

MECHANICS OF COMPOSITE STRUCTURES

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Contents

<i>Preface</i>	<i>page xi</i>
<i>List of Symbols</i>	<i>xiii</i>
1 Introduction	1
2 Displacements, Strains, and Stresses	3
2.1 Strain–Displacement Relations	4
2.2 Equilibrium Equations	6
2.3 Stress-Strain Relationships	8
2.3.1 Generally Anisotropic Material	8
2.3.2 Monoclinic Material	11
2.3.3 Orthotropic Material	14
2.3.4 Transversely Isotropic Material	19
2.3.5 Isotropic Material	20
2.4 Plane–Strain Condition	22
2.4.1 Free End – Generally Anisotropic Material	28
2.4.2 Free End – Monoclinic Material	30
2.4.3 Free End – Orthotropic, Transversely Isotropic, or Isotropic Material	34
2.4.4 Built-In Ends – Generally Anisotropic Material	35
2.4.5 Built-In Ends – Monoclinic Material	36
2.4.6 Built-In Ends – Orthotropic, Transversely Isotropic, or Isotropic Material	38
2.5 Plane-Stress Condition	38
2.6 Hygrothermal Strains and Stresses	44
2.6.1 Plane-Strain Condition	47
2.6.2 Plane-Stress Condition	47
2.7 Boundary Conditions	47
2.8 Continuity Conditions	48
2.9 Stress and Strain Transformations	49
2.9.1 Stress Transformation	50
2.9.2 Strain Transformation	52
2.9.3 Transformation of the Stiffness and Compliance Matrices	53

2.10 Strain Energy	55
2.10.1 The Ritz Method	55
2.11 Summary	56
2.11.1 Note on the Compliance and Stiffness Matrices	56
3 Laminated Composites	63
3.1 Laminate Code	63
3.2 Stiffness Matrices of Thin Laminates	65
3.2.1 The Significance of the $[A]$, $[B]$, and $[D]$ Stiffness Matrices	72
3.2.2 Stiffness Matrices for Selected Laminates	74
4 Thin Plates	89
4.1 Governing Equations	90
4.1.1 Boundary Conditions	92
4.1.2 Strain Energy	92
4.2 Deflection of Rectangular Plates	93
4.2.1 Pure Bending and In-Plane Loads	93
4.2.2 Long Plates	94
4.2.3 Simply Supported Plates – Symmetrical Layup	100
4.2.4 Plates with Built-In Edges – Orthotropic and Symmetrical Layup	107
4.3 Buckling of Rectangular Plates	112
4.3.1 Simply Supported Plates – Symmetrical Layup	112
4.3.2 Plates with Built-In and Simply Supported Edges – Orthotropic and Symmetrical Layup	118
4.3.3 Plates with One Free Edge – Orthotropic and Symmetrical Layup	124
4.3.4 Plates with Rotationally Restrained Edges – Orthotropic and Symmetrical Layup	127
4.3.5 Long Plates	132
4.4 Free Vibration of Rectangular Plates	141
4.4.1 Long Plates	141
4.4.2 Simply Supported Plates – Symmetrical Layup	144
4.4.3 Plates with Built-In and Simply Supported Edges – Orthotropic and Symmetrical Layup	149
4.5 Hygrothermal Effects	151
4.5.1 Change in Thickness Due to Hygrothermal Effects	161
4.6 Plates with a Circular or an Elliptical Hole	163
4.7 Interlaminar Stresses	166
5 Sandwich Plates	169
5.1 Governing Equations	170
5.1.1 Boundary Conditions	172
5.1.2 Strain Energy	173
5.1.3 Stiffness Matrices of Sandwich Plates	174
5.2 Deflection of Rectangular Sandwich Plates	178
5.2.1 Long Plates	178
5.2.2 Simply Supported Sandwich Plates – Orthotropic and Symmetrical Layup	182

5.3 Buckling of Rectangular Sandwich Plates	185
5.3.1 Long Plates	185
5.3.2 Simply Supported Plates – Orthotropic and Symmetrical Layup	187
5.3.3 Face Wrinkling	190
5.4 Free Vibration of Rectangular Sandwich Plates	196
5.4.1 Long Plates	196
5.4.2 Simply Supported Plates – Orthotropic and Symmetrical Layup	199
6 Beams	203
6.1 Governing Equations	203
6.1.1 Boundary Conditions	205
6.1.2 Stiffness Matrix	205
6.1.3 Compliance Matrix	209
6.1.4 Replacement Stiffnesses	210
6.2 Rectangular, Solid Beams Subjected to Axial Load and Bending	210
6.2.1 Displacements – Symmetrical Layup	211
6.2.2 Displacements – Unsymmetrical Layup	213
6.2.3 Stresses and Strains	214
6.3 Thin-Walled, Open-Section Orthotropic or Symmetrical Cross-Section Beams Subjected to Axial Load and Bending	217
6.3.1 Displacements of T-Beams	217
6.3.2 Displacements of L-Beams	221
6.3.3 Displacements of Arbitrary Cross-Section Beams	226
6.3.4 Stresses and Strains	233
6.4 Thin-Walled, Closed-Section Orthotropic Beams Subjected to Axial Load and Bending	243
6.5 Torsion of Thin-Walled Beams	248
6.5.1 Thin Rectangular Cross Section	248
6.5.2 Open-Section Orthotropic Beams	250
6.5.3 Closed-Section Orthotropic Beams – Single Cell	252
6.5.4 Closed-Section Orthotropic Beams – Multicell	260
6.5.5 Restrained Warping – Open-Section Orthotropic Beams	261
6.5.6 Restrained Warping – Closed-Section Orthotropic Beams	264
6.6 Thin-Walled Beams with Arbitrary Layup Subjected to Axial Load, Bending, and Torsion	265
6.6.1 Displacements of Open- and Closed-Section Beams	267
6.6.2 Stresses and Strains in Open- and Closed-Section Beams	268
6.6.3 Centroid	271
6.6.4 Restrained Warping	271
6.7 Transversely Loaded Thin-Walled Beams	274
6.7.1 Beams with Orthotropic Layup or with Symmetrical Cross Section	276
6.7.2 Beams with Arbitrary Layup	280
6.7.3 Shear Center	283
6.8 Stiffened Thin-Walled Beams	288

