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MEMBRANE CAPS UNDER HYDROSTATIC PRESSURE*

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Abstract. It is shown that the exact nonlinear theory for a rotationally symmetric membrane cap deformed by hydrostatic pressure is statically determinant. A small strain theory is obtained without any assumptions on the relative magnitudes of the displacements. This small strain theory can be reduced to a single second-order ordinary differential equation for the determination of the radial stress. A linear shallow cap theory is obtained and solved explicitly for the case of the shallow spherical cap.

1. Introduction. The purpose of this paper is to discuss the stresses, strains, and displacements which occur in a membrane cap when subjected to a hydrostatic pressure.

The equations governing the rotationally symmetric deformation of a membrane cap consist of three sets of relations: (1) the strain-displacement equations, (2) the equilibrium equations, and (3) the constitutive laws. If the membrane cap consists of a surface which is generated by rotating a curve $z = Z(r)$ about the z-axis the equations (1), (2), and (3) take the form

$$\mathcal{E}_r = \frac{2u' + (u')^2 + 2z'w' + (w')^2}{2m^2}, \tag{1.1a}$$

$$\mathcal{E}_\theta = \frac{u}{r} + \frac{1}{2}\left(\frac{u}{r}\right)^2, \tag{1.1b}$$

$$\frac{d}{dr}\left\{\frac{(r+u)(1+u')\sigma_r}{[(1+u')^2 + (z'+w')^2]^{1/2}}\right\} - \sigma_\theta[(1+u')^2 + (z'+w')^2]^{1/2} - \frac{rmP(z'+w')}{h[(1+u')^2 + (z'+w')^2]^{1/2}} = 0, \tag{1.2a}$$

$$\frac{d}{dr}\left\{\frac{(r+u)(z'+w')\sigma_r}{[(1+u')^2 + (z'+w')^2]^{1/2}}\right\} + \frac{rmP(1+u')}{[(1+u')^2 + (z'+w')^2]^{1/2}} = 0, \tag{1.2b}$$

$$\mathcal{E}_r = \mathcal{E}_r(\sigma_r, \sigma_\theta), \qquad \mathcal{E}_\theta = \mathcal{E}_\theta(\sigma_r, \sigma_\theta). \tag{1.3}$$

($' = d/dr$). u and w are the displacements in the radial and z directions, σ_r and σ_θ are the radial and circumferential stresses, $\mathcal{E}_r$ and $\mathcal{E}_\theta$ are the radial and circumferential

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strains and h is the thickness of the membrane. P is the pressure per unit undeformed area whose direction is normal to the deformed surface and

$$m^2 = 1 + (z')^2. \tag{1.4}$$

The equations (1.1), (1.2), and (1.3) are the exact nonlinear membrane equations (cf. Sec. 7). For the boundary conditions on (1.1), (1.2), and (1.3) we will prescribe either radial displacement

$$u(a) = \mu \tag{1.5}$$

or radial stress

$$\sigma_r(a) = \sigma \tag{1.6}$$

(a is the radius of the base of the cap). In either case we will require that all quantities be finite at $r = 0$ and

$$w(a) = 0. \tag{1.7}$$

Approximate theories for the deformation of membrane caps have been suggested by Bromberg and Stoker [1] and Reissner [2]. The theory described in [2] has been used by Goldberg [3] to obtain a numerical solution for the spherical cap.

In [4] it was shown that the exact "normal" pressure theory, i.e., pressure whose direction is normal to the undeformed surface, can be reduced to a single, second-order, nonlinear, ordinary differential equation for the determination of a quantity related to the radial stress. This reduction is possible regardless of the constitutive laws. The fact that the exact theory can be described by a single equation makes it possible to obtain an approximate theory for small strains without the necessity of making *ad hoc* assumptions on the relative magnitudes of the displacements.

