MECHANICAL PROPERTIES OF A LAMINATE DESIGNED TO BE ISOTROPIC

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MECHANICAL PROPERTIES OF A LAMINATE DESIGNED TO BE ISOTROPIC[1]

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Summary

This report presents the results of 36 compression tests and 24 flexural tests of a laminated panel made of orthotropic laminations but designed to be isotropic in the plane of the laminate. The panel was made of a glass-fabric and a polyester resin. Tests at various angles of loading showed that the laminate was isotropic in both elastic and strength properties.

The mathematical development leading to the design of the test laminate is given in the appendix. Also included are equations for predicting the elastic and strength properties of isotropic laminates from the properties of the individual laminations.

Introduction

Mathematical analysis shows that it is possible to make an isotropic material from orthotropic components by relatively simple orientation of the component materials. Materials such as glass fabrics are highly orthotropic in

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[2]Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

nature, and laminates made with parallel or perpendicular orientation of the fabric vary greatly in their directional mechanical properties. In some applications of reinforced laminates, it is desirable to have uniform properties in the plane of the laminate; that is, an isotropic panel.

This study was conducted at the Forest Products Laboratory during 1952, in cooperation with the ANC-17 Panel on Plastics for Aircraft. A single laminated panel was fabricated, and compression and bending tests were made to verify the mathematical analysis. This report presents the mathematical development and the test results.

Isotropic Laminate Defined

The word isotropic, in the usual sense of the word, implies the same properties in all directions. This report deals only with the properties in the plane of the laminate or plate, and the properties in the direction normal to the surface are not necessarily the same. Isotropic, then, as discussed in this report, applies only to the properties in the plane of the panel.

Mathematical Derivation

The complete mathematical development is given in the appendix, and only a discussion of the equations pertaining directly to the isotropic laminate will be made in the main body of this report. The development is divided into two main parts: (1) elastic properties, and (2) strength properties.

Analysis of the elastic properties show that if orthotropic laminations are laid up at an angle θ to each other, so that $\theta = \frac{\pi}{n}$ (where $\underline{n}$ is an integer greater than 2), and there are an equal number of laminations in each direction, then the material is elastically isotropic. If $\underline{n}$ is equal to 3, then $\theta = 60°$; if $\underline{n}$ is equal to 4, $\theta = 45°$; and so on. The equations that show this isotropic relationship are equations (11), (12), and (14) of the appendix. It will be noted that the elastic properties of the isotropic laminate are independent of the angle $(\frac{\pi}{n})$ chosen.

Analysis of the strength properties takes into account the restraint imposed on each of the individual laminations by the adjacent ones. In the final equations, equation (18) for a direct stress and equation (21) for shear, two or more values of a strength will be obtained. If progressive failure took place immediately after the weakest laminae reached their theoretical strength, then the lowest calculated value would be the strength value to use. Experience with materials has shown, however, that this is not usually so because of the redistribution of stress. The theoretical strength of the isotropic laminate would therefore be expected to lie between the minimum and average of these values. Actually, these theoretical strength values are approximations, since they assume that the elastic relationships hold until the material fails.

Description of Material

Since the angles between individual laminations of an isotropic panel must be $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{5}$, or smaller, the first combination, $\theta = \frac{\pi}{3} = 60°$, was selected for this study. Such an assembly has the greatest angle between the individual laminations and would be expected to show any nonisotropic tendencies more readily than assemblies having smaller values of θ.

A single laminated panel was made up of 24 plies of 181-114 glass fabric and a typical resin of the polyester (styrene-alkyd) type, resin 2. The sheets of fabric were 38 inches square and were laid up at 60 degrees to each other, with an equal number of plies in each direction (fig. 1). The outer ply of each face was laid up at 0° to the abritrary axis, plies 2 and 23 at 60°, plies 3 and 22 at 120°, plies 4 and 21 at 0°, and so on, with the twelfth and thirteenth plies being at 120° and parallel to each other. The assembly was cured in a press at a pressure of 14 pounds per square inch for 90 minutes at a temperature gradually increasing from 220° to 250° F.

After pressing, the panel was trimmed, measured, and weighed, and the average values of thickness, resin content, and specific gravity were determined Barcol hardness readings were also made. This general information on the panel is included in table 1.

Methods of Test

All specimens were conditioned for 2 weeks prior to testing in an atmosphere at a temperature of 75° F. and a relative humidity of 50 percent.

Compression specimens were 1 inch wide, 4 inches long, and of the thickness of the laminate. They were loaded on their 1-inch ends and restrained from buckling by means of the apparatus illustrated in figure 2, which is described elsewhere.[3] Load was applied with a testing machine, employing a spherical head, at a head-speed of 0.012 inch per minute. Load-deformation readings were taken at regular increments of load until failure. Strain increments were measured over a 2-inch gage length with a pair of Martin's mirror gages reading to 0.00001 inch. The specimens failed suddenly when the maximum load was reached, by a combination of transverse shear and crushing that was sometimes followed by some delamination of the specimen.

Flexural specimens were 1/2 inch wide, 6 inches long, and of the thickness of the laminate, and were tested flatwise by center-loading over a span of 4 inches, figure 3. Contact edges of the end supports were of 1/8-inch radius, and the center loading piece had a radius of 3/8 inch. Load was applied at a head speed of 0.08 inch per minute, which corresponds to a unit rate of fiber strain of about 0.007 per inch of outer fiber length per minute. Deflection was measured to the nearest 0.001 inch with a dial gage having its spindle in contact with the bottom of the specimen at the center. Simultaneous readings of load and deflection were taken until the specimen failed at the maximum load. Slight compression failures were evident shortly before the maximum load was reached, but at maximum load the specimen failed markedly in the tension side.

Six compression specimens and four flexural specimens were tested at five angles, 90°, 82.5°, 75°, 67.5°, and 60°, to the warp direction of the outer plies. These angles represent nonsymmetrical ply orientation. Similar compression and flexural tests were made parallel to the warp direction of the outer ply to check the values at 60°, since the properties at 0° and 60° should be equal because of symmetry.

Presentation of Data

The average, minimum, and maximum values of the six compression specimens tested at each of the six angles are given in table 1. Results of the flexural tests are given in table 2.

Figure 1 is a sketch showing how the individual plies of fabric were oriented to make up the isotropic laminate. Figures 2 and 3 show the methods of test.

The mechanical properties from compression tests, from table 1, are plotted in figure 4. A value three times the standard deviation of each group of test values was computed and the range, added and subtracted from the average, is shown by the bracketed line.

Average properties from flexural tests are plotted in figure 5, based on the values from table 2.

[3] A. S. T. M. Designation D805-47, "Methods of Testing Plywood, Veneer, and Other Wood-base Materials." 1947.

Discussion of Results

In the panel tested, a direct stress applied at 0°, 60°, or 120° to the warp direction of the outer plies should result in the same mechanical properties because of symmetry. Similarly, the properties at 30°, 90°, and 150°, and at any other combination of angles 60° apart, should be the same because of symmetry. For this reason, five angles, spanning a total of 30°, were chosen to determine the properties of the panel and to substantiate the isotropic relationship. The tests at 0° were made to show the relationship at two symmetrical axes, namely, 0° and 60°.