6.9	Buckling of Beams	290
6.9.1	Beams Subjected to Axial Load (Flexural–Torsional Buckling)	291
6.9.2	Lateral–Torsional Buckling of Orthotropic Beams with Symmetrical Cross Section	296
6.9.3	Local Buckling	300
6.10	Free Vibration of Beams (Flexural–Torsional Vibration)	306
6.10.1	Doubly Symmetrical Cross Sections	306
6.10.2	Beams with Symmetrical Cross Sections	309
6.10.3	Beams with Unsymmetrical Cross Sections	309
6.11	Summary	312
7	Beams with Shear Deformation	313
7.1	Governing Equations	314
7.1.1	Strain–Displacement Relationships	315
7.1.2	Force–Strain Relationships	315
7.1.3	Equilibrium Equations	320
7.1.4	Summary of Equations	320
7.1.5	Boundary Conditions	321
7.2	Stiffnesses and Compliances of Beams	321
7.2.1	Shear Stiffnesses and Compliances of Thin-Walled Open-Section Beams	322
7.2.2	Shear Stiffnesses and Compliances of Thin-Walled Closed-Section Beams	325
7.2.3	Stiffnesses of Sandwich Beams	326
7.3	Transversely Loaded Beams	329
7.4	Buckling of Beams	334
7.4.1	Axially Loaded Beams with Doubly Symmetrical Cross Sections (Flexural and Torsional Buckling)	335
7.4.2	Axially Loaded Beams with Symmetrical or Unsymmetrical Cross Sections (Flexural–Torsional Buckling)	341
7.4.3	Lateral–Torsional Buckling of Beams with Symmetrical Cross Section	345
7.4.4	Summary	346
7.5	Free Vibration of Beams	347
7.5.1	Beams with Doubly Symmetrical Cross Sections	347
7.5.2	Beams with Symmetrical or Unsymmetrical Cross Sections	356
7.5.3	Summary	359
7.6	Effect of Shear Deformation	359
8	Shells	365
8.1	Shells of Revolution with Axisymmetrical Loading	367
8.2	Cylindrical Shells	368
8.2.1	Membrane Theory	368
8.2.2	Built-In Ends	370
8.2.3	Temperature – Built-In Ends	379
8.3	Springback	380
8.3.1	Springback of Cylindrical Shells	380
8.3.2	Doubly Curved Shells	384

8.4	Buckling of Shells	384
8.4.1	Buckling of Cylinders	387
9	Finite Element Analysis	395
9.1	Three-Dimensional Element	396
9.2	Plate Element	397
9.3	Beam Element	397
9.4	Sublaminates	398
9.4.1	Step 1. Elements of $[J]$ due to In-Plane Stresses	400
9.4.2	Step 2. Elements of $[J]$ due to Out-of-Plane Normal Stresses	403
9.4.3	Step 3. Elements of $[J]$ due to Out-of-Plane Shear Stresses	405
9.4.4	Step 4. The Stiffness Matrix	407
10	Failure Criteria	411
10.1	Quadratic Failure Criterion	413
10.1.1	Orthotropic Material	414
10.1.2	Transversely Isotropic Material	420
10.1.3	Isotropic Material	421
10.1.4	Plane-Strain and Plane-Stress Conditions	422
10.1.5	Proportional Loading – Stress Ratio	423
10.2	“Maximum Stress” Failure Criterion	425
10.3	“Maximum Strain” Failure Criterion	426
10.4	Plate with a Hole or a Notch	430
10.4.1	Plate with a Circular Hole	431
10.4.2	Plate with a Notch	434
10.4.3	Characteristic Length	434
11	Micromechanics	436
11.1	Rule of Mixtures	436
11.1.1	Longitudinal Young Modulus E_1	438
11.1.2	Transverse Young Modulus E_2	439
11.1.3	Longitudinal Shear Modulus G_{12}	439
11.1.4	Transverse Shear Modulus G_{23}	440
11.1.5	Longitudinal Poisson Ratio ν_{12}	441
11.1.6	Transverse Poisson Ratio ν_{23}	442
11.1.7	Thermal Expansion Coefficients	443
11.1.8	Moisture Expansion Coefficients	445
11.1.9	Thermal Conductivity	446
11.1.10	Moisture Diffusivity	447
11.1.11	Specific Heat	448
11.2	Modified Rule of Mixtures	448
11.3	Note on the Micromechanics Models	449
	Appendix A. Cross-Sectional Properties of Thin-Walled Composite Beams	453
	Appendix B. Buckling Loads and Natural Frequencies of Orthotropic Beams with Shear Deformation	461
	Appendix C. Typical Material Properties	464
	<i>Index</i>	469

List of Symbols

We have used, wherever possible, notation standard in elasticity, structural analysis, and composite materials. We tried to avoid duplication, although there is some repetition of those symbols that are used only locally. In the following list we have not included those symbols that pertain only to the local discussion. Below, we give a verbal description of each symbol and, when appropriate, the number of the equation in which the symbol is first used.

