MECHANICS OF COMPOSITE STRUCTURES

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Contents

List of Symbols

We have used, wherever possible, notation standard in elasticity, structural analysis, and composite materials. We tried to avoid duplication, although there is some repetition of those symbols that are used only locally. In the following list we have not included those symbols that pertain only to the local discussion. Below, we give a verbal description of each symbol and, when appropriate, the number of the equation in which the symbol is first used.

Latin letters

A	area
A^{iso}	tensile stiffness of an isotropic laminate (Eq. 3.42)
$[A]$, A_{ij}	tensile stiffness of a laminate (Eqs. 3.18, 3.19)
$[a]$, a_{ij}	inverse of the $[A]$ matrix for symmetric laminates (Eq. 3.29)
$[B]$, B_{ij}	stiffness of a laminate (Eqs. 3.18, 3.19)
$[C]$, C_{ij}	3D stiffness matrix in the x_1, x_2, x_3 coordinate system (Eq. 2.22)
$[\overline{C}]$, $\overline{C}_{ij}$	3D stiffness matrix in the x, y, z coordinate system (Eq. 2.19)
c	moisture concentration (Eq. 2.154); core thickness (Fig. 5.2)
$[D]$, D_{ij}	bending stiffness of a laminate (Eqs. 3.18, 3.19)
$[D]^*$, D_{ij}^*	reduced bending stiffness of a laminate (Eq. 4.1)
D^{iso}	bending stiffness of an isotropic laminate (Eq. 3.42)
D, $\overline{D}$, $\widehat{D}$	parameters (Table 6.2, page 222, Eq. 6.157)
$[d]$, d_{ij}	inverse of the $[D]$ matrix for symmetrical laminates (Eq. 3.30)
d, d^{t}, d^{b}	distances for sandwich plates (Fig. 5.2)
E_1, E_2, E_3	Young's moduli in the x_1, x_2, x_3 coordinate system (Table 2.5)
$[E]$	stiffness matrix in the FE calculation (Eq. 9.4)
$\widehat{EA}$	tensile stiffness of a beam (Eq. 6.8)
$\widehat{EI}$	bending stiffness of a beam (Eq. 6.8)
$\widehat{EI}_\omega$	warping stiffness of a beam (Eq. 6.244)
F_i, F_{ij}	strength parameters in the quadratic failure criterion (Eq. 10.2)

f_{ij}	constants in the quadratic failure criterion (Eq. 10.25)
f, f_{ij}	frequency (Eq. 4.190)
f_x, f_y, f_z,	body forces per unit volume (Eq. 2.13)
G_{23}, G_{13}, G_{12}	shear moduli in the x_1, x_2, x_3 coordinate system (Table 2.5)
$\widehat{GI}_{\mathrm{t}}$	torsional stiffness of a beam (Eq. 6.8)
h	plate thickness
h_{b}, h_{t}	distances of the bottom and top surfaces of a plate from the reference plane (Eq. 3.9)
i_ω	polar radius of gyration (Eq. 6.340)
$[J]$	inverse of the material stiffness matrix $[E]$ (Eq. 9.16)
K	number of layers in a laminate; number of wall segments; stiffness parameter of a plate (Eq. 4.153)
$\widetilde{k}$	rotational spring constant (Eq. 4.149)
k	equivalent length factor (Eq. 6.340)
L_x, L_y	dimensions of a plate
L	length; number of cells in a multicell beam (Eq. 6.222)
L_i, L_i^{f}	load and failure load (Eq. 10.42)
l_x, l_x^{o}	half buckling length (Eq. 4.142), half buckling length corresponding to the lowest buckling load of a long plate (Eq. 4.173)
M_x, M_y, M_{xy}	bending and twist moments per unit length acting on a laminate (Eq. 3.9)
M_x^{ht}, M_y^{ht}, M_{xy}^{ht}	hygrothermal moments per unit length (Eq. 4.247)
$\widehat{M}_y$, $\widehat{M}_z$	bending moments acting on a beam (Fig. 6.2)
$\widehat{M}_\omega$	bimoment acting on a beam (Eq. 6.232)
N_x, N_y, N_{xy}	in-plane forces per unit length acting on a laminate (Eq. 3.9)
N_{x0}, N_{y0}, N_{xy0}	in-plane compressive forces per unit length (Eq. 4.109)
N_x^{ht}, N_y^{ht}, N_{xy}^{ht}	hygrothermal forces per unit length (Eq. 4.246)
$N_{x,\,\mathrm{cr}}$	buckling load of a uniaxially loaded plate (Eq. 4.141)
$\widehat{N}$	axial force acting on a beam (Fig. 6.2)
$\widehat{N}_{\mathrm{cr}}$, $\widehat{N}_{\mathrm{cr}}^{\mathrm{B}}$	buckling load and buckling load due to bending deformation (Eq. 6.337)
$\widehat{N}_{\mathrm{cr}y}$, $\widehat{N}_{\mathrm{cr}z}$	buckling load in the x–z and x–y planes, respectively (Eqs. 6.337, 7.110)
$\widehat{N}_{\mathrm{cr}\psi}$	buckling load under torsional buckling (Eqs. 6.337, 7.110)
$[P]$, $[\overline{P}]$	stiffness matrix of a beam (Eqs. 6.2, 6.250). Without bar refers to the centroid; with bar to an arbitrarily chosen coordinate system
p	transverse load per unit area; distance between the origin and the tangent of the wall of a beam (Eq. 6.190)
p_x, p_y, p_z	axial and transverse loads (per unit length) acting on a beam (Fig. 6.1); surface forces per unit area (Eq. 2.166)
$[Q]$, Q_{ij}	2D plane-stress stiffness matrix in the x_1, x_2 coordinate system (Eq. 2.134)

