MEDIAN STRUCTURES ON ASYMPTOTIC CONES AND HOMOMORPHISMS INTO MAPPING CLASS GROUPS

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Abstract. We prove that every asymptotic cone of a mapping class group has a bi-Lipschitz equivariant embedding into a product of real trees, with image a median subspace. We deduce several applications of this, one of which is that a group with Kazhdan's property (T) can only have finitely many pairwise non-conjugate homomorphisms into a mapping class group.

Contents

1. Introduction

Mapping class groups $\mathcal{MCG}(S)$ of surfaces S are very interesting geometric objects, whose geometry is not yet completely understood. Aspects of their geometry are especially striking when compared with lattices in semi-simple Lie groups. Mapping class groups are known to have properties in common with both lattices in rank 1 and lattices in higher rank semi-simple groups. It is known, for example, that the intersection pattern of quasi-flats in $\mathcal{MCG}(S)$ is reminiscent of intersection patterns

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of quasi-flats in uniform lattices of higher rank semi-simple groups [BKMM08]. On the other hand, the pseudo-Anosov elements in $\mathcal{MCG}(S)$ are 'rank 1 elements', i.e. the cyclic subgroups generated by them are quasi-geodesics satisfying the Morse property ([FLM01], [Beh04], [DMS]).

Non-uniform lattices in rank one semi-simple groups are (strongly) relatively hyperbolic [Far98]. As mapping class groups act by isometries on their complex of curves, and the latter are hyperbolic [MM99], mapping class groups are weakly relatively hyperbolic with respect to parabolic subgroups consisting of stabilizers of multicurves. On the other hand, mapping class groups are not relatively hyperbolic with respect to any collection of subgroups or even metrically with respect to any collection of subsets ([AAS07], [BDM]). Still, $\mathcal{MCG}(S)$ share further properties with relatively hyperbolic groups. A subgroup of $\mathcal{MCG}(S)$ either contains a pseudo-Anosov element or is parabolic, that is stabilizes a (multi-)curve in S [Iva92]. A similar property is one of the main "rank 1" properties of relatively hyperbolic groups [DS05].

Another form of rank 1 phenomenon, which is also a weaker version of relative hyperbolicity, is existence of cut-points in the asymptotic cones. It is well known that the asymptotic cones of hyperbolic groups are $\mathbb{R}$-trees, i.e. every point is a cut-point. Any asymptotic cone of a relatively hyperbolic group is a tree-graded space [DS05], that is it contains a collection of proper geodesic subspaces, called *pieces*, such that every two pieces intersect by at most one point and every simple loop in the asymptotic cone is contained in one piece. In a tree-graded space, every intersection point between two distinct pieces is obviously a cut-point. Conversely, every geodesic metric space with cut-points is tree-graded with respect to maximal subspaces without cut-points. When taking the quotient of a tree-graded space $\mathbb{F}$ with respect to the closure of the equivalence relation 'two points are in the same piece' (this corresponds, roughly, to shrinking all pieces to points) one obtains, unsurprisingly, a real tree $T_{\mathbb{F}}$.

It was proved in [Beh06] that the asymptotic cones of mapping class groups are tree-graded. The minimal (under inclusion) pieces of the tree-graded structure are described in [BKMM08] (see Theorem 4.2 and Proposition 5.5 in this paper). The canonical projection of the asymptotic cone $\mathcal{AM}(S)$ of a mapping class group onto the asymptotic cone $\mathcal{AC}(S)$ of the corresponding complex of curves (which is a real tree, since the complex of curves is hyperbolic) is a composition between the projection of $\mathcal{AM}(S)$ seen as a tree-graded space onto the quotient tree described above, which we denote by T_S, and a projection of T_S onto $\mathcal{AC}(S)$. The second projection has large pre-images of singletons, and is therefore very far from being injective (see Remark 4.3).

Asymptotic cones of mapping class groups have been used, in particular, to prove quasi-isometric rigidity of mapping class groups ([BKMM08], [Ham05]) and to prove the Brock-Farb rank conjecture that the rank of every quasi-flat in $\mathcal{MCG}(S)$ does not exceed $\xi(S) = 3g + p - 3$, where g is the genus of the surface S and p is the number of punctures ([BM07], [Ham05]). Many useful results about the structure of asymptotic cones of mapping class groups can be found in [BKMM08, Ham05, BM07].

In this paper, we continue the study of asymptotic cones of mapping class groups, and show that the natural metric on every asymptotic cone of $\mathcal{MCG}(S)$ can be deformed in an equivariant and bi-Lipschitz way, so that the new metric space is inside an ℓ_1-product of $\mathbb{R}$-trees and is a median space. To this end, we use the projection of the mapping class group onto mapping class groups of subsurfaces (see Section 2.5.1) to define the projection of an asymptotic cone $\mathcal{AM}(S)$ onto ultralimits $\mathcal{M}(\mathbf{U})$ of sequences of mapping class groups of subsurfaces $\mathbf{U} = (U_n)^\omega$. An ultralimit $\mathcal{M}(\mathbf{U})$ is isometric to an asymptotic cone of $\mathcal{MCG}(Y)$ with Y a fixed subsurface, the latter is a tree-graded space, hence $\mathcal{M}(\mathbf{U})$ has a projection onto a real tree $T_\mathbf{U}$ obtained by shrinking pieces to points as described above. These projections allow us to construct an embedding of $\mathcal{AM}(S)$.

Theorem 1.1 (Theorem 4.16, Theorem 4.25)**.** *The map $\psi\colon \mathcal{AM} \to \prod_\mathbf{U} T_\mathbf{U}$ whose components are the canonical projections of $\mathcal{AM}$ onto $T_\mathbf{U}$ is a bi-Lipschitz map, when $\prod_\mathbf{U} T_\mathbf{U}$ is endowed with the ℓ^1-metric. Its image $\psi(\mathcal{AM})$ is a median space. Moreover ψ maps limits of hierarchy paths onto geodesics in $\prod_\mathbf{U} T_\mathbf{U}$.*

The bi-Lipschitz equivalence between the limit metric on $\mathcal{AM}$ and the pull-back of the ℓ^1-metric on $\prod_\mathbf{U} T_\mathbf{U}$ yields a distance formula in the asymptotic cone, similar to the Masur–Minsky distance formula for the marking complex [MM00].

The embedding ψ allows us to give in Section 4.2 an alternative proof of the Brock-Farb conjecture. The proof essentially follows the ideas outlined in [Beh04]. We prove that the covering dimension of the asymptotic cones of $\mathcal{MCG}(S)$ does not exceed $\xi(S)$ by showing that for every compact subset K of the asymptotic cone and every $\epsilon > 0$ there exists an ϵ-map f from K to a product of finitely many $\mathbb{R}$-trees X (i.e., a continuous map with diameter of $f^{-1}(x)$ at most ϵ for every $x \in X$) such that $f(K)$ is of dimension at most $\xi(S)$. This, by a standard statement from dimension theory, implies that the dimension of the asymptotic cone is at most $\xi(S)$.

One of the typical "rank 1" properties of groups is the following result essentially due to Bestvina [Bes88] and Paulin [Pau88]: if a group A has infinitely many injective homomorphisms $\phi_1, \phi_2, ...$ into a hyperbolic group G which are pairwise non-conjugate in G, then A splits over a virtually abelian subgroup. The reason for this is that A acts on the asymptotic cone of G (which is an $\mathbb{R}$-tree) by the natural action:

$$a \cdot (x_i) = (\phi_i(a) x_i). \tag{1}$$

Similar statements hold for relatively hyperbolic groups (see [OP98], [Gro04a], [Gro04b], [Gro05], [DS07], [BS08]).

It is easy to see that this statement does not hold for mapping class groups. Indeed, consider the right angled Artin group B corresponding to a finite graph Γ (B is generated by the set X of vertices of Γ subject to commutativity relations: two generators commute if and only if the corresponding vertices are adjacent in Γ). There clearly exists a surface S and a collection of curves X_S in one-to-one correspondence with X such that two curves α, β from X_S are disjoint if and only if the corresponding vertices in X are adjacent. Let d_α, $\alpha \in X$, be the Dehn twist corresponding to the curve α. Then every map $X \to \mathcal{MCG}(S)$ such that $\alpha \mapsto d_\alpha^{k_\alpha}$ for some integer k_α extends to a homomorphism $B \to \mathcal{MCG}(S)$. Clearly one can choose integers k_α so that these homomorphisms are not pairwise conjugate in $\mathcal{MCG}(S)$. But the group B does not necessarily split over any "nice" (say, abelian, small, etc.) subgroup.

Nevertheless, if A has infinitely many pairwise non-conjugate homomorphisms into a mapping class group $\mathcal{MCG}(S)$, then it acts naturally as in (1) on an asymptotic cone of $\mathcal{MCG}(S)$. Since $\mathcal{MCG}(S)$ is a tree-graded space, we apply the theory of actions of groups on tree-graded spaces from [DS07]. We prove (Corollary 5.16) that in this case either A is virtually abelian, it splits over a virtually abelian subgroup, or the action (1) fixes a piece of the asymptotic cone setwise.

We also prove that unless A is virtually abelian, the action (1) of A has unbounded orbits (see Section 6). This and the fact that a group with property (T) cannot act on a median space with unbounded orbits ([NR97], [NR98], [CDH], [Nic08]) allows us to apply our results in the case when A has property (T).

Theorem 1.2 (Theorem 6.2)**.** *A group with property (T) has at most finitely many pairwise non-conjugate homomorphisms into a mapping class group.*

Daniel Groves also announced a proof for Theorem 1.2.

A result similar to Theorem 1.2 is that given a one-ended group A, there are finitely many pairwise non-conjugate injective homomorphisms $A \to \mathcal{MCG}(S)$ such that every non-trivial element in A has a pseudo-Anosov image ([Bow07a], [DF07], [Bow07b]). Both this result and Theorem 1.2 should be seen as evidence that there are few subgroups (if any) with these properties in the mapping class group of a surface. We recall that the mapping class group itself does not have property (T) [And07].

Organization of the paper. In Section 2 we recall results on asymptotic cones, complexes of curves, and mapping class groups, while in Section 3 we recall properties of tree-graded metric spaces and prove new results on groups of isometries of such spaces. In Section 4 we prove Theorem 1.1, and we give a new proof that the dimension of an asymptotic cone of $\mathcal{MCG}(S)$ is at most $\xi(S)$. In Section 5 we describe further the asymptotic cones of $\mathcal{MCG}(S)$ and deduce that for groups not virtually abelian nor splitting over a virtually abelian subgroup sequences of pairwise non-conjugate homomorphism into $\mathcal{MCG}(S)$ induce an action on the asymptotic cone fixing a piece (Corollary 5.16). In Section 6 we prove Theorem 1.2.

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2. Background

2.1. **Asymptotic cones.** A *non-principal ultrafilter* ω over a countable set I is a finitely additive measure on the class $\mathcal{P}(I)$ of subsets of I, such that each subset has measure either 0 or 1 and all finite sets have measure 0. Since we only use non-principal ultrafilters, the word non-principal will be omitted in what follows.

If a statement $P(i)$ is satisfied for all i in a set J with $\omega(J) = 1$, then we say that $P(i)$ holds ω*–a.s.*

Given a sequence of sets $(X_n)_{n\in I}$ and an ultrafilter ω, the *ultraproduct corresponding to* ω, $\Pi X_n/\omega$, consists of equivalence classes of sequences $(x_n)_{n\in I}$, with $x_n \in X_n$, where two sequences (x_n) and (y_n) are identified if $x_n = y_n$ ω–a.s. The equivalence class of a sequence $x = (x_n)$ in $\Pi X_n/\omega$ is denoted either by x^ω or by $(x_n)^\omega$. In particular, if all X_n are equal to the same X, the ultraproduct is called the *ultrapower* of X and it is denoted by $\Pi X/\omega$.

If G_n, $n \geq 1$, are groups then $\Pi G_n/\omega$ is again a group with the multiplication law $(x_n)^\omega (y_n)^\omega = (x_n y_n)^\omega$.

If $\Re$ is a relation on X, then one can define a relation $\Re_\omega$ on $\Pi X/\omega$ by setting $(x_n)^\omega \Re_\omega (y_n)^\omega$ if and only if $x_n \Re y_n$ ω-almost surely.

Lemma 2.1. *Let ω be an ultrafilter and let (X_i) be a sequence of sets which ω–a.s. have cardinality at most N. Then the ultraproduct $\Pi X_i/\omega$ has cardinality at most N.*

For every sequence of points $(x_n)_{n\in I}$ in a topological space X, its *ω-limit*, denoted by $\lim_\omega x_n$, is a point x in X such that every neighborhood U of x contains x_n for ω-almost every n. If a metric space X is Hausdorff and (x_n) is a sequence in X, then when the ω–limit, $\lim_\omega x_n$, exists, it is unique. In a compact metric space every sequence has an ω–limit [Bou65].

Definition 2.2 (ω–limit of metric spaces)**.** Let (X_n, dist_n), $n \in I$, be a sequence of metric spaces and let ω be an ultrafilter over I. Consider the ultraproduct $\Pi X_n/\omega$. For every two points $x = (x_n)^\omega, y = (y_n)^\omega$ in $\Pi X_n/\omega$ let

$$D(x, y) = \lim_\omega \mathrm{dist}_{X_n}(x_n, y_n)\,.$$

Consider an *observation point* $e = (e_n)^\omega$ in $\Pi X_n/\omega$ and define $\Pi_e X_n/\omega$ to be the subset of $\Pi X_n/\omega$ consisting of elements which are finite distance from e with respect to D. The function D is a pseudo-metric on $\Pi_e X_n/\omega$, that is, it satisfies all the properties of a metric except $D(x, y) = 0 \Rightarrow x = y$.

The *ω-limit of the metric spaces* (X_n, dist_n) *relative to the observation point* e is the metric space obtained from $\Pi_e X_n/\omega$ by identifying all pairs of points x, y with $D(x, y) = 0$; this space is denoted $\lim_\omega(X_n, e)$. The equivalence class of a sequence (x_n) in $\lim_\omega(X_n, e)$ is denoted by $\lim_\omega x_n$.

Note that if $e, e' \in \Pi X_n/\omega$ and $D(e, e') < \infty$ then $\lim_\omega(X_n, e) = \lim_\omega(X_n, e')$.

Definition 2.3 (asymptotic cone)**.** Let (X, dist) be a metric space, ω be an ultrafilter over a set I, $e = (e_n)^\omega$ be an observation point. Consider a sequence of numbers $d = (d_n)_{n\in I}$ called *scaling constants* satisfying $\lim_\omega d_n = \infty$.

The space $\lim_\omega(X, \frac{1}{d_n}\mathrm{dist}, e)$ is called an *asymptotic cone of* X. It is denoted by $\mathrm{Con}^\omega(X; e, d)$.

Note that if X is a group G endowed with a word metric then $\Pi_1 G/\omega$ is a subgroup of the ultrapower of G.

Definition 2.4. For a sequence $(x_n)_{n\in I}$ of points in (X, dist), we use the notation $\langle x_n \rangle$ to denote the equivalence class of points in $\mathrm{Con}^\omega(X; e, d)$, i.e., all sequences (y_n) for which $\lim_\omega \frac{1}{d_n}\mathrm{dist}(x_n, y_n) = 0$. When the particular representative is unimportant we denote points of the cone by boldface letters, i.e., $\boldsymbol{x}$.

For a sequence $(A_n)_{n\in I}$ of subsets of (X, dist), we similarly write $\langle A_n \rangle$ to denote the subset of $\mathrm{Con}^\omega(X; e, d)$ consisting of all the elements $\langle x_n \rangle$ such that $\lim_\omega \frac{1}{d_n}\mathrm{dist}(x_n, A_n) = 0$. Notice that if $\lim_\omega \frac{\mathrm{dist}(e_n, A_n)}{d_n} = \infty$ then the set $\lim_\omega (A_n)$ is empty.

Any asymptotic cone of a metric space is a complete metric space [dDW84]. The same proof gives that $\boldsymbol{A} = \langle A_n \rangle$ is always a closed subset of the asymptotic cone $\mathrm{Con}^\omega(X; e, d)$.

Remark 2.5. (1) Let G be a finitely generated group endowed with a word metric. The group $\Pi_1 G/\omega$ acts on $\mathrm{Con}^\omega(G; 1, d)$ transitively by isometries:

$$(g_n)^\omega \lim_\omega (x_n) = \lim_\omega (g_n x_n)\,.$$

(2) Given an arbitrary sequence of observation points x, the group $x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ acts transitively by isometries on the asymptotic cone $\mathrm{Con}^\omega(G; x, d)$. In particular, every asymptotic cone of G is homogeneous.

Convention 2.6. By the above remark, when we consider an asymptotic cone of a finitely generated group, it is no loss of generality to assume that the observation point $\boldsymbol{e}$ is $(1)^\omega$. We shall do this unless explicitly stated otherwise.

2.2. **The complex of curves.** Throughout, $S = S_{g,p}$ will denote a compact connected orientable surface of genus g and with p boundary components. Subsurfaces $Y \subset S$ will always be considered to be essential (i.e., such that their fundamental group injects into the fundamental group of S), also they will not be assumed to be proper unless explicitly stated. We will often measure the complexity of a surface by $\xi(S_{g,p}) = 3g + p - 3$; this complexity is additive under disjoint union. Surfaces and curves are always considered up to homotopy unless explicitly stated otherwise; we refer to a pair of curves (surfaces, etc) intersecting if they intersect independently of the choice of representatives.

The (1-skeleton of the) *complex of curves* of a surface S, denoted by $\mathcal{C}(S)$, is defined as follows. The set of vertices of $\mathcal{C}(S)$, denoted by $\mathcal{C}_0(S)$, is the set of homotopy classes of essential non-peripheral simple closed curves on S.

When $\xi(S) > 1$, two vertices are connected by an edge if the corresponding curves can be realized disjointly on S.

Recall that a *multicurve* on S is a simplex in $\mathcal{C}(S)$.

If $\xi(S) = 1$ then two vertices are connected by an edge if they can be realized so that they intersect in the minimal possible number of points on the surface S (i.e., 1 if $S = S_{1,1}$ and 2 if $S = S_{0,4}$). If $\xi(S) = 0$ then $S = S_{1,0}$ or $S = S_{0,3}$. In the first case we do the same as for $\xi(S) = 1$ and in the second case the complex is empty since this surface doesn't support any essential simple closed curve. The complex is also empty if $\xi(S) \leq -2$.

Finally if $\xi(S) = -1$, then S is an annulus. This case is of interest to us only when the annulus is a subsurface of a surface S'. In this case we define $\mathcal{C}(S)$ by looking in the annular cover $\tilde{S}'$ in which S lifts homeomorphically. We use the compactification of the hyperbolic plane as the closed unit disk to obtain a closed annulus $\hat{S}'$. We define the vertices of $\mathcal{C}(S)$ to be the homotopy classes of arcs connecting the two boundary components of $\hat{S}'$, where the homotopies are required to fix the endpoints. We define a pair of vertices to be connected by an edge if they have representatives which can be realized with disjoint interior. Giving edges an Euclidean metric of a fixed length, in [MM00] it is proven that this space is quasi-isometric to $\mathbb{Z}$.

A fundamental result on the curve complex is the following:

Theorem 2.7 (Masur–Minsky; [MM99])**.** *For any surface S, the complex of curves of S is an infinite-diameter δ-hyperbolic space (as long as it is non-empty), with δ depending only on $\xi(S)$.*

2.3. **Projection on the complex of curves of a subsurface.** We shall need the natural projection of a curve (multicurve) γ onto an essential subsurface $Y \subseteq S$ defined in [MM00]. By definition, the projection $\pi_{\mathcal{C}(Y)}(\gamma)$ is a possibly empty subset of $\mathcal{C}(Y)$.

The definition of this projection is from [MM00, § 2] and is given below. Roughly speaking, the projection consists of all closed curves of the intersections of γ with Y together with all the arcs of $\gamma \cap Y$ combined with parts of the boundary of Y (to form closed curves).

Definition 2.8. Fix an essential subsurface $Y \subset S$. Given an element $\gamma \in \mathcal{C}(S)$ we define the projection $\pi_{\mathcal{C}(Y)}(\gamma) \in 2^{\mathcal{C}(Y)}$ as follows.

(1) If either $\gamma \cap Y = \emptyset$, or Y is an annulus and γ is its core curve (i.e., γ is the unique homotopy class of essential simple closed curve in Y), then we define $\pi_{\mathcal{C}(Y)}(\gamma) = \emptyset$.
(2) If Y is an annulus which transversally intersects γ, then we define $\pi_{\mathcal{C}(Y)}(\gamma)$ to be the element of $\mathcal{C}(Y)$ defined by γ as in the definition of the curve complex of an annulus.
(3) In all remaining cases, consider the arcs and simple closed curves obtained by intersecting Y and γ. We define $\pi_{\mathcal{C}(Y)}(\gamma)$ to be the set of vertices in $\mathcal{C}(Y)$ consisting of:
    - the simple closed curves in the collection above;
    - the simple closed curves obtained by taking arcs from the above collection union a subarc of ∂Y (i.e., those curves which can be obtained by resolving the arcs in $\gamma \cap Y$ to curves).

One of the basic properties is the following result of Masur–Minsky (see [MM00, Lemma 2.3] for the original proof, in [Min03] the bound was corrected from 2 to 3).

Lemma 2.9 ([MM00])**.** *If Δ is a multicurve in S for which every constituent homotopy class of curve intersects Y nontrivially, then* $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\Delta)) \leq 3$.

Notation 2.10. Given Δ, Δ' a pair of multicurves in $\mathcal{C}(S)$, for brevity we often write $\mathrm{dist}_{\mathcal{C}(Y)}(\Delta, \Delta')$ instead of $\mathrm{dist}_{\mathcal{C}(Y)}(\pi_Y(\Delta), \pi_Y(\Delta'))$.

2.4. **Mapping class groups.** The *mapping class group*, $\mathcal{MCG}(S)$, of a surface S of finite type, is the quotient of the group of homeomorphisms of S by the subgroup isotopic to the identity. Since the mapping class group of a surface of finite type is finitely generated, we may consider a word metric on the group - this metric is unique up to quasi-isometries. Note that $\mathcal{MCG}(S)$ acts on $\mathcal{C}(S)$ by simplicial automorphisms (in particular by isometries) and with finite quotient, and that the family of tight geodesics is invariant with respect to this action.

Recall that according to the Nielsen-Thurston classification, any element $g \in \mathcal{MCG}(S)$ satisfies one of the following three properties:

(1) g has finite order;
(2) there exists a multicurve Δ in $\mathcal{C}(S)$ invariant by g (in this case g is called *reducible*);
(3) g is pseudo-Anosov.

We call an element $g \in \mathcal{MCG}(S)$ *pure* if there exists a multicurve Δ (possibly empty) invariant by g and such that g does not permute the connected components of $S \setminus \Delta$, and it induces on each component of $S \setminus \Delta$ either a pseudo-Anosov or the identity map. In particular every pseudo-Anosov is pure.

Theorem 2.11. *(*[Iva92, Corollary 1.8], [Iva02, Theorem 7.1.E]*) Consider the homomorphism from $\mathcal{MCG}(S)$ to the finite group* $\mathrm{Aut}(H_1(S, \mathbb{Z}/k\mathbb{Z}))$ *defined by the action of diffeomorphisms on homology.*

If $k \geq 3$ then the kernel $\mathcal{MCG}_k(S)$ of the homomorphism is composed only of pure elements, in particular it is torsion free.

We now show that versions of some of the previous results hold for the ultrapower $\mathcal{MCG}(S)^\omega$ of a mapping class group. The elements in $\mathcal{MCG}(S)^\omega$ can also be classified into finite order, reducible and pseudo-Anosov elements, according to the case into which their components fall ω-almost surely. Similarly, one may define pure elements. Note that non-trivial pure elements both in $\mathcal{MCG}(S)$ and in its ultrapower are of infinite order.

Theorem 2.11 implies the following statements.

Lemma 2.12. (1) *The ultrapower $\mathcal{MCG}(S)^\omega$ contains a finite index normal subgroup $\mathcal{MCG}(S)^\omega_p$ which consists only of pure elements.*

(2) *The orders of finite subgroups in the ultrapower $\mathcal{MCG}(S)^\omega$ are bounded by a constant $N = N(S)$.*

Proof. (1) The homomorphism in Theorem 2.11 for $k \geq 3$ induces a homomorphism from $\mathcal{MCG}(S)^\omega$ to a finite group whose kernel consists only of pure elements.

(2) Since any finite subgroup of $\mathcal{MCG}(S)^\omega$ has trivial intersection with $\mathcal{MCG}(S)^\omega_p$ it follows that it injects into the quotient group, hence its cardinality is at most the index of $\mathcal{MCG}(S)^\omega_p$. □

2.5. **The marking complex.** For most of the sequel, we do not work with the mapping class group directly, but rather with a particular quasi-isometric model called the *marking graph*, $\mathcal{M}(S)$, which is defined as follows.

The vertices of the marking graph are called *markings*. Each marking $\mu \in \mathcal{M}(S)$ consists of the following pair of data:

- *base curves*: a multicurve consisting of $3g+p-3$ components, i.e., a maximal simplex in $\mathcal{C}(S)$. This collection is denoted base(μ).
- *transversal curves*: to each curve $\gamma \in$ base(μ) is associated an essential curve in the complex of curves of the annulus with core curve γ with a certain compatibility condition. More precisely, letting T denote the complexity 1 component of $S \setminus \bigcup_{\alpha \in \mathrm{base}\,\mu, \alpha \neq \gamma} \alpha$, the transversal curve to γ is any curve $t(\gamma) \in \mathcal{C}(T)$ with $\mathrm{dist}_{\mathcal{C}(T)}(\gamma, t(\gamma)) = 1$; since $t(\gamma) \cap \gamma \neq \emptyset$, the curve $t(\gamma)$ is a representative of a point in the curve complex of the annulus about γ, i.e., $t(\gamma) \in \mathcal{C}(\gamma)$.

We define two vertices $\mu, \nu \in \mathcal{M}(S)$ to be connected by an edge if either of the two conditions hold:

(1) *Twists*: μ and ν differ by a Dehn twist along one of the base curves. That is, base$(\mu) =$ base(ν) and all their transversal curves agree except for about one element $\gamma \in$ base$(\mu) =$ base(ν) where $t_\mu(\gamma)$ is obtained from $t_\nu(\gamma)$ by twisting once about the curve γ.

(2) *Flips*: The base curves and transversal curves of μ and ν agree except for one pair $(\gamma, t(\gamma)) \in \mu$ for which the corresponding pair consists of the same pair but with the roles of base and transversal reversed. Note that the second condition to be a marking requires that each transversal curve intersects exactly one base curve, but the Flip move may violate this condition. It is shown in [MM00, Lemma 2.4], that there is a finite set of natural ways to resolve this issue, yielding a finite (in fact uniformly bounded) number

of flip moves which can be obtained by flipping the pair $(\gamma, t(\gamma)) \in \mu$; an edge connects each of these possible flips to μ.

The following result is due to Masur–Minsky [MM00].

Theorem 2.13. *The graph $\mathcal{M}(S)$ is locally finite and the mapping class group acts cocompactly and properly discontinuously on it. In particular the mapping class group of S endowed with a word metric is quasi-isometric to $\big(\mathcal{M}(S), \mathrm{dist}_{\mathcal{M}(S)}\big)$.*

Notation 2.14. In what follows we denote the simplicial distance on $\mathcal{M}(S)$ either by $\mathrm{dist}_{\mathcal{M}(S)}$ or simply by $\mathrm{dist}_{\mathcal{M}}$, when there is no possibility of confusion.

The subsurface projections introduced in Section 2.2 allow one to consider the projection of a marking on S to the curve complex of a subsurface $Y \subseteq S$. Given a marking $\mu \in \mathcal{M}(S)$ we define $\pi_{\mathcal{C}(S)}(\mu)$ to be $\mathrm{base}\,\mu$. More generally, given a subsurface $Y \subset S$, we define $\pi_{\mathcal{C}(Y)}(\mu) = \pi_{\mathcal{C}(Y)}(\mathrm{base}(\mu))$, if Y is not an annulus about an element of $\mathrm{base}(\mu)$; if Y is an annulus about an element $\gamma \in \mathrm{base}(\mu)$, then we define $\pi_{\mathcal{C}(Y)}(\mu) = t(\gamma)$, the transversal curve to γ.

Notation 2.15. For two markings $\mu, \nu \in \mathcal{M}(S)$ we often use the following standard simplification of notation:

$$\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu) = \mathrm{dist}_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\mu), \pi_{\mathcal{C}(Y)}(\nu)).$$

Remark 2.16. By Remark 2.9, for every marking μ and every subsurface $Y \subseteq S$, the diameter of the projection of μ into $\mathcal{C}(Y)$ is at most 3. This implies that the difference between $\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu)$ as defined above and the Hausdorff distance between $\pi_{\mathcal{C}(Y)}(\mu)$ and $\pi_{\mathcal{C}(Y)}(\nu)$ in $\mathcal{C}(Y)$ is at most six.

Hierarchies. In the marking complex, there is an important family of quasi-geodesics called *hierarchy paths* which have several useful geometric properties. The concept of hierarchy was first developed by Masur–Minsky in [MM00], which the reader may consult for further details. We now recall those aspects of the construction which we shall use in the sequel.

Given two subsets $A, B \subset \mathbb{R}$, a map $f\colon X \to Y$ is said to be *coarsely increasing* if there exists a constant D such that for each $a + D < b$, we have that $f(a) \leq f(b)$. Similarly, we define *coarsely decreasing* and *coarsely monotonic*. We say a map between quasi-geodesics is coarsely monotonic if it defines a coarsely monotonic map between suitable nets in their domain.

We say a quasi-geodesic $\mathfrak{g}$ in $\mathcal{M}(S)$ is $\mathcal{C}(U)$*–monotonic* for some $U \subset S$ if one can associate a geodesic $\mathfrak{t}_U$ in $\mathcal{C}(U)$ which *shadows* $\mathfrak{g}$ in the sense that $\mathfrak{t}_U$ is a path from a vertex of $\pi_U(\mathrm{base}(\mu))$ to a vertex of $\pi_U(\mathrm{base}(\nu))$ and there is a coarsely monotonic map $v\colon\ \mathfrak{g} \to \mathfrak{t}_U$ such that $v(\rho)$ is a vertex in $\pi_U(\mathrm{base}(\rho))$ for every vertex $\rho \in \mathfrak{g}$.

Any pair of points $\mu, \nu \in \mathcal{M}(S)$ are connected by at least one hierarchy path. Hierarchy paths are quasi-geodesics with uniform constants depending only on the surface S. One of the important properties of hierarchy paths is that they are $\mathcal{C}(U)$-monotonic for every $U \subseteq S$ and moreover the geodesic onto which they project is a tight geodesic.