From compression tests along the five nonsymmetrical axes, it may be seen that the values of modulus of elasticity and proportional limit values are in good agreement with the average. Modulus of elasticity values are within 1 percent of the average. Proportional limit values scatter somewhat, but this is to be expected since the values are difficult to determine. There seems to be some slight tendency for the compressive strength to vary with direction; yet the average of each group is within 4 percent of the composite average. Considering the variations found between individual specimens of reinforced plastic laminates, the correlation appears to be good. The value of three times the standard deviation of a group is sometimes considered as the range of variability and experimental error. From figure 4 it may be seen that the composite average is within the range of all group averages plus or minus three times the standard deviation, except for strength at 60°.

A comparison of the compressive properties at 60° and 0° shows excellent correlation at these two symmetrical axes.

Flexural tests are not a good indication of the isotropic properties unless there are a great number of plies. In other words, the direction of the outer plies might have a considerable effect in flexure, while under direct stress they would have no more effect than the inner plies. Four specimens were tested at each of the six angles. From table 2 and figure 3 it may be seen that there was very good correlation between the five nonsymmetrical axes. Also, the correlation between 60° and 0° was good.

Previous tests of 181-114 parallel laminates[4] showed very marked directional properties in compression, as well as in tension and shear. The tests made for this study show that the directional differences can be practically made to disappear by relatively simple orientation of the individual laminae.

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[4] Werren, Fred and Norris, C. B. Directional Properties of Glass-fabric-base Plastic Laminates of Sizes That Do Not Buckle. Forest Products Laboratory Report No. 1803. April 1949.

Comparison of Theoretical and Test Values

The method of lay-up for an isotropic laminate made of orthotropic laminations was found from the mathematical development. Experimental data from one panel made according to this development showed excellent agreement between the angles tested within the elastic range. For proportional limit and strength the agreement was also good. The mathematical development also shows methods of predicting properties of isotropic laminates based on the properties of comparable parallel laminates, and a discussion of these properties follows.

The mechanical properties used in calculating the theoretical values were taken from a previous report on parallel-laminated plastic laminates.[5] These 181-114 laminates were made by laminating techniques comparable to those used with the isotropic panel. Poisson's ratio values used for the orthotropic laminate were $\mu_{\alpha\beta} = 0.16$ and $\mu_{\beta\alpha} = 0.18$, based on tests in another study.[4]

By use of equation (14), the theoretical modulus of elasticity of the isotropic panel is 2.67×10^6 pounds per square inch. This is about 8 percent higher than the average modulus observed in test, which was 2.46×10^6 pounds per square inch. In going back to the data of the original report, however, it was noted that $E_0 = E_\alpha$ varied from 2.98×10^6 to 3.60×10^6 pounds per square inch and $E_{90} = E_\beta$ varied from 3.06×10^6 to 3.23×10^6 pounds per square inch. Considering the variation between the individual specimens, the averages of which were used in calculating the theoretical modulus of elasticity, the correlation between the theoretical and test values appears reasonable.

The theory, as developed, is applicable to direct stresses applied in the plane of the laminate. By inserting values of flexural moduli of elasticity from tests of the orthotropic material, however, an indication of the flexural modulus of elasticity for the isotropic laminate can be determined. By using average values for the 181-114 parallel laminate from a previous report[5] and inserting in equation (14), as above, the theoretical modulus of elasticity of the isotropic panel was 2.31×10^6 pounds per square inch. The observed average modulus of elasticity in flexure was 2.32×10^6 pounds per square inch, which is excellent correlation between theoretical and test values.

In the strength studies, again by using average values from a previous report,[5] the theoretical compressive strength, based on equation (4), lies somewhere between 28,450 and 36,500 pounds per square inch. The observed average was

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[5] Werren, Fred. Mechanical Properties of Plastic Laminates. Forest Products Laboratory Report No. 1820. February 1951.

33,460 pounds per square inch. Actually, it is believed that the strength values used[5] in arriving at the theoretical values are somewhat lower than would normally be expected. If this is so, it would appear that the actual strength of an isotropic laminate might be nearer the lower theoretical value than near the average.

If theoretical flexural strength is calculated from equation (4), the strength for the isotropic laminate lies between 46,100 and 49,600 pounds per square inch. The observed average value was 45,580 pounds per square inch, which is near the minimum theoretical value.

Conclusions

Mathematical development shows that it is possible to make a laminate that is isotropic in the plane of the laminate from orthotropic laminations, which may be done by relatively simple orientation of the laminae. Further, it shows that the elastic and strength properties of the isotropic material can be predicted from a knowledge of the properties of parallel laminates made of the same material.

Based on compression tests of a single laminate, employing 181-114 glass-fabric oriented at one of the possible orientations, the test results show that it is possible to make an elastically isotropic panel. The strength properties were also about the same at the various angles tested. Results of a few flexural tests also show that the test laminate was isotropic in both elastic and strength properties.

The elastic and strength properties from the compression tests were in reasonable agreement with the theoretical values.

No shear tests were made, so the mathematical development was not verified in this study.

Appendix I

The mathematical developments that follow are divided into (1) elastic properties and (2) strength properties of isotropic laminates made from orthotropic laminations. The primary purpose of this study was to prove that such laminates are isotropic in the plane of the laminate, by use of the equations for the elastic properties of orthotropic materials given in "Sandwich Constructions for Aircraft."[6] Also, the mathematics will show methods of predicting the elastic and strength properties of isotropic laminates from the properties of comparable parallel laminates.

The method of mathematical proof given here was suggested by W. S. Ericksen, mathematician, U. S. Forest Products Laboratory.

Elastic Properties

Consider a laminate made up of orthotropic laminae oriented so that the angle between the natural axes of adjacent laminae is θ, and that this angle is taken in a positive sense such that the angle between the first and second laminae is θ, between the first and third laminae is 2θ, and so on. Also, the value of θ is:

$$\theta = \frac{\pi}{n}$$

where $\underline{n}$ is an integer greater than 2. Such laminates will fall into two groups; one group in which $\underline{n}$ is odd and the other in which $\underline{n}$ is even. Two such groups, when $n = 3$ and $n = 4$, are shown in figure 6. The total number of laminations is taken to be $\underline{mn}$ so that $\underline{m}$ is the number of laminae oriented in one direction.

It is evident, from symmetry, that these laminates are orthotropic. The direction of the natural axis of one set of $\underline{m}$ laminae is taken as one natural axis ($\underline{x}$ axis) of the material. The other natural axis ($\underline{y}$ axis) is at right angles to it. Each axis ($\underline{x}$ and $\underline{y}$) is an axis of symmetry.

Intuitively, it is known that an orthotropic material having a set of axes (other than the natural axes) with which are associated elastic properties equal to those associated with the natural axes, is also isotropic.

Assume that the strains $\underline{e_x}$, $\underline{e_y}$, and $\underline{e_{xy}}$ are applied to the laminate and, therefore, that these strains are common to each of the laminae. Equations

[6] Sandwich Construction for Aircraft, Part II, Materials Properties and Design Criteria. ANC-23, May 1951.