$[\overline{Q}], \overline{Q}_{ij}$ 2D plane-stress stiffness matrix in the x, y coordinate system (Eq. 2.126)
$\widehat{Q}_{\text{cr}}$ buckling load resulting in lateral buckling (Eq. 6.359)
q shear flow (Eq. 6.189).
R stiffness parameter (Eq. 3.46)
$\widetilde{R}$ stress ratio (Eq. 10.42)
R_x, R_y, R_{xy} radii of curvatures of a shell (Eq. 8.1)
$[R], R_{ij}$ compliance matrix under plane-strain condition in the x_1, x_2 coordinate system (Eq. 2.79)
$[\overline{R}], \overline{R}_{ij}$ compliance matrix under plane-strain condition in the x, y coordinate system (Eq. 2.65)
$[S], S_{ij}$ 3D compliance matrix in the x_1, x_2, x_3 coordinate system (Eq. 2.23)
$[\overline{S}], \overline{S}_{ij}$ 3D compliance matrix in the x, y, z coordinate system (Eq. 2.21)
$\widehat{S}_{ij}$ shear stiffness of a beam, $i, j = z, y, \omega$ (Eqs. 7.13, 7.36)
$\widetilde{S}_{ij}$ shear stiffness of a plate, $i, j = 1, 2$ (Eq. 5.15)
$\widehat{s}_{ij}$ shear compliance of a beam, $i, j = z, y, \omega$ (Eq. 7.38)
s_1^+, s_2^+, s_3^+ tensile strengths (Eq. 10.13)
s_1^-, s_2^-, s_3^- compression strengths (Eq. 10.13)
s_{23}, s_{13}, s_{12} shear strengths (Eq. 10.15)
$\widehat{T}$ torque acting on a beam (Fig. 6.2)
$\widehat{T}_\omega$ restrained warping-induced torque (Eq. 6.235)
$\widehat{T}_{\text{sv}}$ Saint-Venant torque (Eq. 6.239)
$[T_\sigma]$ 2D stress transformation matrix (Eq. 2.182)
$[\hat{T}_\sigma]$ 3D stress transformation matrix (Eq. 2.179)
$[T_\epsilon]$ 2D strain transformation matrix (Eq. 2.188)
$[\hat{T}_\epsilon]$ 3D strain transformation matrix (Eq. 2.185)
t torque load acting on a beam (Fig. 6.1)
$t^{\text{t}}, t^{\text{b}}$ thicknesses of the top and bottom facesheets (Eq. 5.26)
U strain energy (Eq. 2.200)
U displacement in the x direction; varies with the x and y coordinates only (Eq. 2.50)
u displacement in the x direction
u^{o} displacement of the reference surface in the x direction
u_1, u_2, u_3 displacements in the x_1, x_2, and x_3 direction
V displacement in the y direction; varies with the x and y coordinates only (Eq. 2.51)
$V_{\text{f}}, V_{\text{m}}, V_{\text{v}}$ volume of fibers, matrix, and void
V_x, V_y out-of-plane shear forces per unit length (Eq. 3.10)
$\widehat{V}_y, \widehat{V}_z$ transverse shear forces acting on a beam (Fig. 6.2)
v displacement in the y direction
v^{o} displacement of the reference surface in the y direction
$v_{\text{f}}, v_{\text{m}}, v_{\text{v}}$ volume fraction of fibers, matrix, and void

W	displacement in the z direction; varies with the x and y coordinates only (Eq. 2.52)
$[W]$, $[\overline{W}]$	compliance matrix of a beam (Eq. 6.17). No bar refers to the centroid; bar to an arbitrarily chosen coordinate system
w	deflection in the z direction
$\widetilde{w}$	maximum deflection in the z direction (Eq. 4.29)
w^{o}	deflection of the reference surface in the z direction
w^{B}, w^{S}	deflections due to bending and shear deformations (Eq. 7.85)
y_{c}, z_{c}	coordinates of the centroid of a beam (Eqs. 6.54, 6.73)
y_{sc}, z_{sc}	coordinates of the shear center of a beam (Eq. 6.311)
z_k, z_{k-1}	coordinates of the top and bottom surfaces of the kth ply in a laminate (Eq. 3.20)