A reduction to a single, second-order, differential equation is evidently not possible for the relations (1.1), (1.2), and (1.3). However, it will be shown in Sec. 2 that the theory is statically determinant, i.e., the system can be written as a pair of second-order differential equations for the determination of σ_r and σ_θ. Thus even in this case it is possible to obtain a small strain theory without assuming any conditions on the magnitude of the displacements (cf. Sec. 3). In Sec. 3 it is also shown that the small strain theory does reduce to a single nonlinear second-order ordinary differential equation for the determination of σ_r. In Sec. 4 a nonlinear shallow cap theory is obtained and in Sec. 5 it is shown that if the applied pressure P is "sufficiently small" the small strain, shallow cap theory can be reduced to a linear theory. In Sec. 6 this linear theory is solved explicitly for the shallow spherical cap.

2. Reduction of the equations. It is convenient to write the strain displacement equations in the form

$$m\sqrt{2\mathcal{E}_r + 1} = [(1 + u')^2 + (z' + w')^2]^{1/2}, \tag{2.1a}$$

$$r\sqrt{2\mathcal{E}_\theta + 1} = r + u. \tag{2.1b}$$

Equation (2.1b) implies that

$$1 + u' = (r\sqrt{2\mathcal{E}_\theta + 1})'. \tag{2.2}$$

It is a consequence of (2.1a) that

$$\frac{z'+w'}{[(1+u')^2+(z'+w')^2]^{1/2}} = -\left\{1-\frac{(1+u')^2}{(1+u')^2+(z'+w')^2}\right\}^{1/2} = -\left\{1-\left[\frac{(r\sqrt{2\mathcal{E}_\theta+1})'}{m\sqrt{2\mathcal{E}_r+1}}\right]^2\right\}^{1/2}. \tag{2.3}$$

Combining (2.2) and (2.3) with Eq. (1.2a) we find

$$\frac{d}{dr}\left\{\frac{r\sqrt{2\mathcal{E}_\theta+1}(r\sqrt{2\mathcal{E}_\theta+1})'\sigma_r}{m\sqrt{2\mathcal{E}_r+1}}\right\}-\sigma_\theta m\sqrt{2\mathcal{E}_r+1}+\frac{rmP}{h}\left\{1-\left[\frac{(r\sqrt{2\mathcal{E}_\theta+1})'}{m\sqrt{2\mathcal{E}_r+1}}\right]^2\right\}^{1/2}=0. \tag{2.4}$$

Using the fact that

$$(r\sqrt{2\mathcal{E}_\theta+1})' = \frac{r\mathcal{E}_\theta'+2\mathcal{E}_\theta+1}{\sqrt{2\mathcal{E}_\theta+1}}, \tag{2.5}$$

Eq. (2.4) may be simplified to

$$\frac{d}{dr}\left\{\frac{r(r\mathcal{E}_\theta'+2\mathcal{E}_\theta+1)\sigma_r}{m\sqrt{2\mathcal{E}_r+1}}\right\}-\sigma_\theta m\sqrt{2\mathcal{E}_r+1} + \frac{rP\{m^2(1+2\mathcal{E}_r)(1+2\mathcal{E}_\theta)-(r\mathcal{E}_\theta'+2\mathcal{E}_\theta+1)^2\}^{1/2}}{h\sqrt{1+2\mathcal{E}_r}\sqrt{1+2\mathcal{E}_\theta}}=0. \tag{2.6}$$

A similar procedure may be used to rewrite Eq. (1.2b) in terms of the stresses and strains. Thus combining (2.1), (2.2), and (2.3) with Eq. (1.2b) we find

$$-\frac{d}{dr}\left\{r\sqrt{2\mathcal{E}_\theta+1}\left\{1-\left[\frac{(r\sqrt{2\mathcal{E}_\theta+1})'}{m\sqrt{2\mathcal{E}_r+1}}\right]^2\right\}^{1/2}\sigma_r\right\}+\frac{rP(r\sqrt{2\mathcal{E}_\theta+1})'}{h\sqrt{2\mathcal{E}_r+1}}=0. \tag{2.7}$$

This equation can be rewritten

$$\frac{d}{dr}\left\{\frac{r[m^2(1+2\mathcal{E}_r)(1+2\mathcal{E}_\theta)-(r\mathcal{E}_\theta'+2\mathcal{E}_\theta+1)^2]^{1/2}\sigma_r}{m\sqrt{1+2\mathcal{E}_r}}\right\} - \frac{rP(r\mathcal{E}_\theta'+2\mathcal{E}_\theta+1)}{h\sqrt{1+2\mathcal{E}_r}\sqrt{1+2\mathcal{E}_\theta}}=0. \tag{2.8}$$

In view of the constitutive laws (1.3), Eqs. (2.6) and (2.8) are a pair of second-order, ordinary differential equations for the determination of σ_r and σ_θ.

