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MELLIN TRANSFORM FOR BOEHMIANS

BY

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Abstract

A suitable Boehmian space is constructed to extend the distributional Mellin transform. Mellin transform of a Boehmian is defined as a quotient of analytic functions. We prove that the generalized Mellin transform has all its usual properties. We also discuss the relation between the Mellin transform and the Laplace transform in the context of Boehmians.

1. Introduction

J. Mikusiński and P. Mikusiński [8] introduced Boehmians as a generalization of distributions by the motivation of regular operators [2]. An abstract construction of Boehmian space was given in [9] with two notions of convergence. Thereafter various Boehmian spaces have been defined and also various integral transforms have been extended on them. See [6, 10, 11, 13, 16, 18, 19, 21]. The main objective of introducing an integral transform to the context of Boehmians is to find a Boehmian space which is suitable for defining the integral transform and it is properly larger than the space of distributions where the particular integral transform has been already discussed. If we construct Boehmian space by using distribution space as the *top space* (which contains the numerator sequences) of a Boehmian space, then obviously the Boehmian space contains the distribution space. In certain cases, such Boehmian spaces are even generalizing some other Boehmian spaces. For example, the Boehmian spaces introduced in [5, 6]

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are properly larger than $\mathscr{L}^p$-Boehmians [4] and tempered Boehmians [12] respectively.

In [1, 20], Mellin transform has been discussed on tempered Boehmians and Mellin transformable C^∞-Boehmians respectively. In both papers, the definitions of Mellin transform are erroneous and the Mellin transform on $\mathscr{M}'_{a,b}$ is not at all discussed. The details are given in the last section.

Since the distributional Mellin transform is defined on the space $\mathscr{M}'_{a,b}$, we construct a suitable Boehmian space $\mathscr{B}_{\mathcal{M}}$ which properly contains $\mathscr{M}'_{a,b}$. To discuss the operational properties of the Mellin transform, we also define various operations on the Boehmian space such as multiplication of a Boehmian by a function of the form $(\log x)^k$, multiplication of a Boehmian by a polynomial, derivative of a Boehmian, translation of a Boehmian, change of scale of a Boehmian and change of variable of a Boehmian by the function x^ρ.

We define the Mellin transform of a Boehmian as a quotient of analytic functions, satisfying all the expected operational properties. We also provide the identification between $\mathscr{B}_{\mathcal{M}}$ and the Laplace transformable Boehmian space $\mathscr{B}_{\mathcal{L}}$ [13] and establish the relation between the Mellin transform and the Laplace transform in the context of Boehmian spaces. Finally, we show that $\mathscr{B}_{\mathcal{M}}$ is properly larger than $\mathscr{M}'_{a,b}$.

2. Preliminaries

Let $\mathscr{M}_{a,b}$ be the space of all smooth functions on $I = (0, \infty)$ satisfying

$$\gamma_k(\phi) = \sup\{\zeta_{a,b}(x)x^{k+1}|\phi^{(k)}(x)| : x \in I\} < \infty, k = 0, 1, 2, \ldots \tag{2.1}$$

where

$$\zeta_{a,b}(x) = \begin{cases} x^{-a} & \text{if } 0 < x \leq 1 \\ x^{-b} & \text{if } 1 < x < \infty \end{cases}$$

and a, b are fixed with $0 < a < b < \infty$. The space $\mathscr{M}_{a,b}$ is a Fréchet space with the sequence $(\gamma_k)_{k\in\mathbb{N}_0}$ of semi-norms, where $\mathbb{N}_0$ is the set of all non-negative integers. The dual space $\mathscr{M}'_{a,b}$ of $\mathscr{M}_{a,b}$ is equipped with the weak* topology. We say that $f_n \to f$ as $n \to \infty$ in $\mathscr{M}'_{a,b}$ if $\langle f_n(x), \phi(x)\rangle - \langle f(x), \phi(x)\rangle \to 0$ as $n \to \infty$ for each $\phi \in \mathscr{M}_{a,b}$. A locally integrable function

f on I, with $f(x)/(x\zeta_{a,b}(x))$ is integrable on I, can be identified as a member of $\mathscr{M}'_{a,b}$ by the map

$$\phi \mapsto \int_0^\infty f(x)\phi(x)dx \tag{2.2}$$

For instance, every Schwartz testing function $f \in \mathscr{D}(I)$ can be identified as a regular member of $\mathscr{M}'_{a,b}$. Mellin transform of $f \in \mathscr{M}'_{a,b}$ is defined by

$$(Mf)(s) = \langle f(x), x^{s-1}\rangle,\ \forall s \in \Omega_{a,b} \tag{2.3}$$

where $\Omega_{a,b}$ is the strip $\{s \in \mathbb{C} : a < \text{Re } s < b\}$. It is well known that Mf is an analytic function with a polynomial growth. See [24, p.108]. A Mellin type convolution on $\mathscr{M}'_{a,b}$ is defined by

$$\langle (f \vee g)(x), \phi(x)\rangle = \langle f(x), \langle g(y), \phi(xy)\rangle\rangle, \forall \phi \in \mathscr{M}_{a,b}. \tag{2.4}$$

This convolution is a commutative binary operation on $\mathscr{M}'_{a,b}$. When $f, g \in \mathscr{M}'_{a,b}$ are regular functions or $f \in \mathscr{M}'_{a,b}$ and $g \in \mathscr{D}(I)$, $f \vee g$ is also regular [24, p.118]. In deed,

$$(f \vee g)(x) = \begin{cases} \int_0^\infty f\left(\frac{x}{y}\right) g(y)\frac{1}{y}\, dy, \forall x \in I & \text{if } f \text{ and } g \text{ are regular.} \\ \left\langle f(y), \frac{1}{y} g\left(\frac{x}{y}\right)\right\rangle, \forall x \in I & \text{if } f \in \mathscr{M}'_{a,b},\ g \in \mathscr{D}(I). \end{cases} \tag{2.5}$$

Theorem 2.1. (Inversion) *Let $F(s) = Mf$ for $s \in \Omega_f$. Then, in the sense of convergence in $\mathscr{D}'(I)$,*

$$f(x) = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} F(s)x^{-s}ds,$$

where $a < \sigma < b$.

Definition 2.2. Let $f \in \mathscr{M}'_{a,b}$, $k \in \mathbb{N}_0$, $\alpha = u + iv \in \mathbb{C}$, $r > 0$ and $\rho \in \mathbb{R} \setminus \{0\}$. We define

(1) $L^k : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\langle (L^k f)(x), \phi(x)\rangle = \langle (\log x)^k f(x), \phi(x)\rangle = \langle f(x), (\log x)^k \phi(x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.$$

(2) $P_\alpha : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a-u,b-u}$ by
$$\langle (P_\alpha f)(x), \psi(x)\rangle = \langle x^\alpha f(x), \psi(x)\rangle = \langle f(x), x^\alpha \psi(x)\rangle,\ \forall \psi \in \mathscr{M}_{a-u,b-u}.$$