Latin letters

A	area
A^{iso}	tensile stiffness of an isotropic laminate (Eq. 3.42)
$[A], A_{ij}$	tensile stiffness of a laminate (Eqs. 3.18, 3.19)
$[a], a_{ij}$	inverse of the $[A]$ matrix for symmetric laminates (Eq. 3.29)
$[B], B_{ij}$	stiffness of a laminate (Eqs. 3.18, 3.19)
$[C], C_{ij}$	3D stiffness matrix in the x_1, x_2, x_3 coordinate system (Eq. 2.22)
$[\bar{C}], \bar{C}_{ij}$	3D stiffness matrix in the x, y, z coordinate system (Eq. 2.19)
c	moisture concentration (Eq. 2.154); core thickness (Fig. 5.2)
$[D], D_{ij}$	bending stiffness of a laminate (Eqs. 3.18, 3.19)
$[D]^*, D_{ij}^*$	reduced bending stiffness of a laminate (Eq. 4.1)
D^{iso}	bending stiffness of an isotropic laminate (Eq. 3.42)
D, \bar{D}, \widehat{D}	parameters (Table 6.2, page 222, Eq. 6.157)
$[d], d_{ij}$	inverse of the $[D]$ matrix for symmetrical laminates (Eq. 3.30)
d, d^t, d^b	distances for sandwich plates (Fig. 5.2)
E_1, E_2, E_3	Young's moduli in the x_1, x_2, x_3 coordinate system (Table 2.5)
$[E]$	stiffness matrix in the FE calculation (Eq. 9.4)
\widehat{EA}	tensile stiffness of a beam (Eq. 6.8)
\widehat{EI}	bending stiffness of a beam (Eq. 6.8)
\widehat{EI}_ω	warping stiffness of a beam (Eq. 6.244)
F_i, F_{ij}	strength parameters in the quadratic failure criterion (Eq. 10.2)

f_{ij}	constants in the quadratic failure criterion (Eq. 10.25)
f, f_{ij}	frequency (Eq. 4.190)
$f_x, f_y, f_z,$	body forces per unit volume (Eq. 2.13)
G_{23}, G_{13}, G_{12}	shear moduli in the x_1, x_2, x_3 coordinate system (Table 2.5)
\widehat{GI}_t	torsional stiffness of a beam (Eq. 6.8)
h	plate thickness
h_b, h_t	distances of the bottom and top surfaces of a plate from the reference plane (Eq. 3.9)
i_ω	polar radius of gyration (Eq. 6.340)
$[J]$	inverse of the material stiffness matrix $[E]$ (Eq. 9.16)
K	number of layers in a laminate; number of wall segments; stiffness parameter of a plate (Eq. 4.153)
\tilde{k}	rotational spring constant (Eq. 4.149)
k	equivalent length factor (Eq. 6.340)
L_x, L_y	dimensions of a plate
L	length; number of cells in a multicell beam (Eq. 6.222)
L_i, L_i^f	load and failure load (Eq. 10.42)
l_x, l_x^o	half buckling length (Eq. 4.142), half buckling length corresponding to the lowest buckling load of a long plate (Eq. 4.173)
M_x, M_y, M_{xy}	bending and twist moments per unit length acting on a laminate (Eq. 3.9)
$M_x^{ht}, M_y^{ht}, M_{xy}^{ht}$	hygrothermal moments per unit length (Eq. 4.247)
$\widehat{M}_y, \widehat{M}_z$	bending moments acting on a beam (Fig. 6.2)
\widehat{M}_ω	bimoment acting on a beam (Eq. 6.232)
N_x, N_y, N_{xy}	in-plane forces per unit length acting on a laminate (Eq. 3.9)
N_{x0}, N_{y0}, N_{xy0}	in-plane compressive forces per unit length (Eq. 4.109)
$N_x^{ht}, N_y^{ht}, N_{xy}^{ht}$	hygrothermal forces per unit length (Eq. 4.246)
$N_{x, cr}$	buckling load of a uniaxially loaded plate (Eq. 4.141)
\widehat{N}	axial force acting on a beam (Fig. 6.2)
$\widehat{N}_{cr}, \widehat{N}_{cr}^B$	buckling load and buckling load due to bending deformation (Eq. 6.337)
$\widehat{N}_{cry}, \widehat{N}_{crz}$	buckling load in the x - z and x - y planes, respectively (Eqs. 6.337, 7.110)
$\widehat{N}_{cr\psi}$	buckling load under torsional buckling (Eqs. 6.337, 7.110)
$[P], [\bar{P}]$	stiffness matrix of a beam (Eqs. 6.2, 6.250). Without bar refers to the centroid; with bar to an arbitrarily chosen coordinate system
p	transverse load per unit area; distance between the origin and the tangent of the wall of a beam (Eq. 6.190)
p_x, p_y, p_z	axial and transverse loads (per unit length) acting on a beam (Fig. 6.1); surface forces per unit area (Eq. 2.166)
$[Q], Q_{ij}$	2D plane-stress stiffness matrix in the x_1, x_2 coordinate system (Eq. 2.134)

$[\bar{Q}], \bar{Q}_{ij}$	2D plane-stress stiffness matrix in the x, y coordinate system (Eq. 2.126)
\hat{Q}_{cr}	buckling load resulting in lateral buckling (Eq. 6.359)
q	shear flow (Eq. 6.189).
R	stiffness parameter (Eq. 3.46)
\tilde{R}	stress ratio (Eq. 10.42)
R_x, R_y, R_{xy}	radii of curvatures of a shell (Eq. 8.1)
$[R], R_{ij}$	compliance matrix under plane-strain condition in the x_1, x_2 coordinate system (Eq. 2.79)
$[\bar{R}], \bar{R}_{ij}$	compliance matrix under plane-strain condition in the x, y coordinate system (Eq. 2.65)
$[S], S_{ij}$	3D compliance matrix in the x_1, x_2, x_3 coordinate system (Eq. 2.23)
$[\bar{S}], \bar{S}_{ij}$	3D compliance matrix in the x, y, z coordinate system (Eq. 2.21)
\hat{S}_{ij}	shear stiffness of a beam, $i, j = z, y, \omega$ (Eqs. 7.13, 7.36)
\tilde{S}_{ij}	shear stiffness of a plate, $i, j = 1, 2$ (Eq. 5.15)
\hat{s}_{ij}	shear compliance of a beam, $i, j = z, y, \omega$ (Eq. 7.38)
s_1^+, s_2^+, s_3^+	tensile strengths (Eq. 10.13)
s_1^-, s_2^-, s_3^-	compression strengths (Eq. 10.13)
s_{23}, s_{13}, s_{12}	shear strengths (Eq. 10.15)
\hat{T}	torque acting on a beam (Fig. 6.2)
\hat{T}_ω	restrained warping-induced torque (Eq. 6.235)
\hat{T}_{sv}	Saint-Venant torque (Eq. 6.239)
$[T_\sigma]$	2D stress transformation matrix (Eq. 2.182)
$[\hat{T}_\sigma]$	3D stress transformation matrix (Eq. 2.179)
$[T_\epsilon]$	2D strain transformation matrix (Eq. 2.188)
$[\hat{T}_\epsilon]$	3D strain transformation matrix (Eq. 2.185)
t	torque load acting on a beam (Fig. 6.1)
t^t, t^b	thicknesses of the top and bottom facesheets (Eq. 5.26)
U	strain energy (Eq. 2.200)
U	displacement in the x direction; varies with the x and y coordinates only (Eq. 2.50)
u	displacement in the x direction
u^0	displacement of the reference surface in the x direction
u_1, u_2, u_3	displacements in the x_1, x_2 , and x_3 direction
V	displacement in the y direction; varies with the x and y coordinates only (Eq. 2.51)
V_f, V_m, V_v	volume of fibers, matrix, and void
V_x, V_y	out-of-plane shear forces per unit length (Eq. 3.10)
\hat{V}_y, \hat{V}_z	transverse shear forces acting on a beam (Fig. 6.2)
v	displacement in the y direction
v^0	displacement of the reference surface in the y direction
v_f, v_m, v_v	volume fraction of fibers, matrix, and void