3. Small strain theory. If the strains are small, i.e. $|\mathcal{E}_r| << 1$ and $|\mathcal{E}_\theta| << 1$, the appropriate choice for the constitutive equations (1.3) is Hooke's law

$$E\mathcal{E}_r = \sigma_r - \nu\sigma_\theta, \tag{3.1a}$$

$$E\mathcal{E}_\theta = \sigma_\theta - \nu\sigma_r \tag{3.1b}$$

where E is Young's modulus and ν is the Poisson ratio.

The equilibrium equations (2.6) and (2.8) are greatly simplified by the small strain assumption. Equation (2.6) becomes

$$\frac{d}{dr}\left\{\frac{r\sigma_r}{m}\right\} - m\sigma_\theta + \frac{rP}{h}[m^2(1+2(\mathcal{E}_r+\mathcal{E}_\theta)) - (1+2(r\mathcal{E}_\theta'+2\mathcal{E}_\theta))]^{1/2} = 0 \tag{3.2}$$

and (2.8) is replaced by

$$\frac{d}{dr}\left\{\frac{r}{m}[m^2(1+2(\mathcal{E}_r+\mathcal{E}_\theta)) - (1+2(r\mathcal{E}_\theta'+2\mathcal{E}_\theta))]^{1/2}\right\} - \frac{rP}{h} = 0. \tag{3.3}$$

We note that neglecting either $\mathcal{E}_r$ or $\mathcal{E}_\theta$ in relation to m^2-1 in either (3.2) or (3.3) would be incorrect since $m^2-1=(z')^2$ which may itself be small and in fact $z'=0$ at $r=0$ if the undeformed surface is smooth.

Equation (3.3) implies that

$$\frac{r}{m}[m^2(1+2\mathcal{E}_r)(1+2\mathcal{E}_\theta) - (1+2(r\mathcal{E}_\theta'+2\mathcal{E}_\theta))]^{1/2}\sigma_r = \frac{F}{h} \tag{3.4}$$

where

$$F(r) = \int_0^r \tau P(\tau)\,d\tau. \tag{3.5}$$

Combining (3.4) and (3.2) we find

$$\frac{d}{dr}\left(\frac{r\sigma_r}{m}\right) - m\sigma_\theta + \frac{mPF}{h^2\sigma_r} = 0. \tag{3.6}$$

It simplifies the notation if we introduce the dimensionless stresses

$$\Sigma_r = \frac{\sigma_r}{E}, \qquad \Sigma_\theta = \frac{\sigma_\theta}{E}. \tag{3.7}$$

The Hooke's law (3.1) becomes

$$\mathcal{E}_r = \Sigma_r - \nu\Sigma_\theta, \tag{3.8a}$$

$$\mathcal{E}_\theta = \Sigma_\theta - \nu\Sigma_r \tag{3.8b}$$

while (3.4) and (3.6) become

$$m^2 - 1 + 2m^2(\mathcal{E}_r+\mathcal{E}_\theta) - 2r\mathcal{E}_\theta' - 4\mathcal{E}_\theta = \frac{m^2F^2}{r^2h^2E^2\Sigma_r^2}, \tag{3.9}$$

$$\left(\frac{r\Sigma_r}{m}\right)' - m\Sigma_\theta + \frac{mPF}{h^2E^2\Sigma_r} = 0. \tag{3.10}$$

The strains may be eliminated from Eq. (3.9) using (3.8), so that (3.9) becomes

$$m^2-1+(2m^2(1-\nu)+4\nu)\Sigma_r+(2m^2(1-\nu)-4)\Sigma_\theta-2r\Sigma_\theta'+2\nu r\Sigma_r' = \frac{m^2F^2}{r^2h^2E^2\Sigma_r^2}. \tag{3.11}$$

