Membrane viewpoint on black holes: Properties and evolution of the stretched horizon

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This paper derives the "membrane formalism" for black holes. The membrane formalism rewrites the standard mathematical theory of black holes in a language and notation which (we hope) will facilitate research in black-hole astrophysics: The horizon of a black hole is replaced by a surrogate "stretched horizon," which is viewed as a 2-dimensional membrane that resides in 3-dimensional space and evolves in response to driving forces from the external universe. This membrane, following ideas of Damour and Znajek, is regarded as made from a 2-dimensional viscous fluid that is electrically charged and electrically conducting and has finite entropy and temperature, but cannot conduct heat. The interaction of the stretched horizon with the external universe is described in terms of familiar laws for the horizon's fluid, e.g., the Navier-Stokes equation, Ohm's law, a tidal-force equation, and the first and second laws of thermodynamics. Because these laws have familiar forms, they are likely to help astrophysicists understand intuitively and compute quantitatively the behaviors of black holes in complex external environments. Previous papers have developed and elucidated electromagnetic aspects of the membrane formalism for time-independent rotating holes. This paper derives the full formalism for dynamical, evolving holes, with one exception: In its present form the formalism is not equipped to handle horizon caustics, where new generators attach themselves to the horizon.

I. INTRODUCTION

Previous papers in this series[1–3] have introduced a new viewpoint on black-hole physics, the "membrane viewpoint," which treats the horizon of a black hole as a physical, 2-dimensional membrane that lives in and evolves in 3-dimensional space. This viewpoint was developed by combining the Znajek[4]-Damour[5–7] "bubble" formalism for the hole's horizon, viewed as a null 3-surface in 4-dimensional spacetime, with a $3+1$ split of spacetime into space plus time.

The membrane viewpoint has the goal of providing astrophysicists with mental pictures, physical intuition, computational techniques, and other research tools (i.e., a "paradigm" in the sense of Kuhn[8]) which will facilitate analyses of the interactions of black holes with complex astrophysical environments.

The previous papers in this series[1–3] dealt, primarily, with the interaction of a quasistationary (slowly evolving), axisymmetric (Kerr) black hole with electromagnetic fields. In this paper we extend the membrane formalism to include (i) fully dynamical holes, (ii) gravitational interactions of the horizon with external matter and infalling matter and fields, and (iii) a detailed, kinematical description of the structure and evolution of the horizon. However, we shall not attempt to deal with horizon caustics (which occur only in extremely dynamical situations), at which new generators attach themselves to the horizon.

The purpose of this paper is to derive the membrane formalism from the standard general relativistic theory of black holes. As a result, this paper is written in the language of mathematical relativity and will be easily accessible only to people who are fluent in that language. The resulting formalism, however, is intended to be valuable to people who find mathematical relativity difficult and alien. Such people can get a feeling for the formalism by browsing Secs. I and VI of this paper; but for a full understanding of the formalism they must await the completion of a long, pedagogical treatise that we and our colleagues are preparing.[9,10]

As preparation for our derivation of the membrane formalism, we shall present in the next two sections a qualitative overview of the $3+1$ split and the "stretching of the horizon," which underly the formalism.

The notation in this paper generally follows that of Misner, Thorne, and Wheeler[11] (MTW); in particular c and G are taken to be unity throughout, and the metric signature $-+++$ is used. The index H will denote a horizon quantity and will be placed for convenience as a superscript or subscript with no difference in meaning.

A. The $3+1$ split underlying the membrane formalism

Figure 1 shows the $3+1$ split that underlies the membrane formalism, for the special case of a hole with negligible rotation. If the hole were rotating, the diagram would be qualitatively the same, except for an added "barber-pole twist" of the dashed world lines and the hor-