Greek letters

α	parameter describing shear deformation (Eq. 7.253)
α_i	parameter describing shear deformation, $i = w, \psi, N, \omega$ (Eq. 7.244)
$[\alpha]$, α_{ij}	compliance matrix of a laminate (Eq. 3.23)
α, β	parameters describing buckled shape of a shell (Eq. 8.78)
$\widehat{\alpha}_{ij}$	compliances for closed-section beams (Eq. 6.156)
$\widetilde{\alpha}_i$, $\widetilde{\alpha}_{ij}$	thermal expansion coefficients (Eqs. 2.153, 2.158)
β, λ	parameters in the displacements of a cylinder (Eq. 8.30)
$[\beta]$, β_{ij}	compliance matrix of a laminate (Eq. 3.23)
$\overline{\beta}_{ij}$	compliance of symmetrical cross-section beams (Table 6.2)
$\widehat{\beta}_{ij}$	compliance of closed-section beams (Eq. 6.156)
$\widetilde{\beta}_i$, $\widetilde{\beta}_{ij}$	moisture expansion coefficients in the x, y, z directions (Eqs. 2.154, 2.159)
β_1	property of the cross section (Eq. 6.360)
γ_y, γ_z	shear strain in a beam in the x–y and x–z planes (Eq. 7.2)
γ_{yz}, γ_{xz}, γ_{xy}	engineering shear strain in the x, y, z coordinate system (Eq. 2.9)
γ_{23}, γ_{13}, γ_{12}	engineering shear strain in the x_1, x_2, x_3 coordinate system
Δh	change in thickness (Eq. 4.282)
ΔT	temperature change (Eq. 2.153)
$[\delta]$, δ_{ij}	compliance matrix of a laminate (Eq. 3.23)
$\widehat{\delta}_{ij}$	compliance of closed-section beams (Eq. 6.157)
$\overline{\epsilon}_x, \ldots$	average strains in a sublaminate (Eq. 9.14)
ϵ_x, ϵ_y, ϵ_z	engineering normal strains in the x, y, z coordinate system
ϵ_1, ϵ_2, ϵ_3	engineering normal strains in the x_1, x_2, x_3 coordinate system
ϵ_x^{o}, ϵ_y^{o}, γ_{xy}^{o}	strains of the reference surface
$\epsilon_x^{\mathrm{o,ht}}$, $\epsilon_y^{\mathrm{o,ht}}$, $\gamma_{xy}^{\mathrm{o,ht}}$	hygrothermal strains in a laminate (Eq. 4.250)
ζ	parameter of restraint (Eq. 4.152)
Θ	polar moment of mass (Eq. 6.411)

Θ_k	ply orientation
ϑ	rate of twist (Eq. 6.1)
$\vartheta^{\mathrm{B}}, \vartheta^{\mathrm{S}}$	rate of twist due to bending and shear deformation (Eq. 7.5)
$\kappa_x, \kappa_y, \kappa_{xy}$	curvatures of the reference surface (Eq. 3.8)
$\kappa_x^{\mathrm{ht}}, \kappa_y^{\mathrm{ht}}, \kappa_{xy}^{\mathrm{ht}}$	hygrothermal curvatures of a laminate (Eq. 4.250)
$\lambda, \lambda_{\mathrm{cr}}, \lambda_{ij}$	load parameter (Eq. 4.109); buckling load parameter (Eq. 4.121); eigenvalue (Eq. 4.225)
$\mu_{Bi}, \mu_{Gi}, \mu_{\mathrm{Si}}$	parameters in the calculation of natural frequencies (Eqs. 6.398, 6.400, 7.203)
ν_{ij}	Poisson's ratio
ξ, η, ζ	coordinates attached to the wall of a beam (Fig. 6.13)
ξ, ξ'	parameters in the expressions of the buckling loads of plates with rotationally restrained edges (Eq. 4.151)
π_{p}	potential energy (Eq. 2.204)
ρ_x, ρ_y, ρ_z	radius of curvature in the y–z , x–z, and x–y planes (Eq. 2.45)
ρ_1, ρ_2, ρ_3	radius of curvature in the x_2–x_3 , x_1–x_3, and x_1–x_2 planes (Eq. 2.53)
$\rho_{\mathrm{comp}}, \rho_{\mathrm{f}}, \rho_{\mathrm{m}}$	densities of composite, fiber, and matrix
ρ	mass per unit area or per unit length
$\sigma_1, \sigma_2, \sigma_3$	normal stresses in the x_1, x_2, x_3 coordinate system
$\sigma_x, \sigma_y, \sigma_z$	normal stresses in the x, y, z coordinate system
$\overline{\sigma}$	average stress
$\tau_{23}, \tau_{13}, \tau_{12}$	shear stresses in the x_1, x_2, x_3 coordinate system
$\tau_{yz}, \tau_{xz}, \tau_{yx}$	shear stresses in the x, y, z coordinate system
χ_{xz}, χ_{yz}	rotation of the normal of a plate in the x–z and x–y planes (Eqs. 3.2 and 5.1)
χ_y, χ_z	rotation of the cross section of a beam in the x–y and x–z planes (Eq. 7.2)
ψ	angle of rotation of the cross section about the beam axis (twist) (Fig. 6.3)
Ψ	bending stiffness of an unsymmetrical long plate (Eq. 4.52)
Ω	potential energy of the external loads (Eq. 2.203)
ω	circular frequency (Eq. 4.190)
$\omega^{\mathrm{B}}, \omega^{\mathrm{S}}$	circular frequency of a beam due to bending and shear deformation (Eq. 7.198)
ω_y, ω_z	circular frequency of a freely vibrating beam in the x–z and x–y planes, respectively (Eq. 6.398)
ω_ψ	circular frequency of a freely vibrating beam under torsional vibration (Eq. 6.400)
$\varrho, \widetilde{\varrho}, \bar{\varrho}, \widehat{\varrho}$	distances between the new and the old reference surfaces (Eqs. 3.47, 6.105, 6.107, A.3)