Σ_θ may be eliminated between Eqs. (3.10) and (3.11). The final result is

$$\begin{aligned} m^2 - 1 + 2(m^2+(2-m^2)\nu)\Sigma_r + 2(m^2-2-m^2\nu)\left(\frac{1}{m}\left(\frac{r\Sigma_r}{m}\right)' + \left(\frac{PF}{h^2E^2\Sigma_r}\right)\right) \\ - 2r\left[\left(\frac{1}{m}\left(\frac{r\Sigma_r}{m}\right)'\right)' + \left(\frac{PF}{h^2E^2\Sigma_r}\right)'\right] \\ + 2\nu r\Sigma_r' = \frac{m^2F^2}{r^2h^2E^2\Sigma_r^2}. \end{aligned} \tag{3.12}$$

Equation (3.12) is a single second-order differential equation for the determination of Σ_r. Once Σ_r is determined from (3.12), Σ_θ is determined from (3.10), $\mathcal{E}_r$ and $\mathcal{E}_\theta$ are determined from (3.8), u is determined from (2.1b), and w from (2.1a).

4. The shallow cap theory. Equation (3.12) is greatly simplified if the cap is shallow. In order to make this assumption explicit we write

$$Z(r) = \delta\zeta(r) \tag{4.1}$$

where $0 < \delta << 1$. It is convenient to redefine the pressure and radial stress by

$$P = \delta^3 p, \tag{4.2}$$

$$\Sigma_r = \delta^2 S_r. \tag{4.3}$$

Equation (4.2) implies

$$F = \delta^3 f = \delta^3 \int_0^r \tau p(\tau)\, d\tau. \tag{4.4}$$

Of course, implicit in Eq. (4.2) is the assumption that the applied pressure is small.

The shallow cap theory is obtained by placing Eqs. (4.1)–(4.4) into Eq. (3.2) and keeping the lowest-order terms in δ. In any case, it is found that

$$r^2 S_r'' + 3rS_r' + \frac{f^2}{2r^2h^2E^2S_r^2} = \frac{(\zeta')^2}{2}. \tag{4.5}$$

The dimensionless circumferential stress Σ_θ can be obtained from Eq. (3.10). In particular, if we define

$$\Sigma_\theta = \delta^2 S_\theta, \tag{4.6}$$

Eqs. (4.1)–(4.4) and (3.10) yield

$$S_\theta = (rS_r)', \tag{4.7}$$

as the lowest-order term in δ. If we define

$$\mathcal{E}_r = \delta^2 e_r, \qquad \mathcal{E}_\theta = \delta^2 e_\theta, \tag{4.8}$$

Eqs. (3.8) imply

$$e_r = S_r - \nu S_\theta, \qquad e_\theta = S_\theta - \nu S_r. \tag{4.9}$$

The displacements u and w are determined from (1.1). Define U and W by

$$u = \delta^2 U, \qquad w = \delta W. \tag{4.10}$$

Combining (4.10) and (1.1) we find

$$U = re_\theta, \tag{4.11}$$

$$W' = -\zeta' - [(\zeta')^2 - 2U' + 2e_r]^{1/2} \tag{4.12}$$

up to higher-order terms in δ.

The edge conditions on (4.5) are determined by either (1.5) (the displacement condition) or (1.6) (the stress condition). In particular, if we define

$$\mu = \delta^2 M, \tag{4.13}$$

$$\frac{\sigma}{E} = \delta^2 S, \tag{4.14}$$

the boundary condition (1.5) can be written

$$(rS_r' + (1-\nu)S_r)_{r=a} = \frac{M}{a}, \tag{4.15}$$

and (1.6) becomes

$$S_r(a) = S. \tag{4.16}$$

We also require that $S_r'(0) = 0$.