W	displacement in the z direction; varies with the x and y coordinates only (Eq. 2.52)
$[W], [\bar{W}]$	compliance matrix of a beam (Eq. 6.17). No bar refers to the centroid; bar to an arbitrarily chosen coordinate system
w	deflection in the z direction
\tilde{w}	maximum deflection in the z direction (Eq. 4.29)
w^o	deflection of the reference surface in the z direction
w^B, w^S	deflections due to bending and shear deformations (Eq. 7.85)
y_c, z_c	coordinates of the centroid of a beam (Eqs. 6.54, 6.73)
y_{sc}, z_{sc}	coordinates of the shear center of a beam (Eq. 6.311)
z_k, z_{k-1}	coordinates of the top and bottom surfaces of the k th ply in a laminate (Eq. 3.20)

Greek letters

α	parameter describing shear deformation (Eq. 7.253)
α_i	parameter describing shear deformation, $i = w, \psi, N, \omega$ (Eq. 7.244)
$[\alpha], \alpha_{ij}$	compliance matrix of a laminate (Eq. 3.23)
α, β	parameters describing buckled shape of a shell (Eq. 8.78)
$\hat{\alpha}_{ij}$	compliances for closed-section beams (Eq. 6.156)
$\tilde{\alpha}_i, \tilde{\alpha}_{ij}$	thermal expansion coefficients (Eqs. 2.153, 2.158)
β, λ	parameters in the displacements of a cylinder (Eq. 8.30)
$[\beta], \beta_{ij}$	compliance matrix of a laminate (Eq. 3.23)
$\bar{\beta}_{ij}$	compliance of symmetrical cross-section beams (Table 6.2)
$\hat{\beta}_{ij}$	compliance of closed-section beams (Eq. 6.156)
$\tilde{\beta}_i, \tilde{\beta}_{ij}$	moisture expansion coefficients in the x, y, z directions (Eqs. 2.154, 2.159)
β_1	property of the cross section (Eq. 6.360)
γ_y, γ_z	shear strain in a beam in the x - y and x - z planes (Eq. 7.2)
$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	engineering shear strain in the x, y, z coordinate system (Eq. 2.9)
$\gamma_{23}, \gamma_{13}, \gamma_{12}$	engineering shear strain in the x_1, x_2, x_3 coordinate system
Δh	change in thickness (Eq. 4.282)
ΔT	temperature change (Eq. 2.153)
$[\delta], \delta_{ij}$	compliance matrix of a laminate (Eq. 3.23)
$\hat{\delta}_{ij}$	compliance of closed-section beams (Eq. 6.157)
$\bar{\epsilon}_x, \dots$	average strains in a sublaminde (Eq. 9.14)
$\epsilon_x, \epsilon_y, \epsilon_z$	engineering normal strains in the x, y, z coordinate system
$\epsilon_1, \epsilon_2, \epsilon_3$	engineering normal strains in the x_1, x_2, x_3 coordinate system
$\epsilon_x^o, \epsilon_y^o, \gamma_{xy}^o$	strains of the reference surface
$\epsilon_x^{o,ht}, \epsilon_y^{o,ht}, \gamma_{xy}^{o,ht}$	hygrothermal strains in a laminate (Eq. 4.250)
ζ	parameter of restraint (Eq. 4.152)
Θ	polar moment of mass (Eq. 6.411)