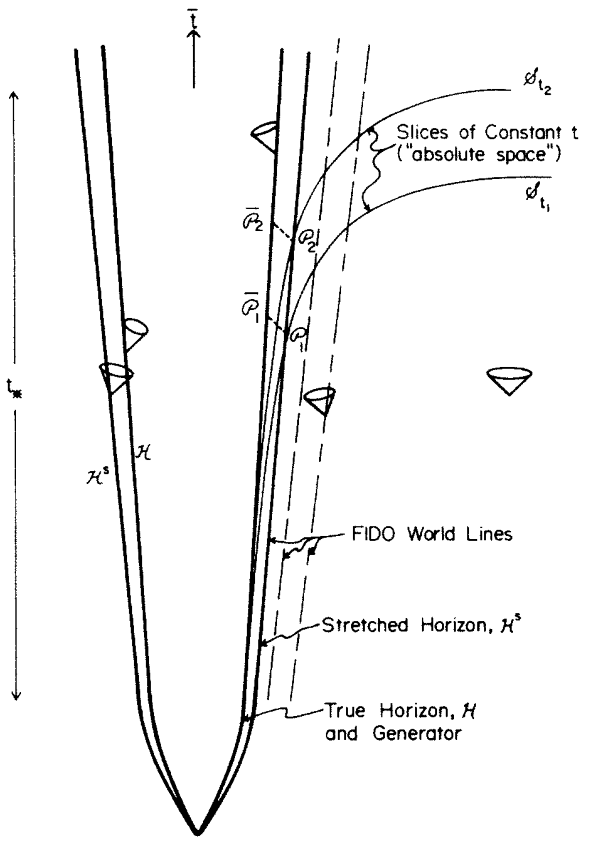

FIG. 1. A spacetime diagram showing the 3 + 1 split that underlies the membrane viewpoint. For details see text.

izon generators (cf. p. 881 of MTW).

In the spacetime diagram of Fig. 1 the time coordinate $\bar{t}$ plotted vertically is an Eddington-Finkelstein type of time; i.e., it is a time coordinate that gives the 3-dimensional horizon $\mathcal{H}$ (innermost solid line) a nearly vertical slope, makes the light cones have the indicated forms, and agrees with the proper time of static observers far from the hole—cf. p. 829 of MTW. The hole's horizon $\mathcal{H}$ is created at the bottom of the diagram by the rapid gravitational collapse of a star (not shown); and it thereafter grows slowly, on a time scale $t_* \gg (G/c^3) \times$ (mass of hole), due to accretion of matter and to gravitational and electromagnetic interactions with external matter and fields (not shown).

The 3 + 1 split is based on a family of hypersurface-orthogonal fiducial observers (FIDO's). The world lines of several FIDO's are shown dashed in Fig. 1, along with two of the spacelike hypersurfaces, $\mathcal{S}_{t_1}$ and $\mathcal{S}_{t_2}$, that are orthogonal to the FIDO world lines. These hypersurfaces are surfaces of constant "universal time" t (not equal to $\bar{t}$); and in the membrane viewpoint one thinks of them as a single, time-evolving, 3-dimensional, "absolute space" in which physics take place as time t passes. Physics in this absolute space is characterized by 3-dimensional tensor, vector, and scalar fields which represent physical quantities measured by the FIDO's—e.g., the FIDO-measured electric and magnetic fields **E** and **B** (or E_j and B_j in component notation).

The FIDO's and universal time of the membrane viewpoint are chosen in accord with the following rules: (i) Far from the hole the FIDO's are at rest in the hole's asymptotic rest frame and universal time t, FIDO proper time τ, and Eddington-Finkelstein-type time $\bar{t}$ all agree; (ii) wherever spacetime is nearly stationary and axisymmetric (e.g., nearly Kerr), the FIDO's are nearly "zero-angular-momentum observers"[12] (ZAMO's) and universal time t is nearly Boyer-Lindquist-type time (e.g., Sec. 33.2 of MTW); (iii) very near the hole the FIDO's "snuggle up to" the horizon in such a way that (a) their world lines cover the entire spacetime outside the true horizon, (b) their world lines remain always timelike, and (c) their world lines asymptotically approach the horizon's null generators as one asymptotically approaches the horizon. (For the precise meaning of "asymptotic" in this context see Sec. III, below.)