Lemma 2.17 (Masur–Minsky; [MM00], Lemma 6.2)**.** *There exists a constant $M = M(S)$ such that, if Y is an essential proper subsurface of S and μ, ν are two markings in $\mathcal{M}(S)$ satisfying* $\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu) > M$*, then any hierarchy path $\mathfrak{g}$ connecting μ to ν contains a marking ρ such that the multicurve* $\mathrm{base}(\rho)$ *includes* *the multicurve ∂Y. Furthermore, there exists a vertex v in the geodesic $\mathfrak{t}_{\mathfrak{g}}$ shadowed by $\mathfrak{g}$ for which $v \in \mathrm{base}(\rho)$, and hence satisfying $Y \subseteq S \setminus v$.*

Definition 2.18. Given a constant $K \geq M(S)$, where $M(S)$ is the constant from Lemma 2.17, and a pair of markings μ, ν, the subsurfaces $Y \subseteq S$ for which $\mathrm{dist}_{\mathcal{C}(Y)}(\mu,\nu) > K$ are called the *K-large domains* for the pair (μ,ν). We say that a hierarchy path *contains a domain* $Y \subseteq S$ if Y is a $M(S)$–large domain between some pair of points on the hierarchy path. Note that for every such domain, the hierarchy contains a marking whose base contains ∂Y.

The following useful lemma is one of the basic ingredients in the structure of hierarchy paths; it is an immediate consequence of [MM00, Theorem 4.7] and the fact that a chain of nested subsurfaces has length at most $\xi(S)$.

Lemma 2.19. *Let $\mu, \nu \in \mathcal{M}(S)$ and let ξ be the complexity of S. Then for every $M(S)$-large domain Y in a hierarchy path $[\mu,\nu]$ there exist at most 2ξ domains that are $M(S)$-large and that contain Y, in that path.*

A pair of subsurfaces Y,Z are said to *overlap* if $Y \cap Z \neq \emptyset$ and none of the two subsurfaces is a subsurface of the other.

Theorem 2.20. (Projection estimates). *There exists a constant D depending only on the topological type of a surface S such that for any two overlapping subsurfaces Y and Z in S, with $\xi(Y) \neq 0 \neq \xi(Z)$, and for any $\mu \in \mathcal{M}(S)$*

$$\mathrm{dist}_{\mathcal{C}(Y)}(\partial Z, \mu) > D \implies \mathrm{dist}_{\mathcal{C}(Z)}(\partial Y, \mu) \leq D\,.$$

Convention 2.21. In what follows we assume that the constant $M = M(S)$ from Lemma 2.17 is larger than the constant D from Theorem 2.20.

Notation 2.22. Let $a > 1$, b, x, y be positive real numbers. We write $x \leq_{a,b} y$ if

$$x \leq ay + b.$$

We write $x \approx_{a,b} y$ if and only if $x \leq_{a,b} y$ and $y \leq_{a,b} x$.

Notation 2.23. Let $K, N > 0$ be real numbers. We define $\{\!\{N\}\!\}_K$ to be N if $N > K$ and 0 otherwise.

The following result is fundamental in studying the metric geometry of the marking complex. It provides a way to compute distance in the marking complex from the distance in the curve complexes of the large domains.

Theorem 2.24 (Masur–Minsky; [MM00])**.** *If $\mu, \nu \in \mathcal{M}(S)$, then there exists a constant $K(S)$, depending only on S, such that for each $K > K(S)$ there exists $a \geq 1$ and $b \geq 0$ for which:*

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu,\nu) \approx_{a,b} \sum_{Y \subseteq S} \{\!\{\mathrm{dist}_{\mathcal{C}(Y)}(\pi_Y(\mu), \pi_Y(\nu))\}\!\}_K\,. \tag{2}$$

We now define an important collection of subsets of the marking complex.

Notation 2.25. Let Δ be a simplex in $\mathcal{C}(S)$. We define $\mathcal{Q}(\Delta)$ to be the set of elements of $\mathcal{M}(S)$ whose bases contain Δ.

Remark 2.26. Note that $\mathcal{Q}(\Delta)$ is quasi-isometric to a coset of a stabilizer in $\mathcal{MCG}$ of a multicurve with same topological type as Δ (i.e., topological type of its complement in S). To see this, fix a collection $\Gamma_1, ..., \Gamma_n$ of multicurves where each topological type of multicurve is represented exactly once in this list. Given any multicurve Δ, fix an element $f \in \mathcal{MCG}$ for which $f(\Gamma_i) = \Delta$, for the appropriate $1 \leq i \leq n$. Now, up to a bounded Hausdorff distance, we have an identification of $\mathcal{Q}(\Delta)$ with $f\text{stab}(\Gamma_i)$ given by the natural quasi-isometry between $\mathcal{M}(S)$ and $\mathcal{MCG}(S)$.

Marking projections.

2.5.1. *Projection on the marking complex of a subsurface.* Given any subsurface $Z \subset S$, we define a projection $\pi_{\mathcal{M}(Z)} \colon \mathcal{M}(S) \to 2^{\mathcal{M}(Z)}$, which sends elements of $\mathcal{M}(S)$ to subsets of $\mathcal{M}(Z)$. Given any $\mu \in \mathcal{M}(S)$ we build a marking on Z in the following way. Choose an element $\gamma_1 \in \pi_Z(\mu)$, and then recursively choose γ_n from $\pi_{Z \setminus \cup_{i<n} \gamma_i}(\mu)$, for each $n \leq \xi(Z)$. Now take these γ_i to be the base curves of a marking on Z. For each γ_i we define its transversal $t(\gamma_i)$ to be an element of $\pi_{\gamma_i}(\mu)$. This process yields a marking, see [Beh06] for details.

Arbitrary choices were made in this construction, but it is proven in [Beh06] that there is a uniform constant depending only on $\xi(S)$, so that given any $Z \subset S$ and any μ any two choices in building $\pi_{\mathcal{M}(Z)}(\mu)$ lead to elements of $\mathcal{M}(Z)$ which are at most a constant distance apart. Thus, in the sequel the choices will be irrelevant.

Remark 2.27. Given two nested subsurfaces $Y \subset Z \subset S$ the projection of an arbitrary marking μ onto $C(Y)$ is at uniformly bounded distance from the projection of $\pi_{\mathcal{M}(Z)}(\mu)$ onto $C(Y)$. This follows from the fact that in the choice of $\pi_{\mathcal{M}(Z)}(\mu)$ one can start with a curve that determines the projection of μ onto $C(Y)$.

A similar argument implies that $\pi_{\mathcal{M}(Y)}(\mu)$ is at uniformly bounded distance from $\pi_{\mathcal{M}(Y)}\left(\pi_{\mathcal{M}(Z)}(\mu)\right)$.

An easy consequence of the distance formula in Theorem 2.24 is the following.

Corollary 2.28. *There exist $A \geq 1$ and $B \geq 0$ depending only on S such that for any subsurface $Z \subset S$ and any two markings $\mu, \nu \in \mathcal{M}(S)$ the following holds:*

$$\text{dist}_{\mathcal{M}(Z)}\left(\pi_{\mathcal{M}(Z)}(\mu)\,,\,\pi_{\mathcal{M}(Z)}(\nu)\right) \leq_{A,B} \text{dist}_{\mathcal{M}(S)}(\mu,\nu)\,.$$

2.5.2. *Projection on a set $\mathcal{Q}(\Delta)$.* Given a marking μ and a multicurve Δ, the projection $\pi_{\mathcal{M}(S\setminus\Delta)}(\mu)$ can be defined as in Section 2.5.1. This allows one to construct a point $\mu' \in \mathcal{Q}(\Delta)$ which is closest to μ. See [BM07] for details. The marking μ' is obtained by taking the union of the (possibly partial collection of) base curves Δ with transversal given by $\pi_\Delta(\mu)$ together with the base curves and transversals given by $\pi_{\mathcal{M}(S\setminus\Delta)}(\mu)$. Note that the construction of μ' requires, for each subsurface W determined by the multicurve Δ, the construction of a projection $\pi_{\mathcal{M}(W)}(\mu)$. As explained in Section 2.5.1 each $\pi_{\mathcal{M}(W)}(\mu)$ is determined up to uniformly bounded distance in $\mathcal{M}(W)$, thus μ' is well defined up to a uniformly bounded ambiguity depending only on the topological type of S.

3. Tree-graded metric spaces

3.1. **Preliminaries.** A subset A in a geodesic metric space X is called *geodesic* if every two points in A can be joined by a geodesic contained in A.

Definition 3.1. ([DS05], [Dru]) Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic proper subsets, called *pieces*. We say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$ if the following two properties are satisfied:

(T_1) Every two different pieces have at most one point in common.
(T_2) Every simple non-trivial geodesic triangle in $\mathbb{F}$ is contained in one piece.

When there is no risk of confusion as to the set $\mathcal{P}$, we simply say that $\mathbb{F}$ is *tree-graded*.

Remarks 3.2. The above definition of tree-graded is now considered to be standard, but it differs slightly from the original definition as given in [DS05]. The difference is that there is no longer the hypothesis that the union of the pieces cover the entire space (cf. [Dru]). Note that in the case of asymptotic cones of groups, or indeed in any situation where a group acts transitively on the tree-graded space, it is easy to prove that the set of pieces cover $\mathbb{F}$ anyways.

All properties of tree-graded spaces in [DS05, §2.1] hold with the new definition 3.1, as none of the proofs uses the property that pieces cover the space. In particular one has the following results.

Proposition 3.3 ([DS05], Proposition 2.17)**.** *Under the assumption that $\mathcal{P}$ covers $\mathbb{F}$, property (T_2) can be replaced by the following property:*

> *(T_2') for every topological arc $\mathfrak{c} : [0, d] \to \mathbb{F}$ and $t \in [0, d]$, let $\mathfrak{c}[t - a, t + b]$ be a maximal sub-arc of $\mathfrak{c}$ containing $\mathfrak{c}(t)$ and contained in one piece. Then every other topological arc with the same endpoints as $\mathfrak{c}$ must contain the points $\mathfrak{c}(t - a)$ and $\mathfrak{c}(t + b)$.*

Note that any complete geodesic metric space with a global cut-point provides an example of a tree-graded metric space [DS05, Lemma 2.30].

Convention 3.4. In what follows when speaking about cut-points we always mean global cut-points.

Lemma 3.5. *(*[DS05]*, Lemma 2.30) Let X be a complete geodesic metric space containing at least two points and let $\mathcal{C}$ be a non-empty set of cut-points in X.*

The set $\mathcal{P}$ of all maximal path connected subsets that are either singletons or such that none of their cut-points belongs to $\mathcal{C}$ is a set of pieces for a tree-graded structure on X.

Moreover the intersection of any two distinct pieces from $\mathcal{P}$ is either empty or a point from $\mathcal{C}$.

Lemma 3.6 ([DS05], §2.1)**.** *Let x be an arbitrary point in $\mathbb{F}$ and let T_x be the set of points $y \in \mathbb{F}$ which can be joined to x by a topological arc intersecting every piece in at most one point.*

The subset T_x is a real tree and a closed subset of $\mathbb{F}$, and every topological arc joining two points in T_x is contained in T_x. Moreover, for every $y \in T_x$, $T_y = T_x$.

Definition 3.7. A subset T_x as in Lemma 3.6 is called a *transversal tree* in $\mathbb{F}$.

A geodesic segment, ray or line contained in a transversal tree is called a *transversal geodesic*.

Throughout the rest of the section, $(\mathbb{F}, \mathcal{P})$ is a tree-graded space.

The following statement is an immediate consequence of [DS05, Corollary 2.11].

Lemma 3.8. *Let A and B be two pieces in $\mathcal{P}$. There exist a unique pair of points $a \in A$ and $b \in B$ such that any topological arc joining A and B contains a and b. In particular* $\operatorname{dist}(A, B) = \operatorname{dist}(a, b)$.

For every tree-graded space there can be defined a canonical $\mathbb{R}$–tree quotient [DS07].

Notation: Let x, y be two arbitrary points in $\mathbb{F}$. We define $\widetilde{\mathrm{dist}}(x,y)$ to be $\mathrm{dist}(x,y)$ minus the sum of lengths of non-trivial sub-arcs which appear as intersections of one (any) geodesic $[x,y]$ with pieces.

The function $\widetilde{\mathrm{dist}}(x,y)$ is well defined (independent of the choice of a geodesic $[x,y]$), symmetric, and it satisfies the triangle inequality.

The relation $\approx$ defined by

$$x \approx y \text{ if and only if } \widetilde{\mathrm{dist}}(x,y) = 0\,, \tag{3}$$

is a closed equivalence relation.

Lemma 3.9. *(*[DS07]*)*

1. *The quotient $T = \mathbb{F}/\approx$ is an $\mathbb{R}$–tree with respect to the metric induced by $\widetilde{\mathrm{dist}}$.*
2. *Every geodesic in $\mathbb{F}$ projects onto a geodesic in T. Conversely, for every non-trivial geodesic $\mathfrak{g}$ in T there exists a non-trivial geodesic $\mathfrak{p}$ in $\mathbb{F}$ such that its projection on T is $\mathfrak{g}$*
3. *If $x \neq y$ are in the same transversal tree of $\mathbb{F}$ then $\mathrm{dist}(x,y) = \widetilde{\mathrm{dist}}(x,y)$. In particular, $x \not\approx y$. Thus every transversal tree projects into T isometrically.*

Following [DS07, Definitions 2.6 and 2.9], given a topological arc $\mathfrak{g}$ in $\mathbb{F}$, we define the *set of cut-points on* $\mathfrak{g}$, which we denote by $\mathrm{Cutp}\,(\mathfrak{g})$, as the set of all points of $\mathfrak{g}$ which are global cut-points. Equivalently, this is the subset of $\mathfrak{g}$ which is complementary to the union of all the interiors of sub-arcs appearing as intersections of $\mathfrak{g}$ with pieces. Given two points x, y in $\mathbb{F}$, we define the *set of cut-points separating x and y*, which we denote by $\mathrm{Cutp}\,\{x,y\}$, as the set of cut-points of some (any) topological arc joining x and y.

3.2. **Isometries of tree-graded spaces.**

For all the results on tree-graded metric spaces that we use in what follows we refer to [DS05], mainly to Section 2 in that paper.

Lemma 3.10. *Let x, y be two distinct points, and assume that $\mathrm{Cutp}\,\{x,y\}$ does not contain a point at equal distance from x and y. Let a be the farthest from x point in $\mathrm{Cutp}\,\{x,y\}$ with $\mathrm{dist}(x,a) \leq \frac{\mathrm{dist}(x,y)}{2}$, and let b be the farthest from y point in $\mathrm{Cutp}\,\{x,y\}$ with $\mathrm{dist}(y,b) \leq \frac{\mathrm{dist}(x,y)}{2}$. Then there exists a unique piece P containing $\{a,b\}$, and P contains all points at equal distance from x and y.*

Proof. Since $\mathrm{Cutp}\,\{x,y\}$ does not contain a point at equal distance from x and y it follows that $a \neq b$. The choice of a, b implies that $\mathrm{Cutp}\,\{a,b\} = \{a,b\}$, whence $\{a,b\}$ is contained in a piece P, and P is the unique piece with this property, by property (T_1) of a tree-graded space.

Let $m \in \mathbb{F}$ be such that $\mathrm{dist}(x,m) = \mathrm{dist}(y,m) = \frac{\mathrm{dist}(x,y)}{2}$. Then any union of geodesics $[x,m] \cup [m,y]$ is a geodesic. By property (T_2') of a tree-graded space, $a \in [x,m], b \in [m,y]$. The sub-geodesic $[a,m] \cup [m,b]$ has endpoints in the piece P therefore by strong convexity of pieces it is entirely contained in P. □

Definition 3.11. Let x, y be two distinct points. If $\mathrm{Cutp}\,\{x,y\}$ contains a point at equal distance from x and y then we call that point the *middle cut point* of x, y.

If $\text{Cutp}\{x,y\}$ does not contain such a point then we call the piece defined in Lemma 3.10 the *middle cut piece* of x,y.

If $x=y$ then we say that x,y have the middle cut point x.

Let P,Q be two distinct pieces, and let $x\in P$ and $y\in Q$ be the unique pair of points minimizing the distance. The *middle cut point* (or *middle cut piece*) of P,Q is the middle cut point (respectively, the middle cut piece) of x,y.

If $P=Q$ then we say that P,Q have the middle cut piece P.

Lemma 3.12. *Let g be an isometry of a tree-graded space $\mathbb{F}$ with bounded orbits and permuting pieces.*

(1) *If x is a point such that $gx\neq x$ then g fixes the middle cut point or the middle cut piece of x,gx.*
(2) *If P is a piece such that $gP\neq P$ then g fixes the middle cut point or the middle cut piece of P,gP.*

Proof. (1) Let e be the farthest from gx point in $\text{Cutp}\{x,gx\}\cap\text{Cutp}\{gx,g^2x\}$ and let $d=\text{dist}(x,gx)>0$.

(a) Assume that x,gx have a middle cut point $m\in\text{Cutp}\{x,gx\}$. If the intersection $\text{Cutp}\{x,gx\}\cap\text{Cutp}\{gx,g^2x\}$ contains m then $gm=m$. We argue by contradiction and assume that $gm\neq m$, whence $\text{Cutp}\{x,gx\}\cap\text{Cutp}\{gx,g^2x\}$ does not contain m. Then $\text{dist}(gx,e)=\frac{d}{2}-\epsilon$ for some $\epsilon>0$.

Assume that $\mathfrak{p}=[x,e]\sqcup[e,g^2x]$ is a topological arc. If $e\in\text{Cutp}\,\mathfrak{p}$ then $e\in\text{Cutp}\{x,g^2x\}$. It follows that $\text{dist}(x,g^2x)=\text{dist}(x,e)+\text{dist}(e,g^2x)=2(\frac{d}{2}+\epsilon)=d+2\epsilon$. An induction argument will then give that $\text{Cutp}\{x,g^nx\}$ contains $e,ge,...,g^{n-2}e$, hence that $\text{dist}(x,g^nx)=d+2\epsilon(n-1)$. This contradicts the hypothesis that the orbits of g are bounded.

If $e\notin\text{Cutp}\,\mathfrak{p}$ then there exists $a\in\text{Cutp}\{x,e\}$ and $b\in\text{Cutp}\{e,g^2x\}$ such that a,e are the endpoints of a non-trivial intersection of any geodesic $[x,gx]$ with a piece P, and e,b are the endpoints of a non-trivial intersection of any geodesic $[gx,g^2x]$ with the same piece P. Note that $a\neq b$ otherwise the choice of e would be contradicted. Also, since $m\in\text{Cutp}\{x,e\}$ and $m\neq e$ it follows that $m\in\text{Cutp}\{x,a\}$. Similarly, $gm\in\text{Cutp}\{g^2x,b\}$. It follows that $\text{dist}(x,a)$ and $\text{dist}(g^2x,b)$ are at least $\frac{d}{2}$. Since $[x,a]\sqcup[a,b]\sqcup[b,g^2x]$ is a geodesic it follows that $\text{dist}(x,g^2x)\geq\text{dist}(x,a)+\text{dist}(a,b)+\text{dist}(b,g^2x)\geq d+\text{dist}(a,b)$. An induction argument gives that $[x,a]\sqcup\bigsqcup_{k=0}^{n-2}\left([g^ka,g^kb]\sqcup[g^kb,g^{k+1}a]\right)\sqcup[g^{n-1}a,g^nx]$ is a geodesic. This implies that $\text{dist}(x,g^nx)\geq d+(n-1)\text{dist}(a,b)$, contradicting the hypothesis that g has bounded orbits.

Assume that $\mathfrak{p}=[x,e]\sqcup[e,g^2x]$ is not a topological arc. Then $[x,e]\cap[e,g^2x]$ contains a point $y\neq e$. According to the choice of e, y is either not in $\text{Cutp}\{x,e\}$ or not in $\text{Cutp}\{e,g^2x\}$, and $\text{Cutp}\{e,y\}$ must be $\{e,y\}$. It follows that y,e are in the same piece P. If we consider the endpoints of the (non-trivial) intersections of any geodesics $[x,e]$ and $[e,g^2x]$ with the piece P, a,e and respectively e,b, then we are in the case discussed previously.

(b) Assume that x,gx have a middle cut piece Q. Then there exist two points i,o in $\text{Cutp}\{x,gx\}$, the entrance and respectively the exit point of any geodesic $[x,gx]$ in the piece Q, such that the middlepoint of $[x,gx]$ is in the interior of $[i,o]$ sub-geodesic of $[x,gx]$ (for any choice of the geodesic $[x,gx]$).

If $e\in\text{Cutp}\{x,i\}$ then g stabilizes $\text{Cutp}\{e,ge\}$ and, since g is an isometry, g fixes the middle cut piece Q of e,ge. Assume on the contrary that $gQ\neq Q$. Then

$e \in \mathrm{Cutp}\,\{o, gx\}$. If $\mathfrak{p} = [x,e] \sqcup [e, g^2x]$ is a topological arc and $e \in \mathrm{Cutp}\,\mathfrak{p}$ then as in (a) we may conclude that g has an unbounded orbit. If either $e \notin \mathrm{Cutp}\,\mathfrak{p}$ or $\mathfrak{p} = [x,e] \sqcup [e, g^2x]$ is not a topological arc then as in (a) we may conclude that there exist $a \in \mathrm{Cutp}\,\{x, e\}$ and $b \in \mathrm{Cutp}\,\{e, g^2x\}$ such that a, b, e are pairwise distinct and contained in the same piece P.

Assume that $e = o$. Then $a = i$ and $P = Q$. By hypothesis $P \neq gQ$. It follows that any geodesic $[b, g^2x]$ intersects gQ, whence $ga, ge \in \mathrm{Cutp}\,\{b, g^2x\}$. Since $[x,a] \sqcup [a,b] \sqcup [b, g^2x]$ is a geodesic we have that $\mathrm{dist}(x, g^2x) = \mathrm{dist}(x,a) + \mathrm{dist}(a,b) + \mathrm{dist}(b, g^2x) \geq \mathrm{dist}(x,a) + \mathrm{dist}(a,b) + \mathrm{dist}(ga, g^2x) = d + \mathrm{dist}(a,b)$. An inductive argument gives that for any $n \geq 1$, the union of geodesics $[x,a] \sqcup \bigsqcup_{k=0}^{n-2} \left([g^ka, g^kb] \sqcup [g^kb, g^{k+1}a]\right) \sqcup [g^{n-1}a, g^nx]$ is a geodesic, hence that $\mathrm{dist}(x, g^nx) \geq d + (n-1)\mathrm{dist}(a,b)$, contradicting the hypothesis that g has bounded orbits.

Assume that $e \neq o$. If $e = gi$, hence $b = go$ and $P = gQ$ then an argument as before gives that for every $n \geq 1$, $\mathrm{dist}(x, g^nx) \geq d + (n-1)\mathrm{dist}(a,b)$, a contradiction.

If $e \notin \{gi, o\}$ then $i, o \in \mathrm{Cutp}\,\{x, a\}$ and $gi, go \in \mathrm{Cutp}\,\{b, g^2x\}$, whence both $\mathrm{dist}(x,a)$ and $\mathrm{dist}(b, g^2x)$ are larger than $\frac{d}{2}$. It follows that the union $[x,a] \sqcup \bigsqcup_{k=0}^{n-2} \left([g^ka, g^kb] \sqcup [g^kb, g^{k+1}a]\right) \sqcup [g^{n-1}a, g^nx]$ is a geodesic, therefore that $\mathrm{dist}(x, g^nx)$ is at least $d + (n-1)\mathrm{dist}(a,b)$, contradiction.

(2) Let $x \in P$ and $y \in gP$ be the pair of points realizing the distance. Assume that $gx \neq y$. Then $[x,y] \sqcup [y, gx]$ is a geodesic, for every geodesics $[x,y]$ and $[y, gx]$. Similarly, $[x,y] \sqcup [y, gx] \sqcup [gx, gy] \sqcup [gy, g^2x]$ is a geodesic. An easy induction argument gives that $\bigsqcup_{k=0}^{n-1} [g^kx, g^ky] \sqcup [g^ky, g^{k+1}x]$ is a geodesic. In particular $\mathrm{dist}(x, g^nx) \geq n\mathrm{dist}(y, gx)$ contradicting the hypothesis that g has bounded orbits.

It follows that $y = gx$. If $gx = x$ then we are done. If $gx \neq x$ then we apply (1). □

Lemma 3.13. *Let $g_1, ..., g_n$ be isometries of a tree-graded space permuting pieces and generating a group with bounded orbits. Then $g_1, ..., g_n$ have a common fixed point or (set-wise) fixed piece.*

Proof. We argue by induction on n. For $n = 1$ it follows from Lemma 3.12. Assume the conclusion holds for n and take $g_1, ..., g_{n+1}$ isometries generating a group with bounded orbits and permuting pieces. By the induction hypothesis $g_1, ..., g_n$ fix either a point x or a piece P.

Assume they fix a point x, and that $g_{n+1}x \neq x$. Assume that $x, g_{n+1}x$ have a middle cut point m. Then $g_{n+1}m = m$ by Lemma 3.12, (1). For every $i \in \{1, ..., n\}$, $g_{n+1}g_ix = g_{n+1}x$ therefore $g_{n+1}g_im = m$, hence $g_im = m$. If $x, g_{n+1}x$ have a middle cut piece Q then it is shown similarly that $g_1, ..., g_{n+1}$ fix Q setwise. In the case when $g_1, ..., g_n$ fix a piece P, Lemma 3.12 also allows to prove that $g_1, ..., g_{n+1}$ fix the middle cut point or middle cut piece of $P, g_{n+1}P$. □

4. Asymptotic cones of mapping class groups

4.1. **Distance formula in asymptotic cones of mapping class groups.** Fix an arbitrary asymptotic cone $\mathcal{AM}(S) = \mathrm{Con}^\omega(\mathcal{M}(S); (x_n), (d_n))$ of $\mathcal{M}(S)$.

We fix a point ν_0 in $\mathcal{M}(S)$ and define the map $\mathcal{MCG}(S) \to \mathcal{M}(S)$, $g \mapsto g\nu_0$, which according to Theorem 2.13 is a quasi-isometry. There exists a sequence $g_0 = (g_n^0)$ in $\mathcal{MCG}(S)$ such that $x_n = g_n^0\nu_0$, which we shall later use to discuss the ultrapower of the mapping class group.

Notation: By Remark 2.5, the group $g_0^\omega(\Pi_1\mathcal{MCG}(S)/\omega)(g_0^\omega)^{-1}$ acts transitively by isometries on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(S);(x_n),(d_n))$. We denote this group by $\mathcal{GM}$.

Definition 4.1. A path in $\mathcal{AM}$ obtained by taking an ultralimit of hierarchy paths, is, by a slight abuse of notation, also called a *hierarchy path*.

It was proven in [Beh06] that the asymptotic cone $\mathcal{AM}$ has cut-points and is thus a tree-graded space. Since $\mathcal{AM}$ is tree-graded, one can define the collection of pieces in the tree-graded structure of $\mathcal{AM}$ as the collection of maximal subsets in $\mathcal{AM}$ without cut-points. That set of pieces can be described as follows [BKMM08, § 7], where the equivalence to the third item below is an implicit consequence of the proof in [BKMM08] of the equivalence of the first two items.

Theorem 4.2 (Behrstock-Kleiner-Minsky-Mosher [BKMM08])**.** *Fix a pair of points $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathcal{AM}(S)$. If $\xi(S)\geq 2$, then the following are equivalent.*

1. *No point of $\mathcal{AM}(S)$ separates $\boldsymbol{\mu}$ from $\boldsymbol{\nu}$.*
2. *There exist points $\boldsymbol{\mu}',\boldsymbol{\nu}'$ arbitrarily close to $\boldsymbol{\mu},\boldsymbol{\nu}$, resp., for which there exists representative sequences $\langle\mu_n'\rangle,\langle\nu_n'\rangle$ satisfying*
$$\lim_\omega d_{\mathcal{C}(S)}(\mu_n',\nu_n')<\infty.$$
3. *For every hierarchy path $\boldsymbol{H}=\langle h_n\rangle$ connecting $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ there exists points $\boldsymbol{\mu}',\boldsymbol{\nu}'$ on $\boldsymbol{H}$ which are arbitrarily close to $\boldsymbol{\mu},\boldsymbol{\nu}$, resp., and for which there exists representative sequences $\langle\mu_n'\rangle,\langle\nu_n'\rangle$ with μ_n',ν_n' on h_n satisfying*
$$\lim_\omega d_{\mathcal{C}(S)}(\mu_n',\nu_n')<\infty.$$

Thus for every two points μ,ν in the same piece of $\mathcal{AM}$ there exists a sequence of pairs $\boldsymbol{\mu^{(k)}},\boldsymbol{\nu^{(k)}}$ such that:
- $\epsilon_k=\max\left(\mathrm{dist}_{\mathrm{Con}^\omega(\mathcal{M}(S);(x_n),(d_n))}\left(\boldsymbol{\mu},\boldsymbol{\mu^{(k)}}\right),\ \mathrm{dist}_{\mathrm{Con}^\omega(\mathcal{M}(S);(x_n),(d_n))}\left(\boldsymbol{\nu},\boldsymbol{\nu^{(k)}}\right)\right)$ goes to zero;
- $D^{(k)}=\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\mu_n^{(k)},\nu_n^{(k)})$ is finite for every $k\in\mathbb{N}$.

Remark 4.3. The projection of the marking complex $\mathcal{M}(S)$ onto the complex of curves $\mathcal{C}(S)$ induces a Lipschitz map from $\mathcal{AM}(S)$ onto the asymptotic cone $\mathcal{AC}(S)=\mathrm{Con}^\omega(\mathcal{C}(S);(\pi_{\mathcal{C}(S)}x_n),(d_n))$. By Theorem 4.2, pieces in $\mathcal{AM}(S)$ project onto singletons in $\mathcal{AC}(S)$, therefore two points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}(S)$ such that $\boldsymbol{\mu}\approx\boldsymbol{\nu}$ in the sense of (3) project onto the same point in $\mathcal{AC}(S)$. Thus the projection $\mathcal{AM}(S)\to\mathcal{AC}(S)$ induces a projection of $T_S=\mathcal{AM}(S)/\approx$ onto $\mathcal{AC}(S)$. The latter projection is not a bijection. This can be seen by taking for instance a sequence (γ_n) of geodesics in $\mathcal{C}(S)$ with one endpoint in $\pi_{\mathcal{C}(S)}(x_n)$ and of length $\sqrt{d_n}$, and considering elements g_n in $\mathcal{MCG}(S)$ obtained by performing $\sqrt{d_n}$ Dehn twists around each curve in γ_n consecutively. The projections of the limit points $\langle x_n\rangle$ and $\langle g_nx_n\rangle$ onto T_S are distinct, while their projections onto $\mathcal{AC}(S)$ coincide.

Let $\mathcal{U}$ be the set of all subsurfaces of S and let $\Pi\mathcal{U}/\omega$ be its ultrapower. For simplicity we denote by $\mathbf{S}$ the element in $\Pi\mathcal{U}/\omega$ given by the constant sequence (S). We define, for every $\mathbf{U}=(U_n)^\omega\in\Pi\mathcal{U}/\omega$, its complexity $\xi(\mathbf{U})$ to be $\lim_\omega\xi(U_n)$.