Θ_k	ply orientation
ϑ	rate of twist (Eq. 6.1)
ϑ^B, ϑ^S	rate of twist due to bending and shear deformation (Eq. 7.5)
$\kappa_x, \kappa_y, \kappa_{xy}$	curvatures of the reference surface (Eq. 3.8)
$\kappa_x^{ht}, \kappa_y^{ht}, \kappa_{xy}^{ht}$	hygrothermal curvatures of a laminate (Eq. 4.250)
$\lambda, \lambda_{cr}, \lambda_{ij}$	load parameter (Eq. 4.109); buckling load parameter (Eq. 4.121); eigenvalue (Eq. 4.225)
$\mu_{Bi}, \mu_{Gi}, \mu_{Si}$	parameters in the calculation of natural frequencies (Eqs. 6.398, 6.400, 7.203)
ν_{ij}	Poisson's ratio
ξ, η, ζ	coordinates attached to the wall of a beam (Fig. 6.13)
ξ, ξ'	parameters in the expressions of the buckling loads of plates with rotationally restrained edges (Eq. 4.151)
π_p	potential energy (Eq. 2.204)
ρ_x, ρ_y, ρ_z	radius of curvature in the $y-z$, $x-z$, and $x-y$ planes (Eq. 2.45)
ρ_1, ρ_2, ρ_3	radius of curvature in the x_2-x_3 , x_1-x_3 , and x_1-x_2 planes (Eq. 2.53)
$\rho_{comp}, \rho_f, \rho_m$	densities of composite, fiber, and matrix
ρ	mass per unit area or per unit length
$\sigma_1, \sigma_2, \sigma_3$	normal stresses in the x_1, x_2, x_3 coordinate system
$\sigma_x, \sigma_y, \sigma_z$	normal stresses in the x, y, z coordinate system
$\bar{\sigma}$	average stress
$\tau_{23}, \tau_{13}, \tau_{12}$	shear stresses in the x_1, x_2, x_3 coordinate system
$\tau_{yz}, \tau_{xz}, \tau_{yx}$	shear stresses in the x, y, z coordinate system
χ_{xz}, χ_{yz}	rotation of the normal of a plate in the $x-z$ and $x-y$ planes (Eqs. 3.2 and 5.1)
χ_y, χ_z	rotation of the cross section of a beam in the $x-y$ and $x-z$ planes (Eq. 7.2)
ψ	angle of rotation of the cross section about the beam axis (twist) (Fig. 6.3)
Ψ	bending stiffness of an unsymmetrical long plate (Eq. 4.52)
Ω	potential energy of the external loads (Eq. 2.203)
ω	circular frequency (Eq. 4.190)
ω^B, ω^S	circular frequency of a beam due to bending and shear deformation (Eq. 7.198)
ω_y, ω_z	circular frequency of a freely vibrating beam in the $x-z$ and $x-y$ planes, respectively (Eq. 6.398)
ω_ψ	circular frequency of a freely vibrating beam under torsional vibration (Eq. 6.400)
$\varrho, \tilde{\varrho}, \bar{\varrho}, \hat{\varrho}$	distances between the new and the old reference surfaces (Eqs. 3.47, 6.105, 6.107, A.3)

Displacements, Strains, and Stresses

We consider composite materials consisting of continuous or discontinuous fibers embedded in a matrix. Such a composite is heterogeneous, and the properties vary from point to point. On a scale that is large with respect to the fiber diameter, the fiber and matrix properties may be averaged, and the material may be treated as homogeneous. This assumption, commonly employed in macromechanical analyses of composites, is adopted here. Hence, the material is considered to be quasi-homogeneous, which implies that the properties are taken to be the same at every point. These properties are not the same as the properties of either the fiber or the matrix but are a combination of the properties of the constituents.

In this chapter, equations are presented for calculating the displacements, stresses, and strains when the structure undergoes only small deformations and the material behaves in a linearly elastic manner.

Continuous fiber-reinforced composite materials (and structures made of such materials) often have easily identifiable preferred directions associated with fiber orientations or symmetry planes. It is therefore convenient to employ two coordinate systems: a local coordinate system aligned, at a point, either with the fibers or with axes of symmetry, and a global coordinate system attached to a fixed reference point (Fig. 2.1). In this book the local and global Cartesian coordinate systems are designated respectively by x_1, x_2, x_3 and the x, y, z axes. In the x, y, z directions the displacements at a point A are denoted by u, v, w , and in the x_1, x_2, x_3 directions by u_1, u_2, u_3 (Fig. 2.2).

In the x, y, z coordinate system the normal stresses are denoted by σ_x, σ_y , and σ_z and the shear stresses by τ_{yz}, τ_{xz} , and τ_{xy} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_x, \epsilon_y, \epsilon_z$ and $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$, respectively.

In the x_1, x_2, x_3 coordinate system the normal stresses are denoted by σ_1, σ_2 , and σ_3 and the shear stresses by τ_{23}, τ_{13} , and τ_{12} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_1, \epsilon_2, \epsilon_3$, and $\gamma_{23}, \gamma_{13}, \gamma_{12}$, respectively. The symbol γ represents engineering shear strain that is twice the tensorial shear strain, $\gamma_{ij} = 2\epsilon_{ij}$ ($i, j = x, y, z$ or $i, j = 1, 2, 3$).

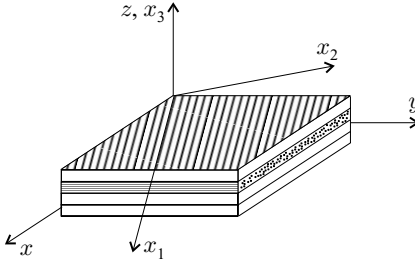


Figure 2.1: The global x, y, z and local x_1, x_2, x_3 coordinate systems.

A stress is taken to be positive when it acts on a positive face in the positive direction. According to this definition, all the stresses shown in Figure 2.3 are positive.

The preceding stress and strain notations, referred to as engineering notations, are used throughout this book. Other notations, most notably tensorial and contracted notations, can frequently be found in the literature. The stresses and strains in different notations are summarized in Tables 2.1 and 2.2.

2.1 Strain–Displacement Relations

We consider a Δx long segment that undergoes a change in length, the new length being denoted by $\Delta x'$. From Figure 2.4 it is seen that

$$u + \Delta x' = \Delta x + \left(u + \frac{\partial u}{\partial x} \Delta x \right), \quad (2.1)$$

where u and $u + \frac{\partial u}{\partial x} \Delta x$ are the displacements of points A and B , respectively, in the x direction. Accordingly, the normal strain in the x direction is

$$\epsilon_x = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}. \quad (2.2)$$

Similarly, in the y and z directions the normal strains are

$$\epsilon_y = \frac{\partial v}{\partial y} \quad (2.3)$$

$$\epsilon_z = \frac{\partial w}{\partial z}, \quad (2.4)$$

where v and w are the displacements in the y and z directions, respectively.

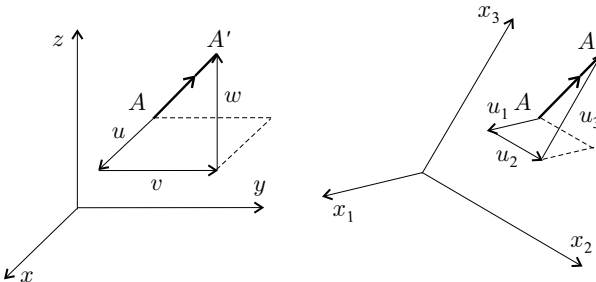


Figure 2.2: The x, y, z and x_1, x_2, x_3 coordinate systems and the corresponding displacements.