B. The stretched horizon

The above choice of FIDO's has great advantages over other possible choices: for example, the FIDO's cover the entire exterior of the hole and never fall in, and for a nearly stationary hole the FIDO's are nearly stationary. But a price is paid for these advantages: They require that the FIDO world lines asymptote to the null horizon generators; and this in turn forces the FIDO's orthogonal hypersurfaces of constant time $\mathcal{S}_t$ ("absolute space") to dip deep down into the past near the hole and thereby asymptote to the horizon (Fig. 1). For example, in a spacetime region that is Schwarzschild (nonrotating, static hole of mass M), the Eddington-Finkelstein time $\bar{t}$ at Schwarzschild radial coordinate location r and universal time t is

$$\bar{t} = t + 2M \ln(r/2M - 1) \tag{1.1}$$

(p. 829 of MTW); so $\bar{t} \to -\infty$ as $r \to 2M$ at fixed universal time t.

This dip into the past has pathological consequences that are nicely illustrated by the black hole of Fig. 1, which distant observers today (at $t=t_2$) see to be nonrotating, to have mass $M=10^8 M_\odot$ (for example), and to be slowly evolving with an astrophysically realistic mass-doubling time $t_* \simeq 10^8$ yr. Suppose that today (at fixed universal time $t=t_2$ of Fig. 1) one mathematically approaches the hole's horizon $r=2M \simeq 3\times 10^8$ km by moving inward through absolute space (inward on the hypersurface $\mathcal{S}_{t_2}$) starting from some radius, say $r=4M$, at which general relativistic effects are not too strong. (The starting radius influences $\Delta\bar{t}$ below, but becomes less and less important as one approaches closer and closer to the horizon.) Then Eq. (1.1) implies that one will see, plastered into the region between $r-2M=100$ microns and $r-2M=2$ microns, the near-horizon structure of fields and matter laid down there $\Delta\bar{t}=10$ to 11 hours ago. Beneath this, at $r-2M=2\times 10^{-18}$ to 4×10^{-20} cm, one will see the structure characteristic of $\Delta\bar{t}=20$–21 hours ago. These structures will be laid down one after another like ancient sediment deposits on the bottom of the sea. Ultimately, at $r-2M\sim\exp(-10^{12})$ cm, one will be probing so far into the past, $\Delta\bar{t}\sim t_*/3\sim 3\times 10^7$ yr, that one will see the hole's mass to be significantly different from that measured at larger radii.

This complex, multilayered, near-horizon structure can have no significant influence on the future ($t>t_2$) evolution of matter and fields outside the hole. It is entirely relic history, and we can ignore it with impunity in our computations of future evolution.

The membrane viewpoint "sweeps under the rug" this multilayered structure by "stretching the hole's horizon" to cover it up.[1,3] More specifically, in the membrane formalism one introduces a timelike "stretched horizon" ($\mathscr{H}^S$ of Fig. 1) which lies just outside the true horizon—far enough outside that the bulk of the irrelevant relic structure in absolute space $\mathscr{S}_t$ is hidden beneath it, but near enough to the true horizon that there is no significant evolution of the infalling matter and fields, as seen by freely falling observers, as they fall from the stretched horizon to the true horizon. This stretched horizon acts as a surrogate for the true horizon in the membrane formalism: Its properties and the properties of fields at location $\mathscr{P}_1$ in Fig. 1 are nearly identical to those same properties on the true horizon at the point $\overline{\mathscr{P}}_1$ (which is connected to $\mathscr{P}_1$ by an ingoing null ray; dotted line of Fig. 1). As the true horizon evolves in response to interactions with the external universe, the stretched horizon evolves in almost identically the same way. And whereas, in the usual approach to black-hole physics, the true horizon acts as the boundary of the external universe, in the membrane approach the stretched horizon acts as the boundary. Thus it is that throughout this paper the phrase "at the horizon" is almost always equivalent to "at the stretched horizon."