We say that an element $\mathbf{U}=(U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ is a *subsurface* of another element $\mathbf{Y}=(Y_n)^\omega$, and we denote it by $\mathbf{U}\subset\mathbf{Y}$ if ω-almost surely $U_n\subset Y_n$. An element $\mathbf{U}=(U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ is said to be a *strict subsurface* of another element $\mathbf{Y}=(Y_n)^\omega$, denoted $\mathbf{U} \subsetneq \mathbf{Y}$, if ω-almost surely $U_n \subsetneq Y_n$; equivalently $\mathbf{U} \subsetneq \mathbf{Y}$ if and only if $\mathbf{U} \subset \mathbf{Y}$ and $\xi(\mathbf{Y}) - \xi(\mathbf{U}) \geq 1$.

For every $\mathbf{U} = (U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ consider the ultralimit of the marking complexes of U_n with their own metric $\mathcal{M}^\omega\mathbf{U} = \lim_\omega(\mathcal{M}(U_n), (1), (d_n))$. Since there exists a surface U' such that ω-almost surely U_n is homeomorphic to U', the ultralimit $\mathcal{M}^\omega\mathbf{U}$ is isometric to the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(U'), (d_n))$. Consequently $\mathcal{M}^\omega\mathbf{U}$ is a tree-graded metric space.

Notation 4.4. We denote by $T_\mathbf{U}$ the quotient tree $\mathcal{M}^\omega\mathbf{U}/\approx$, as constructed in Lemma 3.9. We denote by $\mathrm{dist}_\mathbf{U}$ the metric on $\mathcal{M}^\omega\mathbf{U}$. We abuse notation slightly, by writing $\widetilde{\mathrm{dist}}_\mathbf{U}$ to denote both the pseudo-metric on $\mathcal{M}^\omega\mathbf{U}$ defined at the end of Section 3, and the metric this induces on $T_\mathbf{U}$.

Notation 4.5. We denote by $\mathcal{Q}(\partial\mathbf{U})$ the ultralimit $\lim_\omega(\mathcal{Q}(\partial U_n))$ in the asymptotic cone $\mathcal{AM}$, taken with respect to the basepoint obtained by projecting the base points we use for $\mathcal{AM}$ projected to $\mathcal{Q}(\partial\mathbf{U})$.

There exists a natural projection map $\pi_{\mathcal{M}^\omega\mathbf{U}}$ from $\mathcal{AM}$ to $\mathcal{M}^\omega\mathbf{U}$ sending any element $\boldsymbol{\mu} = \langle\mu_n\rangle$ to the element of $\mathcal{M}^\omega\mathbf{U}$ defined by the sequence of projections of μ_n onto $\mathcal{M}(U_n)$. This projection induces a well-defined projection between asymptotic cones with the same rescaling constants by Corollary 2.28.

Notation 4.6. For simplicity, we write $\mathrm{dist}_\mathbf{U}(\boldsymbol{\mu},\boldsymbol{\nu})$ and $\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu},\boldsymbol{\nu})$ to denote the distance, and the pseudo-distance in $\mathcal{M}^\omega\mathbf{U}$ between the images under the projection maps, $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\nu})$.

We denote by $\mathrm{dist}_{C(\mathbf{U})}(\boldsymbol{\mu},\boldsymbol{\nu})$ the ultralimit $\lim_\omega \frac{1}{d_n}\mathrm{dist}_{C(U_n)}(\mu_n,\nu_n)$.

The following is from [Beh06, Theorem 6.5 and Remark 6.3].

Lemma 4.7 (Behrstock [Beh06])**.** *Given a point $\boldsymbol{\mu}$ in $\mathcal{AM}$, the transversal tree $T_{\boldsymbol{\mu}}$ as defined in Definition 3.7 contains the set*

$$\{\boldsymbol{\nu} \mid \mathrm{dist}_\mathbf{U}(\boldsymbol{\mu},\boldsymbol{\nu}) = 0\,,\ \forall \mathbf{U} \subsetneq \mathbf{S}\}\,.$$

Corollary 4.8. *For any two distinct points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}$ there exists at least one subsurface $\mathbf{U}$ in $\Pi\mathcal{U}/\omega$ such that $\mathrm{dist}_\mathbf{U}(\boldsymbol{\mu},\boldsymbol{\nu}) > 0$ and for every strict subsurface $\mathbf{Y} \subsetneq \mathbf{U}$, $\mathrm{dist}_{\mathcal{M}^\omega(\mathbf{Y})}(\boldsymbol{\mu},\boldsymbol{\nu}) = 0$. In particular $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\nu})$ are in the same transversal tree.*

Proof. Indeed, every chain of nested subsurfaces of S contains at most ξ (the complexity of S) elements. It implies that the same is true for chains of subsurfaces $\mathbf{U}$ in $\Pi\mathcal{U}/\omega$. In particular, $\Pi\mathcal{U}/\omega$ with the inclusion order satisfies the descending chain condition. It remains to apply Lemma 4.7. □

Lemma 4.9. *There exists a constant t depending only on $\xi(S)$ such that for every $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ in $\mathcal{AM}$ and $\mathbf{U} \in \Pi\mathcal{U}/\omega$ the following inequality holds*

$$\mathrm{dist}_{C(\mathbf{U})}(\boldsymbol{\mu},\boldsymbol{\nu}) \leq t\,\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu},\boldsymbol{\nu})\,.$$

Proof. The inequality involves only the projections of $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ onto $\mathcal{M}^\omega(\mathbf{U})$. Also ω-almost surely U_n is homeomorphic to a fixed surface U, hence $\mathcal{M}^\omega(\mathbf{U})$ is isometric to some asymptotic cone of $\mathcal{M}(U)$, and it suffices to prove the inequality for $\mathbf{U}$ the constant sequence $\mathbf{S}$.

Let $([\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k])_{k\in K}$ be the set of non-trivial intersections of a geodesic $[\boldsymbol{\mu},\boldsymbol{\nu}]$ with pieces in $\mathcal{AM}$. Then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu},\boldsymbol{\nu}) = \mathrm{dist}(\boldsymbol{\mu},\boldsymbol{\nu}) - \sum_{k\in K}\mathrm{dist}(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$. For any $\epsilon > 0$

there exists a finite subset J in K such that $\sum_{k\in K\setminus J}\operatorname{dist}(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)\leq\epsilon$. According to Theorem 4.2 for every $k\in K$ there exist $\boldsymbol{\alpha}'_k=\langle\alpha'_{k,n}\rangle$ and $\boldsymbol{\beta}'_k=\langle\beta'_{k,n}\rangle$ for which $[\boldsymbol{\alpha}'_k,\boldsymbol{\beta}'_k]\subset[\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k]$ and such that:

(1) $\lim_\omega \operatorname{dist}_{C(S)}(\alpha'_{k,n},\beta'_{k,n})<\infty$
(2) $\sum_{k\in K}\operatorname{dist}(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)-2\epsilon\leq\sum_{k\in J}\operatorname{dist}(\boldsymbol{\alpha}'_k,\boldsymbol{\beta}'_k)$.

The second item above follows since Theorem 4.2 yields that $\boldsymbol{\alpha}'_k,\boldsymbol{\beta}'_k$ can be chosen so that $\sum_{k\in J}\operatorname{dist}(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$ is arbitrarily close to $\sum_{k\in J}\operatorname{dist}(\boldsymbol{\alpha}'_k,\boldsymbol{\beta}'_k)$, and since the contributions from those entries indexed by $K-J$ are less than ϵ.

Assume that $J=\{1,2,...m\}$ and that the points $\boldsymbol{\alpha}'_1,\boldsymbol{\beta}'_1,\boldsymbol{\alpha}'_2,\boldsymbol{\beta}'_2,....,\boldsymbol{\alpha}'_m,\boldsymbol{\beta}'_m$ appear on the geodesic $[\boldsymbol{\mu},\boldsymbol{\nu}]$ in that order.

By the triangle inequality

$$\operatorname{dist}_{C(S)}(\mu_n,\nu_n)\leq\operatorname{dist}_{C(S)}(\mu_n,\alpha'_{1,n})+\operatorname{dist}_{C(S)}(\beta'_{m,n},\nu_n)+ \sum_{j=1}^{m}\operatorname{dist}_{C(S)}(\alpha'_{j,n},\beta'_{j,n})+\sum_{j=1}^{m-1}\operatorname{dist}_{C(S)}(\beta'_{j,n},\alpha'_{j+1,n})\,. \tag{4}$$

Above we noted $\lim_\omega\sum_{j=1}^m\operatorname{dist}_{C(S)}(\alpha'_{j,n},\beta'_{j,n})<\infty$, thus if we rescale the above inequality by $\frac{1}{d_n}$ and take the ultralimit, we obtain:

$$\operatorname{dist}_{C(\mathbf{S})}(\boldsymbol{\mu},\boldsymbol{\nu})\leq\operatorname{dist}_{C(\mathbf{S})}(\boldsymbol{\mu},\boldsymbol{\alpha}'_1)+\operatorname{dist}_{C(\mathbf{S})}(\boldsymbol{\beta}'_m,\boldsymbol{\nu})+\sum_{j=1}^{m-1}\operatorname{dist}_{C(\mathbf{S})}(\boldsymbol{\beta}'_j,\boldsymbol{\alpha}'_{j+1}). \tag{5}$$

The distance formula implies that up to some multiplicative constant t, the right hand side of equation (5) is at most $\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\alpha}'_1)+\sum_{j=1}^{m-1}\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\beta}'_j,\boldsymbol{\alpha}'_{j+1})+\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\beta}'_m,\boldsymbol{\nu})$, which is equal to $\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})-\sum_{j=1}^m\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\alpha}'_j,\boldsymbol{\beta}'_j)$. Since above we noted that $\sum_{k\in K}\operatorname{dist}(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)-2\epsilon\leq\sum_{j\in J}\operatorname{dist}(\boldsymbol{\alpha}'_k,\boldsymbol{\beta}'_k)$, it follows that

$$\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})-\sum_{j=1}^m\operatorname{dist}_{\mathbf{S}}(\boldsymbol{\alpha}'_j,\boldsymbol{\beta}'_j)\leq\widetilde{\operatorname{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})+2\epsilon.$$

Thus, we have shown that for every $\epsilon>0$ we have $\operatorname{dist}_{C(\mathbf{S})}(\boldsymbol{\mu},\boldsymbol{\nu})\leq t\widetilde{\operatorname{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})+2t\epsilon$. This completes the proof. □

Lemma 4.10. *Let μ,ν be two markings in $\mathcal{M}(S)$ at $\mathcal{C}(S)$-distance s. Then there exist at most $s+1$ proper subsurfaces $S_1,...,S_{s+1}$ of S such that*

$$\operatorname{dist}_{\mathcal{M}(S)}(\mu,\nu)\leq C\sum_i\operatorname{dist}_{\mathcal{M}(S_i)}(\mu,\nu)+Cs+D \tag{6}$$

for some constants C,D depending only on S.

Proof. Since $\operatorname{dist}_{\mathcal{C}(S)}(\mu,\nu)=s$, every hierarchy path $\mathfrak{p}$ between μ,ν shadows a tight geodesic in $\mathcal{C}(S)$ with $s+1$ vertices (curves) $\alpha_1,...,\alpha_{s+1}$. By Lemma 2.17, if $Y\subsetneq S$ is a proper subsurface which yields a term in the distance formula (see Theorem 2.24) for $\operatorname{dist}_{\mathcal{M}(S)}(\mu,\nu)$, then there exists at least one (and at most 3) $i\in\{1,...,s+1\}$ for which $Y\cap\alpha_i=\emptyset$. Hence, any such Y occurs in the distance formula for $\operatorname{dist}_{\mathcal{M}(S_i)}(\mu,\nu)$, where $S_i=S\setminus\alpha_i$. Every term which occurs in the distance formula for $\operatorname{dist}_{\mathcal{M}(S)}(\mu,\nu)$, except for the $\operatorname{dist}_{\mathcal{C}(S)}(\mu,\nu)$ term, has a corresponding term (up to bounded multiplicative and additive errors) in the distance formula for at least one of the $\operatorname{dist}_{\mathcal{M}(S_i)}(\mu,\nu)$. Since $\operatorname{dist}_{\mathcal{C}(S)}(\mu,\nu)=s$, up to the additive and multiplicative bounds occuring in the distance formula this term in the distance formula for $\mathrm{dist}_{\mathcal{M}(S)}(\mu,\nu)$ is bounded above by s up to a bounded multiplicative and additive error. This implies the proof that Equation 6 holds. □

Notation 4.11. For any subset $F \subset \Pi\mathcal{U}/\omega$ we define the map $\psi_F \colon \mathcal{AM} \to \prod_{\mathbf{U}\in F} T_{\mathbf{U}}$, where for each $\mathbf{U} \in F$ the map $\psi_{\mathbf{U}} \colon \mathcal{AM} \to T_{\mathbf{U}}$ is the canonical projection of $\mathcal{AM}$ onto $T_{\mathbf{U}}$. In the particular case when F is finite equal to $\{\mathbf{U}_1, ..., \mathbf{U}_k\}$ we also use the notation $\psi_{\mathbf{U}_1,..,\mathbf{U}_k}$.

Lemma 4.12. *Let $\mathfrak{h} \subset \mathcal{AM}$ denote the ultralimit of a sequence of quasi-geodesics in $\mathcal{M}$ each of which is $\mathcal{C}(U)$–monotonic for every $U \subseteq S$ with the quasi-geodesics and monotonicity constants are all uniform over the sequence.*

Any path $\mathfrak{h} \colon [0,a] \to \mathcal{AM}$, as above, projects onto a geodesic $\mathfrak{g} \colon [0,b] \to T_{\mathbf{U}}$ such that $\mathfrak{h}(0)$ projects onto $\mathfrak{g}(0)$ and, assuming both $\mathfrak{h}$ and $\mathfrak{g}$ are parameterized by arc length, the map $[0,a] \to [0,b]$ defined by the projection is non-decreasing.

Proof. Fix a path $\mathfrak{h}$ satisfying the hypothesis of the lemma. It suffices to prove that for every $\boldsymbol{x}, \boldsymbol{y}$ on $\mathfrak{h}$ and every $\boldsymbol{\mu}$ on $\mathfrak{h}$ between $\boldsymbol{x}$ and $\boldsymbol{y}$, $\psi_{\mathbf{U}}(\boldsymbol{\mu})$ is on the geodesic joining $\psi_{\mathbf{U}}(\boldsymbol{x})$ to $\psi_{\mathbf{U}}(\boldsymbol{y})$ in $T_{\mathbf{U}}$. If the contrary would hold then there would exist $\boldsymbol{\nu}, \boldsymbol{\rho}$ on $\mathfrak{h}$ with $\boldsymbol{\mu}$ between them, such that $\psi_{\mathbf{U}}(\boldsymbol{\nu}) = \psi_{\mathbf{U}}(\boldsymbol{\rho}) \neq \psi_{\mathbf{U}}(\boldsymbol{\mu})$. Without loss of generality we may assume that $\boldsymbol{\nu}, \boldsymbol{\rho}$ are the endpoints of $\mathfrak{h}$. We denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the sub-arcs of $\mathfrak{h}$ of endpoints $\boldsymbol{\nu}, \boldsymbol{\mu}$ and respectively $\boldsymbol{\mu}, \boldsymbol{\rho}$.

The projection $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h})$ is by Corollary 2.28 a continuous path joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$ and containing $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$.

According to [DS07, Lemma 2.19] a geodesic $\bar{\mathfrak{g}}_1$ joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$ projects onto the geodesic $[\psi_{\mathbf{U}}(\boldsymbol{\nu}), \psi_{\mathbf{U}}(\boldsymbol{\mu})]$ in $T_{\mathbf{U}}$. Moreover the set $\mathrm{Cutp}\, \bar{\mathfrak{g}}_1$ of cut-points of $\bar{\mathfrak{g}}_1$ in the tree-graded space $\mathcal{M}(\mathbf{U})$ projects onto $[\psi_{\mathbf{U}}(\boldsymbol{\nu}), \psi_{\mathbf{U}}(\boldsymbol{\mu})]$. By properties of tree-graded spaces [DS05] the continuous path $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h}_1)$ contains $\mathrm{Cutp}\, \bar{\mathfrak{g}}_1$.

Likewise if $\bar{\mathfrak{g}}_2$ is a geodesic joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$ then the set $\mathrm{Cutp}\, \bar{\mathfrak{g}}_2$ projects onto $[\psi_{\mathbf{U}}(\boldsymbol{\mu}), \psi_{\mathbf{U}}(\boldsymbol{\rho})]$, which is the same as the geodesic $[\psi_{\mathbf{U}}(\boldsymbol{\mu}), \psi_{\mathbf{U}}(\boldsymbol{\rho})]$ reversed, and the path $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h}_2)$ contains $\mathrm{Cutp}\, \bar{\mathfrak{g}}_2$. This implies that $\mathrm{Cutp}\, \bar{\mathfrak{g}}_1 = \mathrm{Cutp}\, \bar{\mathfrak{g}}_2$ and that there exists $\boldsymbol{\nu}'$ on $\mathfrak{h}_1$ and $\boldsymbol{\rho}'$ on $\mathfrak{h}_2$ such that $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}') = \pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho}')$ and $\psi_{\mathbf{U}}(\boldsymbol{\nu}') = \psi_{\mathbf{U}}(\boldsymbol{\rho}') \neq \psi_{\mathbf{U}}(\boldsymbol{\mu})$. Without loss of generality we assume that $\boldsymbol{\nu}' = \boldsymbol{\nu}$ and $\boldsymbol{\rho}' = \boldsymbol{\rho}$.

Since $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\mu}) > 0$ it follows that $\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}, \boldsymbol{\mu}) > 0$. Since by construction μ_n is on a path joining ν_n and ρ_n satisfying the hypotheses of the lemma, we know that up to a uniformly bounded additive error we have $\mathrm{dist}_{\mathcal{C}(Y)}(\nu_n, \mu_n) \leq \mathrm{dist}_{\mathcal{C}(Y)}(\nu_n, \rho_n)$ for every $Y \subset U_n$. It then follows from the distance formula that for some positive constant C that

$$\frac{1}{C}\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}, \boldsymbol{\rho}).$$

In particular, this implies that $\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}, \boldsymbol{\rho}) > 0$, contradicting the fact that $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}) = \pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$. □

The following is an immediate consequence of Lemma 4.12 since by construction hierarchy paths satisfy the hypothesis of the lemma.

Corollary 4.13. *Every hierarchy path in $\mathcal{AM}$ projects onto a geodesic in $T_{\mathbf{U}}$ for every subsurface $\mathbf{U}$ as in Lemma 4.12.*

Notation 4.14. Let F, G be two finite subsets in the asymptotic cone $\mathcal{AM}$, and let K be a fixed constant larger than the constant $M(S)$ from Lemma 2.17.

We denote by $\mathcal{Y}(F,G)$ the set of elements $\mathbf{U} = (U_n)^\omega$ in the ultrapower $\Pi\mathcal{U}/\omega$ such that for any two points $\boldsymbol{\mu} = \lim_\omega(\mu_n) \in F$ and $\boldsymbol{\nu} = \lim_\omega(\nu_n) \in G$, the subsurfaces U_n are ω-almost surely K-large domains for the pair (μ_n, ν_n), in the sense of Definition 2.18.

If $F = \{\boldsymbol{\mu}\}$ and $G = \{\boldsymbol{\nu}\}$ then we simplify the notation to $\mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

Lemma 4.15. *Let $\boldsymbol{\mu}, \boldsymbol{\nu}$ be two points in $\mathcal{AM}$ and let $\mathbf{U} = (U_n)^\omega$ be an element in $\Pi\mathcal{U}/\omega$. If $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$, then $\lim_\omega\left(\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n)\right) = \infty$ (and thus $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$). In particular the following holds.*

$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{\mathbf{U} \in \Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$$

Proof. We establish the result by proving the contrapositive; thus we assume that $\lim_\omega\left(\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n)\right) < \infty$. Theorem 4.2 then implies that $\pi_{\mathcal{M}^\omega \mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega \mathbf{U}}(\boldsymbol{\nu})$ are in the same piece, hence $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 0$. □

We are now ready to prove a distance formula in the asymptotic cones.

Theorem 4.16 (distance formula for asymptotic cones)**.** *For every $\boldsymbol{\mu}, \boldsymbol{\nu}$ in $\mathcal{AM}$*

(7)
$$\frac{1}{E}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \le \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \le \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \le E\, \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})\,.$$

The constant E depends only on the constant K used to define $\mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$, and on the complexity $\xi(S)$.

Proof. Let us prove by induction on the complexity of S that

(8)
$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > \frac{1}{E}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})$$

for some $E > 1$. Let $\boldsymbol{\mu}, \boldsymbol{\nu}$ be two distinct elements in $\mathcal{AM}$. If $\mathcal{M}(S)$ is hyperbolic, then $\mathcal{AM}$ is a tree; hence there are no non-trivial subsets without cut-points and thus in this case we have $\widetilde{\mathrm{dist}}_{\mathbf{S}} = \mathrm{dist}_{\mathbf{S}}$. This gives the base for the induction.

We may assume that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < \frac{1}{3}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Otherwise we would have that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$, which implies by Lemma 4.15 that $\mathbf{S} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$, and we would be done by choosing $E = 3$.

Since $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ is obtained from $\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ by removing $\sum_{i \in I} \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$, where $[\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i], i \in I$, are all the non-trivial intersections of a geodesic $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ with pieces, it follows that there exists $F \subset I$ finite such that $\sum_{i \in F} \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i) \ge \frac{1}{2}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. For simplicity assume that $F = \{1, 2, ..., m\}$ and that the intersections $[\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i]$ appear on $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ in the increasing order of their index. According to Proposition 3.3, (T_2'), the points $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i$ also appear on any path joining $\boldsymbol{\mu}, \boldsymbol{\nu}$. Therefore, without loss of generality, for the rest of the proof we will assume write $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ to denote a hierarchy path, and $[\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i]$ sub-paths of it (this is a slight abuse of notation since hierarchy paths are not geodesics). By Theorem 4.2, there exist $[\boldsymbol{\alpha}_i', \boldsymbol{\beta}_i'] \subset [\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i]$ for every $i \in F$ with the following properties:

- there exists a number s such that
$$\lim_{\omega} \text{dist}_{C(S)}(\alpha'_{i,n}, \beta'_{i,n}) < s\,,\ \forall i = 1, ..., m\,;$$
- $\sum_{i=1}^{m} \text{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i) > \frac{1}{3}\text{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

Let $l = m(s+1)$. By Lemma 4.10 there exists a sequence of proper subsurfaces $Y_1(n), ..., Y_l(n)$ such that ω-almost surely:

$$\sum_{i=1}^{m} \text{dist}_{\mathcal{M}(S)}(\alpha'_{i,n}, \beta'_{i,n}) \leq C \sum_{i=1}^{m} \sum_{j=1}^{l} \text{dist}_{\mathcal{M}(Y_j(n))}(\alpha'_{i,n}, \beta'_{i,n}) + Csm + Dm\,. \tag{9}$$

Let $\boldsymbol{Y}_j$ be the element in $\Pi\mathcal{U}/\omega$ given by the sequence of subsurfaces $(Y_j(n))$.

Rescaling (9) by $\frac{1}{d_n}$, passing to the ω-limit and applying Lemma 2.1 we deduce that

$$\frac{1}{3}\text{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq C \sum_{i=1}^{m} \sum_{j=1}^{l} \text{dist}_{\boldsymbol{Y}_j}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i)\,. \tag{10}$$

Then, as the complexity of $\boldsymbol{Y}_j$ is smaller than the complexity of S, according to the induction hypothesis the second term in (10) is at most

$$CE \sum_{i=1}^{m} \sum_{j=1}^{l} \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i), \mathbf{U} \subseteq \boldsymbol{Y}_j} \widetilde{\text{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i)\,.$$

Lemmas 4.12 and 4.15 imply that the non-zero terms in the latter sum correspond to subsurfaces $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$, and that the sum is at most $CEl \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\text{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. We have thus obtained that $\frac{1}{3}\text{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq CEl \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\text{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

The inequality

$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\text{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \text{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$$

immediately follows from the definition of $\widetilde{\text{dist}}$.

In remains to prove the inequality:

$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \text{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < E\text{dist}(\boldsymbol{\mu}, \boldsymbol{\nu}). \tag{11}$$

It suffices to prove (11) for every possible finite sub-sum of the left hand side of (11). Note that this would imply also that the set of $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $\text{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ is countable, since it implies that the set of $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $\text{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > \frac{1}{k}$ has cardinality at most $kE\text{dist}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

Let $\mathbf{U}_1, ..., \mathbf{U}_m$ be elements in $\mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ represented by sequences $(U_{i,n})$ of large domains of hierarchy paths connecting μ_n and ν_n, $i = 1, ..., m$.

By definition, the sum

$$\text{dist}_{\mathbf{U}_1}(\boldsymbol{\mu}, \boldsymbol{\nu}) + ... + \text{dist}_{\mathbf{U}_m}(\boldsymbol{\mu}, \boldsymbol{\nu}) \tag{12}$$

is equal to

$$\lim\nolimits_{\omega}\left(\frac{\text{dist}_{\mathcal{M}(U_{1,n})}(\mu_n, \nu_n)}{d_n}\right) + ... + \lim\nolimits_{\omega}\left(\frac{\text{dist}_{\mathcal{M}(U_{m,n})}(\mu_n, \nu_n)}{d_n}\right) \tag{13}$$
$$= \lim_{\omega} \frac{1}{d_n}\left[\text{dist}_{\mathcal{M}(U_{1,n})}(\mu_n, \nu_n) + ... + \text{dist}_{\mathcal{M}(U_{m,n})}(\mu_n, \nu_n)\right]\,.$$

Using the distance formula, we can write

$$\text{(14)} \qquad \begin{aligned} &\mathrm{dist}_{\mathcal{M}(U_{1,n})}(\mu_n,\nu_n)+...+\mathrm{dist}_{\mathcal{M}(U_{m,n})}(\mu_n,\nu_n) \leq_{a,b} \\ &\sum_{V\subseteq U_{1,n}}\mathrm{dist}_{C(V)}(\mu_n,\nu_n)+...+\sum_{V\subseteq U_{m,n}}\mathrm{dist}_{C(V)}(\mu_n,\nu_n). \end{aligned}$$

Since each $U_{i,n}$ is a large domain in the hierarchy connecting μ_n and ν_n, and since for each fixed n all of the $U_{i,n}$ are different ω-a.s. we can apply Lemma 2.19, and conclude that each summand occurs in the right hand side of (14) at most 2ξ times (where ξ is denoting $\xi(S)$). Hence we can bound the right hand side of (14) from above by

$$2\xi\sum_{V\subseteq S}\mathrm{dist}_{C(V)}(\mu_n,\nu_n)\leq_{a,b} 2\xi\,\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,\nu_n).$$

Therefore the right hand side in (13) does not exceed

$$2\xi\lim_\omega\left(\frac{1}{d_n}(a\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,\nu_n)+b)\right)=2a\xi\,\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu}),$$

proving (11). □

Notation: Let $\boldsymbol{\mu}^0$ be a fixed point in $\mathcal{AM}$ and for every $\mathbf{U}\in\Pi\mathcal{U}/\omega$ let $\boldsymbol{\mu}^0_{\mathbf{U}}$ be the image of $\boldsymbol{\mu}^0$ by canonical projection on $T_{\mathbf{U}}$. In $\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}$ we consider the subset $\mathcal{T}_0'=\left\{(x_{\mathbf{U}})\in\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}\ ;\ x_{\mathbf{U}}\neq\boldsymbol{\mu}^0_{\mathbf{U}}\text{ for countably many }\mathbf{U}\in\Pi\mathcal{U}/\omega\right\}$, and $\mathcal{T}_0=\left\{(x_{\mathbf{U}})\in\mathcal{T}_0'\ ;\ \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega}\widetilde{\mathrm{dist}}_{\mathbf{U}}\left(x_{\mathbf{U}},\boldsymbol{\mu}^0_{\mathbf{U}}\right)<\infty\right\}$. We will always consider $\mathcal{T}_0$ endowed with the ℓ^1 metric.

The following is an immediate consequence of Theorem 4.16 and Lemma 4.15.

Corollary 4.17. *Consider the map $\psi\colon\mathcal{AM}\to\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}$ whose components are the canonical projections of $\mathcal{AM}$ onto $T_{\mathbf{U}}$. This map is a bi-Lipschitz homeomorphism onto its image in $\mathcal{T}_0$.*

Proposition 4.18. *Let $\mathfrak{h}\subset\mathcal{AM}$ denote the ultralimit of a sequence of quasigeodesics in $\mathcal{M}$ each of which is $\mathcal{C}(U)$–monotonic for every $U\subseteq S$ with the quasigeodesics and monotonicity constants are all uniform over the sequence. Then $\psi(\mathfrak{h})$ is a geodesic in $\mathcal{T}_0$.*

In particular, for any hierarchy path $\mathfrak{h}\subset\mathcal{AM}$, its image under ψ is a geodesic in $\mathcal{T}_0$.

The first statement of this proposition is a direct consequence of the following lemma, which is an easy exercise in elementary topology. The second statement is a consequence of the first.

Lemma 4.19. *Let $(X_i,\mathrm{dist}_i)_{i\in I}$ be a collection of metric spaces. Fix a point $x=(x_i)\in\prod_{i\in I}X_i$, and consider the subsets*

$$S_0'=\left\{(y_i)\in\prod_{i\in I}X_i : y_i\neq x_i\text{ for countably many }i\in I\right\}$$

and $S_0=\left\{(y_i)\in S_0' : \sum_{i\in I}\mathrm{dist}_i\left(y_i,x_i\right)<\infty\right\}$ endowed with the ℓ^1 distance $\mathrm{dist}=\sum_{i\in I}\mathrm{dist}_i$.

Let $\mathfrak{h}\colon [0,a] \to S_0$ *be a non-degenerate parameterizations of a topological arc. For each* $i \in I$ *assume that* $\mathfrak{h}$ *projects onto a geodesic* $\mathfrak{h}_i\colon [0,a_i] \to X_i$ *such that* $\mathfrak{h}(0)$ *projects onto* $\mathfrak{h}_i(0)$ *and the function* $\varphi_i = d_i(\mathfrak{h}_i(0), \mathfrak{h}_i(t))\colon [0,a] \to [0,a_i]$ *is a non-decreasing function. Then* $\mathfrak{h}[0,a]$ *is a geodesic in* (S_0, dist)*.*

Notation: We write $\widetilde{\mathrm{dist}}$ to denote the ℓ^1 metric on $\mathcal{T}_0$. We abuse notation slightly by also using $\widetilde{\mathrm{dist}}$ to denote both its restriction to $\psi(\mathcal{AM})$ and for the metric on $\mathcal{AM}$ which is the pull-back via ψ of $\widetilde{\mathrm{dist}}$. We have that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{\mathbf{U} \in \Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ for every $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{AM}$, and that $\widetilde{\mathrm{dist}}$ is bi-Lipschitz equivalent to $\mathrm{dist}_{\mathcal{AM}}$, according to Theorem 4.16.

Note that the canonical map $\mathcal{AM} \to \prod_{\mathbf{U} \in \Pi\mathcal{U}/\omega} \mathcal{C}(\mathbf{U})$, whose components are the canonical projections of $\mathcal{AM}$ onto $\mathcal{C}(\mathbf{U})$, ultralimit of complexes of curves, factors through the above bi-Lipschitz embedding. These maps were studied in [Beh04], where among other things it was shown that this canonical map is not a bi-Lipschitz embedding.