2.11(B) apply to each lamina. The value of ϕ for each lamina is given by:

$$\phi = j\theta = \frac{\pi j}{n}, \text{ where } j = 0, 1, 2 --- (n-1)$$

and where the integer $\underline{j}$ determines the direction (related to the $\underline{x}$ axis) of the natural axis of the particular group of laminae considered, and ϕ is taken positively from the α axis to the $\underline{x}$ axis. The direct stresses, $\bar{f}$, in the laminate are the averages of the stresses in the individual laminae. Equations 2.11(B)6 yield:

$$\bar{f}_x = \frac{1}{n} \sum_{j=0}^{j=(n-1)} f_x = \frac{1}{n} e_x \sum b_{11} + \frac{1}{n} e_y \sum b_{12} + \frac{1}{n} e_{xy} \sum b_{13}$$

$$\bar{f}_y = \frac{1}{n} \sum_{j=0}^{j=(n-1)} f_n = \frac{1}{n} e_x \sum b_{21} + \frac{1}{n} e_y \sum b_{22} + \frac{1}{n} e_{xy} \sum b_{23} \quad (1)$$

$$\bar{f}_{xy} = \frac{1}{n} \sum_{j=0}^{j=(n-1)} f_{xy} = \frac{1}{n} e_x \sum b_{31} + \frac{1}{n} e_y \sum b_{32} + \frac{1}{n} e_{xy} \sum b_{33}$$

in which:

$$b_{11} = \frac{1}{\lambda} \left[E_\alpha \cos^4 \phi + E_\beta \sin^4 \phi + (2E_\alpha \mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}) \sin^2 \phi \cos^2 \phi \right]$$

$$b_{22} = \frac{1}{\lambda} \left[E_\beta \cos^4 \phi + E_\alpha \sin^4 \phi + (2E_\alpha \mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}) \sin^2 \phi \cos^2 \phi \right]$$

$$b_{33} = \frac{1}{\lambda} \left[(E_\alpha + E_\beta - 2E_\alpha \mu_{\beta\alpha}) \sin^2 \phi \cos^2 \phi + \lambda G_{\alpha\beta} (\cos^2 \phi - \sin^2 \phi)^2 \right]$$

$$b_{21} = b_{12} = \frac{1}{\lambda} \left[(E_\alpha + E_\beta - 4\lambda G_{\alpha\beta}) \sin^2 \phi \cos^2 \phi + E_\alpha \mu_{\beta\alpha} (\cos^4 \phi + \sin^4 \phi) \right]$$

$$b_{31} = b_{13} = \frac{1}{\lambda} \left[(E_\beta - E_\alpha \mu_{\beta\alpha} - 2\lambda G_{\alpha\beta}) \sin^3 \phi \cos \phi - (E_\alpha - E_\alpha \mu_{\beta\alpha} - 2\lambda G_{\alpha\beta}) \sin \phi \cos^3 \phi \right]$$

$$b_{32} = b_{23} = \frac{1}{\lambda}\left[(E_\beta - E_\alpha \mu_{\beta\alpha} - 2\lambda G_{\alpha\beta}) \sin\phi \cos^3\phi - (E_\alpha - E_\alpha \mu_{\beta\alpha} - \right.$$

$$\left. 2\lambda G_{\alpha\beta}) \sin^3\phi \cos\phi\right]$$

$$\lambda = 1 - \mu_{\alpha\beta}\,\mu_{\beta\alpha}$$

To make these summations it will be necessary to sum the expressions:

$$\cos^4 \frac{\pi j}{n}$$

$$\sin^4 \frac{\pi j}{n}$$

$$\sin^2 \frac{\pi j}{n} \cos^2 \frac{\pi j}{n}$$

$$\sin^3 \frac{\pi j}{n} \cos \frac{\pi j}{n}$$

$$\sin \frac{\pi j}{n} \cos^3 \frac{\pi j}{n}$$

To effect the summations consider the equation:

$$\left[\cos \frac{2\pi j}{n} + i \sin \frac{2\pi j}{n}\right]^n = 1 \qquad (2)$$

in which $i = \sqrt{-1}$

The expression in the bracket is a root of the equation; n roots are obtained as j takes the values 0, 1, 2, --- (n-1). Writing Z for the bracket, equation (2) becomes:

$$Z^n - 1 = 0 \qquad (3)$$

This equation may be written in the form:

$$(Z - r_0)(Z - r_1)(Z - r_2) \text{------} (Z - r_{(n-1)}) = 0 \qquad (4)$$

where r_0, r_1, etc., are the n roots of the equation.

Equation (4) may be expanded to:

$$Z^n - (r_0 + r_1 + \cdots + r_{(n-1)})\, Z^{(n-1)} + (r_0 r_1 + r_0 r_2 + \cdots\ r_{(n-2)} r_{(n-1)})\, Z^{(n-2)}$$

$$+ \cdots \quad -Z^0 = 0 \tag{5}$$

By comparison of equation (5) with equation (3), from which it was derived:

$$r_0 + r_1 + \cdots + r_{(n-1)} = \sum_{j=0}^{j=(n-1)} r_j = 0 \tag{6}$$

Thus:

$$\sum_{j=0}^{j=(n-1)} \left[\cos \frac{2\pi j}{n} + i \sin \frac{2\pi j}{n} \right] = 0$$

and because the real and imaginary parts of this summation must each be identically zero:

$$\sum_{j=0}^{j=(n-1)} \cos \frac{2\pi j}{n} = \sum_{j=0}^{j=(n-1)} \sin \frac{2\pi j}{n} = 0 \qquad n > 2 \tag{7}$$

Further, from equation (6):

$$(r_0 + r_1 + \cdots + r_{(n-1)})^2 = 0$$

which may be expanded to:

$$r_0^2 + r_1^2 + \cdots\ r_{(n-1)}^2 + 2(r_0 r_1 + r_0 r_2 + \cdots\ r_{(n-2)} r_{(n-1)}) = 0 \tag{8}$$

Note: Equation (6) holds for all values of n except 1. The conclusions drawn from equation (8) hold for all values of n except 1 and 2.

By again comparing equation (5) with (3), the parenthetical expression in equation (8) is found to be zero; therefore the sum of the first n terms of equation (8) is also equal to zero. This sum may be expressed by:

$$\sum_{j=0}^{j=(n-1)} r_j^2 = \sum_{j=0}^{j=(n-1)} \left[\cos\frac{4\pi j}{n} + i\ \sin\frac{4\pi j}{n}\right] = 0$$

Thus:

$$\sum_{j=0}^{j=(n-1)} \cos\frac{4\pi j}{n} = \sum_{j=0}^{j=(n-1)} \sin\frac{4\pi j}{n} = 0 \qquad (9)$$

Equations (7) and (9) are then used in making the required summations.