5. The linear shallow cap theory. We are interested in solutions of Eq. (4.5) which have the property that $S_r \to 0$ as $p \to 0$. If this is to be the case, Eq. (4.5) requires that

$$\lim_{p\to 0} \frac{f^2}{r^2h^2E^2S_r^2} = (\zeta')^2. \tag{5.1}$$

Equivalently,

$$\frac{f}{rhES_r} = -\zeta' + v = -\zeta'\left(1 - \frac{v}{\zeta'}\right) \tag{5.2}$$

where $|v/\zeta'| << 1$ when p is sufficiently small. Thus we find

$$S_r = \frac{f}{rhE\zeta'(1-(v/\zeta'))} \tag{5.3}$$

or, keeping only terms up to first order in v/ζ',

$$S_r = \frac{f}{rhE\zeta'}\left(1 + \frac{v}{\zeta'}\right). \tag{5.4}$$

Combining (5.2) and (5.4) with Eq. (4.5) and keeping first-order terms, we obtain a linear equation for the determination of S_r,

$$r^2\left(\frac{f}{rhE\zeta'}\left(1+\frac{v}{\zeta'}\right)\right)'' + 3r\left(\frac{f}{rhE\zeta'}\left(1+\frac{v}{\zeta'}\right)\right)' + \zeta' v = 0. \tag{5.5}$$

6. The shallow spherical cap. The shallow cap theory is easily specialized to the spherical cap. For simplicity we will treat the constant pressure case so that (cf. (4.4))

$$f = \frac{1}{2}r^2 p. \tag{6.1}$$

The equation of the spherical cap is

$$z = \sqrt{b^2 - r^2} - \sqrt{b^2 - a^2}, \quad 0 \le r \le a, \tag{6.2}$$

so that

$$z' = \frac{\delta(r/a)}{[1 - \delta^2(r/a)^2]^{1/2}} \tag{6.3}$$

where $\delta = a/b << 1$. In any case

$$\zeta' = -r/a. \tag{6.4}$$

In (6.4) higher-order terms in δ have been neglected.

Equations (6.1) and (6.4) can be combined with Eq. (5.5) to yield the linear equation

$$\rho^2\frac{d^2}{d\rho^2}\left(\frac{v}{\rho}\right) + 3\rho\frac{d}{d\rho}\left(\frac{v}{\rho}\right) - \frac{2hE}{pa}\rho v = 0 \tag{6.5}$$

where we have introduced the new independent variable

$$\rho = r/a. \tag{6.6}$$

Equation (6.5) can be rewritten in the form

$$\rho^2 \ddot{v} + \rho \dot{v} - (1 + \lambda^2 \rho^2) v = 0 \tag{6.7}$$

$(\cdot = d/d\rho)$ with

$$\lambda^2 = \frac{2hE}{pa}. \tag{6.8}$$

The solution of (6.7) which is finite at $\rho = 0$ is

$$v = AI_1(\lambda\rho) \tag{6.9}$$

where I_1 is the modified Bessel function of order one and A is a constant to be determined by the boundary condition (4.15) or (4.16). The dimensionless radial stress is given by (cf. (5.4))

$$S_r = \frac{2}{\lambda^2}\left(1 - \frac{A}{\rho} I_1(\lambda\rho)\right). \tag{6.10}$$

S_θ is determined from (4.7),

$$S_\theta = \frac{d}{d\rho}(\rho S_r) = \frac{2}{\lambda^2}(1 - A\lambda I_1'(\lambda\rho)). \tag{6.11}$$

The displacements are determined from (4.11) and (4.12),

$$\frac{U}{a} = \frac{2\rho}{\lambda^2}\left[(1-\nu) - A(\lambda I_1'(\lambda\rho) - \frac{\nu}{\rho} I_1(\lambda\rho)\right], \tag{6.12}$$

$$\dot{W} = \rho - [\rho^2 + 2A\rho I_1(\lambda\rho)]^{1/2}. \tag{6.13}$$

As a special case we have considered the case $U(a) = 0$, i.e., zero displacement at the boundary. For the boundary conditions, Eq. (6.12) implies

$$A = \frac{1-\nu}{\lambda I_1'(\lambda) - \nu I_1(\lambda)}. \tag{6.14}$$

In Figs. 6.1 and 6.2, $\lambda^2 S_r$, $\lambda^2 S_\theta$, $\lambda^2 U/a$, and $-\lambda \dot{W}$ are plotted for various values of λ and $\nu = 0.3$.