4.2. **Dimension of asymptotic cones of mapping class groups.**

Remark 4.20. Let $\mathbf{U}$ and $\mathbf{V}$ be two elements in $\Pi\mathcal{U}/\omega$ such that either $\mathbf{U}, \mathbf{V}$ overlap or $\mathbf{U} \subsetneq \mathbf{V}$. Then $\mathcal{Q}(\partial\mathbf{U})$ projects onto $T_{\mathbf{V}}$ in a unique point.

Proof. Indeed, $\mathcal{Q}(\partial\mathbf{U})$ projects into $\mathcal{Q}(\pi_{\mathcal{M}^\omega\mathbf{V}}(\partial\mathbf{U}))$, which is contained in one piece of $\mathcal{M}^\omega\mathbf{V}$, hence it projects onto one point in $T_{\mathbf{V}}$. □

Theorem 4.21. *Consider a pair* $\mathbf{U}, \mathbf{V}$ *in* $\Pi\mathcal{U}/\omega$*.*

(1) *If* $\mathbf{U} \cap \mathbf{V} = \emptyset$ *then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is* $T_{\mathbf{U}} \times T_{\mathbf{V}}$*.*

(2) *If* $\mathbf{U}$ *and* $\mathbf{V}$ *overlap then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is*
$$(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}}) \ ,$$
where u *is the point in* $T_{\mathbf{V}}$ *onto which projects* $\mathcal{Q}(\partial\mathbf{U})$ *and* v *is the point in* $T_{\mathbf{U}}$ *onto which projects* $\mathcal{Q}(\partial\mathbf{V})$ *(see Remark 4.20);*

(3) *If* $\mathbf{U} \subsetneq \mathbf{V}$*,* $u \in T_{\mathbf{V}}$ *is the point onto which projects* $\mathcal{Q}(\partial\mathbf{U})$ *and* $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i \in I} \mathcal{C}_i$ *is the decomposition into connected components then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is*
$$(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i \in I} (\{t_i\} \times \mathcal{C}_i) \,,$$
where t_i *are points in* $T_{\mathbf{U}}$*.*

Proof. Case (1) is obvious. We prove (2). Let $\boldsymbol{\mu}$ be a point in $\mathcal{AM}$ whose projection on $T_{\mathbf{U}}$ is different from v. Then $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \partial\mathbf{V}) > 0$ which implies that $\lim_\omega \mathrm{dist}_{C(U_n)}(\mu_n, \partial V_n) = +\infty$. Theorem 2.20 implies that ω-almost surely $\mathrm{dist}_{C(V_n)}(\mu_n, \partial U_n) \le D$. Hence $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu})$ is in the same piece of $\mathcal{M}^\omega\mathbf{V}$ as $\mathcal{Q}(\pi_{\mathcal{M}^\omega\mathbf{V}}(\partial\mathbf{U}))$, so $\boldsymbol{\mu}$ projects on $T_{\mathbf{V}}$ in u. The set of $\boldsymbol{\mu}$ in $\mathcal{AM}$ projecting on $T_{\mathbf{U}}$ in v contains $\mathcal{Q}(\partial\mathbf{V})$, hence their projection on $T_{\mathbf{V}}$ is surjective.

We now prove (3). As before the set of $\boldsymbol{\mu}$ projecting on $T_{\mathbf{V}}$ in u contains $\mathcal{Q}(\partial\mathbf{U})$, hence it projects on $T_{\mathbf{U}}$ surjectively.

For every $i \in I$ we choose $\boldsymbol{\mu}_i \in \mathcal{AM}$ whose projection on $T_{\mathbf{V}}$ is in $\mathcal{C}_i$. Then every $\boldsymbol{\mu}$ with projection on $T_{\mathbf{V}}$ in $\mathcal{C}_i$ has the property that any topological arc joining $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu}_i)$ does not intersect the piece containing $\mathcal{Q}(\partial\mathbf{U})$. Otherwise by property (T_2') of tree-graded spaces the geodesic joining $\pi_{\mathcal{M}^\omega \mathbf{V}}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}^\omega \mathbf{V}}(\boldsymbol{\mu}_i)$ in $\mathcal{M}^\omega \mathbf{V}$ would intersect the same piece, and since geodesics in $\mathcal{M}^\omega \mathbf{V}$ project onto geodesics in $T_{\mathbf{V}}$ [DS07] the geodesic in $T_{\mathbf{V}}$ joining the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ would contain u. This would contradict the fact that both projections are in the same connected component $\mathcal{C}_i$.

Take $\langle \mu_n \rangle$ representatives of $\boldsymbol{\mu}$ and $\langle \mu_n^i \rangle$ representatives of $\boldsymbol{\mu}_i$. The above and Lemma 2.17 imply that ω-almost surely $\mathrm{dist}_{C(U_n)}(\mu_n, \mu_n^i) \leq M$, hence the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ onto $\mathcal{M}^\omega \mathbf{U}$ are in the same piece. Therefore the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ onto $T_{\mathbf{U}}$ coincide. Thus all elements in $\mathcal{C}_i$ project in $T_{\mathbf{U}}$ in the same point t_i which is the projection of $\boldsymbol{\mu}_i$. □

Remark 4.22. Note that in cases (2) and (3) the image of $\psi_{\mathbf{U},\mathbf{V}}$ has dimension 1. In case (3) this is due to the Hurewicz-Morita-Nagami Theorem [Nag83, Theorem III.6].

We shall need the following classical Dimension Theory result:

Theorem 4.23 ([Eng95])**.** *Let K be a compact metric space. If for every $\epsilon > 0$ there exists an ϵ-map $f : K \to X$ (i.e., a continuous map with diameter of $f^{-1}(x)$ at most ϵ for every $x \in X$) such that $f(K)$ is of dimension at most n, then K has dimension at most n.*

We give another proof of the following theorem:

Theorem 4.24 (Dimension Theorem [BM07])**.** *Every locally compact subset of the asymptotic cone of the mapping class group of a surface S has dimension at most $\xi(S)$.*

Proof. Since every subset of the asymptotic cone is itself a metric space, it is paracompact. This implies that every locally compact subset of the asymptotic cone is a free union of σ-compact subspaces [Dug66, Theorem 7.3, p. 241]. Thus, it suffices to prove that every compact subset in $\mathcal{AM}$ has dimension at most $\xi(S)$. Let K be such a compact subset. For simplicity we see it as a subset of $\psi(\mathcal{AM}) \subset \Pi_{\mathbf{V} \in \Pi\mathcal{U}/\omega} T_{\mathbf{V}}$.

Fix $\epsilon > 0$. Let N be a finite $\frac{\epsilon}{4}$–net for $(K, \widetilde{\mathrm{dist}})$, i.e. a finite subset such that $K = \bigcup_{a \in N} B_{\widetilde{\mathrm{dist}}}\left(a, \frac{\epsilon}{4}\right)$. There exists a finite subset $J_\epsilon \subset \Pi\mathcal{U}/\omega$ such that for every $a, b \in N$, $\sum_{\mathbf{U} \not\in J_\epsilon} \widetilde{\mathrm{dist}}_{\mathbf{U}}(a,b) < \frac{\epsilon}{2}$. Then for every $x, y \in K$, $\sum_{\mathbf{U} \not\in J_\epsilon} \widetilde{\mathrm{dist}}_{\mathbf{U}}(x,y) < \epsilon$. In particular this implies that the projection $\pi_{J_\epsilon} \colon \Pi_{\mathbf{V} \in \Pi\mathcal{U}/\omega} T_{\mathbf{V}} \to \Pi_{\mathbf{V} \in J_\epsilon} T_{\mathbf{V}}$ restricted to K is an ϵ-map.

We now prove that for every finite subset $J \subset \Pi\mathcal{U}/\omega$ the projection $\pi_J(K)$ has dimension at most $\xi(S)$, by induction on the cardinality of J. This will finish the proof, due to Theorem 4.23.

If the subsurfaces in J are pairwise disjoint then the cardinality of J is at most $3g + p - 3$ and thus the dimension bound follows. So, suppose we have a pair of subsurfaces $\mathbf{U}, \mathbf{V}$ in J which are not disjoint: then they are either nested or overlapping. We deal with the two cases separately.

Suppose $\mathbf{U}, \mathbf{V} \in J$ overlap. Then according to Theorem 4.21, $\psi_{\mathbf{U},\mathbf{V}}(\mathcal{AM})$ is $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$, hence we can write $K = K_{\mathbf{U}} \cup K_{\mathbf{V}}$, where $\pi_{\mathbf{U},\mathbf{V}}(K_{\mathbf{U}}) \subset T_{\mathbf{U}} \times \{u\}$ and $\pi_{\mathbf{U},\mathbf{V}}(K_{\mathbf{V}}) \subset \{v\} \times T_{\mathbf{V}}$. Now $\pi_J(K_{\mathbf{U}}) = \pi_{J \setminus \{\mathbf{U}\}}(K_{\mathbf{U}}) \times \{u\} \subset \pi_{J \setminus \{\mathbf{U}\}}(K) \times \{u\}$, which is of dimension at most $\xi(S)$ by induction hypothesis. Likewise $\pi_J(K_{\mathbf{V}}) = \pi_{J \setminus \{\mathbf{V}\}}(K_{\mathbf{V}}) \times \{v\} \subset \pi_{J \setminus \{\mathbf{V}\}}(K) \times \{v\}$ is of dimension at most $\xi(S)$. It follows that $\pi_J(K)$ is of dimension at most $\xi(S)$.

Assume that $\mathbf{U} \subsetneq \mathbf{V}$. Let u be the point in $T_{\mathbf{V}}$ onto which projects $\mathcal{Q}(\partial \mathbf{U})$ and $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i\in I} \mathcal{C}_i$ the decomposition into connected components. By Theorem 4.21, $\psi_{\mathbf{U},\mathbf{V}}(\mathcal{AM})$ is $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$, where t_i are points in $T_{\mathbf{U}}$. We prove that $\pi_J(K)$ is of dimension at most $\xi(S)$ by means of Theorem 4.23. Let $\delta > 0$. We shall construct a 2δ-map on $\pi_J(K)$ with image of dimension at most $\xi(S)$. Let N be a finite δ-net of $(K, \widetilde{\mathrm{dist}})$. There exist $i_1, ..., i_m$ in I such that $\pi_{\mathbf{U},\mathbf{V}}(N)$ is contained in $\mathfrak{T} = (T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{j=1}^m(\{t_{i_j}\} \times \mathcal{C}_{i_j})$. The set $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$ endowed with the ℓ^1-metric is a tree and $\mathfrak{T}$ is a subtree in it. We consider the nearest point retraction map

$$\mathrm{retr} : (T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i) \to \mathfrak{T}$$

which is moreover a contraction. This defines a contraction

$$\mathrm{retr}_J : \psi_J(\mathcal{AM}) \to \psi_{J\setminus\{\mathbf{U},\mathbf{V}\}}(\mathcal{AM}) \times \mathfrak{T}\,,\ \mathrm{retr}_J = \mathrm{id} \times \mathrm{retr}\,.$$

The set $\pi_J(K)$ splits as $K_{\mathfrak{T}} \sqcup K'$, where $K_{\mathfrak{T}} = \pi_J(K) \cap \pi^{-1}_{\mathbf{U},\mathbf{V}}(\mathfrak{T})$ and K' is its complementary set. Every $x \in K'$ has $\pi_{\mathbf{U},\mathbf{V}}(x)$ in some $\{t_i\} \times \mathcal{C}_i$ with $i \in I\setminus\{i_1, ..., i_m\}$. Since there exists $n \in N$ such that x is at distance smaller than δ from $\pi_J(n)$, it follows that $\pi_{\mathbf{U},\mathbf{V}}(x)$ is at distance smaller than δ from $\mathfrak{T}$, hence at distance smaller that δ from $(t_i, u) = \mathrm{retr}\,(\pi_{\mathbf{U},\mathbf{V}}(x))$. We conclude that $\mathrm{retr}(\pi_{\mathbf{U},\mathbf{V}}(K')) \subset \{t_i \mid i \in I\} \times \{u\} \cap \pi_{\mathbf{U}}(K) \times \{u\}$, hence $\mathrm{retr}_J(K') \subset \pi_{J\setminus\{\mathbf{V}\}}(K) \times \{u\}$, which is of dimension at most $\xi(S)$ by the induction hypothesis.

By definition $\mathrm{retr}_J(K_{\mathfrak{T}}) = K_{\mathfrak{T}}$. The set $K_{\mathfrak{T}}$ splits as $K_{\mathbf{U}} \sqcup \bigsqcup_{j=1}^m K_j$, where $K_{\mathbf{U}} = \pi_J(K) \cap \pi^{-1}_{\mathbf{U},\mathbf{V}}(T_{\mathbf{U}} \times \{u\})$ and $K_j = \pi_J(K) \cap \pi^{-1}_{\mathbf{U},\mathbf{V}}(\{t_{i_j}\} \times \mathcal{C}_{i_j})$. Now $K_{\mathbf{U}} \subset \pi_{J\setminus\{\mathbf{V}\}}(K) \times \{u\}$, while $K_j \subset \pi_{J\setminus\{\mathbf{U}\}}(K) \times \{t_{i_j}\}$ for $j = 1, .., m$, hence by the induction hypothesis they have dimension at most $\xi(S)$. Consequently $K_{\mathfrak{T}}$ has dimension at most $\xi(S)$.

We have obtained that the map retr_J restricted to $\pi_J(K)$ is a 2δ-map with image $K_{\mathfrak{T}} \cup \mathrm{retr}_J(K')$ of dimension at most $\xi(S)$. It follows that $\pi_J(K)$ is of dimension at most $\xi(S)$. □

4.3. **Median structure.** More can be said about the structure of $\mathcal{AM}$ endowed with $\widetilde{\mathrm{dist}}$. We recall that a *median space* is a metric space for which, given any triple of points, there exists a unique *median point*, that is a point which is simultaneously between any two points in that triple. A point x is said to be *between* two other points a, b in a metric space (X, dist) if $\mathrm{dist}(a, x)+\mathrm{dist}(x, b) = \mathrm{dist}(a, b)$. See [CDH] for details.

Theorem 4.25. *The asymptotic cone $\mathcal{AM}$ endowed with the metric $\widetilde{\mathrm{dist}}$ is a median space. Moreover hierarchy paths (i.e., ultralimits of hierarchy paths) are geodesics in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$.*

The second statement follows from Proposition 4.18. Note that the first statement is equivalent to that of $\psi(\mathcal{AM})$ being a median subspace of the median space $(\mathcal{T}_0, \widetilde{\mathrm{dist}})$. The proof is done in several steps.

Lemma 4.26. *Let $\boldsymbol{\nu}$ in $\mathcal{AM}$ and $\boldsymbol{\Delta} = (\Delta_n)^\omega$, where Δ_n is a multicurve. Let $\boldsymbol{\nu}'$ be the projection of $\boldsymbol{\nu}$ on $\mathcal{Q}(\boldsymbol{\Delta})$. Then for every subsurface $\mathbf{U}$ such that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$ the distance $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}') = 0$.*

Proof. The projection of $\boldsymbol{\nu}$ on $\mathcal{Q}(\boldsymbol{\Delta})$ is defined as limit of projections described in Section 2.5.2. Since $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$ the subsurface $\mathbf{U} = (U_n)^\omega$ is contained in a component, $\mathbf{V} = (V_n)^\omega$, of $S - \boldsymbol{\Delta} = (S - \Delta_n)^\omega$. The marking ν'_n, by construction, does not differ from the intersection of ν_n with V_n, and since $U_n \subseteq V_n$ the same is true for U_n, hence $\mathrm{dist}_{C(U_n)}(\nu_n, \nu'_n) = O(1)$. On the other hand, if $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}') > 0$ then by Lemma 4.15 $\lim_\omega \mathrm{dist}_{C(U_n)}(\boldsymbol{\nu}_n, \boldsymbol{\nu}'_n) = +\infty$, whence a contradiction. □

Lemma 4.27. *Let $\boldsymbol{\nu} = \langle \nu_n \rangle$ and $\boldsymbol{\rho} = \langle \rho_n \rangle$ be two points in $\mathcal{AM}$, let $\boldsymbol{\Delta} = (\Delta_n)^\omega$, where Δ_n is a multicurve, and let $\boldsymbol{\nu}', \boldsymbol{\rho}'$ be the respective projections of $\boldsymbol{\nu}, \boldsymbol{\rho}$ on $\mathcal{Q}(\boldsymbol{\Delta})$. Assume there exist $\mathbf{U}_1 = (U_n^1)^\omega, ..., \mathbf{U}_k = (U_n^k)^\omega$ subsurfaces such that $\Delta_n = \partial U_n^1 \cup ... \cup \partial U_n^k$, and $\mathrm{dist}_{C(U_n^i)}(\nu_n, \rho_n) > M$ ω-almost surely for every $i = 1, ..., k$, where M is the constant in Lemma 2.17.*

Then for every $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ hierarchy paths joining $\boldsymbol{\nu}, \boldsymbol{\nu}'$ respectively $\boldsymbol{\nu}', \boldsymbol{\rho}'$ and $\boldsymbol{\rho}', \boldsymbol{\rho}$, the path $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$.

Proof. Let $\mathbf{V} \in \Pi\mathcal{U}/\omega$ be an arbitrary subsurface. According to Lemma 4.12, $\psi_{\mathbf{V}}(\mathfrak{h}_i)$, $i = 1, 2, 3$, is a geodesic in $T_{\mathbf{V}}$. We shall prove that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$.

There are two cases: either $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}$ or $\mathbf{V} \pitchfork \boldsymbol{\Delta}$. In the first case, by Lemma 4.26 the projections $\psi_{\mathbf{V}}(\mathfrak{h}_1)$ and $\psi_{\mathbf{V}}(\mathfrak{h}_3)$ are singletons, and there is nothing to prove.

Assume now that $\mathbf{V} \pitchfork \boldsymbol{\Delta}$. Then $\mathbf{V} \pitchfork \partial\mathbf{U}_i$ for some $i \in \{1, ..., k\}$.

We have that ω-almost surely

$$\mathrm{dist}_{C(U_n)}(\nu'_n, \rho'_n) \le \mathrm{dist}_{C(U_n)}(\nu'_n, \Delta_n) + \mathrm{dist}_{C(U_n)}(\rho'_n, \Delta_n) = O(1)\,.$$

Lemma 4.15 then implies that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}', \boldsymbol{\rho}') = 0$. Hence, $\psi_{\mathbf{V}}(\mathfrak{h}_2)$ reduces to a singleton x which is the projection onto $T_{\mathbf{V}}$ of both $\mathcal{Q}(\boldsymbol{\Delta})$ and $\mathcal{Q}(\partial\mathbf{U}_i)$. It remains to prove that $\psi_{\mathbf{V}}(\mathfrak{h}_1)$ and $\psi_{\mathbf{V}}(\mathfrak{h}_3)$ have in common only x. Assume on the contrary that they are two geodesic with a common non-trivial sub-geodesic containing x. Then the geodesic in $T_{\mathbf{V}}$ joining $\psi_{\mathbf{V}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\rho})$ does not contain x. On the other hand, by hypothesis and Lemma 2.17 any hierarchy path joining $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ contains a point in $\mathcal{Q}(\partial\mathbf{U}_i)$. Lemma 4.12 implies that the geodesic in $T_{\mathbf{V}}$ joining $\psi_{\mathbf{V}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\rho})$ contains x, yielding a contradiction.

Thus $\psi_{\mathbf{V}}(\mathfrak{h}_1) \cap \psi_{\mathbf{V}}(\mathfrak{h}_3) = \{x\}$ and $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$ also in this case.

We proved that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$ for every $\mathbf{V} \in \Pi\mathcal{U}/\omega$. This implies that $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{\boldsymbol{\nu}'\}$ and $\mathfrak{h}_2 \cap \mathfrak{h}_3 = \{\boldsymbol{\rho}'\}$, and that $\mathfrak{h}_1 \cap \mathfrak{h}_3 = \emptyset$ if $\mathfrak{h}_2$ is non-trivial, while if $\mathfrak{h}_2$ reduces to a singleton $\boldsymbol{\nu}'$, $\mathfrak{h}_1 \cap \mathfrak{h}_3 = \{\boldsymbol{\nu}'\}$. Indeed if for instance $\mathfrak{h}_1 \cap \mathfrak{h}_2$ contained a point $\boldsymbol{\mu} \neq \boldsymbol{\nu}'$ then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$ whence $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$ for some subsurface $\mathbf{V}$. It would follow that $\psi_{\mathbf{V}}(\mathfrak{h}_1), \psi_{\mathbf{V}}(\mathfrak{h}_2)$ have in common a non-trivial sub-geodesic, contradicting the proven statement.

Also, if $\mathfrak{h}_1 \cap \mathfrak{h}_3$ contains a point $\boldsymbol{\mu} \neq \boldsymbol{\nu}'$ then for some subsurface $\mathbf{V}$ such that $\mathbf{V} \cap \boldsymbol{\Delta} \neq \emptyset$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$. Since $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\nu}', \boldsymbol{\rho}') = 0$ it follows that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\rho}') > 0$ and that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is not a geodesic in $T_{\mathbf{V}}$. A similar contradiction occurs if $\boldsymbol{\mu} \neq \boldsymbol{\rho}'$. Therefore if $\boldsymbol{\mu}$ is a point in $\mathfrak{h}_1 \cap \mathfrak{h}_3$, then we must have $\boldsymbol{\mu} = \boldsymbol{\nu}' = \boldsymbol{\rho}'$, in particular $\mathfrak{h}_2$ reduces to a point, which is the only point that $\mathfrak{h}_1$ and $\mathfrak{h}_3$ have in common.

Thus in all cases $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a topological arc. Since $\mathfrak{h}_i$ for $i = 1, 2, 3$, each satisfy the hypotheses of Lemma 4.18 and for every $\mathbf{V} \in \Pi\mathcal{U}/\omega$, $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$, it follows that $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ also satisfies the hypotheses of Lemma 4.18. We may therefore conclude that $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$. $\square$

Definition 4.28. A point $\boldsymbol{\mu}$ in $\mathcal{AM}$ is *between* the points $\boldsymbol{\nu}, \boldsymbol{\rho}$ in $\mathcal{AM}$ if for every $\mathbf{U} \in \Pi\mathcal{U}/\omega$ the projection $\psi_{\mathbf{U}}(\boldsymbol{\mu})$ is in the geodesic joining $\psi_{\mathbf{U}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{U}}(\boldsymbol{\rho})$ in $T_{\mathbf{U}}$ (possibly identical to one of its endpoints).

Lemma 4.29. *For every triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$, every choice of a pair $\boldsymbol{\nu}, \boldsymbol{\rho}$ in the triple and every finite subset F in $\Pi\mathcal{U}/\omega$ of pairwise disjoint subsurfaces there exists a point $\boldsymbol{\mu}$ between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu}), \psi_F(\boldsymbol{\rho}), \psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U} \in F} T_{\mathbf{U}}$.*

Proof. Let $F = \{\mathbf{U}_1, ..., \mathbf{U}_k\}$, where $\mathbf{U}_i = \left(U_n^i\right)^\omega$. We argue by induction on k. If $k = 1$ then the statement follows immediately from Lemma 4.12. We assume that the statement is true for all $i < k$, where $k \geq 2$, and we prove it for k.

We consider the multicurve $\Delta_n = \partial U_n^1 \cup \cdots \cup \partial U_n^k$. We denote the set $\{1, 2, ..., k\}$ by I. If for some $i \in I$, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\rho}) = 0$ then the median point of $\psi_{\mathbf{U}_i}(\boldsymbol{\nu}), \psi_{\mathbf{U}_i}(\boldsymbol{\rho}), \psi_{\mathbf{U}_i}(\boldsymbol{\sigma})$ is $\psi_{\mathbf{U}_i}(\boldsymbol{\nu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\rho})$. By the induction hypothesis there exists $\boldsymbol{\mu}$ between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F \setminus \{i\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F \setminus \{i\}}(\boldsymbol{\nu}), \psi_{F \setminus \{i\}}(\boldsymbol{\rho}), \psi_{F \setminus \{i\}}(\boldsymbol{\sigma})$. Since $\psi_{\mathbf{U}_i}(\boldsymbol{\mu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\nu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\rho})$ it follows that the desired statement holds not just for $F \setminus \{i\}$, but for all of F as well.

Assume now that for all $i \in I$, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\rho}) > 0$. Lemma 4.15 implies that $\lim_\omega \mathrm{dist}_{C(U_n^i)}(\nu_n, \rho_n) = \infty$.

Let $\boldsymbol{\nu}', \boldsymbol{\rho}', \boldsymbol{\sigma}'$ be the respective projections of $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ onto $\mathcal{Q}(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta} = (\Delta_n)^\omega$. According to Lemma 4.26, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\nu}') = \widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\rho}, \boldsymbol{\rho}') = \widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = 0$ for every $i \in I$, whence $\psi_F(\boldsymbol{\nu}) = \psi_F(\boldsymbol{\nu}'), \psi_F(\boldsymbol{\rho}) = \psi_F(\boldsymbol{\rho}'), \psi_F(\boldsymbol{\sigma}) = \psi_F(\boldsymbol{\sigma}')$. This and Lemma 4.27 imply that it suffices to prove the statement for $\boldsymbol{\nu}', \boldsymbol{\rho}', \boldsymbol{\sigma}'$. Thus, without loss of generality we may assume that $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ are in $\mathcal{Q}(\boldsymbol{\Delta})$. Also without loss of generality we may assume that $\{U_n^1, U_n^2, ..., U_n^k\}$ are all the connected components of $S \setminus \Delta_n$ and all the annuli with core curves in Δ_n. If not, we may add the missing subsurfaces.

For every $i \in I$ we consider the projections ν_n^i, ρ_n^i of ν_n and, respectively, ρ_n on $\mathcal{M}(U_n^i)$. Let $\mathfrak{g}_n^i$ be a hierarchy path in $\mathcal{M}(U_n^i)$ joining ν_n^i, ρ_n^i and let $\mathfrak{g}^i = \langle \mathfrak{g}_n^i \rangle$ be the limit hierarchy path in $\mathcal{M}(\mathbf{U}_i)$. According to Lemma 4.12, for every $i \in I$ there exists μ_n^i on $\mathfrak{g}_n^i$ such that $\boldsymbol{\mu}^{\boldsymbol{i}} = \langle \mu_n^i \rangle$ projects on $T_{\mathbf{U}_i}$ on the median point of the projections of $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$. Let $\mathfrak{p}_n^i$ and $\mathfrak{q}_n^i$ be the subpaths of $\mathfrak{g}_n^i$ preceding and respectively succeeding μ_n^i on $\mathfrak{g}_n^i$, and let $\mathfrak{p}^i = \langle \mathfrak{p}_n^i \rangle$ and $\mathfrak{q}^i = \langle \mathfrak{q}_n^i \rangle$ be the limit hierarchy paths in $\mathcal{M}(\mathbf{U}_i)$.

Let $\widetilde{\mathfrak{p}}_n^1$ be a path in $\mathcal{M}(S)$ which starts at ν_n and then continues on a path obtained by markings whose restriction to U_1^n are given by $\mathfrak{p}_n^1$ and in the complement of U_1^n are given by the restriction of ν_n. Continue this path by concatenating a path, $\widetilde{\mathfrak{p}}_n^2$, obtained by starting from the terminal point of $\widetilde{\mathfrak{p}}_n^1$ and then continuing by markings which are all the same in the complement of U_n^2 while their restriction to U_n^2 are given by $\mathfrak{p}_n^2$. Similarly, we obtain $\mathfrak{p}_n^j$ is from $\mathfrak{p}_n^{j-1}$ for any $j \leq k$. Note that for any $1 \leq j \leq k$ and any $i \neq j$, the path $\mathfrak{p}_n^j$ restricted to U_n^i is constant. Note that the starting point of $\widetilde{\mathfrak{p}}_n^1 \sqcup \cdots \sqcup \widetilde{\mathfrak{p}}_n^k$ is ν_n and the terminal point is the marking μ_n with the property that it projects on $\mathcal{M}(U_n^i)$ in μ_n^i for every $i \in I$. Now consider $\widetilde{\mathfrak{q}}_n^1$ the path with starting point μ_n obtained following $\mathfrak{q}_n^1$ (and keeping the projections onto U_n^i with $i \in I \setminus \{1\}$ unchanged), then $\widetilde{\mathfrak{q}}_n^2, \dots\ \widetilde{\mathfrak{q}}_n^k$ constructed such that the starting point of $\widetilde{\mathfrak{q}}_n^j$ is the terminal point of $\widetilde{\mathfrak{q}}_n^{j-1}$, and $\widetilde{\mathfrak{q}}_n^j$ is obtained following $\mathfrak{q}_n^j$ (and keeping the projections onto U_n^i with $i \in I \setminus \{j\}$ unchanged).

Let $\widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{j}} = \langle \widetilde{\mathfrak{p}}_n^j \rangle$ and $\widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{j}} = \langle \widetilde{\mathfrak{q}}_n^j \rangle$. We prove that for every subsurface $\mathbf{V} = (V_n)^\omega$ the path $\boldsymbol{\mathfrak{h}} = \widetilde{\boldsymbol{\mathfrak{p}}}^{\mathbf{1}} \sqcup \cdots \sqcup \widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{k}} \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\mathbf{1}} \sqcup \cdots \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{k}}$ projects onto a geodesic in $T_{\mathbf{V}}$. For any $i \neq j$, we have that $\widetilde{\mathfrak{p}}_n^i \cup \widetilde{\mathfrak{q}}_n^i$ and $\widetilde{\mathfrak{p}}_n^j \cup \widetilde{\mathfrak{q}}_n^j$ have disjoint support. Hence for each $i \in I$ we have that the restriction to U_n^i of the entire path is the same as the restriction to U_n^i of $\widetilde{\mathfrak{p}}_n^i \cup \widetilde{\mathfrak{q}}_n^i$. Since the latter is by construction the hierarchy path $\mathfrak{g}_n^i$, if $\mathbf{V} \subset \mathbf{U}_i$ for some $i \in I$, then it follows from Lemma 4.18 that $\boldsymbol{\mathfrak{h}}$ projects to a geodesic in $T_{\mathbf{V}}$. If $\mathbf{V}$ is disjoint from $\mathbf{U}_i$ then all the markings composing $\mathfrak{h}_n'$ have the same intersection with V_n, whence the diameter of $\mathfrak{h}_n$ with respect to $\mathrm{dist}_{C(V_n)}$ must be uniformly bounded. This and Lemma 4.15 implies that $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is a singleton. Lastly, if $\mathbf{V}$ contains or overlaps $\mathbf{U}_i$, then since all the markings in $\mathfrak{h}_n'$ contain ∂U_n^i the diameter of $\mathfrak{h}_n$ with respect to $\mathrm{dist}_{C(V_n)}$ must be uniformly bounded, leading again to the conclusion that $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is a singleton. Thus, the only case when $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is not a singleton is when $\mathbf{V} \subseteq \mathbf{U}_i$.