$$\sum_{j=0}^{j=(n-1)} \cos^4\frac{\pi j}{n} = \sum \frac{3}{8} + \frac{1}{2}\sum \cos\frac{2\pi j}{n} + \frac{1}{8}\sum \cos\frac{4\pi j}{n}$$

and, because of equations (7) and (9)

$$\sum_{j=0}^{j=(n-1)} \cos^4\frac{\pi j}{n} = \frac{3}{8}\,n$$

$$\sum_{j=0}^{j=(n-1)} \sin^4\frac{\pi j}{n} = \sum \cos^4\frac{\pi j}{n} - \sum \cos\frac{2\pi j}{n} = \frac{3}{8}\,n$$

$$\sum_{j=0}^{j=(n-1)} \sin^2\frac{\pi j}{n}\cos^2\frac{\pi j}{n} = \sum \frac{1}{2} - \frac{1}{2}\sum \sin^4\frac{\pi j}{n} - \frac{1}{2}\sum \cos^4\frac{\pi j}{n}$$

$$= \frac{1}{8}\,n$$

$$\sum_{j=0}^{j=(n-1)} \sin^3 \frac{\pi j}{n} \cos \frac{\pi j}{n} = \frac{1}{4} \sum \sin \frac{2\pi j}{n} - \frac{1}{8} \sum \sin \frac{4\pi j}{n} = 0$$

$$\sum_{j=0}^{j=(n-1)} \sin \frac{\pi j}{n} \cos^3 \frac{\pi j}{n} = \frac{1}{4} \sum \sin \frac{2\pi j}{n} - \frac{1}{8} \sum \sin \frac{4\pi j}{n} = 0$$

By using these values, equations (1) become:

$$\bar{f}_x = e_x \frac{1}{\lambda} \left[\frac{3}{8} E_\alpha + \frac{3}{8} E_\beta + \frac{1}{8} (2E_\alpha \mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}) \right]$$

$$+ e_y \frac{1}{\lambda} \left[\frac{1}{8} (E_\alpha + E_\beta - 4\lambda G_{\alpha\beta}) + \frac{3}{4} E_\alpha \mu_{\beta\alpha} \right]$$

$$\bar{f}_y = e_x \frac{1}{\lambda} \left[\frac{1}{8} (E_\alpha + E_\beta - 4\lambda G_{\alpha\beta}) + \frac{3}{4} E_\alpha \mu_{\beta\alpha} \right] \tag{10}$$

$$+ e_y \frac{1}{\lambda} \left[\frac{3}{8} E_\alpha + \frac{3}{8} E_\beta + \frac{1}{8} (2E_\alpha \mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}) \right]$$

$$\bar{f}_{xy} = e_{xy} \frac{1}{\lambda} \left[\frac{1}{8} (E_\alpha + E_\beta - 2E_\alpha \mu_{\beta\alpha}) + \frac{1}{2} \lambda G_{\alpha\beta} \right]$$

It is evident that the material is isotropic in the plane of the laminate because the angle $\frac{2\pi j}{n}$ does not enter in these equations. Also, they are in the form of equations for an isotropic material, which are:

$$\bar{f}_x = \frac{\bar{E}}{\bar{\lambda}} e_x + \frac{\bar{E}\,\bar{\mu}}{\bar{\lambda}} e_y$$

$$\bar{f}_y = \frac{\bar{E}\,\bar{\mu}}{\bar{\lambda}} e_x + \frac{\bar{E}}{\bar{\lambda}} e_y \tag{11}$$

$$\bar{f}_{xy} = \bar{G}\, e_{xy}$$

By comparing equations (10) with (11) we obtain:

$$\frac{\overline{E}}{\overline{\lambda}} = \frac{1}{\lambda}\left[\frac{3}{8}E_\alpha + \frac{3}{8}E_\beta + \frac{1}{8}\left(2E_\alpha\mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}\right)\right]$$

$$\frac{\overline{E}\,\overline{\mu}}{\overline{\lambda}} = \frac{1}{\lambda}\left[\frac{1}{8}\left(E_\alpha + E_\beta - 4\lambda G_{\alpha\beta}\right) + \frac{3}{4}E_\alpha\mu_{\beta\alpha}\right] \tag{12}$$

$$\overline{G} = \frac{1}{\lambda}\left[\frac{1}{8}\left(E_\alpha + E_\beta - 2E_\alpha\mu_{\beta\alpha}\right) + \frac{1}{2}G_{\alpha\beta}\right]$$

By multiplying the first of equations (12) by $\overline{\mu}$ and then equating the right-hand members of the first and second:

$$\overline{\mu} = \frac{E_\alpha + E_\beta + 6E_\alpha\mu_{\beta\alpha} - 4\lambda G_{\alpha\beta}}{3E_\alpha + 3E_\beta + 2E_\alpha\mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}} \tag{13}$$

From the first of equations (12),

$$\overline{E} = \frac{\overline{\lambda}}{\lambda}\left[\frac{3}{8}E_\alpha + \frac{3}{8}E_\beta + \frac{1}{8}\left(2E_\alpha\mu_{\beta\alpha} + 4\lambda G_{\alpha\beta}\right)\right] \tag{14}$$

where $\overline{\lambda} = 1 - \overline{\mu}^2$

The value of the modulus of rigidity of the isotropic laminate ($\overline{G}$) is given by the last of equations (12). It satisfies the well-known equation for isotropic materials

$$\overline{G} = \frac{\overline{E}}{2(1 + \overline{\mu})}$$

It is noteworthy that the elastic properties of the isotropic laminate are independent of the value of the angle $(\frac{\pi}{n})$ chosen.

Strength Properties

If isotropic laminates are made according to the scheme previously discussed, their elastic properties are known. Consider such a laminate subjected to a single direct stress in the direction of the $\underline{x}$ axis. Also, consider a single group of laminae for which

$$\theta = \frac{\pi j}{n}$$

where $j = 0, 1, 2 --- (n-1)$ for the different groups.

When the stress (f_x) is applied, the strains are known to be

$$e_x = e_x$$

$$e_y = -\bar{\mu} e_x$$

$$e_{xy} = 0$$

because the laminate is isotropic.

From equations 2.111 (B)

$$e_\alpha = e_x (\cos^2\theta - \bar{\mu} \sin^2\theta)$$

$$e_\beta = e_x (\sin^2\theta - \bar{\mu} \cos^2\theta) \qquad (15)$$

$$e_{\alpha\beta} = -2e_x(1 - \bar{\mu}) \sin\theta \cos\theta$$

From equations 2.11 (B), $(\phi = 0)$:

$$f_\alpha = \frac{E_\alpha}{\lambda} e_\alpha + \frac{E_\alpha \mu_{\beta\alpha}}{\lambda} e_\beta$$

$$f_\beta = \frac{E_\alpha \mu_{\beta\alpha}}{\lambda} e_\alpha + \frac{E_\beta}{\lambda} e_\beta \qquad (16)$$

$$f_{\alpha\beta} = G_{\alpha\beta} e_{\alpha\beta}$$

The condition for failure in this particular group of laminae is assumed to be:

$$\frac{f_\alpha^{\ 2}}{F_\alpha^{\ 2}} + \frac{f_\beta^{\ 2}}{F_\beta^{\ 2}} + \frac{f_{\alpha\beta}^{\ 2}}{F_{\alpha\beta}^{\ 2}} = 1 \qquad (17)$$

Previous tests[4] have verified this relationship for glass-fabric-base plastic laminates.