If the displacement boundary condition is inhomogeneous, i.e., $U(a) = M$, Eq. (6.12) implies

$$A = \frac{1 - \nu - M\lambda^2/a}{\lambda I_1'(\lambda) - \nu I_1(\lambda)} \tag{6.15}$$

while the boundary condition (4.18)

$$A = \frac{1 - S\lambda^2/2}{I_1(\lambda)}. \tag{6.16}$$

The relations (6.15) or (6.16) enable us to state simple criteria for the membrane to be in tension. In particular, (6.16) implies that both S_r and S_θ are nonnegative if

$$\frac{S\lambda^2}{2} > \frac{\lambda I_1'(\lambda) - \nu I_1(\lambda)}{\lambda I_1'(\lambda)} \tag{6.17}$$

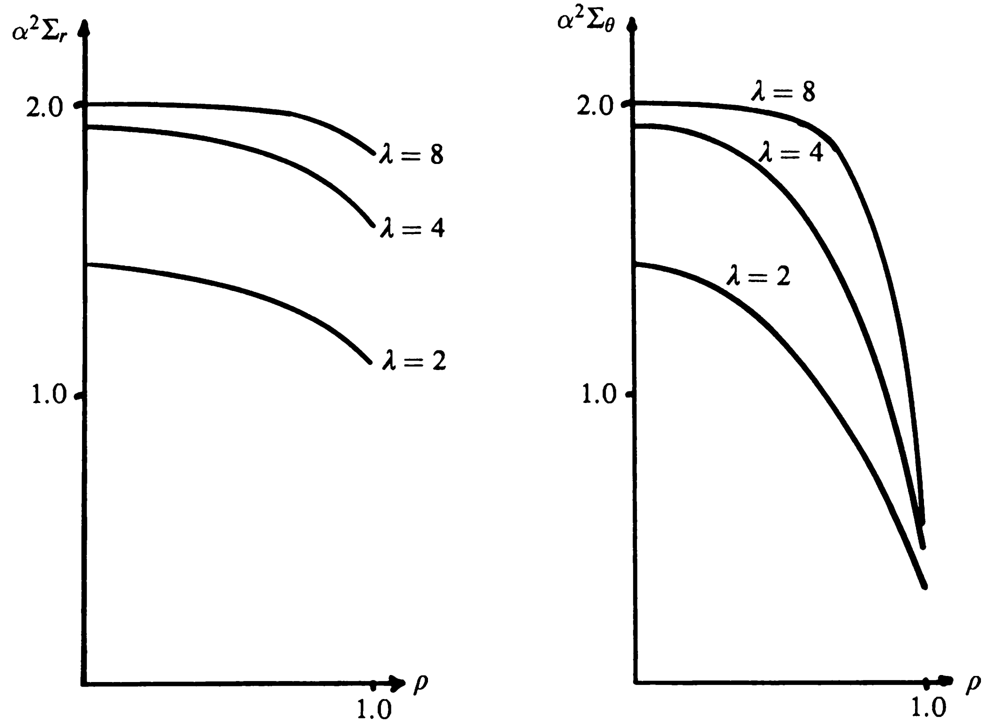

FIG. 6.1.

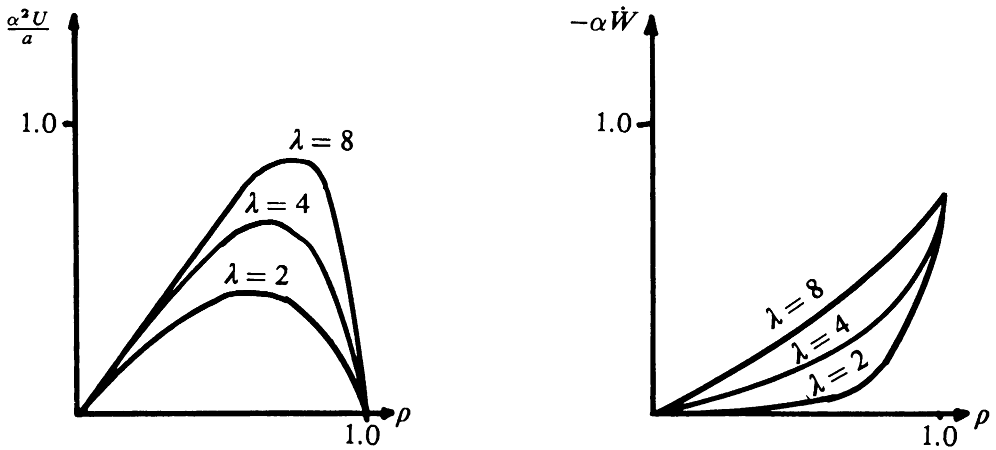

FIG. 6.2.