Now let $\mathbf{V}$ denote an arbitrary subsurface. If it is not contained in any $\mathbf{U}_i$ then $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}})$ is a singleton. The other situation is when $\mathbf{V}$ is contained in some $\mathbf{U}_i$, hence disjoint from all $\mathbf{U}_j$ with $j \neq i$. Then $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}) = \psi_{\mathbf{V}}(\widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{i}} \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{i}}) = \psi_{\mathbf{V}}^i(\boldsymbol{\mathfrak{p}}^i \sqcup \boldsymbol{\mathfrak{q}}^i) = \psi_{\mathbf{V}}^i(\boldsymbol{\mathfrak{g}}^i)$, which is a geodesic. Note that in the last two inequalities the map $\psi_{\mathbf{V}}^i$ is the natural projection of $\mathcal{M}(\mathbf{U}_i)$ onto $T_{\mathbf{V}}$ which exists when $\mathbf{V} \subseteq \mathbf{U}_i$.

Thus we have shown that $\boldsymbol{\mathfrak{h}}$ projects onto a geodesic in $T_{\mathbf{V}}$ for every $\mathbf{V}$, whence $\boldsymbol{\mu}$ is between $\boldsymbol{\nu}, \boldsymbol{\rho}$. By construction, for every $i \in I$, $\psi_{\mathbf{U}_i}(\boldsymbol{\mu})$ is the median point of $\psi_{\mathbf{U}_i}(\boldsymbol{\nu}), \psi_{\mathbf{U}_i}(\boldsymbol{\rho}), \psi_{\mathbf{U}_i}(\boldsymbol{\sigma})$, equivalently $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu})$, $\psi_F(\boldsymbol{\rho})$, $\psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U} \in F} T_{\mathbf{U}}$. □

We now generalize the last lemma by removing the hypothesis that the subsurfaces are disjoint.

Lemma 4.30. *For every triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$, every choice of a pair $\boldsymbol{\nu}, \boldsymbol{\rho}$ in the triple and every finite subset F in $\Pi\mathcal{U}/\omega$ there exists a point $\boldsymbol{\mu}$ in $\mathcal{AM}$ between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu}), \psi_F(\boldsymbol{\rho}), \psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U} \in F} T_{\mathbf{U}}$.*

Proof. We prove the statement by induction on the cardinality of F. When $\operatorname{card} F = 1$ it follows from Lemma 4.12. Assume that it is true whenever $\operatorname{card} F < k$ and consider F of cardinality $k \geq 2$. If the subsurfaces in F are pairwise disjoint then we can apply Lemma 4.29, hence we may assume that there exists a pair of subsurfaces $\mathbf{U}, \mathbf{V}$ in F which either overlap or are nested.

First, assume that $\mathbf{U}, \mathbf{V}$ overlap. Then $\psi_{\mathbf{U},\mathbf{V}}$ is equal to $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$, by Theorem 4.21. We write $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$ to denote the image $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\nu})$ and let $\boldsymbol{\nu}_{\mathbf{U}}$ and $\boldsymbol{\nu}_{\mathbf{V}}$ denote its coordinates (i.e., $\psi_{\mathbf{U}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\nu})$). We use similar notations for $\boldsymbol{\rho}, \boldsymbol{\sigma}$. If the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is not (v, u) then it is either some point (x, u) with $x \in T_{\mathbf{U}} \setminus \{v\}$, or (v, y) with $y \in T_{\mathbf{V}} \setminus \{u\}$. In the first case, by the induction hypothesis, there exists a point $\boldsymbol{\mu}_1$ between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F \setminus \{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F \setminus \{\mathbf{V}\}}(\boldsymbol{\nu}), \psi_{F \setminus \{\mathbf{V}\}}(\boldsymbol{\rho}), \psi_{F \setminus \{\mathbf{V}\}}(\boldsymbol{\sigma})$. In particular, $\psi_{\mathbf{U}}(\boldsymbol{\mu}) = x$, hence $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is a point in $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$ having the first coordinate x. Since there exists only one such point, (x, u), it follows that $\psi_{\mathbf{V}}(\boldsymbol{\mu}) = u$. Thus, for every $\mathbf{Y} \in F$, the point $\psi_{\mathbf{Y}}(\boldsymbol{\mu})$ is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This is equivalent to the fact that $\psi_F(\boldsymbol{\mu})$ is the median point in $\prod_{\mathbf{Y}} T_{\mathbf{Y}}$

of $\psi_F(\boldsymbol{\nu})$, $\psi_F(\boldsymbol{\rho})$ and $\psi_F(\boldsymbol{\sigma})$. A similar argument works when the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is a point (v, y) with $y \in T_{\mathbf{V}} \setminus \{u\}$.

Hence, we may assume that the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is (v, u). Let $\boldsymbol{\mu}_1$ be a point between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\nu}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\rho}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\sigma})$, and let $\boldsymbol{\mu}_2$ be a point between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\nu}), \psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\rho}), \psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\sigma})$. In particular $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_1) = (v, y)$ with $y \in T_{\mathbf{V}}$ and $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_2) = (x, u)$ with $x \in T_{\mathbf{U}}$. Any hierarchy path joining $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ is mapped by $\psi_{\mathbf{U},\mathbf{V}}$ onto a path joining (v, y) and (x, u) in $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$. Therefore it contains a point $\boldsymbol{\mu}$ such that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is (v, u). According to Lemma 4.12 $\boldsymbol{\mu}$ is between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, hence it is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$, moreover for every $\mathbf{Y} \in F \setminus \{\mathbf{U}, \mathbf{V}\}$, $\psi_{\mathbf{Y}}(\boldsymbol{\mu}) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_1) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_2)$, and it is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This and the fact that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}) = (v, u)$ is the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ finish the argument in this case.

We now consider the case that $\mathbf{U} \subsetneq \mathbf{V}$. Let u be the point in $T_{\mathbf{V}}$ which is the projection of $\mathcal{Q}(\partial\mathbf{U})$ and let $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i\in I} \mathcal{C}_i$ be the decomposition into connected components. By Theorem 4.21, the image of $\psi_{\mathbf{U},\mathbf{V}}$ is $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$ where t_i are points in $T_{\mathbf{U}}$. If the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is not in the set $\{(t_i, u) \mid i \in I\}$, then we are done as in the previous case using the induction hypothesis as well as the fact that for such points there are no other points having the same first coordinate or the same second coordinate.

Thus, we may assume that the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is (t_i, u) for some $i \in I$. Let $\boldsymbol{\mu}_1$ be a point between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\nu}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\rho}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\sigma})$, and let $\boldsymbol{\mu}_2$ be a point between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\nu}), \psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\rho}), \psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\sigma})$. In particular $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_1) = (t_i, y)$ with $y \in C_i$ and $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_2) = (x, u)$ with $x \in T_{\mathbf{U}}$. Any hierarchy path joining $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ is mapped by $\psi_{\mathbf{U},\mathbf{V}}$ onto a path joining (t_i, y) and (x, u) in $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$. It contains a point $\boldsymbol{\mu}$ such that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is (t_i, u). By Lemma 4.12, the point $\boldsymbol{\mu}$ is between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, hence in particular it is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$. Moreover, for every $\mathbf{Y} \in F \setminus \{\mathbf{U}, \mathbf{V}\}$, $\psi_{\mathbf{Y}}(\boldsymbol{\mu}) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_1) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_2)$ and hence $\boldsymbol{\mu}$ is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This and the fact that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}) = (t_i, u)$ is the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ finish the argument. □

Proof of Theorem 4.25. Consider an arbitrary triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$. For every $\epsilon > 0$ there exists a finite subset F in $\Pi\mathcal{U}/\omega$ such that $\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) < \epsilon$ for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$. By Lemma 4.30 there exists $\boldsymbol{\mu}$ in $\mathcal{AM}$ between $\boldsymbol{\nu}, \boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu}), \psi_F(\boldsymbol{\rho}), \psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$. The latter implies that for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$,

$$\sum_{\mathbf{U}\in F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) = \sum_{\mathbf{U}\in F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{\mu}) + \sum_{\mathbf{U}\in F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{b}) .$$

Also, since $\boldsymbol{\mu}$ is between $\boldsymbol{\nu}, \boldsymbol{\rho}$ it follows that

$$\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\mu}) < \epsilon \text{ and } \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\rho}) < \epsilon ,$$

whence

$$\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\operatorname{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\sigma}) < 2\epsilon .$$

It follows that for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$,

$$\sum_{\mathbf{U} \in \Pi \mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{\mu}) + \sum_{\mathbf{U} \in \Pi \mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{b}) \leq \sum_{\mathbf{U} \in \Pi \mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) + 3\epsilon\,.$$

That is, $\widetilde{\mathrm{dist}}(\boldsymbol{a}, \boldsymbol{\mu}) + \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{b}) \leq \widetilde{\mathrm{dist}}(\boldsymbol{a}, \boldsymbol{b}) + 3\epsilon$. This and [CDH, Section 2.3] imply that $\psi(\boldsymbol{\mu})$ is at distance at most 5ϵ from the unique median point of $\psi(\boldsymbol{\nu}), \psi(\boldsymbol{\rho}), \psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$.

We have thus proved that for every $\epsilon > 0$ there exists a point $\psi(\boldsymbol{\mu})$ in $\psi(\mathcal{AM})$ at distance at most 5ϵ from the median point of $\psi(\boldsymbol{\nu}), \psi(\boldsymbol{\rho}), \psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$. Now the asymptotic cone $\mathcal{AM}$ is a complete metric space with the metric $\mathrm{dist}_{\mathcal{AM}}$, hence the bi-Lipschitz equivalent metric space $\psi(\mathcal{AM})$ with the metric $\widetilde{\mathrm{dist}}$ is also complete. Since it is a subspace in the complete metric space $\mathcal{T}_0$, it follows that $\psi(\mathcal{AM})$ is a closed subset in $\mathcal{T}_0$. We may then conclude that $\psi(\mathcal{AM})$ contains the unique median point of $\psi(\boldsymbol{\nu}), \psi(\boldsymbol{\rho}), \psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$. □

5. Actions on asymptotic cones of mapping class groups and splitting

5.1. Pieces of the asymptotic cone.

Lemma 5.1. *Let $\langle \mu_n \rangle, \langle \mu_n' \rangle, \langle \nu_n \rangle, \langle \nu_n' \rangle$ be sequences of points in $\mathcal{M}(S)$ for which $\lim_\omega \mathrm{dist}_{\mathcal{M}}(\mu_n, \nu_n) \to \infty$. For every $M > 2K(S)$, where $K(S)$ is the constant in Theorem 2.24 there exists a positive constant $C = C(M) < 1$ so that if*

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \mu_n') + \mathrm{dist}_{\mathcal{M}(S)}(\nu_n, \nu_n') \leq C \mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n)\,,$$

then there exists a sequence of subsurfaces $Y_n \subseteq S$ such that for ω-a.e. n both $\mathrm{dist}_{\mathcal{C}(Y_n)}(\mu_n, \nu_n) > M$ and $\mathrm{dist}_{\mathcal{C}(Y_n)}(\mu_n', \nu_n') > M$.

Proof. Assume that ω-almost surely the sets of subsurfaces

$$\mathcal{Y}_n = \{Y_n \mid \mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) > 2M\} \text{ and } \mathcal{Z}_n = \{Z_n \mid \mathrm{dist}_{C(Z_n)}(\mu_n', \nu_n') > M\}$$

are disjoint. Then for every $Y_n \in \mathcal{Y}_n$, $\mathrm{dist}_{C(Y_n)}(\mu_n', \nu_n') \leq M$, which by the triangle inequality implies that $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n') + \mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n') \geq \mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) - M \geq \frac{1}{2}\mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) > M$. Hence either $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n')$ or $\mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n')$ is larger than $M/2 > K(S)$. Let a, b be the constants appearing in (2) for $K = M/2$, and let A, B be the constants appearing in the same formula for $K' = 2M$. According to the above we may then write

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \mu_n') + \mathrm{dist}_{\mathcal{M}(S)}(\nu_n, \nu_n') \geq_{a,b} \sum_{Y \in \mathcal{Y}_n} \left\{\!\!\left\{ \mathrm{dist}_{\mathcal{C}(Y)}(\mu_n, \mu_n') \right\}\!\!\right\}_K +$$

$$\sum_{Y \in \mathcal{Y}_n} \left\{\!\!\left\{ \mathrm{dist}_{\mathcal{C}(Y)}(\nu_n, \nu_n') \right\}\!\!\right\}_K \geq \frac{1}{4} \sum_{Y \in \mathcal{Y}_n} \mathrm{dist}_{\mathcal{C}(Y)}(\mu_n, \nu_n) \geq_{A,B} \frac{1}{4} \mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n)\,.$$

The coefficient $\frac{1}{4}$ is accounted for by the case when one of the two distances $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n')$ and $\mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n')$ is larger than $K = M/2$ while the other is not.

When C is small enough we thus obtain a contradiction of the hypothesis, hence ω-almost surely $\mathcal{Y}_n \cap \mathcal{Z}_n \neq \emptyset$ □

Definition 5.2. For any $g = (g_n)^\omega \in \mathcal{M}_e^\omega$ let us denote by $U(g)$ the set of points $\boldsymbol{h} \in \mathcal{AM}$ such that for some representative $(h_n)^\omega \in \mathcal{M}_e^\omega$ of $\boldsymbol{h}$,

$$\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(h_n, g_n) < \infty.$$

The set $U(g)$ is called the *g-interior*. This set is non-empty since $g \in U(g)$.

Lemma 5.3. *Let P be a piece in $\mathcal{AM} = \mathrm{Con}^\omega(\mathcal{M}(S); (d_n))$. Let $\boldsymbol{x}, \boldsymbol{y}$ be distinct points in P. Then there exists $g = (g_n)^\omega \in \mathcal{M}_e^\omega$ such that $U(g) \subseteq P$; moreover the intersection of any hierarchy path $[\boldsymbol{x}, \boldsymbol{y}]$ with $U(g)$ contains $[\boldsymbol{x}, \boldsymbol{y}] \setminus \{\boldsymbol{x}, \boldsymbol{y}\}$.*

Proof. Consider arbitrary representatives $(x_n)^\omega, (y_n)^\omega$ of $\boldsymbol{x}$ and respectively $\boldsymbol{y}$, and let $[\boldsymbol{x}, \boldsymbol{y}]$ be the limit of a sequence of hierarchy paths $[x_n, y_n]$. Since $\boldsymbol{x}, \boldsymbol{y} \in P$, there exist sequences of points $\boldsymbol{x}(k) = \langle x_n(k) \rangle, \boldsymbol{y}(k) = \langle y_n(k) \rangle$, $x_n(k), y_n(k) \in [x_n, y_n]$ and a sequence of numbers $C(k) > 0$ such that for ω-almost every n we have:

$$\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), y_n(k)) < C(k)$$

and

$$\begin{array}{c} \mathrm{dist}_{\mathcal{M}(S)}(x_n(k), x_n) < \frac{d_n \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{x}, \boldsymbol{y})}{k}, \\ \mathrm{dist}_{\mathcal{M}(S)}(y_n(k), y_n) < \frac{d_n \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{x}, \boldsymbol{y})}{k}. \end{array} \tag{15}$$

Let $[x_n(k), y_n(k)]$ be the subpath of $[x_n, y_n]$ connecting $x_n(k)$ and $y_n(k)$. Let g_n be the midpoint of $[x_n(n), y_n(n)]$. Then $\langle g_n \rangle \in [\boldsymbol{x}, \boldsymbol{y}]$. Let $g = (g_n)^\omega \in \mathcal{M}_e^\omega$. Let us prove that $U(g)$ is contained in P.

Since $\boldsymbol{x}, \boldsymbol{y} \in P$, it is enough to show that any point $\boldsymbol{z} = \langle z_n \rangle$ from $U(g)$ is in the same piece with $\boldsymbol{x}$ and in the same piece with $\boldsymbol{y}$ (because distinct pieces cannot have two points in common).

By the definition of $U(g)$, we can assume that $\mathrm{dist}_{\mathcal{C}(S)}(z_n, g_n) \le C_1$ for some constant C_1 ω-a.s. For every $k > 0$, $\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), y_n(k)) \le C(k)$, so

$$\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), z_n), \mathrm{dist}_{\mathcal{C}(S)}(y_n(k), z_n) \le C(k) + C_1$$

ω-a.s. By (15) and Theorem 4.2, $\boldsymbol{x} = \langle x_n \rangle, \boldsymbol{z} = \langle z_n \rangle, \boldsymbol{y} = \langle y_n \rangle$ are in the same piece.

Note that $\langle x_n(k) \rangle$ and $\langle y_n(k) \rangle$ are in $U(g)$ for every k. Now let $(x'_n)^\omega, (y'_n)^\omega$ be other representatives of $\boldsymbol{x}$, $\boldsymbol{y}$, and let $x'_n(k), y'_n(k)$ be chosen as above on a sequence of hierarchy paths $[x'_n, y'_n]$. Let $g' = (g'_n)^\omega$, where g'_n is the point in the middle of the hierarchy path $[x'_n, y'_n]$. We show that $U(g') = U(g)$. Indeed, the sequence of quadruples $x_n(k), y_n(k), y'_n(k), x'_n(k)$ satisfies the conditions of Lemma 5.1 for large enough k. Therefore the subpaths $[x_n(k), y_n(k)]$ and $[x'_n(k), y'_n(k)]$ share a large domain ω-a.s. Since the entrance points of these subpaths in this domain are at a uniformly bounded $\mathcal{C}(S)$-distance, the same holds for g_n, g'_n. Hence $U(g') = U(g)$. This completes the proof of the lemma. □

Lemma 5.3 shows that for every two points $\boldsymbol{x}, \boldsymbol{y}$ in a piece P of $\mathcal{AM}$, there exists an interior $U(g)$ depending only on these points and contained in P. We shall denote $U(g)$ by $U(\boldsymbol{x}, \boldsymbol{y})$.

Lemma 5.4. *Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ be three different points in a piece $P \subseteq \mathcal{AM}$. Then*

$$U(\boldsymbol{x}, \boldsymbol{y}) = U(\boldsymbol{y}, \boldsymbol{z}) = U(\boldsymbol{x}, \boldsymbol{z}).$$

Proof. Let $(x_n)^\omega, (y_n)^\omega, (z_n)^\omega$ be representatives of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. Choose hierarchy paths $[x_n, y_n], [y_n, z_n], [x_n, z_n]$. By Theorem 2.24, the hierarchy path $[x_n, y_n]$ shares a large domain with either $[x_n, z_n]$ or $[y_n, z_n]$ for all n ω-a.s. By Lemma 5.3, then $U(\boldsymbol{x}, \boldsymbol{y})$ coincides either with $U(\boldsymbol{x}, \boldsymbol{z})$ or with $U(\boldsymbol{y}, \boldsymbol{z})$. Repeating the argument with $[\boldsymbol{x}, \boldsymbol{y}]$ replaced either by $[\boldsymbol{y}, \boldsymbol{z}]$ or by $[\boldsymbol{x}, \boldsymbol{z}]$, we conclude that all three interiors coincide. □

Proposition 5.5. *Every piece P of the asymptotic cone $\mathcal{AM}$ contains a unique interior $U(g)$, and P is the closure of $U(g)$.*

Proof. Let $U(g)$ and $U(g')$ be two interiors inside P. Let $\boldsymbol{x}, \boldsymbol{y}$ be two distinct points in $U(g)$, $\boldsymbol{z}, \boldsymbol{t}$ be two distinct points in $U(g')$. If $\boldsymbol{y} \neq \boldsymbol{z}$ we apply Lemma 5.4 to the triples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$, and conclude that $U(g) = U(g')$. If $\boldsymbol{y} = \boldsymbol{z}$ we apply Lemma 5.4 to the triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$. The fact that the closure of $U(g)$ is P follows from Lemma 5.3. □

5.2. **Actions and splittings.** We recall a theorem proved by V. Guirardel in [Gui05], that we will use in the sequel.

Definition 5.6. The *height* of an arc in an $\mathbb{R}$–tree with respect to the action of a group G on it is the maximal length of a decreasing chain of sub-arcs with distinct stabilizers. If the height of an arc is zero then it follows that all sub-arcs of it have the same stabilizer. In this case the arc is called *stable*.

The tree T is *of finite height* with respect to the action of some group G if any arc of it can be covered by finitely many arcs with finite height. If the action is minimal and G is finitely generated then this condition is equivalent to the fact that there exists a finite collection of arcs $\mathcal{I}$ of finite height such that any arc is covered by finitely many translates of arcs in $\mathcal{I}$ [Gui05].

Theorem 5.7 (Guirardel [Gui05])**.** *Let Λ be a finitely generated group and let T be a real tree on which Λ acts minimally and with finite height. Suppose that the stabilizer of any non-stable arc in T is finitely generated.*

Then one of the following three situations occurs:

1. *Λ splits over the stabilizer of a non-stable arc or over the stabilizer of a tripod;*
2. *Λ splits over a virtually cyclic extension of the stabilizer of a stable arc;*
3. *T is a line and Λ has a subgroup of index at most 2 that is the extension of the kernel of that action by a finitely generated free abelian group.*

In some cases stability and finite height follow from the algebraic structure of stabilizers of arcs, as the next lemma shows.

Lemma 5.8. *(*[DS07]*) Let G be a finitely generated group acting on an $\mathbb{R}$–tree T with finite of size at most D tripod stabilizers, and (finite of size at most D)-by-abelian arc stabilizers, for some constant D. Then*

1. *an arc with stabilizer of size $> (D+1)!$ is stable;*
2. *every arc of T is of finite height (and so the action is of finite height and stable).*

We also recall the following two well known results due to Bestvina ([Bes88], [Bes02]) and Paulin [Pau88].

Lemma 5.9. *Let Λ and G be two finitely generated groups, let $A = A^{-1}$ be a finite set generating Λ and let* dist *be a word metric on G. Given $\phi_n \colon \Lambda \to G$ an infinite sequence of homomorphisms, one can associate to it a sequence of positive integers defined by*

$$d_n = \inf_{x \in G} \sup_{a \in A} \mathrm{dist}(\phi_n(a)x, x)\,. \tag{16}$$

If (ϕ_n) are pairwise non-conjugate in Γ then $\lim_{n\to\infty} d_n = \infty$.

Remark 5.10. For every $n \in \mathbb{N}$, $d_n = \mathrm{dist}(\phi_n(a_n)x_n, x_n)$ for some $x_n \in \Gamma$ and $a_n \in A$.

Consider an arbitrary ultrafilter ω. According to Remark 5.10, there exists $a \in A$ and $x_n \in G$ such that $d_n = \mathrm{dist}(\phi_n(a)x_n, x_n)$ ω–a.s.

Lemma 5.11. *Under the assumptions of Lemma 5.9, the group Λ acts on the asymptotic cone $\mathcal{K}_\omega = \mathrm{Con}^\omega(G; (x_n), (d_n))$ by isometries, without a global fixed point, as follows:*

$$g \cdot \lim_\omega (x_n) = \lim_\omega (\phi_n(g)x_n) \; . \tag{17}$$

This defines a homomorphism ϕ_ω from Λ to the group $x^\omega(\Pi_1\Gamma/\omega)(x^\omega)^{-1}$ of isometries of $\mathcal{K}_\omega$.

Let S be a surface of complexity $\xi(S)$. When $\xi(S) \leq 1$ the mapping class group $\mathcal{MCG}(S)$ is hyperbolic and the well-known theory on homomorphisms into hyperbolic groups can be applied. Therefore we adopt the following convention for the rest of this section.

Convention 5.12. In what follows we assume that $\xi(S) \geq 2$.

Proposition 5.13. *Suppose that a finitely generated group $\Lambda = \langle A \rangle$ has infinitely many homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$. Let*

$$d_n = \inf_{\mu \in \mathcal{M}(S)} \sup_{a \in A} \mathrm{dist}(\phi_n(a)\mu, \mu)\, , \tag{18}$$

and let μ_n be the point in $\mathcal{M}(S)$ where the above infimum is attained.
Then one of the following two situations occurs:

1. *either the sequence (ϕ_n) defines a non-trivial action of Λ on an asymptotic cone of the complex of curves $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (\ell_n))$,*
2. *or the action by isometries, without a global fixed point, of Λ on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined as in Lemma 5.11 fixes a piece setwise.*

Proof. Let $\ell_n = \inf_{\gamma \in \mathcal{C}(S)} \sup_{a \in A} \mathrm{dist}_{\mathcal{C}(S)}(\phi_n(a)\gamma, \gamma)$. As before, there exists $b_0 \in A$ and $\gamma_n \in \mathcal{C}(S)$ such that $\ell_n = \mathrm{dist}_{\mathcal{C}(S)}(\phi_n(b_0)\gamma_n, \gamma_n)$ ω–a.s.

If $\lim_\omega \ell_n = +\infty$ then the sequence (ϕ_n) defines a non-trivial action of Λ on $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (\ell_n))$.

Assume now that there exists M such that for every $b \in A$, $\mathrm{dist}_{\mathcal{C}(S)}(\phi_n(b)\gamma_n, \gamma_n) \leq M$ ω-almost surely. This implies that for every $g \in \Lambda$ there exists M_g such that $\mathrm{dist}_{\mathcal{C}(S)}(\phi_n(g)\gamma_n, \gamma_n) \leq M_g$ ω-almost surely.

Consider μ_n' the projection of μ_n onto $\mathcal{Q}(\gamma_n)$. A hierarchy path $[\mu_n, \mu_n']$ shadows a tight geodesic $\mathfrak{g}_n$ joining a curve in $\mathrm{base}(\mu_n)$ to a curve in $\mathrm{base}(\mu_n')$, the latter curve being at $C(S)$-distance 1 from γ_n. If the ω-limit of the $C(S)$-distance from μ_n to μ_n' is finite then in follows that for every $b \in A$, $\mathrm{dist}_{\mathcal{C}(S)}(\phi_n(b)\mu_n, \mu_n) = O(1)$ ω-almost surely. Then the action of Λ on $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined by the sequence (ϕ_n) preserves $U((\mu_n)^\omega)$, which is the interior of a piece, hence it fixes a piece setwise. Therefore in what follows we assume that the ω-limit of the $C(S)$-distance from μ_n to μ_n' is infinite.

Let b be an arbitrary element in the set of generators A. Consider a hierarchy path $[\mu_n, \phi_n(b)\mu_n]$. Consider the Gromov product

$$\tau_n(b) = \frac{1}{2}\left[\mathrm{dist}_{\mathcal{C}(S)}(\mu_n, \mu_n') + \mathrm{dist}_{\mathcal{C}(S)}(\mu_n, \phi_n(b)\mu_n) - \mathrm{dist}_{\mathcal{C}(S)}(\mu_n', \phi_n(b)\mu_n)\right] ,$$

and $\tau_n = \max_{b\in A} \tau_n(b)$.

The geometry of quadrangles in hyperbolic geodesic spaces combined with the fact that $\mathrm{dist}_{\mathcal{C}(S)}(\mu_n', \phi_n(b)\mu_n') \leq M$ implies that:

- every element ν_n on $[\mu_n, \mu_n']$ which is at $C(S)$ distance at least $\tau_n(b)$ from μ_n is at $C(S)$-distance $O(1)$ from an element ν_n' on $[\phi_n(b)\mu_n, \phi_n(b)\mu_n']$; it follows that $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n', \phi_n(b)\mu_n') = \mathrm{dist}_{\mathcal{C}(S)}(\nu_n, \mu_n') + O(1)$, therefore $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n', \phi_n(b)\nu_n) = O(1)$ and $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n, \phi_n(b)\nu_n) = O(1)$;
- the element $\rho_n(b)$ which is at $C(S)$ distance $\tau_n(b)$ from μ_n is at $C(S)$-distance $O(1)$ also from an element $\rho_n''(b)$ on $[\mu_n, \phi_n(b)\mu_n]$.

We have thus obtained that for every element ν_n on $[\mu_n, \mu_n']$ which is at $C(S)$ distance at least τ_n from μ_n, $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n, \phi_n(b)\nu_n) = O(1)$ for every $b \in B$ and ω-almost every n. In particular this holds for the point ρ_n on $[\mu_n, \mu_n']$ which is at $C(S)$ distance τ_n from μ_n. Let $a \in A$ be such that $\tau_n(a) = \tau_n$ and $\rho_n = \rho_n(a)$, and let $\rho_n'' = \rho_n''(a)$ be the point on $[\mu_n, \phi_n(a)\mu_n]$ at $C(S)$-distance $O(1)$ from $\rho_n(a)$. It follows that $\mathrm{dist}_{\mathcal{C}(S)}(\rho_n'', \phi_n(b)\rho_n'') = O(1)$ for every $b \in B$ and ω-almost every n. Moreover, since ρ'' is a point on $[\mu_n, \phi_n(a)\mu_n]$, its limit is a point in $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$. It follows that the action of Λ on $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined by the sequence (ϕ_n) preserves $U((\rho_n'')^\omega)$, which is the interior of a piece, hence it fixes a piece setwise. □

Lemma 5.14. *Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ be two distinct points in an asymptotic cone of the complex of curves $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (d_n))$. The stabilizer $\mathrm{stab}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ in the ultrapower $\mathcal{MCG}(S)^\omega$ is the extension of a finite subgroup of cardinality at most $N = N(S)$ by an abelian group.*

Proof. Let $\mathfrak{q}_n$ be a geodesic joining γ_n and γ_n' and let x_n, y_n be points at distance ϵd_n from γ_n and γ_n' respectively, where $\epsilon > 0$ is small enough.

Let $\boldsymbol{g} = (g_n)^\omega$ be an element in $\mathrm{stab}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$. Then

$$\delta_n(\boldsymbol{g}) = \max(\mathrm{dist}_{\mathcal{C}(S)}(\gamma_n, g_n\gamma_n)\, , \, \mathrm{dist}(\gamma_n', g_n\gamma_n'))$$

satisfies $\delta_n(\boldsymbol{g}) = o(d_n)$.

Since $\mathcal{C}(S)$ is a Gromov hyperbolic space it follows that the sub-geodesic of $\mathfrak{q}_n$ with endpoints x_n, y_n is contained in a finite radius tubular neighborhood of $g_n\mathfrak{q}_n$. Since x_n is ω-almost surely at distance $O(1)$ from a point x_n' on $g_n q_n$, define $\ell_x(g_n)$ as $(-1)^\epsilon \mathrm{dist}_{\mathcal{C}(S)}(x_n, g_n x_n)$, where $\epsilon = 0$ if x_n' is nearer to $g_n\mu_n$ than $g_n x_n$ and $\epsilon = 1$ otherwise.

