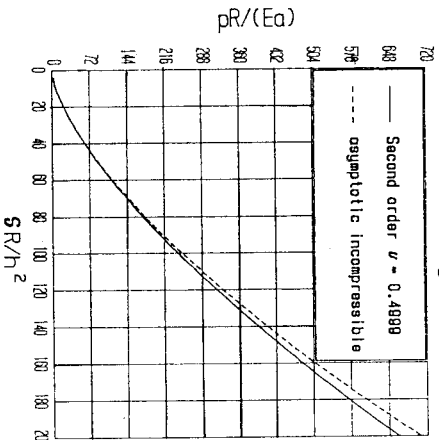


Fig. 3

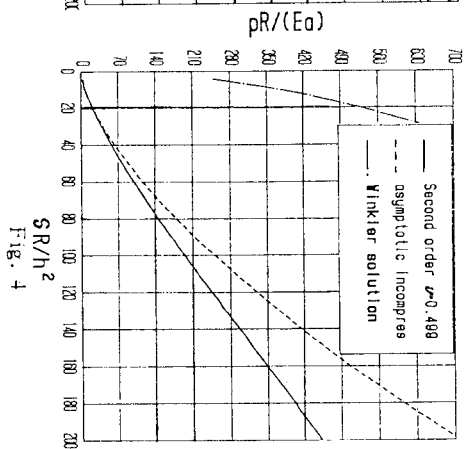


Fig. 4

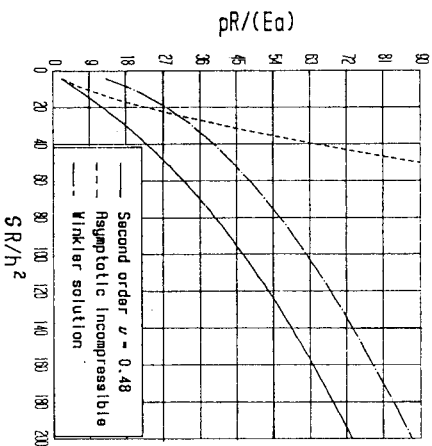


Fig. 5

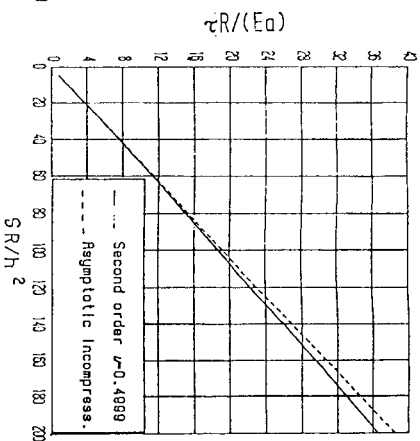


Fig. 6

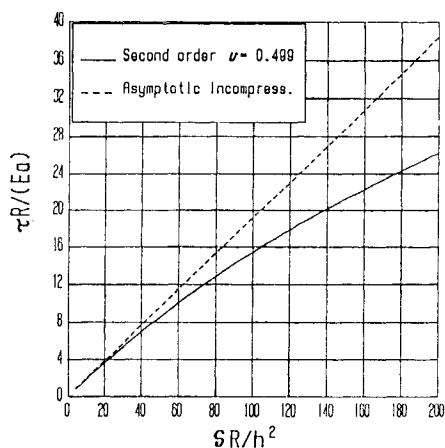


Fig. 7

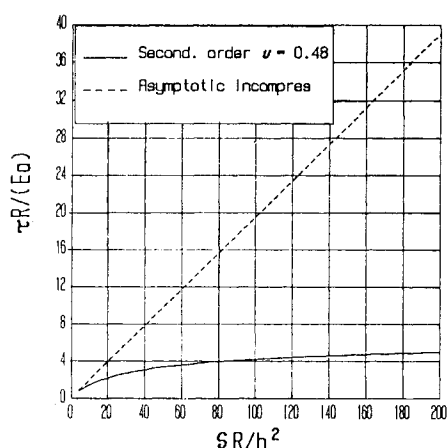


Fig. 8

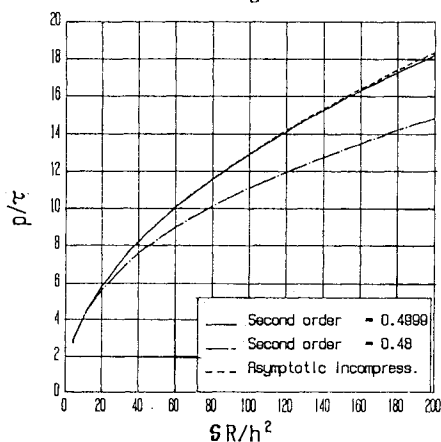


Fig. 9

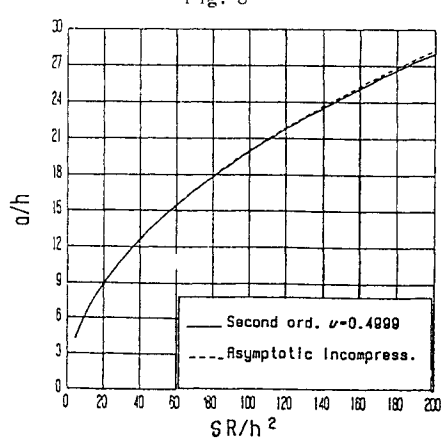


Fig. 10

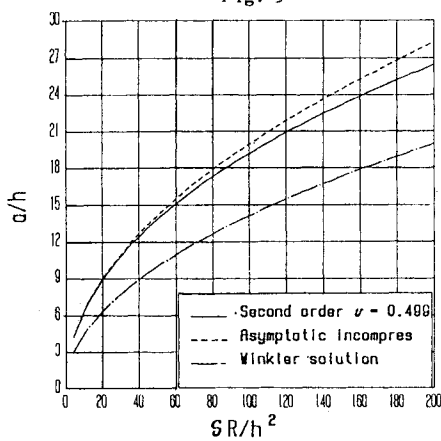


Fig. 11

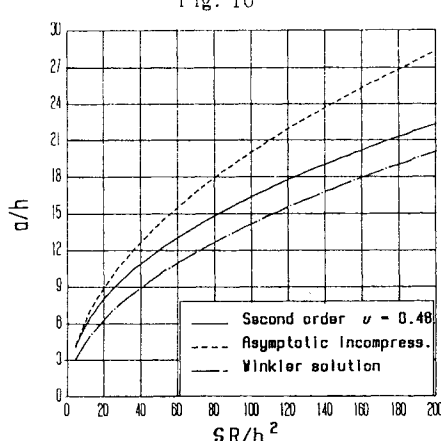


Fig. 12