By substituting the values of strain given by equations (15) in equations (16), then substituting the values of stress given by equations (16) in equation (17) and solving for $\frac{1}{e_x^{\ 2}}$, and, finally, by using the equality

$$\frac{1}{e_x^{\ 2}} = \frac{\overline{E}^2}{\overline{f}_x^{\ 2}} = \frac{\overline{E}^{\ 2}}{\overline{F}_{xj}^{\ 2}}$$

where $\overline{F}_{xj}$ is the direct stress on the laminate at which this particular group of laminae will fail:

$$\frac{1}{\overline{F}_{xj}^{\ 2}} = \frac{E_\alpha^{\ 2}}{\lambda^2 F_\alpha^{\ 2} \overline{E}^{\ 2}} \left[(1 - \overline{\mu}\mu_{\beta\alpha}) \cos^2 \frac{\pi j}{n} + (\mu_{\beta\alpha} - \overline{\mu}) \sin^2 \frac{\pi j}{n}\right]^2 +$$

$$\frac{E_\beta^{\ 2}}{\lambda^2 F_\beta^{\ 2} \overline{E}^{\ 2}} \left[(1 - \overline{\mu}\mu_{\alpha\beta}) \sin^2 \frac{\pi j}{n} + (\mu_{\alpha\beta} - \overline{\mu}) \cos^2 \frac{\pi j}{n}\right]^2 + \qquad (18)$$

$$4 \frac{G_{\alpha\beta}^{\ 2}}{F_{\alpha\beta}^{\ 2} \overline{E}^{\ 2}} (1 - \overline{\mu})^2 \sin^2 \frac{\pi j}{n} \cos^2 \frac{\pi j}{n}$$

By letting $\underline{j}$ take on its successive values, $\underline{n}$ values of $\overline{F}_{xj}$ will be obtained, each giving a stress at which one of the groups of laminae will fail. The strength of the laminate will lie somewhere between the minimum and the average of these values.

The shear strength is obtained in a similar manner. Consider the laminate to be subjected to a shear stress associated with the x and y axes:

$$e_x = 0$$

$$e_y = 0$$

$$e_{xy} = e_{xy}$$

From equations 2.111 (B)

$$e_\alpha = e_{xy} \sin\theta \cos\theta$$

$$e_\beta = - e_{xy} \sin\theta \cos\theta \qquad (19)$$

$$e_{\alpha\beta} = e_{xy} (\cos^2\theta - \sin^2\theta)$$

From equations 2.11 (B) ($\phi = 0$) and equation (19):

$$f_\alpha = e_{xy} \frac{E_\alpha}{\lambda} (1 - \mu_{\beta\alpha}) \sin\theta \cos\theta$$

$$f_\beta = e_{xy} \frac{E_\beta}{\lambda} (\mu_{\alpha\beta} - 1) \sin\theta \cos\theta \qquad (20)$$

$$f_{\alpha\beta} = e_{xy} G_{\alpha\beta} (\cos^2\theta - \sin^2\theta)$$

By using equations (20) in (17), and remembering that

$$\frac{1}{e_{xy}^2} = \frac{\overline{G}^2}{f_{xy}^2} = \frac{\overline{G}^2}{\overline{F}_{xyj}^2} ,$$

then:

$$\frac{1}{F_{xyj}^{\,2}} = \frac{E_\alpha^{\,2}}{\lambda^2 F_\alpha^{\,2} \bar{G}^{\,2}} \left[1 - \mu_{\beta\alpha}\right]^2 \sin^2 \frac{\pi j}{n} \cos^2 \frac{\pi j}{n} +$$

$$\frac{E_\beta^{\,2}}{\lambda^2 F_\beta^{\,2} \bar{G}^{\,2}} \left[1 - \mu_{\alpha\beta}\right]^2 \sin^2 \frac{\pi j}{n} \cos^2 \frac{\pi j}{n} + \qquad (21)$$

$$\frac{G_{\alpha\beta}^{\,2}}{F_{\alpha\beta}^{\,2} \bar{G}^{\,2}} \left(\cos^2 \frac{\pi j}{n} - \sin^2 \frac{\pi j}{n}\right)^2$$

The shear strength of the laminate may be found by using equation (21) in the manner described for equation (17). In this equation, F_α, is a tensile strength and F_β is a compressive strength if the expression $\sin \frac{\pi j}{n} \cos \frac{\pi j}{n}$ is positive and F_α if a compressive strength and F_β is a tensile strength if this expression is negative.

Symbols Used

- E Young's modulus of elasticity.
- e Strain; the subscripts associated with e indicate the axes with which it is associated.
- F Strength.
- f Stress; the subscripts associated with f indicate the part of the laminate in which the stress occurs and the axes with which it is associated.
- G Modulus of rigidity.
- x Axis; as a subscript it denotes direction parallel to x axis.
- y Axis perpendicular to x axis; as a subscript it denotes direction parallel to y axis.
- α Subscript denoting direction of the natural axis of any one of the laminae.
- β Subscript denoting direction of the other natural axis of any one of the laminae.
- μ Poisson's ratio, $\mu_{\alpha\beta}$, is Poisson's ratio of contraction in the β direction to extension in the α direction due to a tensile stress acting in the α direction.
- ϕ Angle measured positively from the natural axis α to the x axis.
- λ One minus the product of two Poisson's ratios.