Let $\ell_x \colon \mathrm{stab}(\mu, \nu) \to \Pi\mathbb{R}/\omega$ defined by $\ell_x(\boldsymbol{g}) = (\ell_x(g_n))^\omega$. It is easy to see that ℓ_x is a quasi-morphism, that is

$$|\ell_x(\boldsymbol{gh}) - \ell_x(\boldsymbol{g}) - \ell_x(\boldsymbol{h})| \leq_\omega O(1)\, . \tag{19}$$

It follows that $|\ell_x([\boldsymbol{g}, \boldsymbol{h}])| \leq_\omega O(1)\, .$

The above and a similar argument for y_n imply that for every commutator, $\boldsymbol{c} = \langle c_n \rangle$, in the stabilizer of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, $\mathrm{dist}_{\mathcal{C}(S)}(x_n, c_n x_n)$ and $\mathrm{dist}_{\mathcal{C}(S)}(y_n, c_n y_n)$ are at most $O(1)$. Lemma 2.1 together with Bowditch's acylindricity result [Bow, Theorem 1.3] imply that the set of commutators of $\mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$ has uniformly bounded cardinality, say, N. Then any finitely generated subgroup G of stab$(\boldsymbol{\mu}, \boldsymbol{\nu})$ has conjugacy classes of cardinality at most N, i.e., G is an FC-group [Neu51]. By [Neu51], the set of all torsion elements of G is finite, and the derived subgroup of G is finite of cardinality $\leq N(S)$ (by Lemma 2.12).

□

Lemma 5.15. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the vertices of a non-trivial tripod in an asymptotic cone of the complex of curves* $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (d_n))$. *The stabilizer* stab$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ *in the ultrapower $\mathcal{MCG}(S)^\omega$ is a finite subgroup of cardinality at most $N = N(S)$.*

Proof. Since $\mathcal{C}(S)$ is δ-hyperbolic, for every $a > 0$ there exists $b > 0$ such that for any triple of points $x, y, z \in \mathcal{C}(S)$ the intersection of the three a-tubular neighborhoods of geodesics $[x, y]$, $[y, z]$, and $[z, a]$ is a set $C_a(x, y, z)$ of diameter at most b.

Let $\boldsymbol{g} = (g_n)^\omega$ be an element in stab$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Then

$$\delta_n(\boldsymbol{g}) = \max(\mathrm{dist}_{\mathcal{C}(S)}(\alpha_n, g_n\alpha_n)\,,\, \mathrm{dist}(\beta_n, g_n\beta_n)\,,\, \mathrm{dist}(\gamma_n, g_n\gamma_n))$$

satisfies $\delta_n(\boldsymbol{g}) = o(d_n)$. While, the distance between each pair of points among α_n, β_n, and γ_n is at least λd_n for some $\lambda > 0$. It follows that if (x_n, y_n) is any of the pairs (α_n, β_n), (α_n, γ_n), (γ_n, β_n), then away from a $o(d_n)$-neighborhood of the endpoints the two geodesics $[x_n, y_n]$ and $[g_n x_n, g_n y_n]$ are uniformly Hausdorff close. This in particular implies that away from a $o(d_n)$-neighborhood of the endpoints, the a-tubular neighborhood of $[g_n x_n, g_n y_n]$ is contained in the A-tubular neighborhood of $[x_n, y_n]$ for some $A > a$. Since $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are the vertices of a non-trivial tripod, for any $a > 0$, $C_a(\alpha_n, \beta_n, \gamma_n)$ is ω-almost surely disjoint of $o(d_n)$-neighborhoods of $\alpha_n, \beta_n, \gamma_n$. The same holds for $\boldsymbol{g\alpha}, \boldsymbol{g\beta}, \boldsymbol{g\gamma}$. It follows that $C_a(g_n\alpha_n, g_n\beta_n, g_n\gamma_n)$ is contained in $C_A(\alpha_n, \beta_n, \gamma_n)$, hence it is at Hausdorff distance at most $B > 0$ from $C_a(\alpha_n, \beta_n, \gamma_n)$.

Thus we may find a point $\tau_n \in [\alpha_n, \beta_n]$ such that $\mathrm{dist}(\tau_n, g_n\tau_n) = O(1)$, while the distance from τ_n to $\{\alpha_n, \beta_n\}$ is at least $2\epsilon d_n$. Let $\eta_n \in [\tau_n, \alpha_n]$ be a point at distance ϵd_n from τ_n. Then $g_n\eta_n \in [g_n\tau_n, g_n\alpha_n]$ is a point at distance ϵd_n from $g\tau_n$. On the other hand, since η_n is at distance at least ϵd_n from α_n it follows that there exists $\eta_n' \in [g_n\tau_n, g_n\alpha_n]$ at distance $O(1)$ from η_n. It follows that η_n' is at distance $\epsilon d_n + O(1)$ from $g_n\tau_n$, hence η_n' is at distance $O(1)$ from $g_n\eta_n$. Thus we obtained that $g_n\eta_n$ is at distance $O(1)$ from η_n. This, the fact that $g_n\tau_n$ is at distance $O(1)$ from τ_n as well, and the fact that $\mathrm{dist}(\tau_n, \eta_n) = \epsilon d_n$, together with Bowditch's acylindricity result [Bow, Theorem 1.3] and Lemma 2.1 imply that the stabilizer stab$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ has uniformly bounded cardinality. □

Corollary 5.16. *Suppose that a finitely generated group $\Lambda = \langle A \rangle$ has infinitely many injective homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$.*

Then one of the following two situations occurs:

(1) *Λ is virtually abelian, or it splits over a virtually abelian subgroup;*
(2) *the action by isometries, without a global fixed point, of Λ on the asymptotic cone* $\mathrm{Con}^\omega(\mathcal{MCG}(S); (\mu_n), (d_n))$ *defined as in Lemma 5.11 fixes a piece setwise.*

Proof. It suffices to prove that case (1) from Proposition 5.13 implies (1) from Corollary 5.13. According to case (1) from Proposition 5.13 the group Λ acts non-trivially on a real tree, by Lemma 5.14 we know that the stabilizers of non-trivial arcs are virtually abelian, and by Lemma 5.15 we know that the stabilizers of non-trivial tripods are finite.

On the other hand, since Λ injects into $\mathcal{MCG}(S)$, it follows immediately from results of Birman–Lubotzky–McCarthy [BLM83] that virtually abelian subgroups in Λ satisfy the ascending chain condition, and are always finitely generated. By Theorem 5.7 we thus have that Λ is either virtually abelian or it splits over a virtually solvable subgroup. One of the main theorems of [BLM83] is that any virtual solvable subgroup of the mapping class group is virtually abelian, finishing the argument. □

Corollary 5.17. *Suppose that a finitely generated group $\Lambda = \langle A\rangle$ with property (T) has infinitely many injective homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$.*

Then the action by isometries, without a global fixed point, of Λ on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{MCG}(S);(\mu_n),(d_n))$ defined as in Lemma 5.11 fixes a piece setwise.

6. Subgroups with property (T)

The following result of Chatterji, Druţu, and Haglund relates property (T) to actions on median spaces:

Theorem 6.1. *([CDH]) A locally compact, second countable group has property (T) if and only if any action by isometries on a metric median space has bounded orbits.*

We shall use in an essential way the "only if" part of this theorem. Note that the first proofs of this implication for countable groups (the case we are in here) are implicitly done by Niblo and Reeves in [NR97] and by Niblo and Roller in [NR98]. The direct implication for countable groups is explicitely formulated for the first time by Nica in [Nic08].

Our main result in this section is the following.

Theorem 6.2. *Let Λ be a group with property (T) and let S be a surface.*

Any collection Φ of homomorphisms $\phi\colon \Lambda \to \mathcal{MCG}(S)$ pairwise non-conjugate in $\mathcal{MCG}(S)$ is finite.

Proof. By contradiction, assume there exists an infinite collection $\Phi = \{\phi_1, \phi_2, ...\}$ of pairwise non-conjugate homomorphisms $\phi_n\colon \Lambda \to \mathcal{MCG}(S)$. Lemma 5.9 implies that given a finite generating set A of Λ, $\lim_{n\to\infty} d_n = \infty$, where

$$d_n = \inf_{\mu\in\mathcal{M}(S)} \sup_{a\in A} \mathrm{dist}(\phi_n(a)\mu, \mu)\,. \tag{20}$$

Since $\mathcal{M}(S)$ is a simplicial complex there exists a vertex $\mu_n^0 \in \mathcal{M}(S)$ such that $d_n = \sup_{a\in A} \mathrm{dist}(\phi_n(a)\mu_n^0, \mu_n^0)$. Using the fact that $\mathcal{MCG}(S)$ and $\mathcal{M}(S)$ are quasi-isometric, we let K denote a compact subset of $\mathcal{M}(S)$ which contains μ_0 and for which $\mathcal{MCG}(S)K = \mathcal{M}(S)$. Now consider $x_n \in \mathcal{MCG}(S)$ such that x_nK contains μ_n^0.

Consider an arbitrary ultrafilter ω. Let $\mathcal{AM} = \mathrm{Con}^\omega(\mathcal{M}(S);(\mu_n^0),(d_n))$. We denote by x^ω the element $(x_n)^\omega$ in the ultrapower of $\mathcal{MCG}(S)$. According to the Remark 2.5, (2), the subgroup $x^\omega\left(\Pi_1\mathcal{MCG}(S)/\omega\right)(x^\omega)^{-1}$ of the ultrapower of $\mathcal{MCG}(S)$ acts transitively by isometries on $\mathcal{AM}$. The action is isometric both with respect to the metric $\mathrm{dist}_{\mathcal{AM}}$ and with respect to the metric $\widetilde{\mathrm{dist}}$.

Notation 6.3. We denote for simplicity the subgroup $x^\omega\left(\Pi_1\mathcal{MCG}(S)/\omega\right)(x^\omega)^{-1}$ by $\mathcal{MCG}(S)^\omega_b$.

We shall say that an element $g=(g_n)^\omega$ in $\mathcal{MCG}(S)^\omega_b$ has a given property (i.e., is pseudo-Anosov, pure, reducible, etc) if and only if ω-almost surely g_n has that property (i.e., is pseudo-Anosov, resp. pure, resp. reducible, etc).

The infinite sequence of homomorphisms (ϕ_n) defines a homomorphism

$$\phi_\omega\colon\ \Lambda\to\mathcal{MCG}(S)^\omega_b\ ,\ \phi_\omega(g)=(\phi_n(g))^\omega\ .$$

This homomorphism defines an isometric action of Λ on $\mathcal{AM}$. Since $(\mathcal{AM},\widetilde{\mathrm{dist}})$ is a median space, Theorem 6.1 implies that this action of Λ on $\mathcal{AM}$ has every orbit bounded. In what follows we deduce from it that this action has a fixed multicurve. This and an inductive argument on the complexity of the surface implies that there is no such infinite sequence of homomorphisms, thereby proving the theorem.

Note that when there is no possibility of confusion, we shall write $g\boldsymbol{\mu}$ instead of $\phi_\omega(g)\boldsymbol{\mu}$, for $g\in\Lambda$ and $\boldsymbol{\mu}$ in $\mathcal{AM}$.

If Λ does not fix setwise a piece in the (most refined) tree-graded structure of $\mathcal{AM}$, then by Proposition 5.13, Λ acts without a global fixed point on an asymptotic cone of the complex of curves, which is a complete $\mathbb{R}$-tree, and so Λ cannot have property (T) [dlHV89]. Thus we may assume that Λ fixes setwise a piece P in $\mathcal{AM}$. Then Λ fixes the interior $U(P)$ of P as well by Proposition 5.5.

The point $\boldsymbol{\mu}^0=\langle\mu_n^0\rangle$ must be in the piece P. If not, the projection $\boldsymbol{\nu}$ of $\boldsymbol{\mu}^0$ on P would be moved by less that 1 by all $a\in A$. Indeed, if $a\boldsymbol{\nu}\neq\boldsymbol{\nu}$ then the concatenation of geodesics $[\boldsymbol{\mu}^0,\boldsymbol{\nu}]\sqcup[\boldsymbol{\nu},a\boldsymbol{\nu}]\sqcup[a\boldsymbol{\nu},a\boldsymbol{\mu}^0]$ is a geodesic according to [DS05].

According to Lemma 2.12 there exists a normal subgroup Λ_p in Λ of index at most $N=N(S)$ such that for every $g\in\Lambda_p$, $\phi_\omega(g)$ is pure and fixes setwise the boundary components of S. We need several intermediate results.

Lemma 6.4. *Let $g=(g_n)^\omega\in\mathcal{MCG}(S)^\omega_b$ be a reducible element in $\mathcal{AM}$, and let $\boldsymbol{\Delta}=\langle\Delta_n\rangle$ be a multicurve such that if $U_n^1,...,U_n^m$ are the connected components of $S\setminus\Delta_n$ and the annuli with core curve in Δ_n then ω-almost surely g_n is a pseudo-Anosov on $U_n^1,...,U_n^k$ and the identity map on $U_n^{k+1},...,U_n^m$, and $\Delta_n=\partial U_n^1\cup...\cup\partial U_n^k$ (the latter condition may be achieved by deleting the boundary between two components onto which g_n acts as identity).*

Then the limit set $Q(\boldsymbol{\Delta})$ appears in the asymptotic cone (i.e. the distance from the basepoint μ_n^0 to $Q(\Delta_n)$ is $O(d_n)$), in particular g fixes the piece containing $Q(\boldsymbol{\Delta})$.

If g fixes a piece P then $U(P)$ contains $Q(\boldsymbol{\Delta})$.

Proof. Consider a point $\boldsymbol{\mu}=\langle\mu_n\rangle$ in $\mathcal{AM}$. Let $D_n=\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,\Delta_n)$. Assume that $\lim_\omega\frac{D_n}{d_n}=+\infty$. Let ν_n be a projection of μ_n onto $Q(\Delta_n)$. Note that for every $i=1,2,...,k$, $\mathrm{dist}_{C(U_n^i)}(\mu_n,g_n\mu_n)=\mathrm{dist}_{C(U_n^i)}(\nu_n,g_n\nu_n)+O(1)$. Therefore when replacing g by some large enough power of it we may ensure that $\mathrm{dist}_{C(U_n^i)}(\mu_n,g_n\mu_n)>M$, where M is the constant from Lemma 2.17, while we still have that $\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,g_n\mu_n)\le Cd_n$. In the cone $\mathrm{Con}^\omega(\mathcal{M}(S);(\mu_n),(D_n))$ we have that $\boldsymbol{\mu}=g\boldsymbol{\mu}$ projects onto $Q(\boldsymbol{\Delta})$ into $\boldsymbol{\nu}=g\boldsymbol{\nu}$, which is at distance 1. This contradicts Lemma 4.27. It follows that $\lim_\omega\frac{D_n}{d_n}<+\infty$.

Assume now that g fixes a piece P and assume that the point $\boldsymbol{\mu}$ considered above is in $U(P)$. Since the previous argument implies that a hierarchy path joining

$\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$ for some large enough k intersects $Q(\partial \mathbf{U}_i)$, where $\mathbf{U}_i = (U_n^i)^\omega$ and $i = 1, 2, ..., k$, and $Q(\boldsymbol{\Delta}) \subset Q(\partial \mathbf{U}_i)$, it follows that $Q(\boldsymbol{\Delta}) \subset U(P)$. □

Notation: Given two points $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ in $\mathcal{AM}$ we denote by $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ the set of subsurfaces $\mathbf{U} \subseteq \mathbf{S}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. Note that $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ is non-empty if and only if $\boldsymbol{\mu} \neq \boldsymbol{\nu}$.

Lemma 6.5. *Let $\boldsymbol{\mu}, \boldsymbol{\nu}$ be two distinct points in the same piece. Then $\mathbf{S} \notin \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$.*

Proof. Assume on the contrary that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 4\epsilon > 0$. By Proposition 5.5 there exist $\boldsymbol{\mu}, \boldsymbol{\nu}$ in $U(P)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\mu}') < \epsilon$ and $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, \boldsymbol{\nu}') < \epsilon$. Then $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}', \boldsymbol{\nu}') > 2\epsilon > 0$ whence $\lim_\omega \left(\mathrm{dist}_{C(S)}(\mu_n', \nu_n')\right) = +\infty$, contradicting the fact that $\boldsymbol{\mu}, \boldsymbol{\nu}$ are in $U(P)$. □

Lemma 6.6. (1) *Let $\boldsymbol{\mu}, \boldsymbol{\nu}$ be two points in $Q(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is a multicurve. Then any $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ has the property that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$.*

(2) *Let $\boldsymbol{\mu}$ be a point outside $Q(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is a multicurve, and let $\boldsymbol{\mu}'$ be the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Then any $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}')$ has the property that $\mathbf{U} \pitchfork \boldsymbol{\Delta}$.*

(3) *Let $\boldsymbol{\mu}$, $Q(\boldsymbol{\Delta})$ and $\boldsymbol{\mu}'$ be as in (2). For every $\boldsymbol{\nu} \in Q(\boldsymbol{\Delta})$ we have that $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \sqcup \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu})$.*

(4) *Let $\boldsymbol{\mu}, \boldsymbol{\nu}$ be two points in $Q(\boldsymbol{\Delta})$. Any geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ joining $\boldsymbol{\mu}, \boldsymbol{\nu}$ is entirely contained in $Q(\boldsymbol{\Delta})$.*

Proof. (1) Indeed if $\mathbf{U} = (U_n)^\omega \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ then $\lim_\omega(\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n)) = \infty$, according to Lemma 4.15. On the other hand if $\mathbf{U} \pitchfork \boldsymbol{\Delta}$ then $\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n) = O(1)$, as the bases of both μ_n and ν_n contain Δ_n.

(2) follows immediately from Lemma 4.26.

(3) According to (1) and (2), $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \cap \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu}) = \emptyset$. The triangle inequality implies that for every $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}')$ either $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}', \boldsymbol{\nu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. But since the former cannot occur it follows that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Likewise we prove that $\mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. The inclusion $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \sqcup \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu})$ follows from the triangle inequality.

(4) follows from the fact that for any point $\boldsymbol{\alpha}$ on a $\widetilde{\mathrm{dist}}$-geodesic joining $\boldsymbol{\mu}, \boldsymbol{\nu}$, $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\nu})$, as well as from (1) and (3). □

Lemma 6.7. *Let g be an element in $\mathcal{MCG}(S)_b^\omega$ fixing two distinct points $\boldsymbol{\mu}$, $\boldsymbol{\nu}$. Then for every subsurface $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ there exists $k \in \mathbb{N}$, $k \geq 1$, such that $g^k\mathbf{U} = \mathbf{U}$.*

In the particular case when g is pure $k = 1$.

Proof. If $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = d > 0$ then for every $i \in \mathbb{N}$, $i \geq 1$, $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = d$. Then there exists $k \geq 1$, k smaller than $\left[\frac{\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu})}{d}\right] + 1$ such that $g^k\mathbf{U} = \mathbf{U}$.

The latter part of the statement follows from the fact that if g is pure any power of it fixes exactly the same subsurfaces as g itself. □

Lemma 6.8. *Let $g \in \mathcal{MCG}(S)_b^\omega$ and $\boldsymbol{\Delta}$ be as in Lemma 6.4. If $\boldsymbol{\mu}$ is fixed by g then $\boldsymbol{\mu} \in Q(\boldsymbol{\Delta})$.*

Proof. Assume on the contrary that $\boldsymbol{\mu} \notin Q(\boldsymbol{\Delta})$, and let $\boldsymbol{\nu}$ be its projection onto $Q(\boldsymbol{\Delta})$. Then $g\boldsymbol{\nu}$ is the projection of $g\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Corollary 2.28 implies that $g\boldsymbol{\nu} = \boldsymbol{\nu}$. By replacing g with some power of it we may assume that the hypotheses of Lemma 4.27 hold. On the other hand, the conclusion of Lemma 4.27 does not hold since the geodesic between μ and ν and the geodesic between $g\mu$ and $g\nu$ coincide. This contradiction proves the lemma. □

Lemma 6.9. *Let $g \in \mathcal{MCG}(S)^{\omega}_{b}$ be a pure element such that $\langle g\rangle$ has bounded orbits in $\mathcal{AM}$, and let $\boldsymbol{\mu}$ be a point such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then for every $k \in \mathbb{Z}\setminus\{0\}$, $g^k\boldsymbol{\mu} \neq \boldsymbol{\mu}$.*

Proof. Case 1. Assume that g is a pseudo-Anosov element.

Case 1.a Assume moreover that $\boldsymbol{\mu}$ is in a piece P stabilized by g. Let $\mathbf{U}$ be a subsurface in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$. As $\boldsymbol{\mu}, g\boldsymbol{\mu}$ are both in P it follows by Lemma 6.5 that $\mathbf{U} \subsetneq \mathbf{S}$.

Assume that the subsurfaces $g^{-i_1}\mathbf{U}$, ..., $g^{-i_k}\mathbf{U}$ are also in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$, where $i_1 < \cdots < i_k$. Let $3\epsilon > 0$ be the minimum of $\widetilde{\mathrm{dist}}_{g^{-i}\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}), i = 0, i_1, ..., i_k$. Since P is the closure of its interior $U(P)$ (Proposition 5.5) there exists $\boldsymbol{\nu} \in U(P)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < \epsilon$. It follows that $\widetilde{\mathrm{dist}}_{g^{-i}\mathbf{U}}(\boldsymbol{\nu}, g\boldsymbol{\nu}) \geq \epsilon$ for $i = 0, i_1, ..., i_k$. Then by Lemma 4.15, $\lim_\omega \mathrm{dist}_{C(g^{-i}U_n)}(\nu_n, g_n\nu_n) = \infty$. Let $\mathfrak{h} = \langle \mathfrak{h}_n\rangle$ be a hierarchy path joining $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$. The above implies that ω-almost surely $\mathfrak{h}_n$ intersects $Q(g^{-i_j}\partial U_n)$, hence there exists a vertex v_n^j on the tight geodesic $\mathfrak{t}_n$ shadowed by $\mathfrak{h}_n$ such that $g^{-i_j}U_n \subseteq S \setminus v_n^j$. In particular $\mathrm{dist}_{\mathcal{C}(S)}(g^{-i_j}\partial U_n, v_n^j) \leq 1$. Since $\boldsymbol{\nu} \in U(P)$ and g stabilizes $U(P)$ it follows that $g\boldsymbol{\nu} \in U(P)$, whence $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n, g_n\nu_n) \leq D = D(g)$ ω-almost surely. In particular the length of the tight geodesic $\mathfrak{t}_n$ is at most $D+2$ ω-almost surely.

According to [Bow, Theorem 1.4], there exists $m = m(S)$ such that ω-almost surely g_n^m preserves a bi-infinite geodesic $\mathfrak{g}_n$ in $C(S)$. To denote g^m we write g_1 for the sequence with terms $g_{1,n}$.

For every curve γ let γ' be a projection of it on $\mathfrak{g}_n$. A standard hyperbolic geometry argument implies that for every $i \geq 1$

$$\mathrm{dist}_{\mathcal{C}(S)}(\gamma, g_{1,n}^{-i}\gamma) \geq \mathrm{dist}_{\mathcal{C}(S)}(\gamma', g_{1,n}^{-i}\gamma') + O(1) \geq i + O(1)\,.$$

The same estimate holds for γ replaced by ∂U_n. Now assume that the maximal power $i_k = mq + r$, where $0 \leq r < m$. Then $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n, g_n^{-mq}\partial U_n) = \mathrm{dist}_{\mathcal{C}(S)}(g_n^{-r}\partial U_n, \partial U_n) \leq 2(D+2) + \mathrm{dist}_{\mathcal{C}(S)}(g_n^{-r}\nu_n, \nu_n) \leq 2(D+2) + rD = D_1$. It follows that $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n, \partial U_n) \geq \mathrm{dist}_{\mathcal{C}(S)}(\partial U_n, g_n^{-mq}\partial U_n) - D_1 \geq q + O(1) - D_1$.

On the other hand $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n, \partial U_n) \leq 2 + \mathrm{dist}_{\mathcal{C}(S)}(v_n^k, v_n^0) \leq D+4$, whence $q \leq D + D_1 + 4 + O(1) = D_2$ and $i_k \leq m(D_2+1)$. Thus the sequence $i_1, ..., i_k$ is bounded, and it has a maximal element. It follows that there exists a subsurface $\mathbf{U}$ in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ such that for every $k > 0$, $\widetilde{\mathrm{dist}}_{g^{-k}\mathbf{U}}(\boldsymbol{\nu}, g\boldsymbol{\nu}) = \widetilde{\mathrm{dist}}_{\mathbf{U}}(g^k\boldsymbol{\nu}, g^{k+1}\boldsymbol{\nu}) = 0$. The triangle inequality in $T_{\mathbf{U}}$ implies that $\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, g^k\boldsymbol{\mu}) > 0$ for every $k \geq 1$. It follows that no power g^k fixes $\boldsymbol{\mu}$.

Case 1.b Assume now that $\boldsymbol{\mu}$ is not contained in any piece fixed by g. By Lemma 3.12 g fixes either the middle cut piece P or the middle cut point m of $\boldsymbol{\mu}, g\boldsymbol{\mu}$.

Assume that $\boldsymbol{\mu}, g\boldsymbol{\mu}$ have a middle cut piece P, and let $\boldsymbol{\nu}, \boldsymbol{\nu}'$ be the endpoints of the intersection with P of any arc joining $\boldsymbol{\mu}, g\boldsymbol{\mu}$. Then $g\boldsymbol{\nu} = \boldsymbol{\nu}'$ hence $g\boldsymbol{\nu} \neq \boldsymbol{\nu}$. By

Case 1.a it then follows that for every $k \neq 0$, $g^k\boldsymbol{\nu} \neq \boldsymbol{\nu}$, and since $[\boldsymbol{\mu},\boldsymbol{\nu}] \sqcup [\boldsymbol{\nu}, g^k\boldsymbol{\nu}] \sqcup [g^k\boldsymbol{\nu}, g^k\boldsymbol{\mu}]$ is a geodesic, it follows that $g^k\boldsymbol{\mu} \neq \boldsymbol{\mu}$.

We now assume that $\boldsymbol{\mu}, g\boldsymbol{\mu}$ have a middle cut point m, fixed by g. Assume that $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ contains a strict subsurface of $\mathbf{S}$. Then the same thing holds for $\mathfrak{U}(\boldsymbol{\mu}, m)$. Let $\mathbf{U} \subsetneq \mathbf{S}$ be an element in $\mathfrak{U}(\boldsymbol{\mu}, m)$.

If $g^k\boldsymbol{\mu} = \boldsymbol{\mu}$ for some $k \neq 0$, since $g^k m = m$ it follows that $g^{kn}(\mathbf{U}) = \mathbf{U}$ for some $n \neq 0$, by Lemma 6.7. But this is impossible, since g is a pseudo-Anosov.

Thus, we may assume that $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \{\mathbf{S}\}$, i.e. that $\boldsymbol{\mu}, g\boldsymbol{\mu}$ are in the same transversal tree.

Let $\mathfrak{g}_n$ be a bi-infinite geodesic in $C(S)$ such that $g_n\mathfrak{g}_n$ is at Hausdorff distance $O(1)$ from $\mathfrak{g}_n$. Let γ_n be the projection of $\pi_{\mathcal{C}(S)}(\mu_n)$ onto $\mathfrak{g}_n$. A hierarchy path $\mathfrak{h} = \langle \mathfrak{h}_n \rangle$ joining μ_n and $g_n\mu_n$ contains two points ν_n, ν_n' such that:

- the sub-path with endpoints μ_n, ν_n is at $C(S)$-distance $O(1)$ from any $C(S)$-geodesic joining $\pi_{\mathcal{C}(S)}(\mu_n)$ and γ_n;
- the sub-path with endpoints $g\mu_n, \nu_n'$ is at $C(S)$-distance $O(1)$ from any $C(S)$-geodesic joining $\pi_{\mathcal{C}(S)}(g\mu_n)$ and $g\gamma_n$;
- if $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n, \nu_n')$ is large enough then the sub-path with endpoints ν_n, ν_n' is at $C(S)$-distance $O(1)$ from $\mathfrak{g}_n$;
- $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n', g\nu_n)$ is $O(1)$.

Let $\boldsymbol{\nu} = \langle \nu_n \rangle$ and $\boldsymbol{\nu}' = \langle \nu_n' \rangle$. The last property above implies that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu}', g\boldsymbol{\nu}) = 0$. Assume that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu}, \boldsymbol{\nu}') > 0$ hence $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu}, g\boldsymbol{\nu}) > 0$. Let $\mathfrak{h}'$ be a hierarchy sub-path with endpoints $\boldsymbol{\nu}, g\boldsymbol{\nu}$. Its projection onto $T_{\mathbf{S}}$ and the projection of $g\mathfrak{h}'$ onto $T_{\mathbf{S}}$ have in common only their endpoint. Otherwise there would exist $\boldsymbol{\alpha}$ on $\mathfrak{h}' \cap g\mathfrak{h}'$ with $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\alpha}, g\boldsymbol{\mu}) > 0$, and such that $\mathrm{Cutp}\,\{\boldsymbol{\alpha}, g\boldsymbol{\mu}\}$ is in the intersection of $\mathrm{Cutp}\,(\mathfrak{h}')$ with $\mathrm{Cutp}\,(g\mathfrak{h}')$. Consider $\boldsymbol{\beta} \in \mathrm{Cutp}\,\{\boldsymbol{\alpha}, g\boldsymbol{\mu}\}$ at equal $\widetilde{\mathrm{dist}}_{\mathbf{S}}$-distance from $\boldsymbol{\alpha}, g\boldsymbol{\mu}$. Take α_n, β_n on $\mathfrak{h}_n'$ and α_n', β_n' on $g\mathfrak{h}_n'$ such that $\boldsymbol{\alpha} = \langle \alpha_n \rangle = \langle \alpha_n' \rangle$ and $\boldsymbol{\beta} = \langle \beta_n \rangle = \langle \beta_n' \rangle$. Since α_n, α_n' and β_n, β_n' are at distance $o(d_n)$ it follows that $\mathfrak{h}_n'$ between α_n, β_n and $g\mathfrak{h}_n'$ between α_n', β_n' share a large domain U_n. Let σ_n and σ_n' be the corresponding points on the two hierarchy sub-paths contained in $Q(\partial U_n)$. The projections of $\mathfrak{h}_n'$ and $g\mathfrak{h}_n'$ onto $C(S)$, both tight geodesics, would contain the points $\pi_{\mathcal{C}(S)}(\sigma_n)$ and $\pi_{\mathcal{C}(S)}(\sigma_n')$ at $\mathrm{dist}_{\mathcal{C}(S)}$-distance $O(1)$ while $\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\sigma_n, g\nu_n) = \infty$ and $\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\sigma_n', g\nu_n) = \infty$. This contradicts the fact that the projection of $\mathfrak{h}_n' \sqcup g\mathfrak{h}_n'$ is at $\mathrm{dist}_{\mathcal{C}(S)}$-distance $O(1)$ from the geodesic $\mathfrak{g}_n$.

We may thus conclude that the projections of $\mathfrak{h}'$ and $g\mathfrak{h}'$ on $T_{\mathbf{S}}$ intersect only in their endpoints. From this fact one can easily deduce by induction that g has unbounded orbits in $T_{\mathbf{S}}$, hence in $\mathbb{F}$.