Previous papers in this series[1–3] derived and studied the boundary conditions which must be imposed on electromagnetic fields at the stretched horizon for the special case of a black hole which, aside from tiny perturbations, is stationary and axisymmetric. Those boundary conditions are essentially identical to the boundary conditions on the true horizon, as developed by Znajek[4] and Damour;[5] and they are most elegantly expressed in the language of Znajek and Damour. The stretched horizon is regarded as endowed with a surface charge that annuls the normal component of electric field (Gauss's law) and a surface current that annuls the tangential magnetic field (Ampère's law); the surface current is proportional to the tangential electric field (Ohm's law with a surface resistivity $R_H=4\pi=377$ ohms per square); charge is conserved at the stretched horizon, with any charge that falls in from the external universe remaining always on the stretched horizon (no penetration into the interior) until it is annihilated by charge of opposite sign; current flowing in the stretched horizon produces Ohmic dissipation (entropy increase) and correspondingly, through the horizon's first law of thermodynamics, produces an increase of the hole's mass; and electromagnetic fields interact with the charge and current of the stretched horizon to produce a Lorentz force and a corresponding change in the hole's angular momentum of rotation. These electromagnetic properties of the stretched horizon, together with the $3+1$ equations of plasma physics in the absolute space outside the stretched horizon, produce an elegant and physically simple description[2,13] of the Blandford-Znajek process[14] by which magnetized, rotating, supermassive black holes might power quasars and active galactic nuclei.

C. Overview of this paper

This paper extends the membrane viewpoint to encompass dynamical (i.e., non-Kerr) black holes, and to include a detailed description of the stretched horizon's kinematic properties and their fluidlike evolution in response to the driving forces of gravity and electromagnetism.

We begin in Sec. II with a review of the kinematics of the true horizon, viewed as a 3-dimensional null surface in 4-dimensional spacetime, and a review of the equations which govern the true horizon's evolution. Then in Sec. III we introduce a $3+1$ split of spacetime in the vicinity of the true horizon, we define the stretched horizon, and we study the manner in which the kinematic properties of the stretched horizon closely mirror those of the true horizon. Sections II and III are couched in the language of standard general relativity without introducing any membranelike physical interpretations of the equations. In Secs. IV and V we develop the membrane interpretation of the formalism, i.e., the "membrane paradigm." Section IV deals with electromagnetic properties of the stretched horizon, extending the formalism of previous papers to dynamical holes. Section V deals with gravitational, mechanical, and thermodynamic properties of the stretched horizon. There we see how the stretched horizon can be regarded as a membrane endowed not only with electric charge and current, but also with the properties of a two-dimensional viscous fluid whose surface-layer stress-energy tensor annuls the horizon's extrinsic curvature; and we see how this fluid intercepts and conserves all energy and momentum that enter the stretched horizon from the outside universe. The stretched horizon's evolution is generated by its laws of momentum conservation (Navier-Stokes equation) and energy conservation (dissipation equation), together with a law describing how the fluid's shear is driven by tidal gravity. This fluid description of the stretched horizon mirrors and augments an analogous description of the true horizon due to Damour.[6,7] Section VI gives a concluding summary of the key equations of the membrane paradigm.

Appendixes A, B, and E present mathematical details which, if left in the body of the paper, would get in the way of the presentation. Appendixes C and D discuss important concepts and issues which are tied to specific parts of the paper, but which cannot be understood fully until later parts of the paper have been read: a *canonical form* for the spacetime metric near the horizon (Appendix C), and the *slicing transformations* that quantify the elements of nonuniqueness in the membrane formalism (Appendix D).

II. THE MATHEMATICAL DESCRIPTION OF THE HORIZON

A. Definitions and notation

Because spacetime will be sectioned in so many different ways in this article, we take the opportunity at the outset to define the various notations that will help the reader to identify what section is being considered. Different conventions for tensorial indices will be used self-consistently for the different manifolds considered: spacetime, the horizon, horizon sections, spacelike and timelike hypersurfaces in spacetime; see Table I.

The 3-dimensional, null, absolute event horizon will be denoted $\mathscr{H}$; and it will be endowed with a well-behaved time coordinate $\bar{t}$. The choice of $\bar{t}$ (which, for the moment, will be arbitrary) is said to determine the "slicing" of the horizon into 2-dimensional spatial manifolds $\mathscr{H}_{\bar{t}}$ at constant $\bar{t}$. The slices $\mathscr{H}_{\bar{t}}$ are coordinatized by spatial coordinates x^a $(a=2,3)$. The full set of coordinates on the horizon is denoted $x^{\bar{A}}$ $(x^{\bar{0}}=\bar{t},x^2,x^3)$.