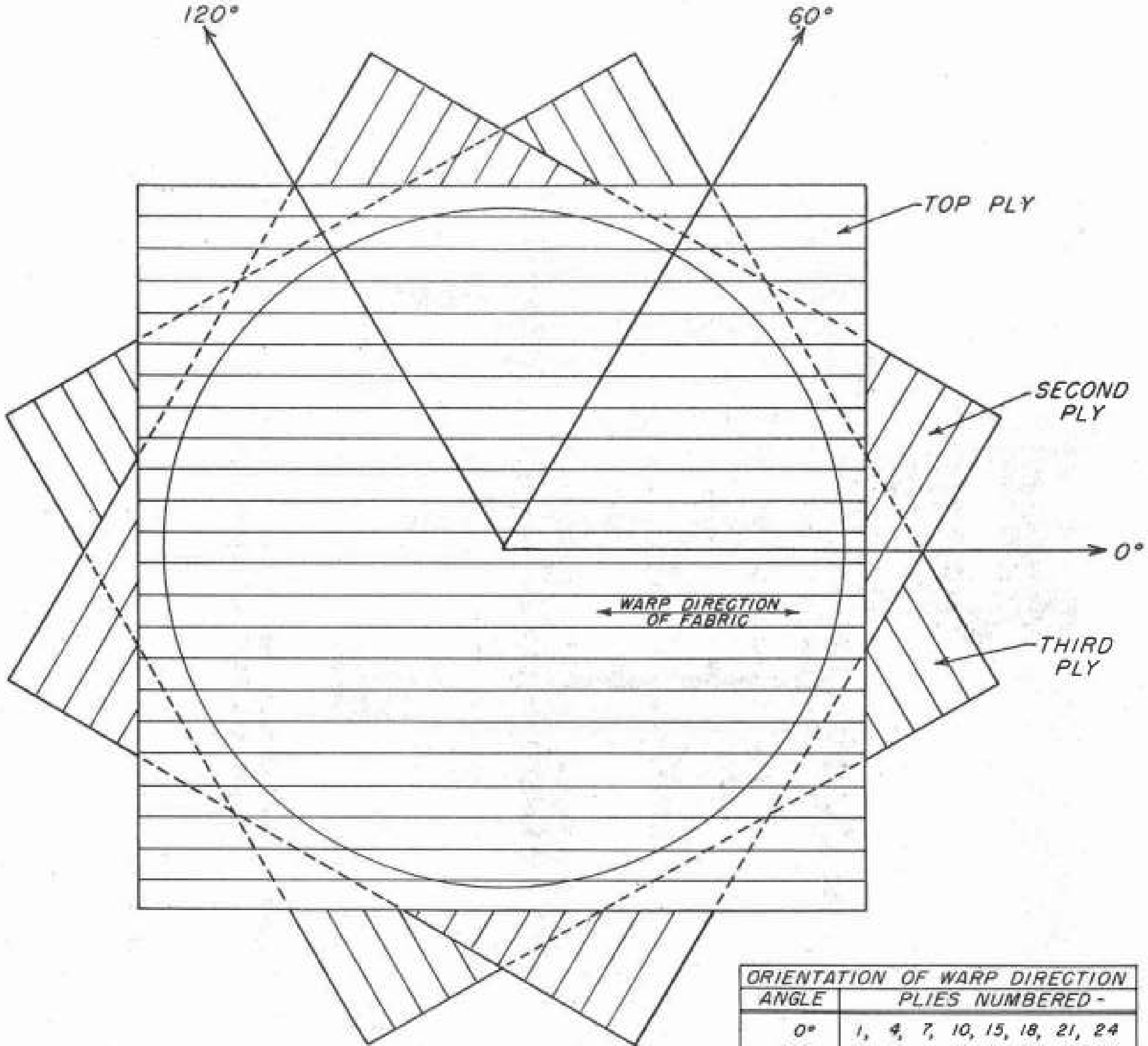
Table 1.--Results of compression tests of a laminate[1] designed to be isotropic. Six specimens tested at each angle

Angle of loading	Value	Modulus of elasticity	Stress at proportional limit	Maximum stress
Degrees		1,000 p.s.i.	P.s.i.	P.s.i.
90	Average	2,462	10,920	33,260
	Minimum	2,408	10,220	31,640
	Maximum	2,578	11,650	34,400
82.5	Average	2,458	10,820	32,580
	Minimum	2,419	10,220	31,120
	Maximum	2,514	11,680	33,700
75	Average	2,434	11,370	32,590
	Minimum	2,324	10,340	29,300
	Maximum	2,506	12,610	34,160
67.5	Average	2,484	10,820	33,970
	Minimum	2,462	9,580	32,280
	Maximum	2,522	12,740	34,800
60	Average	2,489	10,190	34,890
	Minimum	2,470	9,570	33,750
	Maximum	2,529	11,060	35,660
Av. of av.		2,465	10,820	33,460
0	Average	2,520	11,210	34,790
	Minimum	2,470	10,370	32,780
	Maximum	2,581	12,900	35,860

[1] Average properties of cured laminate include: (1) thickness = 0.252 inch; (2) specific gravity = 1.79; (3) resin content = 36.6; and (4) Barcol hardness = 67.

Table 2.--Results of flexural tests of a laminate designed to be isotropic

Specimen No.	Angle of loading	Modulus of elasticity	Stress at proportional limit	Modulus of rupture
	Degrees	1,000 p.s.i.	P.s.i.	P.s.i.
1	90	2,270	14,710	45,600
2		2,250	16,260	45,700
3		2,358	15,580	44,600
4		2,352	19,200	45,500
Av.		2,308	16,440	45,350
1	82.5	2,348	15,150	45,250
2		2,339	16,860	44,610
3		2,363	17,740	47,700
4		2,435	17,340	46,650
Av.		2,371	16,770	46,050
1	75	2,351	17,110	46,580
2		2,228	18,680	45,180
3		2,308	17,320	45,600
4		2,288	18,680	46,120
Av.		2,294	17,950	45,870
1	67.5	2,328	15,180	45,340
2		2,258	16,710	45,470
3		2,332	19,400	46,530
4		2,362	16,980	44,900
Av.		2,320	17,070	45,560
1	60	2,257	18,970	44,980
2		2,308	17,010	44,590
3		2,342	15,550	45,680
4		2,320	16,740	45,010
Av.		2,307	17,070	45,060
Av. of av.		2,320	17,060	45,580
1	0	2,441	17,560	44,890
2		2,459	17,450	44,790
3		2,428	15,360	44,180
4		2,429	17,150	47,040
Av.		2,439	16,880	45,220



ORIENTATION OF WARP DIRECTION	
ANGLE	PLIES NUMBERED -
0°	1, 4, 7, 10, 15, 18, 21, 24
60°	2, 5, 8, 11, 14, 17, 20, 23
120°	3, 6, 9, 12, 13, 16, 19, 22

Z M 92221 F

Figure 1.--Sketch showing orientation of fabric in isotropic laminate.

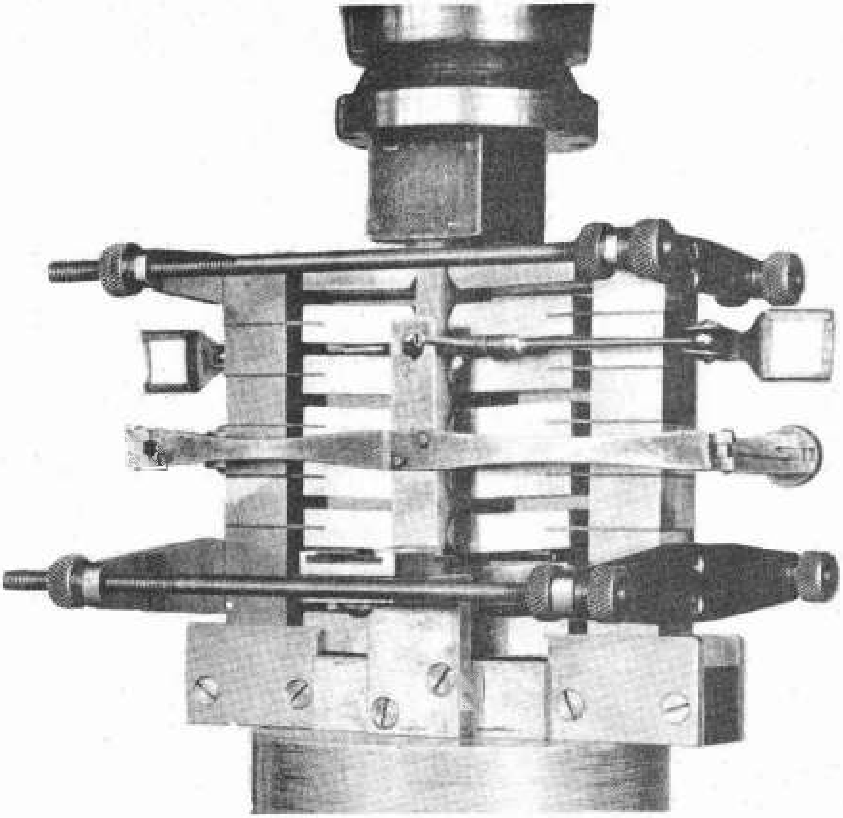


Figure 2.--Compression pack test of the type used in testing plastic laminate specimens.