Assume now that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu}, \boldsymbol{\nu}') = 0$ (hence $\boldsymbol{\nu} = m$) and that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. Let $\boldsymbol{\alpha}$ be the point on the hierarchy path joining $\boldsymbol{\mu}, \boldsymbol{\nu}$ at equal $\widetilde{\mathrm{dist}}_{\mathbf{S}}$-distance from its extremities and let $\boldsymbol{\mathfrak{h}''} = \langle \mathfrak{h}_n'' \rangle$ be the sub-path of endpoints $\boldsymbol{\mu}, \boldsymbol{\alpha}$. All the domains of $\mathfrak{h}_n''$ have $C(S)$ distance to $\mathfrak{g}_n$ going to infinity, likewise for the $C(S)$ distance to any geodesic joining $\pi_{\mathcal{C}(S)}(g^k\mu_n)$ and $\pi_{\mathcal{C}(S)}(g^k\nu_n)$ with $k \neq 0$. It follows that $\mathrm{dist}(\boldsymbol{\mu}, g^k\boldsymbol{\mu}) \geq \mathrm{dist}(\boldsymbol{\mu}, \boldsymbol{\alpha}) > 0$.

Case 2. Assume now that g is a reducible element, and let $\boldsymbol{\Delta} = \langle \Delta_n \rangle$ be a multicurve as in Lemma 6.4. According to the same lemma, $Q(\boldsymbol{\Delta}) \subset U(P)$.

If $\boldsymbol{\mu} \notin Q(\boldsymbol{\Delta})$ then $g^k\boldsymbol{\mu} \neq \boldsymbol{\mu}$ by Lemma 6.8. Assume therefore that $\boldsymbol{\mu} \in Q(\boldsymbol{\Delta})$. The set $Q(\boldsymbol{\Delta})$ can be identified to $\prod_{i=1}^m \mathcal{M}(\mathbf{U}_i)$ and $\boldsymbol{\mu}$ can be therefore identified to $(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_m)$. If for every $i \in \{1, 2, ..., k\}$ the component of g acting on $\mathbf{U}_i$ would fix $\boldsymbol{\mu}_i$ in $\mathcal{M}(\mathbf{U}_i)$ then g would fix $\boldsymbol{\mu}$. This would contradict the hypothesis on g. Thus for some $i \in \{1, 2, ..., k\}$ the corresponding component of g acts on $\mathcal{M}(\mathbf{U}_i)$ as a pseudo-Anosov and does not fix $\boldsymbol{\mu}_i$. According to the first case for every $k \in \mathbb{Z} \setminus \{0\}$ the component of g^k acting on $\mathbf{U}_i$ does not fix $\boldsymbol{\mu}_i$ either, hence g^k does not fix $\boldsymbol{\mu}$. □

Lemma 6.10. *Let $g \in \mathcal{MCG}(S)^\omega_b$ be a pure element, and let $\boldsymbol{\mu} = \langle \mu_n \rangle$ be a point such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. If g is reducible take $\boldsymbol{\Delta} = \langle \Delta_n \rangle$, and $U_n^1, ..., U_n^m$ as in Lemma 6.4, while if g is pseudo-Anosov take $\boldsymbol{\Delta} = \emptyset$ and $\{U_n^1, ..., U_n^m\} = \{S\}$, and by convention $Q(\Delta_n) = \mathcal{M}(S)$. Assume that g is such that for any $\nu_n \in Q(\Delta_n)$, $\mathrm{dist}_{C(U_n^i)}(\nu_n, g_n\nu_n) > D$ ω-almost surely for every $i \in \{1, ..., k\}$, where D is a fixed constant, depending only on $\xi(S)$ (this may be achieved for instance by replacing g with a large enough power of it).*

Then $\mathfrak{U} = \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ splits as $\mathfrak{U}_0 \sqcup \mathfrak{U}_1 \sqcup g\mathfrak{U}_1 \sqcup \mathfrak{P}$, where

- *$\mathfrak{U}_0$ is the set of elements $\mathbf{U} \in \mathfrak{U}$ such that no $g^k\mathbf{U}$ with $k \in \mathbb{Z} \setminus \{0\}$ is in $\mathfrak{U}$,*
- *$\mathfrak{P}$ is the intersection of $\mathfrak{U}$ with $\{\mathbf{U}^1, ..., \mathbf{U}^k\}$, where $\mathbf{U}^j = (U_n^j)^\omega$,*
- *$\mathfrak{U}_1$ is the set of elements $\mathbf{U} \in \mathfrak{U} \setminus \mathfrak{P}$ such that $g^k\mathbf{U} \in \mathfrak{U}$ only for $k = 0, 1$ (hence $g\mathfrak{U}_1$ is the set of elements $\mathbf{U} \in \mathfrak{U} \setminus \mathfrak{P}$ such that $g^k\mathbf{U} \in \mathfrak{U}$ only for $k = 0, -1$).*

Moreover, if either $\mathfrak{U}_0 \neq \emptyset$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$ then the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Proof. Case 1. Assume that g is a pseudo-Anosov with $C(S)$-translation length D, where D is a large enough constant. There exists a bi-infinite axis $\mathfrak{p}_n$ such that $g_n\mathfrak{p}_n$ is at Hausdorff distance $O(1)$ from $\mathfrak{p}_n$. Consider $\mathfrak{h} = \langle \mathfrak{h}_n \rangle$ a hierarchy path joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, such that $\mathfrak{h}_n$ shadows a tight geodesic $\mathfrak{t}_n$. Choose two points γ_n, γ_n' on $\mathfrak{p}_n$ that are nearest to $\pi_{\mathcal{C}(S)}(\mu_n)$, and $\pi_{\mathcal{C}(S)}(g\mu_n)$ respectively. Note that $\mathrm{dist}_{\mathcal{C}(S)}(\gamma_n', g_n\gamma_n) = O(1)$.

Standard arguments concerning the way hyperbolic elements act on hyperbolic metric spaces imply that the geodesic $\mathfrak{t}_n$ is in a tubular neighborhood with radius $O(1)$ of the union of $C(S)$-geodesics $[\pi_{\mathcal{C}(S)}(\mu_n), \gamma_n] \sqcup [\gamma_n, \gamma_n'] \sqcup [\gamma_n', g_n\pi_{\mathcal{C}(S)}(\mu_n)]$. Moreover any point on $\mathfrak{t}_n$ has any nearest point projection on $\mathfrak{p}_n$ at distance $O(1)$ from $[\gamma_n, \gamma_n'] \subset \mathfrak{p}_n$.

Now let $\mathbf{U} = (U_n)^\omega$ be a subsurface in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$, $\mathbf{U} \subsetneq \mathbf{S}$. Assume that for some $i \in \mathbb{Z}$, $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) > 0$. This implies that $\lim_\omega \mathrm{dist}_{C(U_n)}(g_n^j\mu_n, g_n^{j+1}\mu_n) = +\infty$ for $j \in \{0, i\}$, according to Lemma 4.15. In particular, by Lemma 2.17, ∂U_n is at $C(S)$-distance ≤ 1 from a vertex $u_n \in \mathfrak{t}_n$ and $g_n^{-i}\partial U_n$ is at $C(S)$-distance ≤ 1 from a vertex $v_n \in \mathfrak{t}_n$. It follows from the above that ∂U_n and $g_n^{-i}\partial U_n$ have any nearest point projection on $\mathfrak{p}_n$ at distance $O(1)$ from $[\gamma_n, \gamma_n'] \subset \mathfrak{p}_n$. Let x_n be a nearest point projection on $\mathfrak{p}_n$ of ∂U_n. Then $g_n^{-i}x_n$ is a nearest point projection on $\mathfrak{p}_n$ of $g_n^{-i}\partial U_n$. As both x_n and $g_n^{-i}x_n$ are at distance $O(1)$ from $[\gamma_n, \gamma_n']$, they are at distance at most $D + O(1)$ from each other. On the other hand $\mathrm{dist}_{\mathcal{C}(S)}(x_n, g_n^{-i}x_n) = |i|D + O(1)$. For D large enough this implies that $i \in \{-1, 0, 1\}$. Moreover for $i = -1$, ∂U_n projects on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n while $g_n\partial U_n$ projects on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g_n\gamma_n$. This in particular implies that, for D large enough, either $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g\boldsymbol{\mu}, g^2\boldsymbol{\mu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, \boldsymbol{\mu}) > 0$ but not both.

Let $\mathfrak{U}_0 = \mathfrak{U} \setminus (g\mathfrak{U} \cup g^{-1}\mathfrak{U})$. Let $\mathfrak{U}_1 = \left(\mathfrak{U} \cap g^{-1}\mathfrak{U}\right) \setminus \{\mathbf{S}\}$ and $\mathfrak{U}_2 = (\mathfrak{U} \cap g\mathfrak{U}) \setminus \{\mathbf{S}\}$. Clearly $\mathfrak{U} = \mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{P}$, where $\mathfrak{P}$ is either $\emptyset$ or $\{\mathbf{S}\}$. Since $g^{-1}\mathfrak{U} \cap g\mathfrak{U}$ is either empty or $\{\mathbf{S}\}$, $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{P}$ are pairwise disjoint, and $\mathfrak{U}_2 = g\mathfrak{U}_1$.

Assume that $\mathfrak{U}_0$ is non-empty, and let $\mathbf{U}$ be an element in $\mathfrak{U}_0$. Then $d = \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) > 0$ and $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) = 0$ for every $i \in \mathbb{Z} \setminus \{0\}$. Indeed if there existed $i \in \mathbb{Z} \setminus \{0\}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) > 0$ then, by the choice of D large enough, either $i = -1$ or $i = 1$, therefore either $\mathbf{U} \in g\mathfrak{U}_1$ or $\mathbf{U} \in \mathfrak{U}_1$, both contradicting the fact that $\mathbf{U} \in \mathfrak{U}_0$. The triangle inequality then implies that for every $i \leq 0 < j$, $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = d$. Moreover for every $i \leq k \leq j$, by applying g^{-k} to the previous equality we deduce that $\widetilde{\mathrm{dist}}_{g^k\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = d$. Thus for every $i \leq 0 < j$ the distance $\widetilde{\mathrm{dist}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu})$ is at least $\sum_{i \leq k \leq j} \widetilde{\mathrm{dist}}_{g^k\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = (j-i)d$. This implies that the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Assume that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$. Then the distance $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, g\boldsymbol{\mu})$ is at least $|\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) - \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})| = d > 0$. Moreover since $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^k\boldsymbol{\mu}, g^{k+1}\boldsymbol{\mu}) = 0$ for every $k \geq 1$ and $k \leq -2$, it follows that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) = \widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, g\boldsymbol{\mu}) > d$ for every $k, m \geq 1$. We then obtain that for every $\mathbf{V} = g^j\mathbf{U}$ with $j \in \{-k+1, ..., m-1\}$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) > d$. Since $\mathbf{U} \subsetneq \mathbf{S}$ and g is a pseudo-Anosov, it follows that if $i \neq j$ then $g^i\mathbf{U} \neq g^j\mathbf{U}$. Then $\widetilde{\mathrm{dist}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) \geq \sum_{j=-k+1}^{m-1} \widetilde{\mathrm{dist}}_{g^j\mathbf{U}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) \geq (k+m-1)d$. Hence the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Case 2. Assume that g is reducible. Let $\boldsymbol{\nu}$ be the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Consequently $g\boldsymbol{\nu}$ is the projection of $g\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Lemma 6.6 implies that $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) \cup \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu}) \cup \mathfrak{U}(g\boldsymbol{\nu}, g\boldsymbol{\mu})$.

Consider an element $\mathbf{U} \in \mathfrak{U}$, $\mathbf{U} \notin \{\mathbf{U}_1, ...\mathbf{U}_m\}$, and assume that for some $i \in \mathbb{Z} \setminus \{0\}$, $g^i\mathbf{U} \in \mathfrak{U}$. We prove that $i \in \{-1, 0, 1\}$.

Assume that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu})$. Then, since $\lim_\omega \mathrm{dist}_{C(U_n)}(\mu_n, g\mu_n) = +\infty$, it follows that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$ and $\mathbf{U}$ is contained in $\mathbf{U}^j$ for some $j \in \{1, ..., k\}$. Either $\mathbf{U} = \mathbf{U}^j \in \mathfrak{P}$ or $\mathbf{U} \subsetneq \mathbf{U}^j$. In the latter case, an argument as in Case 1 implies that for D large enough $i \in \{-1, 0, 1\}$.

Assume that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Then $\mathbf{U} \pitchfork \boldsymbol{\Delta}$, since $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ do not differ inside the subsurfaces $\mathbf{U}^j$, $j = 1, ..., m$. Since $\boldsymbol{\Delta} = \bigcup_{j=1}^k \partial\mathbf{U}^j$ it follows that for some $j \in \{1, ..., k\}$, $\mathbf{U} \pitchfork \partial\mathbf{U}^j$.

We have that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g^i\boldsymbol{\mu}, g^i\boldsymbol{\nu}) > 0$, hence a hierarchy path joining $g_n^i\mu_n$ and $g_n^i\nu_n$ contains a point β_n in $Q(g_n^i\partial U_n)$.

The hypothesis that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) > 0$ implies that either $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ or $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$. Assume that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. Then a hierarchy path joining μ_n and ν_n also contains a point β_n' in $Q(g_n^i\partial U_n)$.

For the element $j \in \{1, ..., k\}$ such that $\mathbf{U} \pitchfork \partial\mathbf{U}^j$, $\mathrm{dist}_{C(U_n^j)}(\beta_n, \beta_n') = O(1)$ since both β_n and β_n' contain the multicurve ∂U_n. By properties of projections, $\mathrm{dist}_{C(U_n^j)}(\mu_n, \nu_n) = O(1)$ and $\mathrm{dist}_{C(U_n^j)}(g_n^i\mu_n, g_n^i\nu_n) = O(1)$, which implies that $\mathrm{dist}_{C(U_n^j)}(\beta_n, g_n^i\nu_n) = O(1)$ and $\mathrm{dist}_{C(U_n^j)}(\beta_n', \nu_n) = O(1)$. It follows that the distance $\mathrm{dist}_{C(U_n^j)}(g_n^i\nu_n, \nu_n)$ has order $O(1)$. On the other hand $\mathrm{dist}_{C(U_n^j)}(g_n^i\nu_n, \nu_n) > |i|D$. For D large enough this implies that $i = 0$.

Assume that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$. This and the fact that $\widetilde{\mathrm{dist}}_{g\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$ imply as in the previous argument, with $\boldsymbol{\mu}, \boldsymbol{\nu}$ and $\mathbf{U}$ replaced by $g\boldsymbol{\mu}, g\boldsymbol{\nu}$ and $g\mathbf{U}$, that $i = 1$.

The case when $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$ is dealt with similarly. In this case it follows that, if $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ then $i = -1$, and if $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$ then $i = 0$.

We have thus proved that for every $\mathbf{U} \in \mathfrak{U}$, $\mathbf{U} \notin \{\mathbf{U}_1, \ldots \mathbf{U}_m\}$, if for some $i \in \mathbb{Z} \setminus \{0\}$, $g^i\mathbf{U} \in \mathfrak{U}$ then $i \in \{-1, 0, 1\}$. We take $\mathfrak{P} = \mathfrak{U} \cap \{\mathbf{U}_1, \ldots \mathbf{U}_m\}$ and $\mathfrak{U}' = \mathfrak{U} \setminus \mathfrak{P}$. We define $\mathfrak{U}_0 = \mathfrak{U}' \setminus (g\mathfrak{U}' \cup g^{-1}\mathfrak{U}')$. Let $\mathfrak{U}_1 = \mathfrak{U}' \cap g^{-1}\mathfrak{U}'$ and $\mathfrak{U}_2 = \mathfrak{U}' \cap g\mathfrak{U}'$. Clearly $\mathfrak{U} = \mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{P}$. Since $g^{-1}\mathfrak{U}' \cap g\mathfrak{U}'$ is empty, $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{P}$ are pairwise disjoint, and $\mathfrak{U}_2 = g\mathfrak{U}_1$.

If $\mathfrak{U}_0 \neq \emptyset$ then a proof as in Case 1 yields that the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Assume that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$. It follows from the previous argument that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu})$, hence $g\mathbf{U}$ is in the same set. Without loss of generality we may therefore replace $\boldsymbol{\mu}$ by $\boldsymbol{\nu}$ and assume that $\boldsymbol{\mu} \in Q(\Delta)$. In particular $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ is composed only of subsurfaces that do not intersect Δ. We proceed as in Case 1 and prove that the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded. □

Lemma 6.11. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)^\omega_b$ be a pseudo-Anosov fixing a piece P, such that $\langle g \rangle$ has bounded orbits in $\mathcal{AM}$. Assume that ω-almost surely the translation length of g_n on $C(S)$ is larger than a uniformly chosen constant depending only on $\xi(S)$. Then for any point $\boldsymbol{\mu}$ in P and for any hierarchy path $\mathfrak{h}$ connecting $\boldsymbol{\mu}$ and its translate $g\boldsymbol{\mu}$, the isometry g fixes the middlepoint of $\mathfrak{h}$.*

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in P and $\mathfrak{h} = \langle \mathfrak{h}_n \rangle$ a hierarchy path joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, such that $\mathfrak{h}_n$ shadows a tight geodesic $\mathfrak{t}_n$. We may assume that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$, and consider the splitting defined in Lemma 6.10, $\mathfrak{U} = \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}_0 \sqcup \mathfrak{U}_1 \sqcup g\mathfrak{U}_1$. Note that since $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$ are both in the same piece P, $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ cannot contain $\mathbf{S}$, by Lemma 6.5. As $\langle g \rangle$ has bounded orbits, we may assume that $\mathfrak{U}_0$ is empty, and that $\mathfrak{U} = \mathfrak{U}_1 \cup g\mathfrak{U}_1$. We denote $g\mathfrak{U}_1$ also by $\mathfrak{U}_2$. For every $\mathbf{U} \in \mathfrak{U}$ choose a sequence (U_n) representing it, and define $\mathfrak{U}(n)$, $\mathfrak{U}_1(n)$, $\mathfrak{U}_2(n)$ as the set of U_n corresponding to $\mathbf{U}$ in $\mathfrak{U}, \mathfrak{U}_1, \mathfrak{U}_2$ respectively.

Let α_n be the last point on the hierarchy path $\mathfrak{h}_n$ belonging to $Q(\partial U_n)$ for some $U_n \in \mathfrak{U}_1(n)$. Let $\boldsymbol{\alpha} = \langle \alpha_n \rangle$. Assume that $g\boldsymbol{\alpha} \neq \boldsymbol{\alpha}$. For every subsurface $\mathbf{V} = (V_n)^\omega \in \Pi\mathcal{U}/\omega$ such that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha}) > 0$ it follows by the triangle inequality that either $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\mu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g\boldsymbol{\mu}, g\boldsymbol{\alpha}) > 0$. In the first case $\mathbf{V} \in \mathfrak{U}$. If $\mathbf{V} \in \mathfrak{U}_1$ then $\mathbf{V} = (U_n)^\omega$ for one of the chosen sequences (U_n) representing an element in $\mathfrak{U}_1$, whence $\lim_\omega \mathrm{dist}_{C(U_n)}(\alpha_n, g_n\mu_n) = \infty$ and the hierarchy sub-path of $\mathfrak{h}_n$ between α_n and $g_n\mu_n$ has a large intersection with $Q(\partial U_n)$. This contradicts the choice of α_n. Thus in this case we must have that $\mathbf{V} \in g\mathfrak{U}_1$.

We now consider the second case, where $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g\boldsymbol{\mu}, g\boldsymbol{\alpha}) > 0$. Since this condition is equivalent to $\widetilde{\mathrm{dist}}_{g^{-1}\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\alpha}) > 0$ it follows that $g^{-1}\mathbf{V} \in \mathfrak{U}$. Moreover ω-almost surely the hierarchy sub-path of $\mathfrak{h}_n$ between μ_n and α_n has a large intersection with $Q(g_n^{-1}\partial V_n)$.

Define $\mathfrak{p}_n$ and the points γ_n, γ_n' on $\mathfrak{p}_n$ as in Case 1 of the proof of Lemma 6.10. The argument in that proof shows that for every $\mathbf{U} = (U_n)^\omega \in \mathfrak{U}_1$, ω-almost surely ∂U_n has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n while $g_n\partial U_n$ has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g_n\gamma_n$. In particular α_n has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n whence $g_n^{-1}\partial V_n$ has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from

γ_n too. For sufficiently large translation length (i.e., the constant in the hypothesis of the lemma), this implies that $g_n^{-1}\partial V_n$ cannot have a nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g\gamma_n$. Thus ω-almost surely $g_n^{-1}V_n \notin \mathfrak{U}_2(n)$, therefore $g^{-1}\mathbf{V} \notin \mathfrak{U}_2$. It follows that $g^{-1}\mathbf{V} \in \mathfrak{U}_1$, whence $\mathbf{V} \in g\mathfrak{U}_1$.

We have thus obtained that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha}) > 0$ implies that $\mathbf{V} \in g\mathfrak{U}_1$, therefore for every $k \in \mathbb{Z}$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g^k\boldsymbol{\alpha}, g^{k+1}\boldsymbol{\alpha}) > 0$ implies that $\mathbf{V} \in g^{k+1}\mathfrak{U}_1$. Since the collections of subsurfaces $g^i\mathfrak{U}_1$ and $g^j\mathfrak{U}_1$ are disjoint for $i \neq j$ it follows that $\widetilde{\mathrm{dist}}(g^{-i}\boldsymbol{\alpha}, g^j\boldsymbol{\alpha}) = \sum_{k=-i}^{j-1}\sum_{\mathbf{V}\in g^{k+1}\mathfrak{U}_1} \widetilde{\mathrm{dist}}_{\mathbf{V}}(g^{-i}\boldsymbol{\alpha}, g^j\boldsymbol{\alpha}) = \sum_{k=-i}^{j-1}\sum_{\mathbf{V}\in g^{k+1}\mathfrak{U}_1} \widetilde{\mathrm{dist}}_{\mathbf{V}}(g^k\boldsymbol{\alpha}, g^{k+1}\boldsymbol{\alpha}) = (j+i-1)\widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha})$. This implies that the $\langle g\rangle$-orbit of $\boldsymbol{\alpha}$ is unbounded, contradicting our hypothesis.

Therefore $\boldsymbol{\alpha} = g\boldsymbol{\alpha}$. From this, the fact that g acts as an isometry on $(\mathcal{AM}, \widetilde{\mathrm{dist}})$, and the fact that hierarchy paths are geodesics in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$, it follows that $\boldsymbol{\alpha}$ is the middlepoint of $\mathfrak{h}$. □

Lemma 6.12. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)_b^\omega$ be a pseudo-Anosov such that $\langle g\rangle$ has bounded orbits in $\mathcal{AM}$. Assume that ω-almost surely the translation length of g_n on $C(S)$ is larger than a uniformly chosen constant depending only on $\xi(S)$. Then for any point $\boldsymbol{\mu}$ and for any hierarchy path $\mathfrak{h}$ connecting $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, the isometry g fixes the middlepoint of $\mathfrak{h}$.*

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in $\mathcal{AM}$ and assume $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Lemma 3.12, (1), implies that g fixes either the middle cut point or the middle cut piece of $\boldsymbol{\mu}, g\boldsymbol{\mu}$. In the former case we are done. In the latter case consider P the middle cut piece, $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ the entrance and respectively exit points of $\mathfrak{h}$ from P. Then $\boldsymbol{\nu}' = g\boldsymbol{\nu} \neq \boldsymbol{\nu}$ and we may apply Lemma 6.11 to g and $\boldsymbol{\nu}$ to finish the argument. □

Lemma 6.13. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)_b^\omega$ be a pseudo-Anosov. The set of fixed points of g is either empty or it is a convex subset of a transversal tree of $\mathcal{AM}$.*

Proof. Assume there exists a point $\boldsymbol{\mu} \in \mathcal{AM}$ fixed by g. Let $\boldsymbol{\nu}$ be another point fixed by g. Since g is an isometry permuting pieces, this and property (T_2') implies that g fixes every point in $\mathrm{Cutp}\,\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$. If a geodesic (any geodesic) joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ has a non-trivial intersection $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ with a piece then $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are also fixed by g. By Lemma 6.5, $\mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ contains a proper subsurface $\mathbf{U} \subsetneq \mathbf{S}$, and by Lemma 6.7, $g\mathbf{U} = \mathbf{U}$, which is impossible.

It follows that any geodesic joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ intersects all pieces in points. This means that the set of points fixed by g is contained in the transversal tree $T_{\boldsymbol{\mu}}$ (as defined in Definition 3.7). It is clearly a convex subset of $T_{\boldsymbol{\mu}}$. □

Lemma 6.14. *Let $g \in \mathcal{MCG}(S)_b^\omega$ be a reducible element such that $\langle g\rangle$ has bounded orbits in $\mathcal{AM}$, and let $\boldsymbol{\Delta} = \langle \Delta_n\rangle$ and $\mathbf{U}_1 = (U_n^1)^\omega, ..., \mathbf{U}_m = (U_n^m)^\omega$ be the multicurve and the subsurfaces associated to g as in Lemma 6.4. Assume that for any $i \in \{1, 2, ..., k\}$ the distance $\mathrm{dist}_{C(U_i)}(\boldsymbol{\nu}, g\boldsymbol{\nu})$ is larger than some sufficiently large constant, D, depending only on $\xi(S)$.*

Then for any point $\boldsymbol{\mu}$ there exists a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting $\boldsymbol{\mu}$ and its translate $g\boldsymbol{\mu}$, such that the isometry g fixes its middlepoint.

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in $\mathcal{AM}$. By means of Lemma 3.12, (1), we may reduce the argument to the case when $\boldsymbol{\mu}$ is contained in a piece P fixed set-wise by g. Lemma 6.4 implies that $U(P)$ contains $Q(\boldsymbol{\Delta})$. Let $\boldsymbol{\nu}$ be the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. According to Lemma 4.27, if D is large enough then given $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ hierarchy paths connecting respectively $\boldsymbol{\mu}, \boldsymbol{\nu}$, $\boldsymbol{\nu}, g\boldsymbol{\nu}$, and $g\boldsymbol{\nu}, g\boldsymbol{\mu}$, $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$.

If $\boldsymbol{\nu} = g\boldsymbol{\nu}$ then we are done. If not, we apply Lemma 6.12 to g restricted to each $\mathbf{U}^j$ and we find a point $\boldsymbol{\alpha}$ between $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$ fixed by g. Since both $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$ are between $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$ it follows that $\boldsymbol{\alpha}$ is between $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, hence on a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting them. □

Lemma 6.15. *Let $g \in \mathcal{MCG}(S)^{\omega}_{b}$ be a reducible element, and let $\boldsymbol{\Delta} = \langle \Delta_n \rangle$ and $\mathbf{U}_1 = (U^1_n)^{\omega}, ..., \mathbf{U}_m = (U^m_n)^{\omega}$ be the multicurve and the subsurfaces associated to g as in Lemma 6.4.*

If the set $\mathrm{Fix}(g)$ of points fixed by g contains a point $\boldsymbol{\mu}$ then, when identifying $Q(\boldsymbol{\Delta})$ with $\mathcal{M}(\mathbf{U}_1) \times \cdots \times \mathcal{M}(\mathbf{U}_m)$ and correspondingly $\boldsymbol{\mu}$ with a point $(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_m)$, $\mathrm{Fix}(g)$ identifies with $C_1 \times \cdots \times C_k \times \mathcal{M}(\mathbf{U}_{k+1}) \times \cdots \times \mathcal{M}(\mathbf{U}_m)$, where C_i is a convex subset contained in the transversal tree $T_{\boldsymbol{\mu}_i}$.

Proof. This follows immediately from the fact that $\mathrm{Fix}(g) = \mathrm{Fix}(g(1)) \times \cdots \times \mathrm{Fix}(g(m))$, where $g(i)$ is the restriction of g to the subsurface $\mathbf{U}_i$, and from Lemma 6.13. □

Lemma 6.16. *Let g be a pure element with bounded orbits in $\mathcal{AM}$. Let $\boldsymbol{\mu}$ be a point in $\mathcal{AM}$ such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$ and let $\boldsymbol{m}$ be a midpoint of a $\widetilde{\mathrm{dist}}$-geodesic joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, where $\boldsymbol{m}$ is the middle fixed point fixed by g. Then, in the splitting of $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ given by Lemma 6.10, the set $\mathfrak{U}_1$ coincides with $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \setminus \mathfrak{P}$.*

Proof. As g has bounded orbits, we have that $\mathfrak{U}_0 = \emptyset$, according to the last part of the statement of Lemma 6.10.

Since $\boldsymbol{m}$ is on a geodesic joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \cup \mathfrak{U}(\boldsymbol{m}, g\boldsymbol{\mu})$. From the definition of $\mathfrak{U}_1$ it follows that $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \setminus \mathfrak{P}$ is contained in $\mathfrak{U}_1$. Also, if an element $\mathbf{U} \in \mathfrak{U}_1$ would be contained in $\mathfrak{U}(\boldsymbol{m}, g\boldsymbol{\mu})$ then it would follow that $g^{-1}\mathbf{U}$ is also in $\mathfrak{U}$, a contradiction. □

Notation: In what follows, for any reducible element $t \in \mathcal{MCG}(S)^{\omega}_{b}$ we denote by $\boldsymbol{\Delta}_t$ the multicurve associated to t as in Lemma 6.4.

Lemma 6.17. (1) *Let g be a pure element with $\mathrm{Fix}(g)$ non-empty. For every $\boldsymbol{x} \in \mathcal{AM}$ there exists a unique point $\boldsymbol{y} \in \mathrm{Fix}(g)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{x}, \boldsymbol{y}) = \widetilde{\mathrm{dist}}(\boldsymbol{x}, \mathrm{Fix}(g))$.*

(2) *Let g and h be two pure elements not fixing a common multicurve. If $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ are non-empty then there exists a unique pair of points $\boldsymbol{\mu} \in \mathrm{Fix}(g)$ and $\boldsymbol{\nu} \in \mathrm{Fix}(h)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\mathrm{Fix}(g), \mathrm{Fix}(h))$.*

Moreover, for every $\boldsymbol{\alpha} \in \mathrm{Fix}(g)$, $\widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \mathrm{Fix}(h))$, and $\boldsymbol{\nu}$ is the unique point with this property; likewise for every $\boldsymbol{\beta} \in \mathrm{Fix}(h)$, $\widetilde{\mathrm{dist}}(\boldsymbol{\beta}, \boldsymbol{\mu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\beta}, \mathrm{Fix}(g))$ and $\boldsymbol{\mu}$ is the unique point with this property.

Proof. We identify $\mathcal{AM}$ with a subset of the product of trees $\prod_{\mathbf{U} \in \Pi\mathcal{U}/\omega} T_{\mathbf{U}}$. Let g be a pure element with $\mathrm{Fix}(g)$ non-empty. By Lemma 6.7, for any $\mathbf{U}$ such that $g(\mathbf{U}) \neq \mathbf{U}$ we have that the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{U}}$ is a point which we denote

$\boldsymbol{\mu}_{\mathbf{U}}$. If $\mathbf{U}$ is such that $\mathbf{U} \pitchfork \boldsymbol{\Delta}_g$ then the projection of $\mathrm{Fix}(g)$, and indeed of $Q(\boldsymbol{\Delta}_g)$ onto $T_{\mathbf{U}}$ also reduces to a point, by Lemma 6.6, (1). The only surfaces $\mathbf{U}$ such that $g(\mathbf{U}) = \mathbf{U}$ and $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}_g$ are $\mathbf{U}_1, ..., \mathbf{U}_k$ and $\mathbf{Y} \subseteq \mathbf{U}_j$ with $j \in \{k+1, ..., m\}$, where $\mathbf{U}_1, ..., \mathbf{U}_m$ are the subsurfaces determined on $\mathbf{S}$ by $\boldsymbol{\Delta}_g$, g restricted to $\mathbf{U}_1, ..., \mathbf{U}_k$ is a pseudo-Anosov, g restricted to $\mathbf{U}_{k+1}, ..., \mathbf{U}_m$ is identity. By Lemma 6.15, the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{U}_i}$ is a convex tree $C_{\mathbf{U}_i}$, when $i = 1, ..., k$, and the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{Y}}$ with $\mathbf{Y} \subseteq \mathbf{U}_j$ and $j \in \{k+1, ..., m\}$ is $T_{\mathbf{Y}}$.