Displacements, Strains, and Stresses

We consider composite materials consisting of continuous or discontinuous fibers embedded in a matrix. Such a composite is heterogeneous, and the properties vary from point to point. On a scale that is large with respect to the fiber diameter, the fiber and matrix properties may be averaged, and the material may be treated as homogeneous. This assumption, commonly employed in macromechanical analyses of composites, is adopted here. Hence, the material is considered to be quasi-homogeneous, which implies that the properties are taken to be the same at every point. These properties are not the same as the properties of either the fiber or the matrix but are a combination of the properties of the constituents.

In this chapter, equations are presented for calculating the displacements, stresses, and strains when the structure undergoes only small deformations and the material behaves in a linearly elastic manner.

Continuous fiber-reinforced composite materials (and structures made of such materials) often have easily identifiable preferred directions associated with fiber orientations or symmetry planes. It is therefore convenient to employ two coordinate systems: a local coordinate system aligned, at a point, either with the fibers or with axes of symmetry, and a global coordinate system attached to a fixed reference point (Fig. 2.1). In this book the local and global Cartesian coordinate systems are designated respectively by x_1, x_2, x_3 and the x, y, z axes. In the x, y, z directions the displacements at a point A are denoted by u, v, w, and in the x_1, x_2, x_3 directions by u_1, u_2, u_3 (Fig. 2.2).

In the x, y, z coordinate system the normal stresses are denoted by σ_x, σ_y, and σ_z and the shear stresses by τ_{yz}, τ_{xz}, and τ_{xy} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_x, \epsilon_y, \epsilon_z$ and $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$, respectively.

In the x_1, x_2, x_3 coordinate system the normal stresses are denoted by σ_1, σ_2, and σ_3 and the shear stresses by τ_{23}, τ_{13}, and τ_{12} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_1, \epsilon_2, \epsilon_3$, and $\gamma_{23}, \gamma_{13}, \gamma_{12}$, respectively. The symbol γ represents engineering shear strain that is twice the tensorial shear strain, $\gamma_{ij} = 2\epsilon_{ij}$ ($i, j = x, y, z$ or $i, j = 1, 2, 3$).

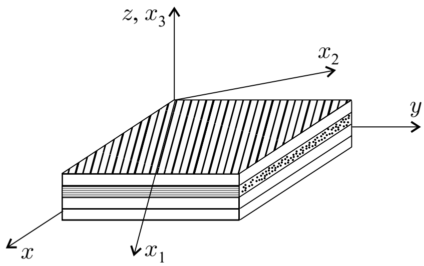


Figure 2.1: The global x, y, z and local x_1, x_2, x_3 coordinate systems.

A stress is taken to be positive when it acts on a positive face in the positive direction. According to this definition, all the stresses shown in Figure 2.3 are positive.

The preceding stress and strain notations, referred to as engineering notations, are used throughout this book. Other notations, most notably tensorial and contracted notations, can frequently be found in the literature. The stresses and strains in different notations are summarized in Tables 2.1 and 2.2.

2.1 Strain–Displacement Relations

We consider a Δx long segment that undergoes a change in length, the new length being denoted by $\Delta x'$. From Figure 2.4 it is seen that

$$u + \Delta x' = \Delta x + \left(u + \frac{\partial u}{\partial x}\Delta x\right), \tag{2.1}$$

where u and $u + \frac{\partial u}{\partial x}\Delta x$ are the displacements of points A and B, respectively, in the x direction. Accordingly, the normal strain in the x direction is

$$\epsilon_x = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}. \tag{2.2}$$

Similarly, in the y and z directions the normal strains are

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{2.3}$$

$$\epsilon_z = \frac{\partial w}{\partial z}, \tag{2.4}$$

where v and w are the displacements in the y and z directions, respectively.