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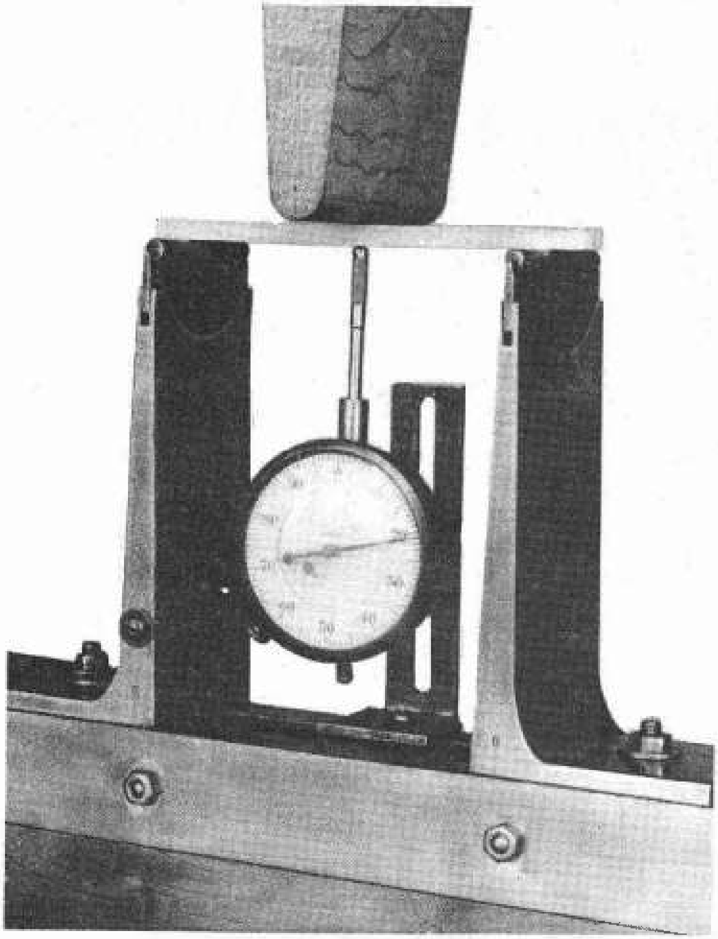

Figure 3.--Flexural test set-up used in testing plastic laminate specimens.

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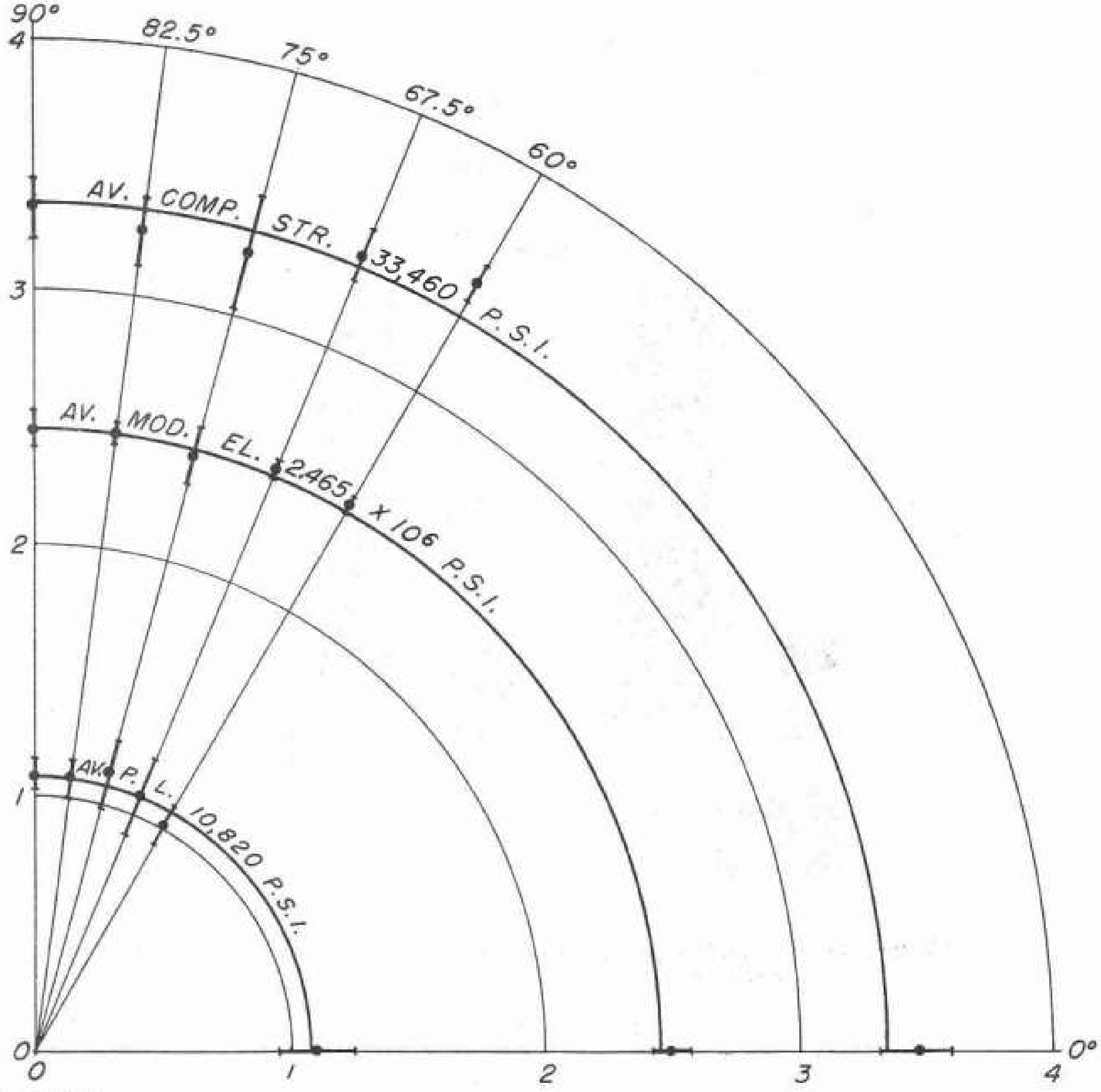

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Figure 4.--Average compressive strength, modulus of elasticity, and stress at proportional limit for compression tests at various angles of loading. Brackets designate 3 times the standard deviation from the average for each angle tested.