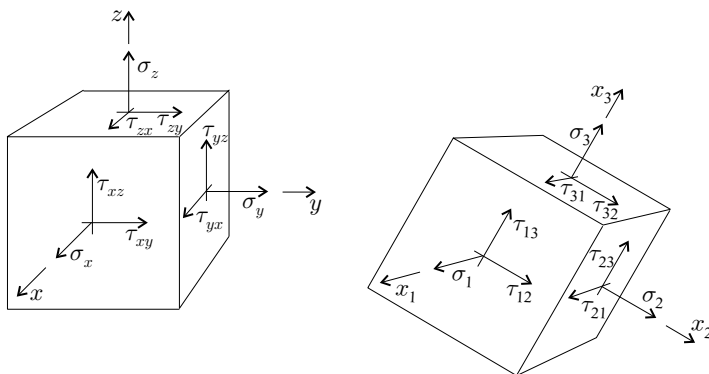


Figure 2.3: The stresses in the global x, y, z and the local x_1, x_2, x_3 coordinate systems.

For angular (shear) deformation the tensorial shear strain is the average change in the angle between two mutually perpendicular lines (Fig. 2.5)

$$\epsilon_{xy} = \frac{\alpha + \beta}{2}. \tag{2.5}$$

For small deformations we have

$$\alpha \approx \tan \alpha = \frac{(v + \frac{\partial v}{\partial x} \Delta x) - v}{\Delta x} = \frac{\partial v}{\partial x}. \tag{2.6}$$

Similarly $\beta = \partial u / \partial y$, and the xy component of the tensorial shear strain is

$$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \tag{2.7}$$

In a similar manner we obtain the following expressions for the ϵ_{yz} and ϵ_{xz} components of the tensorial shear strains:

$$\epsilon_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad \epsilon_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right). \tag{2.8}$$

Table 2.1. Stress notations						
	Normal stress			Shear stress		
x, y, z coordinate system						
Tensorial stress	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yz}	σ_{xz}	σ_{xy}
Engineering stress	σ_x	σ_y	σ_z	τ_{yz}	τ_{xz}	τ_{xy}
Contracted notation	σ_x	σ_y	σ_z	σ_q	σ_r	σ_s
x_1, x_2, x_3 coordinate system						
Tensorial stress	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{13}	σ_{12}
Engineering stress	σ_1	σ_2	σ_3	τ_{23}	τ_{13}	τ_{12}
Contracted notation	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6

Table 2.2. Strain notations (the engineering and contracted notation shear strains are twice the tensorial shear strain)

	Normal strain			Shear strain		
<i>x, y, z</i> coordinate system						
Tensorial strain	ϵ_{xx}	ϵ_{yy}	ϵ_{zz}	ϵ_{yz}	ϵ_{xz}	ϵ_{xy}
Engineering strain	ϵ_x	ϵ_y	ϵ_z	γ_{yz}	γ_{xz}	γ_{xy}
Contracted notation	ϵ_x	ϵ_y	ϵ_z	ϵ_q	ϵ_r	ϵ_s
<i>x₁, x₂, x₃</i> coordinate system						
Tensorial strain	ϵ_{11}	ϵ_{22}	ϵ_{33}	ϵ_{23}	ϵ_{13}	ϵ_{12}
Engineering strain	ϵ_1	ϵ_2	ϵ_3	γ_{23}	γ_{13}	γ_{12}
Contracted notation	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6

The engineering shear strains are twice the tensorial shear strains:

$$\gamma_{yz} = 2\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad (2.9)$$

$$\gamma_{xz} = 2\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (2.10)$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2.11)$$

In the x_1, x_2, x_3 coordinate system the strain–displacement relationships are also given by Eqs. (2.2)–(2.4) and (2.9)–(2.11) with x, y, z replaced by x_1, x_2, x_3 , the subscripts x, y, z by 1, 2, 3, and u, v, w by u_1, u_2, u_3 .

2.2 Equilibrium Equations

The equilibrium equations at a point O are obtained by considering force and moment balances on a small $\Delta x \Delta y \Delta z$ cubic element located at point O . (The point O is at the center of the element, Fig. 2.6.) We relate the stresses at one face to those at the opposite face by the Taylor series. By using only the first term of the Taylor series, force balance in the x direction gives

$$\begin{aligned} & -\sigma_x \Delta z \Delta y - \tau_{zx} \Delta x \Delta y - \tau_{yx} \Delta x \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x \right) \Delta z \Delta y \\ & + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right) \Delta x \Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y \right) \Delta x \Delta z + f_x \Delta x \Delta y \Delta z = 0, \end{aligned} \quad (2.12)$$

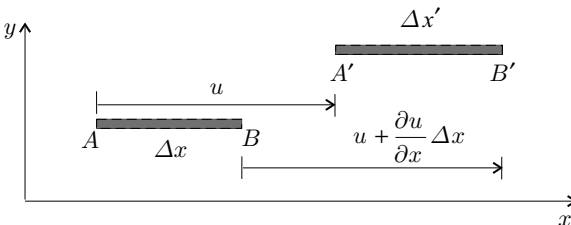


Figure 2.4: Displacement of the AB line segment.

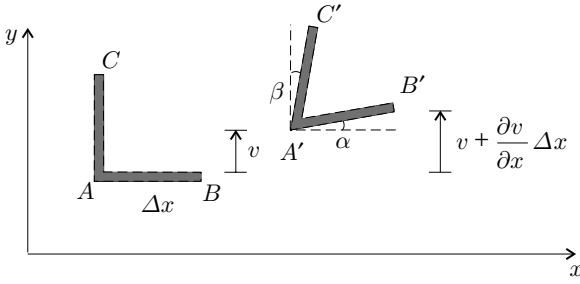


Figure 2.5: Displacement of the ABC segment.

where f_x is the body force per unit volume in the x direction. After simplification, this equation becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0. \quad (2.13)$$

By similar arguments, the equilibrium equations in the y and z directions are

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0, \quad (2.14)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0, \quad (2.15)$$

where f_y and f_z are the body forces per unit volume in the y and z directions.