(3) $D^k : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a+k,b+k}$ by
$$\begin{aligned}\langle (D^k f)(x), \chi(x)\rangle &= \langle f^{(k)}(x), \chi(x)\rangle \\ &= (-1)^k \langle f(x), (D_x^k \chi)(x)\rangle,\ \forall \chi \in \mathscr{M}_{a+k,b+k}.\end{aligned}$$
(4) $(DP_1)^k : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\begin{aligned}\langle ((DP_1)^k f)(x), \phi(x)\rangle &= \langle (D_x x)^k f(x), \phi(x)\rangle \\ &= (-1)^k \langle f(x), ((xD_x)^k \phi)(x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.\end{aligned}$$
(5) $(P_1 D)^k : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\begin{aligned}\langle ((P_1 D)^k f)(x), \phi(x)\rangle &= \langle (xD_x)^k f(x), \phi(x)\rangle \\ &= (-1)^k \langle f(x), ((D_x x)^k \phi)(x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.\end{aligned}$$
(6) $(D^k P_1^k) : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\begin{aligned}\langle ((D^k P_1^k) f)(x), \phi(x)\rangle &= \langle (D_x^k x^k) f(x), \phi(x)\rangle \\ &= (-1)^k \langle f(x), ((x^k D_x^k \phi)(x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.\end{aligned}$$
(7) $(P_1^k D^k) : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\begin{aligned}\langle ((P_1^k D^k) f)(x), \phi(x)\rangle &= \langle (x^k D_x^k) f(x), \phi(x)\rangle \\ &= (-1)^k \langle f(x), ((D_x^k x^k)\phi)(x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.\end{aligned}$$
(8) $M_r : \mathscr{M}'_{a,b} \to \mathscr{M}'_{a,b}$ by
$$\langle (M_r f)(x), \phi(x)\rangle = \langle f(rx), \phi(x)\rangle = \langle f(x), r^{-1}\phi(r^{-1}x)\rangle,\ \forall \phi \in \mathscr{M}_{a,b}.$$
(9) $E_\rho : \mathscr{M}'_{a,b} \to \mathscr{M}'_{c,d}$ by
$$\langle (E_\rho f)(x), \phi(x)\rangle = \langle f(x^\rho), \phi(x)\rangle = \langle f(x), |\rho|^{-1} x^{(1-\rho)/\rho} \phi(x^{-\rho})\rangle,\ \forall \phi \in \mathscr{M}_{c,d},$$
where $c = \rho a,\ d = \rho b$ if $\rho > 0$, $c = \rho b,\ d = \rho a$ if $\rho < 0$.

Theorem 2.3. *Let $f \in \mathscr{M}'_{a,b}$, $\alpha = u + iv \in \mathbb{C}$, $r > 0$, $k \in \mathbb{N}_0$ and $\rho \in \mathbb{R} \setminus \{0\}$ and $F(s)$ be the Mellin transform of f. Then*

(1) $M(L^k f) = D_s^k F(s)$.

(2) $M(P_\alpha f) = F(s + \alpha)$.

(3) $M(D^k f) = (-1)^k (s-k)(s-k+1)\cdots(s-1)F(s-k)$.

(4) $M((DP_1)^k f) = (-1)^k (s-1)^k F(s)$.

(5) $M((P_1 D)^k f) = (-1)^k s^k F(s)$.

(6) $M(D^k P_1^k f) = (-1)^k (s-k)(s-k+1)\cdots(s-1)F(s)$.

(7) $M(P_1^k D^k f) = (-1)^k s(s+1)\cdots(s+k-1)F(s)$.

(8) $M(M_r f) = r^{-s} F(s)$.

(9) $M(E_\rho f) = |\rho|^{-1} F(\rho^{-1} s)$.

Remark 2.4. In the literature, the continuity of the distributional Mellin transform is not discussed.

By introducing a topology on the range of Mellin transform as

$$U \textit{ is open in } M(\mathscr{M}'_{a,b}) \textit{ if } M^{-1}(U) \textit{ is open in } \mathscr{M}'_{a,b},$$

we can get that the Mellin transform becomes a homeomorphism between $\mathscr{M}'_{a,b}$ and $M(\mathscr{M}'_{a,b})$. A similar technique is followed in the context of Hilbert transform on the Schwartz testing function space $\mathscr{D}$. See [14, p.114].

The Mellin transform is closely related with the Laplace transform as follows:
Let $\mathscr{L}_{a,b}$ be the space of all smooth functions ϕ on $(-\infty,\infty)$ with

$$\gamma_k(\phi) = \sup_{-\infty<t<\infty} |\kappa_{a,b}(t) D^k \phi| < \infty,\ k = 0, 1, 2\dots$$

where

$$\kappa_{a,b} = \begin{cases} \mathrm{e}^{at} & 0 \le t < \infty \\ \mathrm{e}^{bt} & -\infty < t < 0 \end{cases}$$

and its dual is denoted by $\mathscr{L}'_{a,b}$.

The distributional Laplace transform on $\mathscr{L}'_{a,b}$ is defined by

$$(\mathcal{L}f)(s) = \langle f(t), \mathrm{e}^{-st}\rangle,\ s \in \Omega_{a,b}.$$

The convolution on $\mathscr{L}'_{a,b}$ is defined as follows.

Definition 2.5. For $g_1, g_2 \in \mathscr{L}'_{a,b}$,

$$\langle (g_1 * g_2)(t), \phi(t)\rangle = \langle g_1(t), \langle g_2(\tau), \phi(t+\tau)\rangle\rangle,\ \phi \in \mathscr{L}_{a,b}.$$

A locally integrable function g such that $g/\kappa_{a,b}$ is absolutely integrable on $(-\infty,\infty)$ can be identified as a regular member of $\mathscr{L}'_{a,b}$. When g_1 and g_2 are regular functions then $g_1 * g_2$ also is regular, which is given by

$$x \mapsto \int_{-\infty}^{\infty} g_1(x-y) g_2(y)\, dy,\ \forall x \in (-\infty,\infty).$$

The following theorems are proved in [24, §4.2 and §4.3].

Theorem 2.6. *The mapping*

$$\theta(x) \mapsto \mathrm{e}^{-t}\theta(\mathrm{e}^{-t}) = \phi(t)$$

is a continuous isomorphism $\mathcal{J}$ *from* $\mathscr{M}_{a,b}$ *onto* $\mathscr{L}_{a,b}$*. The inverse mapping is given by*

$$\phi(t) \mapsto x^{-1}\phi(-\log x) = \theta(x).$$

Theorem 2.7. *If*

$$\langle f(\mathrm{e}^{-t}), \phi(t)\rangle \;=\; \langle f(x), (\mathcal{J}^{-1}\phi)(x)\rangle,\ \forall \phi \in \mathscr{L}_{a,b}$$

and

$$\langle g(-\log x), \theta(x)\rangle \;=\; \langle g(t), (\mathcal{J}\theta)(t)\rangle,\ \forall \theta \in \mathscr{M}_{a,b}$$

then the mapping $f(x) \mapsto f(\mathrm{e}^{-t})$ *is an isomorphism* $\mathcal{I}$ *from* $\mathscr{M}'_{a,b}$ *onto* $\mathscr{L}'_{a,b}$*. The inverse mapping is given by* $g(t) \mapsto g(-\log x)$*.*

Theorem 2.8. *If* $f \in \mathscr{M}'_{a,b}$ *then* $Mf = L(\mathcal{I}f)$*, where* L *is the Laplace transform on* $\mathscr{L}'_{a,b}$*.*

For more details on the distributional Mellin transform we refer the reader to [3, 23], [24, Chapter 4], [15, Chapter 7].

3. Auxiliary Results

First we recall the abstract construction of a Boehmian space. To construct a Boehmian space we need $G, S, \star$ and Δ where G is a topological vector space, S is a subset of G and $\star : G \times S \to G$ satisfying the following conditions.

Let $\alpha, \beta \in G$ and $\zeta, \xi \in S$ be arbitrary.