(1) The point $\boldsymbol{x}$ in $\mathcal{AM}$ is identified to the element $(\boldsymbol{x}_{\mathbf{U}})_{\mathbf{U}}$ in the product of trees $\prod_{\mathbf{U} \in \Pi\mathcal{U}/\omega} T_{\mathbf{U}}$.

For every $i \in \{1, ..., k\}$ we choose the unique point $\boldsymbol{y}_{\mathbf{U}_i}$ in the tree $C_{\mathbf{U}_i}$ realizing the distance from $\boldsymbol{x}_{\mathbf{U}_i}$ to that tree. The point $\boldsymbol{y}_{\mathbf{U}_i}$ lifts to a unique point $\boldsymbol{y}_i$ in the transversal sub-tree C_i.

Let $i \in \{k+1, ..., m\}$ and let $\boldsymbol{y}_{\mathbf{U}_i} = (\boldsymbol{x}_{\mathbf{Y}})_{\mathbf{Y}}$ be the projection of $(\boldsymbol{x}_{\mathbf{U}})_{\mathbf{U}}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$. Now the projection of $\mathcal{AM}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ coincides with the embedded image of $\mathcal{M}(\mathbf{U}_i)$, since for every $\boldsymbol{x} \in \mathcal{AM}$ its projection in $T_{\mathbf{Y}}$ coincides with the projection of $\pi_{\mathcal{M}(\mathbf{U}_i)}(\boldsymbol{x})$. Therefore there exists a unique element $\boldsymbol{y}_i \in \mathcal{M}(\mathbf{U}_i)$ such that its image in $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ is $\boldsymbol{y}_{\mathbf{U}_i}$. Note that the point $\boldsymbol{y}_i$ can also be found as the projection of $\boldsymbol{x}$ onto $\mathcal{M}(\mathbf{U}_i)$.

Let $\boldsymbol{z}$ be an arbitrary point in $\mathrm{Fix}(g)$. For every subsurface $\mathbf{U}$ the point $\boldsymbol{z}$ has the property that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{z}, \boldsymbol{x}) \geq \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{y}, \boldsymbol{x})$. Moreover if $\boldsymbol{z} \neq \boldsymbol{y}$ then there exist at least one subsurface $\mathbf{V}$ with $g(\mathbf{V}) = \mathbf{V}$ and $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}_g$ such that $\boldsymbol{z}_{\mathbf{V}} \neq \boldsymbol{y}_{\mathbf{V}}$. By the choice of $\boldsymbol{y}_{\mathbf{V}}$ it follows that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{z}_{\mathbf{V}}, \boldsymbol{x}_{\mathbf{V}}) > \widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{y}_{\mathbf{V}}, \boldsymbol{x}_{\mathbf{V}})$. Therefore $\widetilde{\mathrm{dist}}(\boldsymbol{z}, \boldsymbol{x}) \geq \widetilde{\mathrm{dist}}(\boldsymbol{y}, \boldsymbol{x})$, and the inequality is strict if $\boldsymbol{z} \neq \boldsymbol{y}$.

(2) Let $\mathbf{V}_1, ..., \mathbf{V}_s$ be the subsurfaces determined on $\mathbf{S}$ by $\boldsymbol{\Delta}_h$, such that h restricted to $\mathbf{V}_1, ..., \mathbf{V}_l$ is a pseudo-Anosov, h restricted to $\mathbf{V}_{l+1}, ..., \mathbf{V}_s$ is identity. The projection of $\mathrm{Fix}(h)$ onto $T_{\mathbf{V}_i}$ is a convex tree $C_{\mathbf{V}_i}$, when $i = 1, ..., l$, the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{Z}}$ with $\mathbf{Z} \subseteq \mathbf{V}_j$ and $j \in \{l+1, ..., s\}$ is $T_{\mathbf{Z}}$, and for any other subsurface $\mathbf{U}$ the projection of $\mathrm{Fix}(h)$ is one point $\boldsymbol{\nu}_{\mathbf{U}}$.

For every $i \in \{1, ..., k\}$ $\mathrm{Fix}(h)$ projects onto a point $\boldsymbol{\nu}_{\mathbf{U}_i}$ by the hypothesis that g, h do not fix a common multicurve (hence a common subsurface). Consider $\boldsymbol{\mu}_{\mathbf{U}_i}$ the nearest to $\boldsymbol{\nu}_{\mathbf{U}_i}$ point in the convex tree $C_{\mathbf{U}_i}$. This point lifts to a unique point $\boldsymbol{\mu}_i$ in the transversal sub-tree C_i. Let $i \in \{k+1, ..., m\}$. On $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ $\mathrm{Fix}(h)$ projects onto a unique point, since it has a unique projection in each $T_{\mathbf{Y}}$. As pointed out already in the proof of (1), the projection of $\mathcal{AM}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ coincides with the embedded image of $\mathcal{M}(\mathbf{U}_i)$. Therefore there exists a unique element $\boldsymbol{\mu}_i \in \mathcal{M}(\mathbf{U}_i)$ such that its image in $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ is $(\boldsymbol{\nu}_{\mathbf{Y}})_{\mathbf{Y}}$. Note that the point $\boldsymbol{\mu}_i$ can also be found as the unique point which is the projection of $\mathrm{Fix}(h)$ onto $\mathcal{M}(\mathbf{U}_i)$ for $i = k+1, ..., m$. We consider the point $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_m) \in C_1 \times \cdots \times C_k \times \mathcal{M}(\mathbf{U}_{k+1}) \times \cdots \times \mathcal{M}(\mathbf{U}_m)$. Let $\boldsymbol{\alpha}$ be an arbitrary point in $\mathrm{Fix}(g)$ and let $\boldsymbol{\beta}$ be an arbitrary point in $\mathrm{Fix}(h)$. For every subsurface $\mathbf{U}$ the point $\boldsymbol{\mu}$ has the property that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leq \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Moreover if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$ then there exist at least one subsurface $\mathbf{V}$ with $g(\mathbf{V}) = \mathbf{V}$ and $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}_g$ such that $\boldsymbol{\alpha}_{\mathbf{V}} \neq \boldsymbol{\mu}_{\mathbf{V}}$. By the choice of $\boldsymbol{\mu}_{\mathbf{V}}$ it follows that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}_{\mathbf{V}}, \boldsymbol{\beta}_{\mathbf{V}}) < \widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}_{\mathbf{V}}, \boldsymbol{\beta}_{\mathbf{V}})$. Therefore $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leq \widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and the inequality is strict if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$.

We construct similarly a point $\boldsymbol{\nu} \in \operatorname{Fix}(h)$. Then $\widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq \widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leq \widetilde{\operatorname{dist}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for any $\boldsymbol{\alpha} \in \operatorname{Fix}(g)$ and $\boldsymbol{\beta} \in \operatorname{Fix}(h)$. Moreover the first inequality is strict if $\boldsymbol{\beta} \neq \boldsymbol{\nu}$, and the second inequality is strict if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$. □

Lemma 6.18. *Let $g \in \mathcal{MCG}(S)^\omega_b$ be a pure element satisfying the hypotheses from Lemma 6.10, and moreover assume that g has bounded orbits, whence* $\operatorname{Fix}(g) \neq \emptyset$*, by Lemmas 6.12 and 6.14. Let $\boldsymbol{\mu}$ be an element such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$ and let $\boldsymbol{\nu}$ be the unique projection of $\boldsymbol{\mu}$ onto* $\operatorname{Fix}(g)$ *defined in Lemma 6.17, (1).*

Then for every $k \in \mathbb{Z} \setminus \{0\}$, $\boldsymbol{\nu}$ is on a geodesic joining $\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$.

Proof. By Lemmas 6.12 and 6.14 there exists $\boldsymbol{m}$ middle of a geodesic joining $\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$ such that $\boldsymbol{m} \in \operatorname{Fix}(g^k)$. By Lemma 6.9, $\operatorname{Fix}(g^k) = \operatorname{Fix}(g)$. Assume that $\boldsymbol{m} \neq \boldsymbol{\nu}$. Then by Lemma 6.17, (1), $\widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < \widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{m})$. Then $\widetilde{\operatorname{dist}}(\boldsymbol{\mu}, g^k\boldsymbol{\mu}) \leq \widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) + \widetilde{\operatorname{dist}}(\boldsymbol{\nu}, g^k\boldsymbol{\mu}) = 2\widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < 2\widetilde{\operatorname{dist}}(\boldsymbol{\mu}, \boldsymbol{m}) = \widetilde{\operatorname{dist}}(\boldsymbol{\mu}, g^k\boldsymbol{\mu})$, which is impossible. □

Lemma 6.19. *Let $g = (g_n)^\omega$ and $h = (h_n)^\omega$ be two pure reducible elements in $\mathcal{MCG}(S)^\omega_b$, such that they do not both fix a multicurve. If a proper subsurface $\mathbf{U}$ has the property that $h(\mathbf{U}) = \mathbf{U}$ then*

1. *$g^m\mathbf{U} \pitchfork \boldsymbol{\Delta}_h$ for $|m| \geq N = N(g)$;*
2. *the equality $h(g^k(\mathbf{U})) = g^k(\mathbf{U})$ can hold only for finitely many $k \in \mathbb{Z}$.*

Proof. (1) Assume by contradiction that $g^m\mathbf{U} \not\pitchfork \boldsymbol{\Delta}_h$ for $|m|$ large. Since $h(\mathbf{U}) = \mathbf{U}$ it follows that $\mathbf{U}$ must overlap a component $\mathbf{V}$ of $\mathbf{S} \setminus \boldsymbol{\Delta}_g$ on which g is a pseudo-Anosov (otherwise $g\mathbf{U} = \mathbf{U}$). If $\boldsymbol{\Delta}_h$ would also intersect $\mathbf{V}$ then the projections of $\Delta_{h,n}$ and of ∂U_n onto the curve complex $C(V_n)$ would be at distance $O(1)$. On the other hand, since $\operatorname{dist}_{C(V_n)}(g^m\partial U_n, \partial U_n) \geq |m| + O(1)$ it follows that for $|m|$ large enough $\operatorname{dist}_{C(V_n)}(g^m\partial U_n, \Delta_{h,n}) > 3$, that is $g^m\partial\mathbf{U}$ would intersect $\boldsymbol{\Delta}_h$, a contradiction. Thus $\boldsymbol{\Delta}_h$ does not intersect $\mathbf{V}$. It follows that $\mathbf{U}$ does not have all boundary components from $\boldsymbol{\Delta}_h$, thus the only possibility for $h(\mathbf{U}) = \mathbf{U}$ to be achieved is that $\mathbf{U}$ is a finite union of subsurfaces determined by $\boldsymbol{\Delta}_h$ and subsurfaces contained in a component of $\mathbf{S} \setminus \boldsymbol{\Delta}_h$ on which h is identity. Since $\mathbf{V}$ intersects $\mathbf{U}$ and not $\boldsymbol{\Delta}_h$, $\mathbf{V}$ intersects only a subsurface $\mathbf{U}_1 \subseteq \mathbf{U}$ restricted to which h is identity, and $\mathbf{V}$ is in the same component of $\mathbf{S} \setminus \boldsymbol{\Delta}_h$ as $\mathbf{U}_1$. Therefore $h\mathbf{V} = \mathbf{V}$, and we also had that $g\mathbf{V} = \mathbf{V}$, a contradiction.

(2) Assume that $h(g^k(\mathbf{U})) = g^k(\mathbf{U})$ holds for infinitely many $k \in \mathbb{Z}$. Without loss of generality we may assume that all k are positive integers and that for all k, $g^k\mathbf{U} \pitchfork \boldsymbol{\Delta}_h$. Up to taking a subsequence of k we may assume that there exist $\mathbf{U}_1, ..., \mathbf{U}_m$ subsurfaces determined by $\boldsymbol{\Delta}_h$ and $1 \leq r \leq m$ such that h restricted to $\mathbf{U}_1, ..., \mathbf{U}_r$ is either a pseudo-Anosov or identity, h restricted to $\mathbf{U}_{r+1}, ..., \mathbf{U}_m$ is identity, and $g^k(\mathbf{U}) = \mathbf{U}_1 \cup ... \cup \mathbf{U}_r \cup \mathbf{V}_{r+1}(k) \cup ... \cup \mathbf{V}_m(k)$, where $\mathbf{V}_j(k) \subsetneq \mathbf{U}_j$ for $j = r+1, ..., m$. The boundary of $g^k(\mathbf{U})$ decomposes as $\partial' S \sqcup \boldsymbol{\Delta}'_h \sqcup \partial_k$, where $\partial' S$ is the part of $\partial g^k(\mathbf{U})$ contained in ∂S, $\boldsymbol{\Delta}'_h$ is the part contained in $\boldsymbol{\Delta}_h$, and ∂_k is the remaining part (coming from the subsurfaces $\mathbf{V}_j(k)$). Up to taking a subsequence and pre-composing with some g^{-k_0} we may assume that $\mathbf{U} = \mathbf{U}_1 \cup ... \cup \mathbf{U}_r \cup \mathbf{V}_{r+1}(0) \cup ... \cup \mathbf{V}_m(0)$ and that g^k do not permute the boundary components. It follows that $\boldsymbol{\Delta}'_h = \emptyset$, hence $\partial_k \neq \emptyset$. Take a boundary curve $\gamma \in \partial_0$. Then $\gamma \in \partial\mathbf{V}_j(0)$ for some $j \in \{r+1, ..., m\}$, and for every k, $g^k\gamma \in \partial\mathbf{V}_j(k) \subset \mathbf{U}_j$, in particular $g^k\gamma \not\pitchfork \boldsymbol{\Delta}_h$. An argument as in (1) yields a contradiction. □

Lemma 6.20. *Let $g = (g_n)^\omega$ and $h = (h_n)^\omega$ be two pure elements in $\mathcal{MCG}(S)^\omega_b$, such that $\langle g, h\rangle$ is composed only of pure elements and its orbits in $\mathcal{AM}$ are bounded. Then g and h fix a point.*

Proof. **(1)** Assume that g and h do not fix a common multicurve. We argue by contradiction and assume that g and h do not fix a point and we shall deduce from this that $\langle g, h\rangle$ has unbounded orbits.

Since g and h do not fix a point, by Lemma 6.17, (2), Fix(g) and Fix(h) do not intersect, therefore the $\widetilde{\text{dist}}$-distance between them is $d > 0$. Let $\boldsymbol{\mu} \in \text{Fix}(g)$ and $\boldsymbol{\nu} \in \text{Fix}(h)$ be the unique pair of points realizing this distance d, according to Lemma 6.17. Possibly by replacing g and h by some powers we may assume that g, h and all their powers have the property that each pseudo-Anosov components has sufficiently large translation lengths in their respective curve complexes.

(1.a) We prove that for every $\boldsymbol{\alpha} \in \mathcal{AM}$ and every $\epsilon > 0$ there exists k such that $g^k(\boldsymbol{\alpha})$ projects onto Fix(h) at distance at most ϵ from $\boldsymbol{\nu}$.

Let $\boldsymbol{\mu}_1$ be the unique projection of $\boldsymbol{\alpha}$ on Fix(g), as defined in Lemma 6.17, (1). According to Lemma 6.18, $\boldsymbol{\mu}_1$ is on a geodesic joining $\boldsymbol{\alpha}$ and $p\boldsymbol{\alpha}$ for every $p \in \langle g\rangle \setminus \{id\}$. Let $\boldsymbol{\nu}_1$ be the unique point on Fix(h) that is nearest to $p(\boldsymbol{\alpha})$, whose existence is ensured by Lemma 6.17, (1).

By Lemma 6.10 $\mathfrak{U}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) = \mathfrak{U}_1^p \sqcup p\mathfrak{U}_1^p \sqcup \mathfrak{P}$. Moreover, by Lemma 6.16, $\mathfrak{U}_1^p = \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\mu}_1) \setminus \mathfrak{P}$, therefore $\mathfrak{U}_1^p$ is independent of the power p. Therefore we shall henceforth denote it simply by $\mathfrak{U}_1$.

Let $\mathbf{U}$ be a subsurface in $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$. If $\mathbf{U}$ is a pseudo-Anosov component of h then the projection of Fix(h) onto $T_\mathbf{U}$ is a subtree $C_\mathbf{U}$, the whole set Fix(g) projects onto a point $\boldsymbol{\mu}_\mathbf{U}$, and $\boldsymbol{\nu}_\mathbf{U}$ is the projection of $\boldsymbol{\mu}_\mathbf{U}$ onto $C_\mathbf{U}$, $(\boldsymbol{\nu}_1)_\mathbf{U}$ is the projection of $(p(\boldsymbol{\alpha}))_\mathbf{U}$ onto $C_\mathbf{U}$, and $\boldsymbol{\nu}_\mathbf{U}$, $(\boldsymbol{\nu}_1)_\mathbf{U}$ are distinct. It follows that the geodesic joining $(\boldsymbol{\mu}_1)_\mathbf{U}$ and $(p(\boldsymbol{\alpha}))_\mathbf{U}$ covers the geodesic joining $\boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U}$, whence $\widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \geq \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$.

If $\mathbf{U}$ is a subsurface of an identity component of h then the projection of Fix(h) onto $T_\mathbf{U}$ is the whole tree $T_\mathbf{U}$, Fix(g) projects onto a unique point $\boldsymbol{\mu}_\mathbf{U} = \boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U} = (p(\boldsymbol{\alpha}))_\mathbf{U}$. It follows that the geodesic joining $(\boldsymbol{\mu}_1)_\mathbf{U}$ and $(p(\boldsymbol{\alpha}))_\mathbf{U}$ is the same as the geodesic joining $\boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U}$, whence $\widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) = \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$.

Thus in both cases $\widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \geq \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) > 0$, in particular $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha}))$. Since g and h do not fix a common subsurface, $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) \cap \mathfrak{P} = \emptyset$, therefore $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) \subset \mathfrak{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \setminus \mathfrak{P} = p\mathfrak{U}_1$. The last equality holds by Lemma 6.16.

Now consider $\mathbf{V}_1, ..., \mathbf{V}_r$ subsurfaces in $\mathfrak{U}_1$ such that the sum

$$\sum_{j=1}^{r} \left(\widetilde{\text{dist}}_{\mathbf{V}_j}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) + \widetilde{\text{dist}}_{g\mathbf{V}_j}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha}))\right) + \sum_{\mathbf{U} \in \mathfrak{P}} \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})))$$

is at least $\widetilde{\text{dist}}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) - \epsilon$.

According to Lemma 6.19, (2), by taking p a large enough power of g we may ensure that $h(p(\mathbf{V}_j)) \neq p(\mathbf{V}_j)$ for every $j = 1, ..., r$. Then

$$\widetilde{\text{dist}}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) = \sum_{\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)} \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) \leq \sum_{\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)} \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \leq$$

$$\sum_{\mathbf{U} \in p\mathfrak{U}_1, \mathbf{U} \neq p\mathbf{V}_j} \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) = \sum_{\mathbf{U} \in p\mathfrak{U}_1, \mathbf{U} \neq p\mathbf{V}_j} \widetilde{\text{dist}}_\mathbf{U}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) \leq \epsilon\,.$$

(1.b) In a similar way we prove that for every $\boldsymbol{\beta} \in \mathcal{AM}$ and every $\delta > 0$ there exists m such that $h^m(\boldsymbol{\beta})$ projects onto $\mathrm{Fix}(g)$ at distance at most δ from $\boldsymbol{\mu}$.

(1.c) We now prove by induction on k that for every $\epsilon > 0$ there exists a word w in g and h such that:

- $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w\boldsymbol{\nu})$ is in the interval $[2kd - \epsilon, 2kd]$;
- $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w\boldsymbol{\nu})$ is in the interval $[(2k-1)d - \epsilon, (2k-1)d]$;
- $w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$.

This will show that the $\boldsymbol{\nu}$-orbit of $\langle g, h\rangle$ is unbounded, contradicting the hypothesis.

Take $k = 1$. Then (1.a) applied to $\boldsymbol{\nu}$ and ϵ implies that there exists a power p of g such that $p\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$. Note that by Lemma 6.18, $\boldsymbol{\mu}$ is the middle of a geodesic joining $\boldsymbol{\nu}, p\boldsymbol{\nu}$, hence $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, p\boldsymbol{\nu}) = 2d$ and $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, p\boldsymbol{\nu}) = d$.

Assume that the statement is true for k, and consider $\epsilon > 0$ arbitrary. The induction hypothesis applied to $\epsilon_1 = \frac{\epsilon}{16}$ produces a word w in g and h. Property (1.b) applied to $\boldsymbol{\beta} = w\boldsymbol{\nu}$ implies that there exists a power h^m such that $h^m w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(g)$ at distance at most $\delta = \frac{\epsilon}{4}$.

The distance $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu})$ is equal to $\widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\nu})$, hence it is in $[2kd - \epsilon_1, 2kd]$. The distance $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu})$ is at most $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu}) + d = (2k+1)d$. Also $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, w\boldsymbol{\nu}) - \widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\mu}) \geq 2(\widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\nu}) - \epsilon_1) - (2k-1)d \geq 2(2kd - 2\epsilon_1) - (2k-1)d = (2k+1)d - 4\epsilon_1 \geq (2k+1)d - \epsilon$.

We apply (1.a) to $\boldsymbol{\alpha} = h^m w\boldsymbol{\nu}$ and ϵ and obtain that for some k, $g^k h^m w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$. Take $w' = g^k h^m w$. We have $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w'\boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, h^m w\boldsymbol{\nu})$, and the latter is in $[(2k+1)d - \epsilon, (2k+1)d]$.

The distance $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w'\boldsymbol{\nu})$ is at most $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w'\boldsymbol{\nu}) + d$, hence at most $(2k+2)d$. Also $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w'\boldsymbol{\nu})$ is at least $\widetilde{\mathrm{dist}}(w'\boldsymbol{\nu}, h^m w\boldsymbol{\nu}) - \widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu}) \geq 2(\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu}) - \delta) - 2kd \geq 2((2k+1)d - 4\epsilon_1 - \delta) - 2kd = (2k+2)d - 8\epsilon_1 - 2\delta = (2k+2)d - \epsilon$.

(2) Let $\boldsymbol{\Delta}$ be a multicurve fixed by both g and h, and let $\mathbf{U}_1, \ldots \mathbf{U}_n$ be the subsurfaces determined by $\boldsymbol{\Delta}$. The restrictions of g and h to each $\mathbf{U}_i$, $g(i)$ and $h(i)$, do not fix any multicurve. By (1), $g(i)$ and $h(i)$ fix a point $\boldsymbol{\nu}_i$ in $\mathcal{M}(\mathbf{U}_i)$. It then follows that g and h fix the point $(\boldsymbol{\nu}_1, \ldots \boldsymbol{\nu}_n) \in \mathcal{M}(\mathbf{U}_1) \times \cdots \times \mathcal{M}(\mathbf{U}_n) = Q(\boldsymbol{\Delta})$. □

Lemma 6.21. *Let $g_1 = (g_n^1)^\omega, \ldots, g_m = (g_n^m)^\omega$ be pure elements in $\mathcal{MCG}(S)_b^\omega$, such that $\langle g_1, \ldots, g_m\rangle$ is composed only of pure elements and its orbits in $\mathcal{AM}$ are bounded. Then $g_1, \ldots, g_m$ fix a point in $\mathcal{AM}$.*

Proof. According to Lemma 3.13 it suffices to prove the following statement: if $g_1, \ldots, g_m$ are pure elements in $\mathcal{MCG}(S)_b^\omega$, such that $\langle g_1, \ldots, g_m\rangle$ is composed only of pure elements, its orbits in $\mathcal{AM}$ are bounded and it fixes set-wise a piece P then $g_1, \ldots, g_m$ fix a point in P. We prove this statement by induction on k. For $k = 1$ and $k = 2$ it follows from Lemma 6.20. Note that if an isometry of a tree-graded space fixes a point x and a piece P then it fixes the projection of x on P.

Assume by induction that the statement is true for k elements, and consider $g_1, \ldots, g_{k+1}$ pure elements in $\mathcal{MCG}(S)_b^\omega$, such that $\langle g_1, \ldots, g_{k+1}\rangle$ is composed only of pure elements, its orbits in $\mathcal{AM}$ are bounded and it fixes set-wise a piece P.

(1) Assume that $g_1, ..., g_{k+1}$ do not fix a common multicurve. By the induction hypothesis $g_1, ..., g_{k-2}, g_{k-1}, g_k$ fixes a point $\boldsymbol{\alpha} \in P$, $g_1, ..., g_{k-2}, g_{k-1}, g_{k+1}$ fixes a point $\boldsymbol{\beta} \in P$ and $g_1, ..., g_{k-2}, g_k, g_{k+1}$ fixes a point $\boldsymbol{\gamma} \in P$. If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are not pairwise distinct then we are done. Assume therefore that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise distinct, and let $\boldsymbol{\mu}$ be their unique median point. Since pieces are convex in tree-graded spaces, $\boldsymbol{\mu} \in P$. For $i \in \{1, ..., k-2\}$, g_i fixes each of the points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, hence it fixes their median point $\boldsymbol{\mu}$.

Assume that $g_{k-1}\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup \mathfrak{U}(\boldsymbol{\alpha}, g_{k-1}\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup g_{k-1}\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. Now $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \subset \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, and since g_{k-1} fixes both $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ it fixes every subsurface $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, by Lemma 6.7. In particular $g_{k-1}\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. Hence $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. A similar argument implies that $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\beta})$. Take $\mathbf{V} \in (\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu})$. Then $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. In particular $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, hence each g_i with $i = 1, 2, ..., k-1$ fixes $\mathbf{V}$, since it fixes the points $\boldsymbol{\beta}, \boldsymbol{\alpha}$. Also $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha})$, whence $g_k\mathbf{V} = \mathbf{V}$. Finally, as $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\beta}) \subset \mathfrak{U}(\boldsymbol{\gamma}, \boldsymbol{\beta})$ it follows that $g_{k+1}\mathbf{V} = \mathbf{V}$. This contradicts the hypothesis that $g_1, ..., g_{k+1}$ do not fix a common multicurve. Note that $\mathbf{V} \subsetneq \mathbf{S}$ by Lemma 6.5, since $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are in the same piece and $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

We conclude that $g_{k-1}\boldsymbol{\mu} = \boldsymbol{\mu}$. Similar arguments imply that $g_k\boldsymbol{\mu} = \boldsymbol{\mu}$ and $g_{k+1}\boldsymbol{\mu} = \boldsymbol{\mu}$.

(2) Assume that $g_1, ..., g_{k+1}$ fix a common multicurve. Let $\boldsymbol{\Delta}$ be this multicurve, and let $\mathbf{U}_1, ...\mathbf{U}_m$ be the subsurfaces determined by $\boldsymbol{\Delta}$. According to Lemma 6.4, $Q(\boldsymbol{\Delta}) \subset U(P)$.

The restrictions of $g_1, ..., g_{k+1}$ to each $\mathbf{U}_i$, $g_1(i), ..., g_{k+1}(i)$, do not fix any multicurve. By Lemma 3.13 either $g_1(i), ..., g_{k+1}(i)$ fix a point $\boldsymbol{\nu}_i$ in $\mathcal{M}(\mathbf{U}_i)$ or they fix set-wise a piece P_i in $\mathcal{M}(\mathbf{U}_i)$. In the latter case, by (1) we may conclude that $g_1(i), ..., g_{k+1}(i)$ fix a point $\boldsymbol{\nu}_i \in P_i$.

It then follows that $g_1, ..., g_{k+1}$ fix the point $(\boldsymbol{\nu}_1, ...\boldsymbol{\nu}_n) \in \mathcal{M}(\mathbf{U}_1) \times \cdots \times \mathcal{M}(\mathbf{U}_m) = Q(\boldsymbol{\Delta}) \subset U(P)$. □

We are now ready to finish the proof of Theorem 6.2.

We argue by induction on the complexity of S. When $\xi(S) \leq 1$ we have that $\mathcal{AM}$ is a complete real tree and Λ acts non-trivially on it, contradicting the fact that Λ has property (T). Assume that we proved the theorem for surfaces with complexity at most k, and assume that $\xi(S) = k+1$.

By Lemma 6.21, Λ_p fixes a point $\boldsymbol{\alpha}$ in $\mathcal{AM}$. Since it also fixes set-wise the piece P, it fixes the unique projection of $\boldsymbol{\alpha}$ to P. Denote this projection by $\boldsymbol{\mu}$.

Since Λ acts on $\mathcal{AM}$ without fixed point it follows that there exists $g \in \Lambda$ such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then $\Lambda_p = g\Lambda_p g^{-1}$ also fixes $g\boldsymbol{\mu}$. Lemma 6.7 implies that Λ_p fixes a subsurface $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$. Since $\boldsymbol{\mu}, g\boldsymbol{\mu}$ are in the piece P, it follows that $\mathbf{U}$ is a proper subsurface of $\mathbf{S}$. Thus Λ_p must fix a multicurve $\partial\mathbf{U}$.

Let $\boldsymbol{\Delta}$ be a maximal multicurve fixed by Λ_p. Assume there exists $g \in \Lambda$ such that $g\boldsymbol{\Delta} \neq \boldsymbol{\Delta}$. Then $\Lambda_p = g\Lambda_p g^{-1}$ also fixes $g\boldsymbol{\Delta}$, contradicting the maximality of $\boldsymbol{\Delta}$. We then conclude that all Λ fixes $\boldsymbol{\Delta}$. It follows that the image of ϕ_ω is in $\mathrm{Stab}(\boldsymbol{\Delta})$, hence ω-almost surely $\phi_n(\Lambda) \subset \mathrm{Stab}(\Delta_n)$. Up to taking a subsequence and conjugating we may assume that $\phi_n(\Lambda) \subset \mathrm{Stab}(\Delta)$ for some fixed multicurve Δ. Let $U_1, ..., U_m$ be the subsurfaces and annuli determined by Δ. Then $\mathrm{Stab}(\Delta)$ is isomorphic to $\mathcal{MCG}(U_1) \times \cdots \times \mathcal{MCG}(U_m)$. Thus we can see ϕ_n as isomorphisms with target $\mathcal{MCG}(U_1) \times \cdots \times \mathcal{MCG}(U_m)$. The inductive hypothesis implies that there are finitely many possibilities for ϕ_n, up to conjugation, contradicting our hypothesis. □

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