A moment balance about an axis parallel to x and passing through the center (point O) gives (Fig. 2.7)

$$\begin{aligned} & \tau_{yz} \Delta x \Delta z \frac{\Delta y}{2} - \tau_{zy} \Delta x \Delta y \frac{\Delta z}{2} \\ & + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} \Delta y \right) \Delta x \Delta z \frac{\Delta y}{2} - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} \Delta z \right) \Delta x \Delta y \frac{\Delta z}{2} = 0. \end{aligned} \quad (2.16)$$

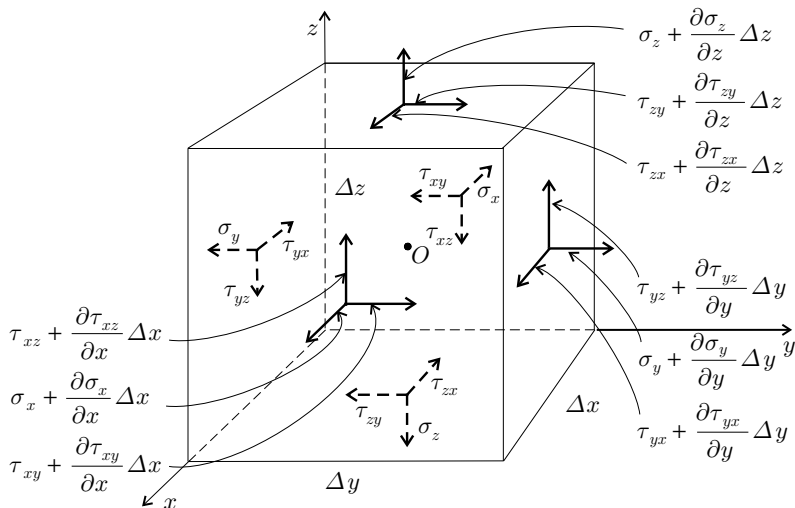


Figure 2.6: Stresses on the $\Delta x \Delta y \Delta z$ cubic element.

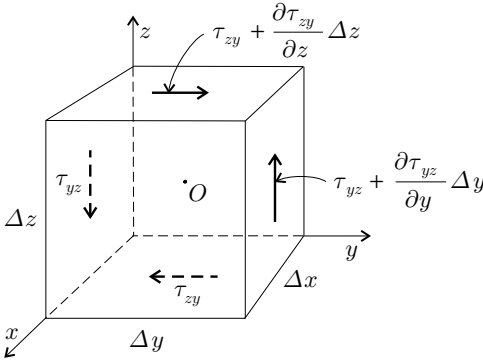


Figure 2.7: Stresses on the $\Delta x \Delta y \Delta z$ cubic element that appear in the moment balance about an axis parallel to x and passing through the center (point O).

By omitting higher order terms, which vanish in the limit $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$, this equation becomes

$$\tau_{yz} = \tau_{zy}. \quad (2.17)$$

Similarly, we obtain the following equalities:

$$\tau_{xz} = \tau_{zx} \quad \tau_{xy} = \tau_{yx}. \quad (2.18)$$

By virtue of Eqs. (2.17) and (2.18), the three equilibrium equations (Eqs. 2.13–2.15) contain six unknowns, namely, the three normal stresses (σ_x , σ_y , σ_z) and the three shear stresses (τ_{yz} , τ_{xz} , τ_{xy}).

In the x_1, x_2, x_3 coordinate system the equilibrium equations are also given by Eqs. (2.13)–(2.15) with x, y, z replaced by x_1, x_2, x_3 and the subscripts x, y, z by 1, 2, 3.

2.3 Stress–Strain Relationships

In a composite material the fibers may be oriented in an arbitrary manner. Depending on the arrangements of the fibers, the material may behave differently in different directions. According to their behavior, composites may be characterized as generally anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic. In the following, we present the stress–strain relationships for these types of materials under linearly elastic conditions.

2.3.1 Generally Anisotropic Material

When there are no symmetry planes with respect to the alignment of the fibers the material is referred to as generally anisotropic. A fiber-reinforced composite material is, for example, generally anisotropic when the fibers are aligned in three nonorthogonal directions (Fig. 2.8).

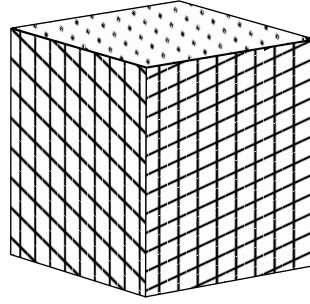


Figure 2.8: Example of a generally anisotropic material.

For a generally anisotropic linearly elastic material, in the x, y, z global coordinate system, the stress–strain relationships are

$$\begin{aligned}
 \sigma_x &= \bar{C}_{11}\epsilon_x + \bar{C}_{12}\epsilon_y + \bar{C}_{13}\epsilon_z + \bar{C}_{14}\gamma_{yz} + \bar{C}_{15}\gamma_{xz} + \bar{C}_{16}\gamma_{xy} \\
 \sigma_y &= \bar{C}_{21}\epsilon_x + \bar{C}_{22}\epsilon_y + \bar{C}_{23}\epsilon_z + \bar{C}_{24}\gamma_{yz} + \bar{C}_{25}\gamma_{xz} + \bar{C}_{26}\gamma_{xy} \\
 \sigma_z &= \bar{C}_{31}\epsilon_x + \bar{C}_{32}\epsilon_y + \bar{C}_{33}\epsilon_z + \bar{C}_{34}\gamma_{yz} + \bar{C}_{35}\gamma_{xz} + \bar{C}_{36}\gamma_{xy} \\
 \tau_{yz} &= \bar{C}_{41}\epsilon_x + \bar{C}_{42}\epsilon_y + \bar{C}_{43}\epsilon_z + \bar{C}_{44}\gamma_{yz} + \bar{C}_{45}\gamma_{xz} + \bar{C}_{46}\gamma_{xy} \\
 \tau_{xz} &= \bar{C}_{51}\epsilon_x + \bar{C}_{52}\epsilon_y + \bar{C}_{53}\epsilon_z + \bar{C}_{54}\gamma_{yz} + \bar{C}_{55}\gamma_{xz} + \bar{C}_{56}\gamma_{xy} \\
 \tau_{xy} &= \bar{C}_{61}\epsilon_x + \bar{C}_{62}\epsilon_y + \bar{C}_{63}\epsilon_z + \bar{C}_{64}\gamma_{yz} + \bar{C}_{65}\gamma_{xz} + \bar{C}_{66}\gamma_{xy}.
 \end{aligned} \tag{2.19}$$