1. $\zeta \star \xi = \xi \star \zeta \in S$; 2. $(\alpha \star \zeta) \star \xi = \alpha \star (\zeta \star \xi)$; 3. $(\alpha + \beta) \star \zeta = \alpha \star \zeta + \beta \star \zeta$;
4. If $\alpha_n \to \alpha$ as$n \to \infty$ in G and $\xi \in S$ then $\alpha_n \star \xi \to \alpha \star \xi$ as $n \to \infty$,

and Δ is a collection of sequences from S satisfying

(a) If $(\xi_n), (\zeta_n) \in \Delta$ then $(\xi_n \star \zeta_n) \in \Delta$.
(b) If $\alpha \in G$ and $(\xi_n) \in \Delta$ then $\alpha \star \xi_n \to \alpha$ in G as $n \to \infty$.

Let $\mathscr{A}$ denote the collection of all pairs of sequences $((\alpha_n), (\xi_n))$ where $\alpha_n \in G$, $\forall n \in \mathbb{N}$ and $(\xi_n) \in \Delta$ satisfying the property

$$\alpha_n \star \xi_m = \alpha_m \star \xi_n,\ \forall\ \ m, n \in \mathbb{N}. \tag{3.1}$$

Each element of $\mathscr{A}$ is called a quotient and is denoted by α_n/ξ_n. Define a relation $\sim$ on $\mathscr{A}$ by

$$\alpha_n/\xi_n \sim \beta_n/\zeta_n \quad \text{if} \quad \alpha_n \star \zeta_m = \beta_m \star \xi_n, \ \forall \ \ m, n \in \mathbb{N}. \tag{3.2}$$

It is easy to verify that $\sim$ is an equivalence relation on $\mathscr{A}$ and hence it decomposes $\mathscr{A}$ into disjoint equivalence classes. Each equivalence class is called a Boehmian and is denoted by $[\alpha_n/\xi_n]$. The collection of all Boehmians is denoted by $\mathscr{B} = \mathscr{B}(G, S, \star, \Delta)$. Every element α of G is identified uniquely as a member of $\mathscr{B}$ by $[(\alpha \star \xi_n)/\xi_n]$ where $(\xi_n) \in \Delta$ is arbitrary.

$\mathscr{B}$ is a vector space with addition and scalar multiplication defined as follows.

- $[\alpha_n/\xi_n] + [\beta_n/\zeta_n] = [(\alpha_n \star \zeta_n + \beta_n \star \xi_n)/(\xi_n \star \zeta_n)]$.
- $c\,[\alpha_n/\xi_n] = [(c\alpha_n)/\xi_n]$.

The operation $\star$ can be extended to $\mathscr{B} \times S$ by the following definition.

Definition 3.1. If $x = [\alpha_n/\xi_n] \in \mathscr{B}$, and $\zeta \in S$ then $x \star \zeta = [(\alpha_n \star \zeta)/\xi_n]$.

Now we recall the δ-convergence on $\mathscr{B}$.

Definition 3.2. [δ-Convergence] We say that $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}$ if there exists a delta sequence (ξ_n) such that $X_n \star \xi_k \in G$, $\forall n, k \in \mathbb{N}$, $X \star \xi_k \in G, \forall k \in \mathbb{N}$ and for each $k \in \mathbb{N}$,

$$X_n \star \xi_k \to X \star \xi_k \text{ as } n \to \infty \text{ in } G.$$

The following lemma states an equivalent statement for δ-convergence.

Lemma 3.3. $X_n \xrightarrow{\delta} X$ as $n \to \infty$ if and only if there exist $\alpha_{n,k}, \alpha_k \in G$ and $(\xi_k) \in \Delta$ such that $X_n = [\alpha_{n,k}/\xi_k]$, $X = [\alpha_k/\xi_k]$ and for each $k \in \mathbb{N}$,

$$\alpha_{n,k} \to \alpha_k \text{ as } n \to \infty \text{ in } G.$$

Now we state and prove some auxiliary results to construct the Boehmian space $\mathscr{B}_{\mathcal{M}} = (\mathscr{M}'_{a,b}, \mathscr{D}(I), \vee, \Delta_1)$. The following lemma is proved in [24, p.119].

Lemma 3.4. *If $f, g \in \mathscr{M}'_{a,b}$ then $M(f \vee g)(s) = (Mf)(s) \cdot (Mg)(s)$, $\forall s \in \Omega_{a,b}$.*

Lemma 3.5. *If $f, g \in \mathscr{M}'_{a,b}$, $\eta, \theta \in \mathscr{D}(I)$ and $\alpha \in \mathbb{C}$ then*

(1) $(f+g) \vee \eta = (f \vee \eta) + (g \vee \eta)$.

(2) $(\alpha f) \vee \eta = \alpha(f \vee \eta)$.

(3) $f \vee g = g \vee f$.

(4) $f \vee (\eta \vee \theta) = (f \vee \eta) \vee \theta$.

(5) $\eta \vee \theta \in \mathscr{D}(I)$.

Proof. The conclusions (1) and (2) are straight forward. The commutativity and associativity of $\vee$ are consequences of the Lemma 3.4 and the fact that Mellin transform on $\mathscr{M}'_{a,b}$ is an injection [23, Theorem 14]. Hence we prove only (5). It is easy to verify that the map $\mathcal{I} : \mathscr{D}(I) \to \mathscr{D}(\mathbb{R})$ given by $\theta(t) \mapsto \theta(\mathrm{e}^{-t})$ is a bijection. We also note that

$$\begin{aligned}
\mathcal{I}(\eta \vee \theta)(t) &= (\eta \vee \theta)(\mathrm{e}^{-t}) \\
&= \int_0^\infty \eta\left(\frac{\mathrm{e}^{-t}}{y}\right) \theta(y) \frac{1}{y}\, dy \\
&= \int_{-\infty}^\infty \eta(\mathrm{e}^{u-t}) \theta(\mathrm{e}^{-u}) du \text{ by putting } y = \mathrm{e}^{-u} \\
&= \int_{-\infty}^\infty (\mathcal{I}\eta)(t-u)(\mathcal{I}\theta)(u)\, du \\
&= ((\mathcal{I}\eta) * (\mathcal{I}\theta))(t),
\end{aligned}$$

Since $(\mathcal{I}\eta) * (\mathcal{I}\theta) \in \mathscr{D}(\mathbb{R})$, we get that $\eta \vee \theta \in \mathscr{D}(I)$. □

Lemma 3.6. *If $f_n \to f$ as $n \to \infty$ in $\mathscr{M}'_{a,b}$ and $\eta \in \mathscr{D}(I)$ then $f_n \vee \eta \to f \vee \eta$ as $n \to \infty$ in $\mathscr{M}'_{a,b}$.*

Proof. Let $\phi \in \mathscr{M}_{a,b}$ be arbitrary. Now

$$\begin{aligned}
&\langle (f_n \vee \eta)(x), \phi(x) \rangle - \langle (f \vee \eta)(x), \phi(x) \rangle \\
&\quad = \langle f_n(x), \langle \eta(y), \phi(xy) \rangle \rangle - \langle f(x), \langle \eta(y), \phi(xy) \rangle \rangle
\end{aligned}$$

$$= \langle (f_n - f)(x), \langle \eta(y), \phi(xy)\rangle\rangle \to 0 \text{ as } n \to \infty,$$

since the function $x \mapsto \langle \eta(y), \phi(xy)\rangle$ is a member of $\mathscr{M}_{a,b}$. □

Definition 3.7. A sequence (η_n) from $\mathscr{D}(I)$ is said to be a δ_1-sequence if it satisfies the following conditions:

(i) $\int_0^\infty \eta_n(x)dx = 1$, $\forall \in \mathbb{N}$.