(note that $\lambda I_1'(\lambda) - \nu I_1(\lambda) > 0$). The relation (6.15) implies that both S_r and S_θ are nonnegative if

$$\frac{M\lambda^2}{a} > (1-\nu) - \frac{\lambda I_1'(\lambda) - \nu I_1(\lambda)}{\min(I_1(\lambda), I_1'(\lambda))}. \tag{6.18}$$

We note that when λ is large (p small)

$$\min(I_1(\lambda), I_1'(\lambda)) = I_1'(\lambda). \tag{6.19}$$

If the condition (6.17) or (6.18) is violated, the solution will contain either radial or circumferential compressions. Since membranes have no resistance to bending, it is not to be expected that they would support compressions. Thus, if condition (6.17) or (6.18) is violated, it is likely that there are other solutions which are not rotationally symmetric.

7. Appendix, equilibrium equations. The strain-displacement equations (1.1) and the constitutive laws (1.3) were obtained in [4]. The equilibrium equations are (cf. [4])

$$\frac{d}{dr}\left\{\frac{(r+u)(1+u')\sigma_r}{[(1+u')^2+(z'+w')^2]^{1/2}}\right\} - \sigma_\theta[(1+u')^2+(z'+w')^2]^{1/2} + \frac{P_1}{h} = 0, \tag{7.1a}$$

$$\frac{d}{dr}\left\{\frac{(r+u)(z'+w')\sigma_r}{[(1+u')^2+(z'+w')^2]^{1/2}}\right\} + \frac{P_2}{h} = 0. \tag{7.1b}$$

The quantities P_1 and P_2 are determined from the relation

$$P_1\mathbf{e}_r + P_2\mathbf{k} = rmP\hat{\mathbf{n}} \tag{7.2}$$

where P is the force per unit undeformed area and $\hat{\mathbf{n}}$ is the unit vector in the direction normal to the deformed surface. $\mathbf{e}_r$ and $\mathbf{k}$ are the unit vectors in the radial and z directions (cylindrical coordinates).

A point on the undeformed surface whose position is given by

$$\mathbf{R} = r\mathbf{e}_r + z\mathbf{k} \tag{7.3}$$

has the position

$$\mathbf{R}^* = (r+u)\mathbf{e}_r + (z+w)\mathbf{k} \tag{7.4}$$

after deformation. Thus the unit normal $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{\dfrac{\partial \mathbf{R}^*}{\partial r} \times \dfrac{\partial \mathbf{R}^*}{\partial \theta}}{\left|\dfrac{\partial \mathbf{R}^*}{\partial r} \times \dfrac{\partial \mathbf{R}^*}{\partial \theta}\right|}. \tag{7.5}$$

It is a consequence of (7.4) that

$$\frac{\partial \mathbf{R}^*}{\partial r} = (1+u')\mathbf{e}_r + (z'+w')\mathbf{k}, \tag{7.6a}$$

$$\frac{\partial \mathbf{R}^*}{\partial \theta} = (r+u)\mathbf{e}_\theta, \tag{7.6b}$$

where $\mathbf{e}_\theta$ is the unit vector in the circumferential direction. Equation (7.5) implies

$$\hat{\mathbf{n}} = \frac{(1+u')\mathbf{k} - (z'+w')\mathbf{e}_r}{[(1+u')^2+(z'+w')^2]^{1/2}}. \tag{7.7}$$

Combining (7.7) and (7.2) we find

$$P_1 = \frac{rmp(z' + w')}{[(1 + u')^2 + (z' + w')^2]^{1/2}}, \tag{7.8a}$$

$$P_2 = \frac{rmP(1 + u')}{[(1 + u')^2 + (z' + w')^2]^{1/2}}. \tag{7.8b}$$

The equilibrium equations (1.2) follow from (7.1) and (7.8).

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