Because we do not attempt to deal with horizon caustics, through each event $\overline{\mathscr{P}}$ on the horizon there passes precisely one horizon generator (null geodesic). We normalize the null tangent $\boldsymbol{l}$ to the generator at $\overline{\mathscr{P}}$ by requiring that

$$\langle d\bar{t},\boldsymbol{l}\rangle=1\ , \tag{2.1}$$

so that the sections $\mathscr{H}_{\bar{t}}$ are Lie dragged by the $\boldsymbol{l}$ congruence. As a consequence of Eq. (2.1) we can write

$$\boldsymbol{l}=\frac{\partial}{\partial\bar{t}}+v^a\frac{\partial}{\partial x^a} \tag{2.2}$$

for some "velocity" v^a, which will *not* play a significant role in this paper. The spatial coordinates x^a can always be chosen (except at caustics) to be constant on generators. For this choice, which we will call "comoving x^a," we have

$$v^a=0 \text{ for comoving } x^a\ . \tag{2.3}$$

The basis $\mathbf{e}_{\bar{A}}$ that spans the horizon will consist of $\mathbf{e}_{\bar{0}}=\boldsymbol{l}$ and $\mathbf{e}_a$ $(a=2,3)$ where $\mathbf{e}_a$ lie in $\mathscr{H}_{\bar{t}}$ and $\boldsymbol{l}\cdot\mathbf{e}_a=0$, but $\mathbf{e}_a$ are not necessarily $\partial/\partial x^a$ (e.g., one might occasionally want to use an orthonormal $\mathbf{e}_a$). For the dual basis $\boldsymbol{\omega}^{\bar{A}}$ we note that $\boldsymbol{\omega}^{\bar{0}}=d\bar{t}$ since $\langle d\bar{t},\boldsymbol{l}\rangle=1$ and since $\mathbf{e}_a$ lies in $\mathscr{H}_{\bar{t}}$ so $\langle d\bar{t},\mathbf{e}_a\rangle=0$. The basis $\mathbf{e}_{\bar{A}}$ will *not* in general be a coordinate basis, and in fact can be a coordinate basis corresponding to $x^{\bar{A}}$ only if the spatial coordinates x^a are comoving [cf. Eqs. (2.2) and (2.3)].

The 2-dimensional metric tensor in $\mathscr{H}_{\bar{t}}$ is denoted $\gamma_{ab}\equiv\mathbf{e}_a\cdot\mathbf{e}_b$ and its inverse is γ^{ab}. Indices of tensors in $\mathscr{H}_{\bar{t}}$ (e.g., σ^H_{ab} to be introduced below) are raised and lowered with this metric. Covariant differentiation with respect to γ_{ab} is denoted by $||$, as in $\sigma^{H\ b}_{a\ \ ||b}$.

Lower case greek indices, $\alpha,\beta,\gamma,\ldots$, range over 0–3 and indicate spacetime coordinates and components. In Sec. III we shall use a time coordinate $x^0=t\equiv$"universal time" (not equal to $\bar{t}$) to slice spacetime $\mathscr{S}$ outside $\mathscr{H}$ into 3-dimensional slices $\mathscr{S}_t$ called absolute space. Indices in absolute space are $i,j,k,\ldots$, ranging over 1,2,3.

B. Kinematic horizon fields and their evolution

The kinematics and evolution of the absolute event horizon $\mathscr{H}$ were first elucidated by Hawking[15,16] and by

TABLE I. Manifolds and index conventions.