(ii) $\int_0^\infty |\eta_n(x)|dx \le J$, $\forall \in \mathbb{N}$ for some $J > 0$.

(iii) Support of $\eta_n \subset (\alpha_n, \beta_n)$, $\forall n \in \mathbb{N}$ where $\alpha_n \to 1$ and $\beta_n \to 1$ as $n \to \infty$.

We denote the collection of all δ_1-sequences as Δ_1.

Lemma 3.8. *If $f \in \mathscr{M}'_{a,b}$ and $(\eta_n) \in \Delta_1$ then $f \vee \eta_n \to f$ as $n \to \infty$ in $\mathscr{M}'_{a,b}$.*

Proof. Let $\phi \in \mathscr{M}_{a,b}$ be arbitrary. Since

$$\langle (f \vee \eta_n)(x), \phi(x)\rangle - \langle f(x), \phi(x)\rangle = \langle f(x), \langle \eta_n(y), \phi(xy)\rangle\rangle - \langle f(x), \phi(x)\rangle,$$

to prove this lemma we shall show that

$$\langle \eta_n(y), \phi(xy)\rangle \to \phi(x) \text{ in } \mathscr{M}_{a,b} \text{ as } n \to \infty. \tag{3.3}$$

Let $\psi_n(x) = \langle \eta_n(y), \phi(xy)\rangle$. First we claim that

$$\psi_n^{(k)}(x) = \langle \eta_n(y), y^k \phi^{(k)}(xy)\rangle, \forall k \in \mathbb{N}. \tag{3.4}$$

We choose $\epsilon > 0$ and $J > 0$ such that support of $\eta_n \subset (\epsilon, J)$. Now for a small $r > 0$ and $h \in (-r, r)$,

$$\frac{\psi_n(x+h) - \psi_n(x)}{h} = \int_\epsilon^J \eta_n(y) \frac{\phi((x+h)y) - \phi(xy)}{h} dy. \tag{3.5}$$

We know that for each $y \in (\epsilon, J)$, $\eta_n(y)\frac{\phi((x+h)y)-\phi(xy)}{h} \to \eta_n(y) y \phi'(xy)$ as $h \to 0$. Using mean-value theorem, we get that

$$\left|\frac{\phi((x+h)y) - \phi(xy)}{h}\right| |\eta_n(y)| \le |y||\phi'(t)||\eta_n(y)| \tag{3.6}$$

where t lies between $(x+h)y$ and xy. As x is fixed, y varies in a compact set, and $h \in (-r, r)$, t varies at the most in a compact set B. Hence the expression on the right hand side of the inequality (3.6) is dominated by the constant

$$J \sup\{|\phi'(t) : t \in B\} \sup\{|\eta_n(y) : \epsilon \leq y \leq J\}$$

Thus by dominated convergence theorem we get that

$$\int_\epsilon^J \eta_n(y) \frac{\phi((x+h)y) - \phi(xy)}{h} dy \to \int_\epsilon^J \eta_n(y) y \phi'(xy) dy \text{ as } h \to 0.$$

Hence our claim holds for $k = 1$ and by induction it holds for each $k \in \mathbb{N}$.

Now by the property (i) of (η_n), we get for $x \in I$,

$$\begin{aligned}
&\zeta_{a,b}(x) x^{k+1} |(\psi_n - \phi)^{(k)}(x)| \\
&= \zeta_{a,b}(x) x^{k+1} \left| \int_0^\infty \eta_n(y) y^k \phi^{(k)}(xy) dy - \int_0^\infty \eta_n(y) \phi^{(k)}(x) dy \right| \\
&\leq \zeta_{a,b}(x) x^{k+1} \int_0^\infty \left| \eta_n(y) \left(y^k \phi^{(k)}(xy) - \phi^{(k)}(x) \right) \right| dy \\
&\leq I_1 + I_2
\end{aligned}$$

where $I_1 = \zeta_{a,b}(x) x^{k+1} \int_0^\infty \left| \eta_n(y) y^k \left(\phi^{(k)}(xy) - \phi^{(k)}(x) \right) \right| dy$ and $I_2 = \zeta_{a,b}(x) x^{k+1} \int_0^\infty \left| \eta_n(y) \left(y^k - 1 \right) \phi^{(k)}(x) \right| dy$. Let support of $\eta_n \subset (\alpha_n, \beta_n)$, where $\alpha_n \to 1, \beta_n \to 1$ as $n \to \infty$. It is easy to verify that if $\alpha_n \leq y \leq \beta_n$ and if $J_{n,k} = \max\{|\alpha_n^k - 1|, |\beta_n^k - 1|\}$, then

$$|y^k - 1| \leq J_{n,k} \text{ for all } n \in \mathbb{N},\ k \in \mathbb{N}_0 \text{ and } J_{n,k} \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Choose $m \in \mathbb{N}$ such that $J_{n,1} < 1$, $\forall n \geq m$. Now for $n \geq m$, using mean value theorem we get

$$\begin{aligned}
I_1 &= \int_{\alpha_n}^{\beta_n} \zeta_{a,b}(x) x^{k+1} \left| \eta_n(y) y^k \left(\phi^{(k)}(xy) - \phi^{(k)}(x) \right) \right| dy \\
&\leq \int_{\alpha_n}^{\beta_n} \zeta_{a,b}(x) x^{k+2} |y-1| y^k |\eta_n(y) \phi^{(k+1)}(x + t(xy - x))| dy \quad \text{for some } t \in (0,1) \\
&\leq \int_{\alpha_n}^{\beta_n} \zeta_{a,b}(x) |y-1| y^k |\eta_n(y)| \gamma_{k+1}(\phi) \left[(1+t(y-1))^{(k+2)} \zeta_{a,b}(x(1+t(y-1))) \right]^{-1} dy
\end{aligned}$$

Since $|y-1| \leq J_{n,1}$, $y^k \leq \beta_n^k$ and $(1+t(y-1))^{-(k+2)} \leq (1-J_{n,1})^{-(k+2)}$, $\forall n \geq m$, the last integral is dominated by

$$\beta_n^k J_{n,1}(1-J_{n,1})^{-(k+2)}\gamma_{k+1}(\phi)\int_{\alpha_n}^{\beta_n} |\eta_n(y)|\zeta_{a,b}(x)\left[\zeta_{a,b}(x(1+t(y-1)))\right]^{-1} dy \tag{3.8}$$

Using the inequality

$$\zeta_{a,b}(x)\left[\zeta_{a,b}(x(1+t(y-1)))\right]^{-1} \leq \begin{cases} (1+J_{n,1})^a & \text{if } 0 < x(1+t(y-1)) \leq 1, x \in I \\ (1+J_{n,1})^b & \text{if } x(1+t(y-1)) > 1, x \in I, \end{cases}$$

and the property (ii) of $(\eta_n) \in \Delta_1$, the expression (3.8) is dominated by

$$\beta_n^k J_{n,1}(1-J_{n,1})^{-(k+2)}\gamma_{k+1}(\phi) J (1+J_{n,1})^b.$$

Since $J_{n,1} \to 0$ as $n \to \infty$ we get that $I_1 \to 0$ as $n \to \infty$.