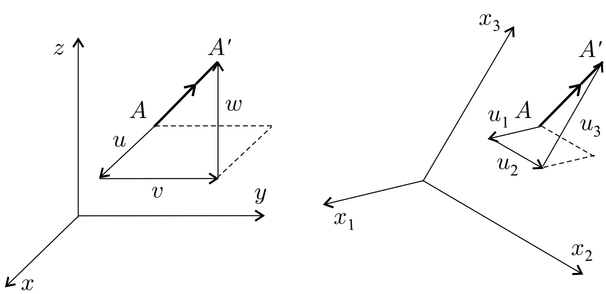


Figure 2.2: The x, y, z and x_1, x_2, x_3 coordinate systems and the corresponding displacements.

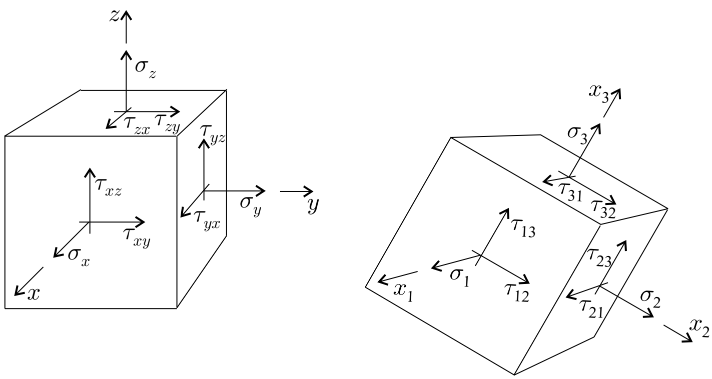


Figure 2.3: The stresses in the global x, y, z and the local x_1, x_2, x_3 coordinate systems.

For angular (shear) deformation the tensorial shear strain is the average change in the angle between two mutually perpendicular lines (Fig. 2.5)

$$\epsilon_{xy} = \frac{\alpha + \beta}{2}. \tag{2.5}$$

For small deformations we have

$$\alpha \approx \tan\alpha = \frac{\left(v + \frac{\partial v}{\partial x}\Delta x\right) - v}{\Delta x} = \frac{\partial v}{\partial x}. \tag{2.6}$$

Similarly $\beta = \partial u/\partial y$, and the xy component of the tensorial shear strain is

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.7}$$

In a similar manner we obtain the following expressions for the ϵ_{yz} and ϵ_{xz} components of the tensorial shear strains:

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{2.8}$$

Table 2.1. Stress notations

	Normal stress			**Shear stress**		
x, y, z coordinate system						
Tensorial stress	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yz}	σ_{xz}	σ_{xy}
Engineering stress	σ_x	σ_y	σ_z	τ_{yz}	τ_{xz}	τ_{xy}
Contracted notation	σ_x	σ_y	σ_z	σ_q	σ_r	σ_s
x_1, x_2, x_3 coordinate system						
Tensorial stress	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{13}	σ_{12}
Engineering stress	σ_1	σ_2	σ_3	τ_{23}	τ_{13}	τ_{12}
Contracted notation	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6

Table 2.2. Strain notations (the engineering and contracted notation shear strains are twice the tensorial shear strain)

	Normal strain			Shear strain		
x, y, z coordinate system						
Tensorial strain	ϵ_{xx}	ϵ_{yy}	ϵ_{zz}	ϵ_{yz}	ϵ_{xz}	ϵ_{xy}
Engineering strain	ϵ_x	ϵ_y	ϵ_z	γ_{yz}	γ_{xz}	γ_{xy}
Contracted notation	ϵ_x	ϵ_y	ϵ_z	ϵ_q	ϵ_r	ϵ_s
x_1, x_2, x_3 coordinate system						
Tensorial strain	ϵ_{11}	ϵ_{22}	ϵ_{33}	ϵ_{23}	ϵ_{13}	ϵ_{12}
Engineering strain	ϵ_1	ϵ_2	ϵ_3	γ_{23}	γ_{13}	γ_{12}
Contracted notation	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6

The engineering shear strains are twice the tensorial shear strains:

$$\gamma_{yz} = 2\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{2.9}$$

$$\gamma_{xz} = 2\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2.10}$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{2.11}$$

In the x_1, x_2, x_3 coordinate system the strain–displacement relationships are also given by Eqs. (2.2)–(2.4) and (2.9)–(2.11) with x, y, z replaced by x_1, x_2, x_3, the subscripts x, y, z by 1, 2, 3, and u, v, w by u_1, u_2, u_3.