Symbol	Meaning	Indices	Covariant differentiation
$\mathscr{S}$	Spacetime (4-dimensional)	$\alpha,\beta,\gamma,\ldots(=0,1,2,3)$	$\nabla\mathbf{V}$, $V^\alpha{}_{;\beta}$
$\mathscr{H}$	Horizon (3-dimensional null)	$\bar{A},\bar{B},\bar{C},\ldots(=0,2,3)$	$V^{\bar{A}}{}_{\mid\bar{B}}$
$\mathscr{H}_{\bar{t}}$	Section of $\mathscr{H}$ at constant $\bar{t}$ (2-dimensional spacelike)	$a,b,c,\ldots(=2,3)$	$V^a{}_{\|b}$
$\mathscr{S}_t$	Section of $\mathscr{S}$ at fixed universal time t; "absolute space" (3-dimensional spacelike)	$i,j,k,\ldots(=1,2,3)$	$V^j{}_{\mid k}$
$\mathscr{H}^S$	Stretched horizon (3-dimensional timelike)	$A,B,C,\ldots(=0,2,3)$	$V^A{}_{\mid B}$
$\mathscr{H}^S{}_t$	Section of stretched horizon (often simply "stretched horizon"); $\mathscr{H}^S\cap\mathscr{S}_t$ (2-dimensional spacelike)	$a,b,c,\ldots(=2,3)$	$V^a{}_{\|b}$

Hawking and Hartle[17] using the Newman-Penrose[18] null-tetrad formalism; and they were translated into tensorial notation by Damour.[6,7] In this section we will review Damour's description of the kinematics and evolution, but without reference to his membranelike interpretations of the equations. The membrane viewpoint will be delayed until Secs. IV and V.

Since l is tangent to a geodesic of spacetime, $\nabla_l l \propto l$ where ∇ is the covariant derivative in 4-dimensional spacetime $\mathscr{S}$. The proportionality constant is called the surface gravity of the horizon, g_H:

$$\nabla_l l \equiv g_H l \ . \tag{2.4}$$

The horizon expansion θ_H and shear σ^H_{ab} are the trace and trace-free parts of the projection of ∇l into $\mathscr{H}_{\bar t}$, i.e.,

$$\theta_H \equiv \theta_a{}^a = \gamma^{ab}\theta_{ab} \ , \tag{2.5a}$$

$$\sigma^H_{ab} \equiv \theta_{ab} - \tfrac{1}{2}\gamma_{ab}\theta_H \ , \tag{2.5b}$$

where

$$\theta_{ab} \equiv (\nabla_a l)\cdot \mathbf{e}_b \equiv l_{b||a} \ . \tag{2.5c}$$

Note that θ_{ab} is symmetric since the generator congruence is rotation free. The "Hajicek field" Ω^H_a describes the part of $\nabla_a l$ given by

$$\Omega^H_a \equiv \langle \omega^{\bar 0}, \nabla_a l \rangle \ . \tag{2.6}$$

Like surface gravity, shear, and expansion, this quantity has played a role in the mathematical theory of black holes in the past.[19] Its primary importance lies in its connection with the total angular momentum J of an axisymmetric hole. If $\mathbf{e}_\phi$, lying in $\mathscr{H}_{\bar t}$, is the Killing vector which generates rotations about the symmetry axis and dA is the area element on the horizon $\mathscr{H}_{\bar t}$, then[20]

$$J = \int_{\mathscr{H}_{\bar t}} (-\Omega^H_\phi / 8\pi)\, dA \ . \tag{2.7}$$

The horizon fields g_H, θ_H, σ^H_{ab}, and Ω^H_a are kinematic in that they describe the nature of the l congruence. Moreover, they are all components of a single geometric entity, the horizon's extrinsic curvature $K_H{}^{\bar B}{}_{\bar A}$, which is defined[21] by

$$\nabla_{\bar A} l \equiv -K_H{}^{\bar B}{}_{\bar A}\mathbf{e}_{\bar B} \ . \tag{2.8}$$

(Damour[6] uses the opposite sign for $K_H{}^{\bar B}{}_{\bar A}$ and calls it the "Weingarten map.") The components $K_H{}^a{}_l \equiv K_H{}^a{}_{\bar 0}$ vanish [cf. Eq. (2.4)] and the other components of $\overleftrightarrow{K}_H$ are

$$K_H{}^l{}_l = -g_H \ , \tag{2.9a}$$

$$K_H{}^l{}_a = -\Omega^H{}_a \ , \tag{2.9b}$$

$$K_H{}^a{}_b = -\sigma^{Ha}{}_b - \tfrac{1}{2}\theta_H \delta^a{}_b \ , \tag{2.9c}$$

$$\theta_H = -K_H{}^a{}_a \ , \quad \sigma^H_{ab} = -\gamma_{ac}K_H{}^c{}_b + \tfrac{1}{2}\gamma_{ab}K_H{}^c{}_c \ . \tag{2.9d}$$