Next we consider

$$\begin{aligned} I_2 &\leq \gamma_k(\phi)\int_{\alpha_n}^{\beta_n} |\eta_n(y)||y^k - 1| dy \\ &\leq J_{n,k}\gamma_k(\phi)\int_0^\infty |\eta_n(y)| dy \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence the lemma follows. □

Remark 3.9. It is interesting to note that the distribution δ_1 is the identity for this Mellin type convolution $\vee$, where δ_1 is defined by $\langle \delta_1, \phi\rangle = \phi(1)$, $\forall \phi \in \mathscr{D}(I)$. This fact and the previous lemma motivate us to call the members of Δ_1 as δ_1-sequences.

Thus the Boehmian space $\mathscr{B}_{\mathcal{M}} = \mathscr{B}(\mathscr{M}'_{a,b}, \mathscr{D}(I), \vee, \Delta_1)$ is constructed. At this juncture, we point out the reason for using a distribution space as G of $\mathscr{B}_{\mathcal{M}}$. Since our objective is to find a Boehmian space, containing $\mathscr{M}'_{a,b}$, each member $f \in \mathscr{M}'_{a,b}$ would be identified by the representative $\frac{f \vee \eta_n}{\eta_n}$ for some $(\eta_n) \in \Delta_1$. Therefore the required G should be containing $f \vee \eta$, $\forall f \in \mathscr{M}'_{a,b}$ and $\forall \eta \in \mathscr{D}(I)$. Though it is known that $f \vee \eta$ is a function, so far it is not discussed what type of function is this. Therefore we prefer to use $\mathscr{M}'_{a,b}$ as G of the required Boehmian space. The advantage of using a function space as G of a Boehmian space is that every Boehmian can be approximated by functions. This can also be achieved in a Boehmian space with G as a distribution space, since each Boehmian is approximated by distributions and each distribution is approximated by functions. In the literature, many Boehmian spaces have been constructed by using distribution spaces and it is well established that they are more comfortable for studying various integral transforms. See [5, 6, 7, 17, 18, 19].

4. Generalized Mellin Transform

First we construct a quotient field of certain analytic functions. Let $H_{a,b}$ be the space of all analytic functions on the strip $\Omega_{a,b}$ consisting of the images of $f \in \mathscr{M}'_{a,b}$. We know that $H_{a,b}$ is a commutative ring with identity with respect to point-wise addition and point-wise multiplication. It is interesting note that it is an integral domain.

Indeed, if $f \cdot g = 0$ on $\Omega_{a,b}$ and if

$$Z(f) = \{s \in \Omega_{a,b} : f(s) = 0\} \text{ and } Z(g) = \{s \in \Omega_{a,b} : g(s) = 0\}$$

then

$$\Omega_{a,b} = Z(f) \cup Z(g).$$

Since $\Omega_{a,b}$ has limit points at least one of $Z(f)$ and $Z(g)$ has limit points. Hence by analytic continuation we get that at least one of f and g is identically zero.

Thus we can construct the quotient field $F_{a,b}$ of $H_{a,b}$ and every element of $F_{a,b}$ is denoted by $\frac{f}{g}$ or $\frac{f(s)}{g(s)}$.

By defining the scalar multiplication by $\alpha\frac{f}{g} = \frac{\alpha f}{g}$ we can make $F_{a,b}$ as an algebra.

Definition 4.1. Let $F = \frac{f(s)}{g(s)} \in H_{a,b}$, $\alpha = u + iv \in \mathbb{C}$, $r > 0$, $\rho \neq 0$ and $q(s)$ is a polynomial. We define

(1) $F' = \frac{f' \cdot g - f \cdot g'}{g \cdot g} \in H_{a,b}$.

(2) $\tau_\alpha F = \frac{f(s-\alpha)}{g(s-\alpha)} \in H_{a-u,b-u}$.

(3) $q(s)F = \frac{q(s)f(s)}{g(s)} \in H_{a,b}$.

(4) $r^{-s}F = \frac{r^{-s}f(s)}{g(s)} \in H_{a,b}$.

(5) $S_\rho F = \frac{f(\rho^{-1}s)}{g(\rho^{-1}s)} \in H_{c,d}$, where $c = \rho a,\ d = \rho b$ if $\rho > 0$, $c = \rho b,\ d = \rho a$ if $\rho < 0$.

It is easy to verify that the above definitions are well defined in $H_{a,b}$.

We introduce a notion of convergence as follows.

Definition 4.2. We say that a sequence (F_n) converges to F in $F_{a,b}$ if there exists f_n, $\forall n \in \mathbb{N}$ and $g \in F_{a,b}$ such that $F_n = \frac{f_n}{g}$, $F = \frac{f}{g}$ and

$$f_n \to f \text{ as } n \to \infty \text{ in } H_{a,b}.$$

Now we are ready to define the generalized Mellin transform.

Definition 4.3. The *generalized Mellin transform* $\mathfrak{M} : \mathscr{B}_{\mathcal{M}} \to F_{a,b}$ is defined by

$$\mathfrak{M}\left([f_n/\eta_n]\right) = \frac{Mf_n}{M\eta_n} \quad \text{for any } n \in \mathbb{N}. \tag{4.1}$$

Remark 4.4. It is easy to verify that $M\eta_n \to 1$ as $n \to \infty$ uniformly on compact subsets of $\Omega_{a,b}$. Hence in view of defining Laplace transform of a Boehmian in [13], $\frac{Mf_n}{M\eta_n}$ can also be viewed as $\lim_{n\to\infty} Mf_n$, where this limit is obtained by uniform convergence on each compact subset of $\Omega_{a,b}$.

By routine procedure, it can be verified that the generalized Mellin transform is well defined, consistent with the distributional Mellin transform on $\mathscr{M}'_{a,b}$, linear and one-to one.

Lemma 4.5. *Let $f, g \in \mathscr{M}'_{a,b}$, $\alpha = u + iv \in \mathbb{C}$, $k \in \mathbb{N}_0$, $r > 0$ and $\rho \in \mathbb{R} \setminus \{0\}$.*

(1) $L(f \vee g) = (Lf) \vee g + f \vee (Lg)$.

(2) $P_\alpha(f \vee g) = (P_\alpha f) \vee (P_\alpha g)$.

(3) $D^k(f \vee g) = (D^k f) \vee (P^k_{-1} g)$.

(4) $(DP_1)^k(f \vee g) = ((DP_1)^k f) \vee g$.

(5) $(P_1 D)^k(f \vee g) = ((P_1 D)^k f) \vee g$.

(6) $(D^kP_1^k)(f\vee g) = ((D^kP_1^k)f)\vee g$.
(7) $(P_1^kD^k)(f\vee g) = ((P_1^kD^k)f)\vee g$.
(8) $M_r(f\vee g) = (M_rf)\vee g$.
(9) $E_\rho(f\vee g) = |\rho|(E_\rho f)\vee (E_\rho g)$.