Equation (2.19) may be written in the form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} & \bar{C}_{26} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} & \bar{C}_{36} \\ \bar{C}_{41} & \bar{C}_{42} & \bar{C}_{43} & \bar{C}_{44} & \bar{C}_{45} & \bar{C}_{46} \\ \bar{C}_{51} & \bar{C}_{52} & \bar{C}_{53} & \bar{C}_{54} & \bar{C}_{55} & \bar{C}_{56} \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{63} & \bar{C}_{64} & \bar{C}_{65} & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \tag{2.20}$$

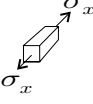
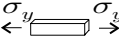

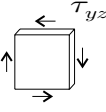
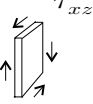
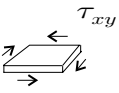
where \bar{C}_{ij} are the elements of the stiffness matrix $[\bar{C}]$ in the x, y, z coordinate system.

Inversion of Eq. (2.20) results in the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_{15} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ \bar{S}_{31} & \bar{S}_{32} & \bar{S}_{33} & \bar{S}_{34} & \bar{S}_{35} & \bar{S}_{36} \\ \bar{S}_{41} & \bar{S}_{42} & \bar{S}_{43} & \bar{S}_{44} & \bar{S}_{45} & \bar{S}_{46} \\ \bar{S}_{51} & \bar{S}_{52} & \bar{S}_{53} & \bar{S}_{54} & \bar{S}_{55} & \bar{S}_{56} \\ \bar{S}_{61} & \bar{S}_{62} & \bar{S}_{63} & \bar{S}_{64} & \bar{S}_{65} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}, \tag{2.21}$$

where \bar{S}_{ij} are the elements of the compliance matrix $[\bar{S}]$ in the x, y, z coordinate system and are defined in Table 2.3 (page 10). In this table tests are illustrated that, in principle, could provide means of determining the different compliance matrix elements.

Table 2.3. The elements of the compliance matrix $[\bar{S}]$ in the x, y, z coordinate system. The elements S_{ij} (without bar) in the x_1, x_2, x_3 coordinate system are obtained by replacing x, y, z by 1, 2, 3 on the right-hand sides of the expressions.

Test	Elements of the compliance matrix	
	$\bar{S}_{11} = \epsilon_x / \sigma_x$	$\bar{S}_{41} = \gamma_{yz} / \sigma_x$
	$\bar{S}_{21} = \epsilon_y / \sigma_x$	$\bar{S}_{51} = \gamma_{xz} / \sigma_x$
	$\bar{S}_{31} = \epsilon_z / \sigma_x$	$\bar{S}_{61} = \gamma_{xy} / \sigma_x$
	$\bar{S}_{12} = \epsilon_x / \sigma_y$	$\bar{S}_{42} = \gamma_{yz} / \sigma_y$
	$\bar{S}_{22} = \epsilon_y / \sigma_y$	$\bar{S}_{52} = \gamma_{xz} / \sigma_y$
	$\bar{S}_{32} = \epsilon_z / \sigma_y$	$\bar{S}_{62} = \gamma_{xy} / \sigma_y$
	$\bar{S}_{13} = \epsilon_x / \sigma_z$	$\bar{S}_{43} = \gamma_{yz} / \sigma_z$
	$\bar{S}_{23} = \epsilon_y / \sigma_z$	$\bar{S}_{53} = \gamma_{xz} / \sigma_z$
	$\bar{S}_{33} = \epsilon_z / \sigma_z$	$\bar{S}_{63} = \gamma_{xy} / \sigma_z$
	$\bar{S}_{14} = \epsilon_x / \tau_{yz}$	$\bar{S}_{44} = \gamma_{yz} / \tau_{yz}$
	$\bar{S}_{24} = \epsilon_y / \tau_{yz}$	$\bar{S}_{54} = \gamma_{xz} / \tau_{yz}$
	$\bar{S}_{34} = \epsilon_z / \tau_{yz}$	$\bar{S}_{64} = \gamma_{xy} / \tau_{yz}$
	$\bar{S}_{15} = \epsilon_x / \tau_{xz}$	$\bar{S}_{45} = \gamma_{yz} / \tau_{xz}$
	$\bar{S}_{25} = \epsilon_y / \tau_{xz}$	$\bar{S}_{55} = \gamma_{xz} / \tau_{xz}$
	$\bar{S}_{35} = \epsilon_z / \tau_{xz}$	$\bar{S}_{65} = \gamma_{xy} / \tau_{xz}$
	$\bar{S}_{16} = \epsilon_x / \tau_{xy}$	$\bar{S}_{46} = \gamma_{yz} / \tau_{xy}$
	$\bar{S}_{26} = \epsilon_y / \tau_{xy}$	$\bar{S}_{56} = \gamma_{xz} / \tau_{xy}$
	$\bar{S}_{36} = \epsilon_z / \tau_{xy}$	$\bar{S}_{66} = \gamma_{xy} / \tau_{xy}$

In the x_1, x_2, x_3 coordinate system the stress–strain relationships are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}, \quad (2.22)$$

where C_{ij} are the elements of the stiffness matrix $[C]$ in the x_1, x_2, x_3 coordinate system.

By inverting Eq. (2.22) we obtain the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}, \quad (2.23)$$

where S_{ij} are the elements of the compliance matrix $[S]$ in the x_1, x_2, x_3 coordinate system.