2.2 Equilibrium Equations

The equilibrium equations at a point O are obtained by considering force and moment balances on a small $\Delta x \Delta y \Delta z$ cubic element located at point O. (The point O is at the center of the element, Fig. 2.6.) We relate the stresses at one face to those at the opposite face by the Taylor series. By using only the first term of the Taylor series, force balance in the x direction gives

$$-\sigma_x \Delta z \Delta y - \tau_{zx} \Delta x \Delta y - \tau_{yx} \Delta x \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta z \Delta y$$
$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z + f_x \Delta x \Delta y \Delta z = 0, \tag{2.12}$$

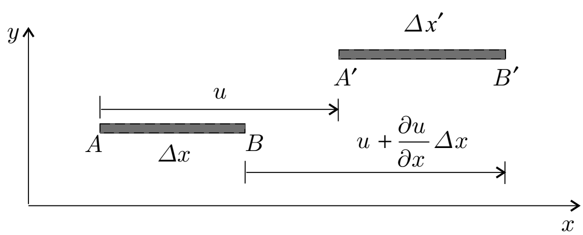


Figure 2.4: Displacement of the AB line segment.

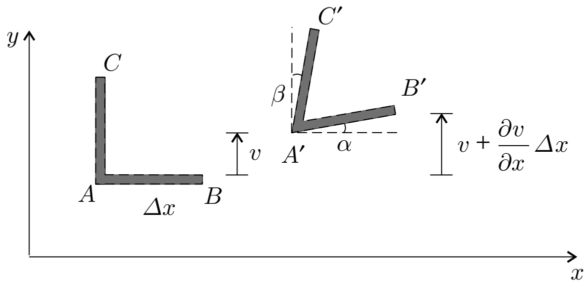


Figure 2.5: Displacement of the *ABC* segment.

where f_x is the body force per unit volume in the x direction. After simplification, this equation becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0. \tag{2.13}$$

By similar arguments, the equilibrium equations in the y and z directions are

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0, \tag{2.14}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0, \tag{2.15}$$

where f_y and f_z are the body forces per unit volume in the y and z directions.

A moment balance about an axis parallel to x and passing through the center (point O) gives (Fig. 2.7)

$$\tau_{yz}\Delta x\Delta z\frac{\Delta y}{2} - \tau_{zy}\Delta x\Delta y\frac{\Delta z}{2} + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\Delta y\right)\Delta x\Delta z\frac{\Delta y}{2} - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\Delta z\right)\Delta x\Delta y\frac{\Delta z}{2} = 0. \tag{2.16}$$

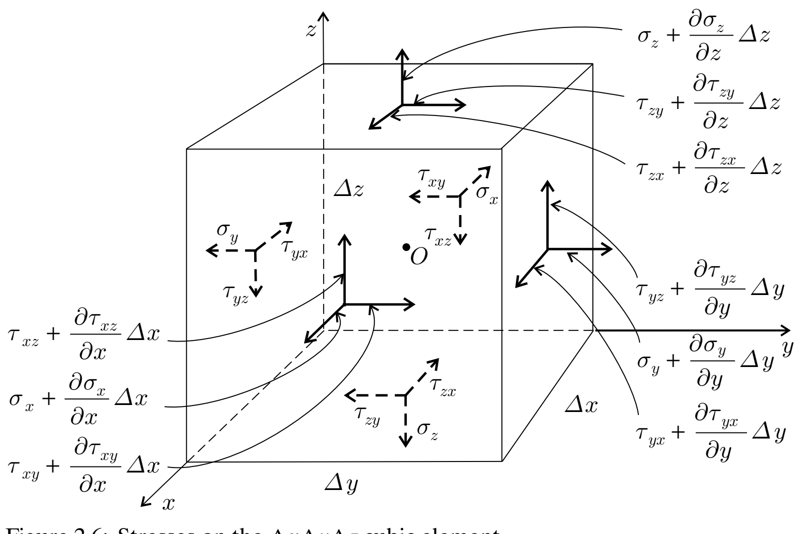


Figure 2.6: Stresses on the $\Delta x\Delta y\Delta z$ cubic element.

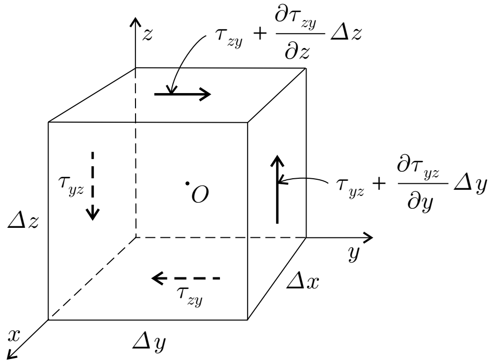


Figure 2.7: Stresses on the $\Delta x \Delta y \Delta z$ cubic element that appear in the moment balance about an axis parallel to x and passing through the center (point O).