Proof. We prove the first result. Simialrly the other results can be proved. Let $\phi\in\mathscr{M}_{a,b}$ be arbitrary.

$$\begin{aligned}\langle (L(f\vee g))(x),\phi(x)\rangle &= \langle (f\vee g)(x),\log x\phi(x)\rangle\\ &= \langle f(x),\langle g(y),\log(xy)\phi(xy)\rangle\\ &= \langle f(x),\langle g(y),(\log x+\log y)\phi(xy)\rangle\\ &= \langle f(x),\log x\langle g(y),\phi(xy)\rangle\rangle + \langle f(x),\langle g(y),\log y\phi(xy)\rangle\rangle\\ &= \langle (Lf)(x),\langle g(y),\phi(xy)\rangle\rangle + \langle f(x),\langle (Lg)(y),\phi(xy)\rangle\rangle\\ &= \langle ((Lf)\vee g)(x),\phi(x)\rangle + \langle (f\vee (Lg))(x),\phi(x)\rangle\end{aligned}$$

Hence the Lemma follows. □

Definition 4.6. Let $X=[f_n/\eta_n], Y=[g_n/\theta_n]\in\mathscr{B}_{\mathcal{M}}$, $k\in\mathbb{N}_0$, $\alpha = u+iv\in\mathbb{C}$, $r>0$ and $\rho\in\mathbb{R}\setminus\{0\}$. We define

(1) $LX = [(Lf_n\vee\eta_n - f_n\vee L\eta_n)/(\eta_n\vee\eta_n)]\in\mathscr{B}_{\mathcal{M}}$.
(2) $P_\alpha X = [(\lambda_nP_\alpha f_n)/(\lambda_nP_\alpha\eta_n)]\in\mathscr{B}_{a-u,b-u}$, where $\lambda_n = (\int_0^\infty x^\alpha\eta_n(x)dx)^{-1}$.
(3) $DX = [(\nu_n(Df_n))/(\nu_n(P_{-1}\eta_n))]\in\mathscr{B}_{a+1,b+1}$, where $\nu_n = \big(\int_0^\infty x^{-1}\eta_n(x)\,dx\big)^{-1}$.
(4) $(DP_1)^kX = \big[((DP_1)^kf_n)/\eta_n\big]\in\mathscr{B}_{\mathcal{M}}$.
(5) $(P_1D)^kX = \big[((P_1D)^kf_n)/\eta_n\big]\in\mathscr{B}_{\mathcal{M}}$.
(6) $(D^kP_1^k)X = \big[((D^kP_1^k)f_n)/\eta_n\big]\in\mathscr{B}_{\mathcal{M}}$.
(7) $(P_1^kD^k)X = \big[((P_1^kD^k)f_n)/\eta_n\big]\in\mathscr{B}_{\mathcal{M}}$.
(8) $M_rX = [(M_rf_n)/\eta_n]\in\mathscr{B}_{\mathcal{M}}$.
(9) $E_\rho X = \big[(|\rho|^{-1}\mu_nE_\rho f_n)/(\mu_nE_\rho\eta_n)\big]\in\mathscr{B}_{c,d}$ where $\mu_n = \big(\int_0^\infty \eta_n(x^\rho)dx\big)^{-1}$ and $c=\rho a,\ d=\rho b$ if $\rho>0$, $c=\rho b,\ d=\rho a$ if $\rho<0$.
(10) $X\vee Y = [(f_n\vee g_n)/(\eta_n\vee\theta_n)]$.

For $k>1$, L^kX and D^kX can be defined recursively by $L^kX = L(L^{k-1}X)$ and $D^kX = D(D^{k-1}X)$ respectively.

Remark 4.7. Because $\eta_n \to \delta_1$ as $n \to \infty$, there exists $m \in \mathbb{N}$ such that $\int_0^\infty x^\alpha \eta_n(x)dx$, $\int_0^\infty \eta_n(x^\rho)dx$ and $\int_0^\infty x^{-1}\eta_n(x)dx$ are not equal to zero for all $n \geq m$. Since $\frac{f_n}{\eta_n} \sim \frac{f_{m+n}}{\eta_{m+n}}$ with out loss of generality we assume that λ_n, μ_n and ν_n exist for all $n \in \mathbb{N}$.

It can be verified that the above operations are well defined, using the Lemma 4.5. Indeed we prove the first operation is well defined.

For any $m, n \in \mathbb{N}$,

$$\begin{aligned}
&Lf_n \vee \eta_n - f_n \vee L\eta_n \vee \eta_m \vee \eta_m \\
&= Lf_n \vee \eta_n \vee \eta_m \vee \eta_m - u_n \vee L\eta_n \vee \eta_m \vee \eta_m \\
&= Lf_n \vee \eta_m \vee \eta_n \vee \eta_m - u_n \vee \eta_m \vee L\eta_n \vee \eta_m. \\
&= L(f_n \vee \eta_m) \vee \eta_n \vee \eta_m - f_n \vee L\eta_m \vee \eta_n \vee \eta_m - f_n \vee \eta_m \vee L\eta_n \vee \eta_m \\
&= L(f_m \vee \eta_n) \vee \eta_n \vee \eta_m - f_m \vee \eta_n \vee L\eta_m \vee \eta_n - f_m \vee \eta_n \vee L\eta_n \vee \eta_m \\
&= L(f_m \vee \eta_n) \vee \eta_n \vee \eta_m - f_m \vee L\eta_n \vee \eta_n \vee \eta_m - f_m \vee L\eta_m \vee \eta_n \vee \eta_n \\
&= Lf_m \vee \eta_n \vee \eta_n \vee \eta_m - f_m \vee L\eta_m \vee \eta_n \vee \eta_n \\
&= (Lf_m \vee \eta_m - f_m \vee L\eta_m) \vee \eta_n \vee \eta_n.
\end{aligned}$$

Hence $(Lf_n \vee \eta_n - f_n \vee L\eta_n)/(\eta_n \vee \eta_n)$ is a quotient.

To prove these definitions are independent of the choice of the representative let $f_n/\eta_n \sim h_n/\xi_n$. Then we have $f_n \vee \xi_m = h_m \vee \eta_n, \forall m, n \in \mathbb{N}$.

In the proof of $(Lf_n \vee \eta_n - f_n \vee L\eta_n)/(\eta_n \vee \eta_n)$ is a quotient, by replacing f_m by h_m and η_m by ξ_m respectively, we get that

$$(Lf_n \vee \eta_n - f_n \vee L\eta_n)/(\eta_n \vee \eta_n) \sim (Lh_n \vee \xi_n - h_n \vee L\xi_n)/(\xi_n \vee \xi_n), \quad (4.2)$$

and hence LX is well defined.

Using Lemma 4.5, we get that, if T is any one of the operations in Definition 4.6, $f \in \mathscr{M}'_{a,b}$ and X is a Boehmian representing f in $\mathscr{B}_{\mathcal{M}}$ then $Tf = TX$. Hence it follows that these definitions are consistent with the definitions on $\mathscr{M}'_{a,b}$.

The following two theorems are straight forward from the Lemma 4.5 and Theorem 2.3.

Theorem 4.8. *If $X, Y \in \mathscr{B}_{\mathcal{M}}$, $k \in \mathbb{N}_0$, $\alpha = u + iv \in \mathbb{C}$, $r > 0$ and $\rho \in \mathbb{R} \setminus \{0\}$ then*

(1) $\mathfrak{M}(L^k X) = (\mathfrak{M}X)^{(k)}$.

(2) $\mathfrak{M}(P_\alpha X) = \tau_\alpha \mathfrak{M}X$.

(3) $\mathfrak{M}(D^k X) = (-1)^k (s-k)(s-k+1)\cdots(s-1)\tau_k \mathfrak{M}X$.

(4) $\mathfrak{M}((DP_1)^k X) = (-1)^k (s-1)^k \mathfrak{M}X$.

(5) $\mathfrak{M}((P_1 D)^k X) = (-1)^k s^k \mathfrak{M}X$.