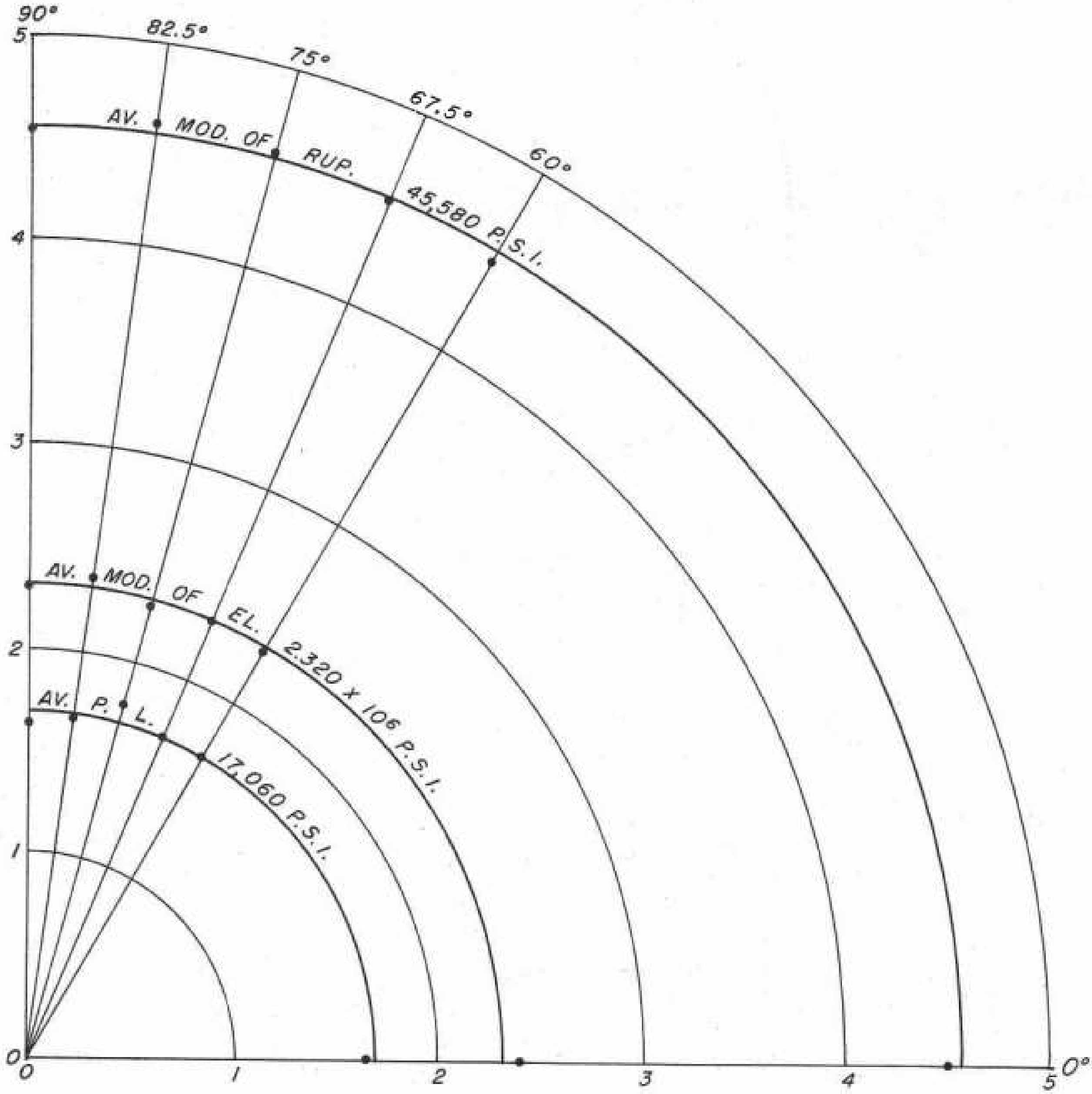

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Figure 5.--Average modulus of rupture, modulus of elasticity, and stress at proportional limit at various angles of test. Each point represents the average from tests of four specimens.

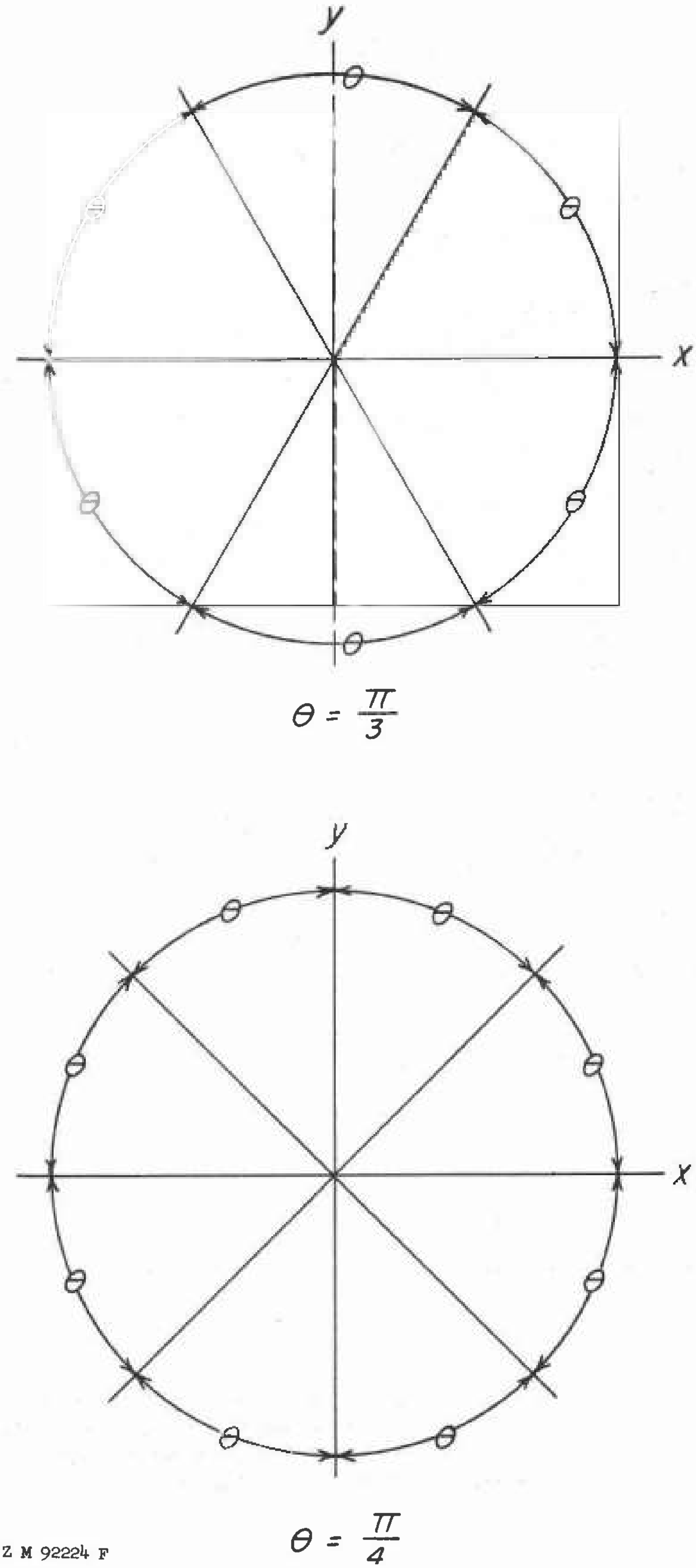

Figure 6.--Two methods of orienting laminae to have isotropic laminate, where $\theta = \frac{\pi}{n}$. Designations for n = 3 and n = 4 are shown.