By omitting higher order terms, which vanish in the limit $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0$, this equation becomes

$$\tau_{yz} = \tau_{zy}. \tag{2.17}$$

Similarly, we obtain the following equalities:

$$\tau_{xz} = \tau_{zx} \qquad \tau_{xy} = \tau_{yx}. \tag{2.18}$$

By virtue of Eqs. (2.17) and (2.18), the three equilibrium equations (Eqs. 2.13–2.15) contain six unknowns, namely, the three normal stresses ($\sigma_x, \sigma_y, \sigma_z$) and the three shear stresses ($\tau_{yz}, \tau_{xz}, \tau_{xy}$).

In the x_1, x_2, x_3 coordinate system the equilibrium equations are also given by Eqs. (2.13)–(2.15) with x, y, z replaced by x_1, x_2, x_3 and the subscripts x, y, z by 1, 2, 3.

2.3 Stress–Strain Relationships

In a composite material the fibers may be oriented in an arbitrary manner. Depending on the arrangements of the fibers, the material may behave differently in different directions. According to their behavior, composites may be characterized as generally anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic. In the following, we present the stress–strain relationships for these types of materials under linearly elastic conditions.

2.3.1 Generally Anisotropic Material

When there are no symmetry planes with respect to the alignment of the fibers the material is referred to as generally anisotropic. A fiber-reinforced composite material is, for example, generally anisotropic when the fibers are aligned in three nonorthogonal directions (Fig. 2.8).

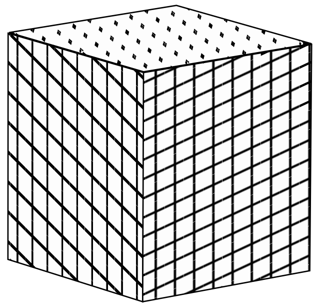


Figure 2.8: Example of a generally anisotropic material.

For a generally anisotropic linearly elastic material, in the x, y, z global coordinate system, the stress–strain relationships are

$$\begin{aligned}
\sigma_x &= \overline{C}_{11}\epsilon_x + \overline{C}_{12}\epsilon_y + \overline{C}_{13}\epsilon_z + \overline{C}_{14}\gamma_{yz} + \overline{C}_{15}\gamma_{xz} + \overline{C}_{16}\gamma_{xy}\\
\sigma_y &= \overline{C}_{21}\epsilon_x + \overline{C}_{22}\epsilon_y + \overline{C}_{23}\epsilon_z + \overline{C}_{24}\gamma_{yz} + \overline{C}_{25}\gamma_{xz} + \overline{C}_{26}\gamma_{xy}\\
\sigma_z &= \overline{C}_{31}\epsilon_x + \overline{C}_{32}\epsilon_y + \overline{C}_{33}\epsilon_z + \overline{C}_{34}\gamma_{yz} + \overline{C}_{35}\gamma_{xz} + \overline{C}_{36}\gamma_{xy}\\
\tau_{yz} &= \overline{C}_{41}\epsilon_x + \overline{C}_{42}\epsilon_y + \overline{C}_{43}\epsilon_z + \overline{C}_{44}\gamma_{yz} + \overline{C}_{45}\gamma_{xz} + \overline{C}_{46}\gamma_{xy}\\
\tau_{xz} &= \overline{C}_{51}\epsilon_x + \overline{C}_{52}\epsilon_y + \overline{C}_{53}\epsilon_z + \overline{C}_{54}\gamma_{yz} + \overline{C}_{55}\gamma_{xz} + \overline{C}_{56}\gamma_{xy}\\
\tau_{xy} &= \overline{C}_{61}\epsilon_x + \overline{C}_{62}\epsilon_y + \overline{C}_{63}\epsilon_z + \overline{C}_{64}\gamma_{yz} + \overline{C}_{65}\gamma_{xz} + \overline{C}_{66}\gamma_{xy}.
\end{aligned} \tag{2.19}$$

Equation (2.19) may be written in the form

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16}\\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26}\\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} & \overline{C}_{34} & \overline{C}_{35} & \overline{C}_{36}\\ \overline{C}_{41} & \overline{C}_{42} & \overline{C}_{43} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46}\\ \overline{C}_{51} & \overline{C}_{52} & \overline{C}_{53} & \overline{C}_{54} & \overline{C}_{55} & \overline{C}_{56}\\ \overline{C}_{61} & \overline{C}_{62} & \overline{C}_{63} & \overline{C}_{64} & \overline{C}_{65} & \overline{C}_{66}\end{bmatrix} \begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}, \tag{2.20}$$

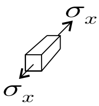
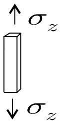
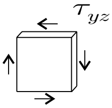
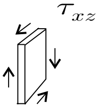
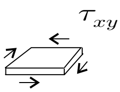
where $\overline{C}_{ij}$ are the elements of the stiffness matrix $[\overline{C}]$ in the x, y, z coordinate system.