(6) $\mathfrak{M}((D^k P_1^k) X) = (-1)^k (s-k)(s-k+1)\cdots(s-1)\mathfrak{M}X$.

(7) $\mathfrak{M}((P_1^k D^k) X) = (-1)^k s(s+1)\cdots(s+k-1)\mathfrak{M}X$.

(8) $\mathfrak{M}(M_r X) = r^{-s}\mathfrak{M}X$.

(9) $\mathfrak{M}(E_\rho X) = |\rho|^{-1} S_\rho \mathfrak{M}X$.

(10) $\mathfrak{M}(X \vee Y) = \mathfrak{M}X \cdot \mathfrak{M}Y$.

Theorem 4.9. Let $X = [f_n/\eta_n], Y = [[g_n/\theta_n] \in \mathscr{B}_{\mathcal{M}}$, $\alpha = u + iv \in \mathbb{C}$, $k \in \mathbb{N}_0$, $r > 0$ and $\rho \in \mathbb{R} \setminus \{0\}$.

(1) $L(X \vee Y) = (LX) \vee Y + X \vee (LY)$.

(2) $P_\alpha(X \vee Y) = (P_\alpha X) \vee (P_\alpha Y)$.

(3) $D(X \vee Y) = (DX) \vee (P_{-1} Y)$.

(4) $(DP_1)^k (X \vee Y) = ((DP_1)^k X) \vee Y$.

(5) $(P_1 D)^k (X \vee Y) = ((P_1 D)^k X) \vee Y$.

(6) $(D^k P_1^k)(X \vee Y) = ((D^k P_1^k) X) \vee Y$.

(7) $(P_1^k D^k)(X \vee Y) = ((P_1^k D^k) X) \vee Y$.

(8) $M_r(X \vee Y) = (M_r X) \vee Y$.

(9) $E_\rho(X \vee Y) = |\rho|(E_\rho X) \vee (E_\rho Y)$.

Theorem 4.10. *The generalized Mellin transform $\mathfrak{M} : \mathscr{B}_{\mathcal{M}} \to F_{a,b}$ is continuous with respect to the δ-convergence.*

Proof. Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}_{\mathcal{M}}$. Then by Lemma 3.3, there exists $f_{n,k}, f_n \in \mathscr{M}'_{a,b}$, $\forall n, k \in \mathbb{N}$ and $(\eta_k) \in \Delta_1$ such that $X_n = [f_{n,k}/\eta_k]$, $X = [f_k/\eta_k]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathscr{M}'_{a,b}.$$

Applying the continuity of the distributional Mellin transform, we get that for each $k \in \mathbb{N}$,

$$Mf_{n,k} \to Mf_k \text{ as } n \to \infty \text{ in } H_{a,b}.$$

But for any $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$, $\mathfrak{M}X_n = \frac{Mf_{n,k}}{M\eta_k}$ and $\mathfrak{M}X = \frac{Mf_k}{M\eta_k}$. Thus it follows that $\mathfrak{M}X_n \to \mathfrak{M}X$ as $n \to \infty$ in $F_{a,b}$. □

Finding the range of this generalized Mellin transform, is an interesting open question.

5. Mellin Transform And Laplace Transform

Two sided Laplace transform is defined and studied in the context of Boehmians by P. Mikusiński, A. Morse and D. Nemzer [13]. For our convenience, we slightly modify the definition of the Boehmian space $\mathscr{B}_{\mathcal{L}}$ introduced in [13] as $\mathscr{B}(\mathscr{L}'_{a,b}, \mathscr{D}(\mathbb{R}), *, \Delta)$, where the operation $* : \mathscr{L}'_{a,b} \times \mathscr{D}(\mathbb{R}) \to \mathscr{L}'_{a,b}$ is the convolution of a distribution and a function, defined by

$$\langle (f * \phi)(t), \psi(t) \rangle = \left\langle f(t), \int_{-\infty}^{\infty} \psi(t)\phi(t+\tau)\, d\tau \right\rangle, \ \forall \psi \in \mathscr{L}_{a,b},$$

and Δ is the collection of all sequences (δ_n) from $\mathscr{D}(\mathbb{R})$, satisfying

(1) $\int_{-\infty}^{\infty} \delta_n(x)dx = 1, \ \forall n \in \mathbb{N}$,

(2) $\int_{-\infty}^{\infty} |\delta_n(x)|dx \le M$ for some $M > 0$ and all $n \in \mathbb{N}$,

(3) $s(\delta_n) = \inf\{\epsilon > 0 : \text{ supp } \delta_n \subset [-\epsilon, \epsilon]\} \to 0$ as $n \to \infty$.

First we justify that this modification does not alter the original Boehmian space $\mathscr{B}_{\mathcal{L}}$. Obviously, the original Boehmian space is contained the altered Boehmian space. If $[g_n/\delta_n] \in \mathscr{B}(\mathscr{L}'_{a,b}, \mathscr{D}(\mathbb{R}), *, \Delta)$ then

$$[g_n/\delta_n] = [(g_n * \delta_n)/(\delta_n * \delta_n)].$$

Here each $g_n * \delta_n$ is a regular function in $\mathscr{L}'_{a,b}$. Hence this change may just increase the collection of representatives of a Boehmian and not the space $\mathscr{B}_{\mathcal{L}}$. Thus $\mathscr{B}_{\mathcal{L}} = \mathscr{B}(\mathscr{L}'_{a,b}, \mathscr{D}(\mathbb{R}), *, \Delta)$.

Now we define the identification between the Boehmian spaces $\mathscr{B}_{\mathcal{M}}$ and $\mathscr{B}_{\mathcal{L}}$. For $[f_n/\eta_n] \in \mathscr{B}_{\mathcal{M}}$ if we put $\mathcal{I}(f_n) = g_n$ and $\mathcal{I}(\eta_n) = \delta_n$, $\forall n \in \mathbb{N}$ then obviously, $g_n \in \mathscr{L}'_{a,b}$ and $\delta_n \in \mathscr{D}(\mathbb{R})$, $\forall n \in \mathbb{N}$. We claim that $(\sigma_n \delta_n) \in \Delta$, where $\sigma_n = \left(\int_0^\infty \frac{\eta_n(x)}{x} dx\right)^{-1}$, $\forall n \in \mathbb{N}$.

(1) For an arbitrary $n \in \mathbb{N}$, by using the change of variable $\mathrm{e}^{-t} = x$, $\sigma_n \int_{-\infty}^\infty \delta_n(t) dt = \sigma_n \int_{-\infty}^\infty \eta_n(\mathrm{e}^{-t}) dt = \sigma_n \int_0^\infty \frac{\eta_n(x)}{x} dx = 1.$

(2) Let support of $\eta_n \subset (\alpha_n, \beta_n)$, $\forall n \in \mathbb{N}$, where (α_n) and (β_n) both converge to 1. Now

$$\begin{aligned} \int_{-\infty}^\infty |\sigma_n \delta_n(t)| dt &= |\sigma_n| \int_{-\infty}^\infty |\eta_n(\mathrm{e}^{-t})| dt \\ &\leq |\sigma_n| \int_0^\infty \left|\frac{\eta_n(x)}{x}\right| dx \\ &= |\sigma_n| \int_{\alpha_n}^{\beta_n} \left|\frac{\eta_n(x)}{x}\right| dx \\ &\leq \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n|}{\alpha_n}\right) \int_0^\infty |\eta_n(x)| dx. \end{aligned}$$

Since (σ_n) and (α_n) converge to 1, $\left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n|}{\alpha_n}\right)$ is finite. Hence the second property of a delta sequence is satisfied by $(\sigma_n \delta_n)$.