Inversion of Eq. (2.20) results in the following strain–stress relationships:

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} & \overline{S}_{15} & \overline{S}_{16}\\ \overline{S}_{21} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} & \overline{S}_{25} & \overline{S}_{26}\\ \overline{S}_{31} & \overline{S}_{32} & \overline{S}_{33} & \overline{S}_{34} & \overline{S}_{35} & \overline{S}_{36}\\ \overline{S}_{41} & \overline{S}_{42} & \overline{S}_{43} & \overline{S}_{44} & \overline{S}_{45} & \overline{S}_{46}\\ \overline{S}_{51} & \overline{S}_{52} & \overline{S}_{53} & \overline{S}_{54} & \overline{S}_{55} & \overline{S}_{56}\\ \overline{S}_{61} & \overline{S}_{62} & \overline{S}_{63} & \overline{S}_{64} & \overline{S}_{65} & \overline{S}_{66}\end{bmatrix} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}, \tag{2.21}$$

where $\overline{S}_{ij}$ are the elements of the compliance matrix $[\overline{S}]$ in the x, y, z coordinate system and are defined in Table 2.3 (page 10). In this table tests are illustrated that, in principle, could provide means of determining the different compliance matrix elements.

Table 2.3. The elements of the compliance matrix $[\overline{S}]$ in the x, y, z coordinate system. The elements S_{ij} (without bar) in the x_1, x_2, x_3 coordinate system are obtained by replacing x, y, z by 1, 2, 3 on the right-hand sides of the expressions.

Test	Elements of the compliance matrix	
	$\overline{S}_{11} = \epsilon_x/\sigma_x$ $\overline{S}_{21} = \epsilon_y/\sigma_x$ $\overline{S}_{31} = \epsilon_z/\sigma_x$	$\overline{S}_{41} = \gamma_{yz}/\sigma_x$ $\overline{S}_{51} = \gamma_{xz}/\sigma_x$ $\overline{S}_{61} = \gamma_{xy}/\sigma_x$
	$\overline{S}_{12} = \epsilon_x/\sigma_y$ $\overline{S}_{22} = \epsilon_y/\sigma_y$ $\overline{S}_{32} = \epsilon_z/\sigma_y$	$\overline{S}_{42} = \gamma_{yz}/\sigma_y$ $\overline{S}_{52} = \gamma_{xz}/\sigma_y$ $\overline{S}_{62} = \gamma_{xy}/\sigma_y$
	$\overline{S}_{13} = \epsilon_x/\sigma_z$ $\overline{S}_{23} = \epsilon_y/\sigma_z$ $\overline{S}_{33} = \epsilon_z/\sigma_z$	$\overline{S}_{43} = \gamma_{yz}/\sigma_z$ $\overline{S}_{53} = \gamma_{xz}/\sigma_z$ $\overline{S}_{63} = \gamma_{xy}/\sigma_z$
	$\overline{S}_{14} = \epsilon_x/\tau_{yz}$ $\overline{S}_{24} = \epsilon_y/\tau_{yz}$ $\overline{S}_{34} = \epsilon_z/\tau_{yx}$	$\overline{S}_{44} = \gamma_{yz}/\tau_{yz}$ $\overline{S}_{54} = \gamma_{xz}/\tau_{yz}$ $\overline{S}_{64} = \gamma_{xy}/\tau_{yz}$
	$\overline{S}_{15} = \epsilon_x/\tau_{xz}$ $\overline{S}_{25} = \epsilon_y/\tau_{xz}$ $\overline{S}_{35} = \epsilon_z/\tau_{xz}$	$\overline{S}_{45} = \gamma_{yz}/\tau_{xz}$ $\overline{S}_{55} = \gamma_{xz}/\tau_{xz}$ $\overline{S}_{65} = \gamma_{xy}/\tau_{xz}$
	$\overline{S}_{16} = \epsilon_x/\tau_{xy}$ $\overline{S}_{26} = \epsilon_y/\tau_{xy}$ $\overline{S}_{36} = \epsilon_z/\tau_{xy}$	$\overline{S}_{46} = \gamma_{yz}/\tau_{xy}$ $\overline{S}_{56} = \gamma_{xz}/\tau_{xy}$ $\overline{S}_{66} = \gamma_{xy}/\tau_{xy}$

In the x_1, x_2, x_3 coordinate system the stress–strain relationships are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}, \tag{2.22}$$

where C_{ij} are the elements of the stiffness matrix $[C]$ in the x_1, x_2, x_3 coordinate system.

By inverting Eq. (2.22) we obtain the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}, \tag{2.23}$$

where S_{ij} are the elements of the compliance matrix $[S]$ in the x_1, x_2, x_3 coordinate system.