(3) The support of $\eta_n \subset (\alpha_n, \beta_n)$ implies that support of $\delta_n \subset (-\log \beta_n, -\log \alpha_n)$. Since $\alpha_n \to 1$ and $\beta_n \to 1$ as $n \to \infty$, our claim follows.

Now for $m, n \in \mathbb{N}$ and for each $\phi \in \mathscr{L}_{a,b}$,

$$\begin{aligned} \langle (\mathcal{I}(f_n \vee \eta_m))(t), \phi(t)\rangle &= \langle (f_n \vee \eta_m)(x), (\mathcal{J}^{-1}\phi)(x)\rangle \\ &= \langle (f_n \vee \eta_m)(x), x^{-1}\phi(-\log x)\rangle \\ &= \langle f_n(x), \langle \eta_m(y), (xy)^{-1}\phi(-\log(xy))\rangle \\ &= \langle f_n(x), x^{-1}\langle \eta_m(y), y^{-1}\phi(-\log x - \log y))\rangle \\ &= \langle (\mathcal{I}f_n)(t), \langle \eta_m(y), y^{-1}\phi(t - \log y))\rangle \\ &= \langle (\mathcal{I}f_n)(t), \langle (\mathcal{I}\eta_m)(\tau), \phi(t+\tau))\rangle \\ &= \langle (((\mathcal{I}f_n) * (\mathcal{I}\eta_m))(t), \phi(t)\rangle. \end{aligned}$$

Hence $((\mathcal{I}f_n), (\mathcal{I}\eta_n))$ is a quotient whenever (f_n, η_n) is a quotient. Thus if we denote $\mathscr{I}([f_n/\eta_n]) = [(\mathcal{I}f_n)/(\mathcal{I}\eta_n)]$ then $\mathscr{I} : \mathscr{B}_{\mathcal{M}} \to \mathscr{B}_{\mathcal{L}}$, and its inverse is given by $\mathscr{I}^{-1}([g_n/\delta_n]) = [(\mathcal{I}^{-1}g_n)/(\mathcal{I}^{-1}\delta_n)]$. We know that $\mathscr{B}_{\mathcal{L}}$ contains an Boehmian Y which is no longer a member of $\mathscr{D}'(\mathbb{R})$. Then $\mathscr{I}^{-1}Y \in \mathscr{B}_{\mathcal{M}}$ is not representing any member of $\mathscr{D}'(I)$ and hence, $\mathscr{M}'_{a,b} \subsetneqq \mathscr{B}_{\mathcal{M}}$.

Let $\mathcal{L}$ denote the Laplace transform on Boehmians in [13]. Using Therorem 2.8 and Remark 4.4, the relation between $\mathcal{L}$ and $\mathfrak{M}$ is given by, for each $[f_n/\eta_n] \in \mathscr{B}_{\mathcal{M}}$,

$$\begin{aligned}\mathcal{L}(\mathscr{I}[f_n/\eta_n]) &= \mathcal{L}([(\mathcal{I}f_n)/(\mathcal{I}\eta_n)] = \lim_{n\to\infty} L(\mathcal{I}f_n)\\ &= \lim_{n\to\infty} Mf_n \qquad = \frac{Mf_n}{M\eta_n}\\ &= \mathfrak{M}[f_n/\eta_n].\end{aligned}$$

6. Comparative Study

In [1], Mellin transform $\hat{F}$ of a tempered Boehmian $F = [f_n/\phi_n]$ is defined by the limit of $\{\hat{f}_n\}$ in $\mathscr{D}'$, where $\hat{f}_n$ is the Mellin transform of f_n. According to [1, (1.3)], the Mellin transform of a function $f(x)$ is defined by $\mathcal{M}\{f(x); s\} = F\{f(e^x); is\}$, where F is the Fourier transform.

We recall that tempered Boehmians [11] are constructed by taking numerator sequence from the space of slowly increasing functions. We shall show that Mellin transform of a slowly increasing function is not always a member of $\mathscr{D}'$ but a member of $\mathscr{Z}'$. For example, we consider $f(x) = x^2$, since it is a polynomial, obviously $f(x)$ is a slowly increasing function. Then $f(e^x) = e^{x^2}$, which is a distribution and not an ultra distribution. See [22, Problem 4]. As a consequence, Fourier transform of e^{x^2} is an ultra distribution and not a distribution. Therefore it is not possible to expect that $(\hat{f}_n)$ is a sequence in $\mathscr{D}'$, when $[f_n/\phi_n]$ is a tempered Boehmian.

In the following theorem, the operational properties of Mellin transform is discussed, in which the first property is not consistent with the corresponding property of distributional Mellin transform. See Theorem 2.3(3).

Theorem 6.1. [[1]] Let $F = [f_n/\phi_n] \in B_{\mathscr{I}}$ and $G = [g_n/\gamma_n] \in B_S$. Then

(a) $(\partial/\partial x_m F\hat{)} = (-s)\hat{F}(is)$,

(b) $\hat{G}$ is an infinitely differentiable function,

(c) $(F * G\hat{)} = \hat{F}\hat{G}$ and

(d) $\hat{F}\hat{\phi}_n = \hat{f}_n$ for all $n \in N$.

Since the convolutions and the delta sequences used in $\mathscr{B}_{\mathcal{M}}$ and $B_{\mathscr{I}}$ are different, it is not possible to say one space is contained in the other. However, we can say that present work is a better extension of Mellin transform on $\mathscr{M}'_{a,b}$ to the context of Bohemians than that in [1], since in the latter, Mellin transform on $\mathscr{M}'_{a,b}$ is not discussed and not all the properties of Mellin transform are proved.

In [20], a C^∞-Boehmian $[f_n/\phi_n]$ (see [9]), is called Mellin transformable,

(1) if each f_n is Mellin transformable,

(2) there exists a non-empty strip $\Omega \subseteq \cap_n\{s \in \mathbb{C} : \text{ Re } s < \beta_{f_n}\}$, where $\beta_f = \sup\{s \in \mathbb{R} : \int_0^\infty |f(t)|t^{s-1}dt < \infty\}$.

Here to define the Mellin transform, first it is claimed that if $[f_n/\delta_n]$ is a Mellin transformable Boehmian then (Mf_n) converges to an analytic function uniformly on every compact subsets of a suitable region. In the proof of this theorem, the following two unproven statements are used.

- $M(f * \phi) = (Mf)(M\phi)$, where $(f * \phi)(x) = \int f(x-y)\phi(y)\, dy$.
- $M(\phi_n) \to 1$ as $n \to \infty$ where (ϕ_n) is a sequence from $\mathscr{D}$, with $\int \phi_n(x)dx = 1$, $\forall n$, $\int |\phi_n(x)|dx \leq M, \forall n$, for some $M > 0$ and supp $\phi_n \to 0$ as $n \to \infty$.

The first statement is not true, for the following reason. If $f, g \in \mathscr{D}(I)$, then $f \vee g$ and $f * g$ both are defined, and obviously $f \vee g \neq f * g$. It is proved that $M(f \vee g) = (Mf)(Mg)$ [24, p.119]. If $M(f * g) = (Mf)(Mg)$ were true then we get $M(f \vee g) = M(f * g)$. Using the injectivity of the Mellin transform M, it follows that $f * g = f \vee g$, which is a contradiction. Hence the existence of the Mellin transform in [20] is not justified.

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