It is straightforward to verify that in comoving coordinates, where $l = (\partial/\partial \bar t)_{x^a}$,

$$K^H_{ab} \equiv \gamma_{ac}K_H{}^c{}_b = -\tfrac{1}{2}\partial\gamma_{ab}/\partial\bar t \tag{2.10a}$$

so that [cf. Eq. (2.9c)]

$$\theta_H = \frac{\partial}{\partial \bar t}\ln\sqrt{\gamma} \ , \quad \sigma^H_{ab} = \frac{1}{2}\left[\frac{\partial\gamma_{ab}}{\partial \bar t} - \theta_H\gamma_{ab}\right] . \tag{2.10b}$$

The kinematic horizon fields are tensors in $\mathscr{H}_{\bar t}$. We can describe their evolution with passing time $\bar t$ using a covariant time derivative $D_{\bar t}$ defined with respect to spacetime parallel transport. More specifically, if Ψ_{ab} is a tensor in $\mathscr{H}_{\bar t}$, then $D_{\bar t}\Psi_{ab}$ is also a tensor in $\mathscr{H}_{\bar t}$ which, viewed as a 4-tensor in spacetime, is given by

$$D_{\bar t}\Psi_{\alpha\beta} \equiv \Psi_{\mu\nu;\lambda} l^\lambda \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \ . \tag{2.11}$$

Here $\gamma^\alpha{}_\beta$, the 2-metric $\gamma^a{}_b$ viewed as a 4-tensor, plays the role of a projection operator, projecting into $\mathscr{H}_{\bar t}$. In terms of the time derivative $D_{\bar t}$ the evolution of the horizon expansion is governed by the "focusing equation"

$$D_{\bar t}\theta_H = g_H\theta^H - \tfrac{1}{2}\theta_H{}^2 - \sigma^H_{ab}\sigma^{ab}_H - 8\pi T_{\bar 0\bar 0} \ , \tag{2.12}$$

the evolution of the shear is governed by the "tidal-force equation"

$$D_{\bar t}\sigma^H_{ab} + (\theta_H - g_H)\sigma^H_{ab} = -C_{a\bar 0 b\bar 0} \ , \tag{2.13}$$

and the evolution of the Hajicek field is governed by the Hajicek equation

$$D_{\bar t}\Omega^H_a + (\sigma_a^{H\ c} + \tfrac{1}{2}\delta^c_a\theta_H)\Omega^H_c + \theta_H\Omega^H_a = (g_H + \tfrac{1}{2}\theta_H)_{,a} - \sigma_a^{H\ b}{}_{||b} + 8\pi T_{\bar 0 a} \ . \tag{2.14}$$

Here $C_{\bar A\bar B\bar C\bar D}$ and $T_{\bar A\bar B}$ are the components, along the horizon's basis vectors, of the Weyl curvature tensor of spacetime and of the stress-energy tensor of matter and fields falling through the horizon; and in Eqs. (2.12) and (2.14) the Einstein field equations have been assumed. Equations (2.12)–(2.14) can be verified by straightforward manipulation, but are most easily derived in the Newman-Penrose formalism; see Appendix A. The focusing equation (2.12) and tidal-force equation (2.13) are Sachs's optical scalar equations[22] in tensorial notation.

C. Slicing transformations

For a stationary, axisymmetric black hole there is a preferred choice of the horizon time coordinate $\bar t$ [Eqs. (2.22) and (2.23) below]; but for a dynamical, nonaxisymmetric hole the time slicing is somewhat arbitrary. It will be important later to understand how the horizon fields change when one changes the time slicing. For simplicity we shall *use comoving coordinate bases throughout this section.*

Consider two different slicings of the horizon, $\bar t$ and $\bar t'$; and for both slicings use the same comoving spatial coordinates so that

$$\bar t = \bar t(\bar t', x^{a'}) \ , \quad \bar t' = \bar t'(\bar t, x^a) \ , \tag{2.15a}$$

$$x^{a'} = x^a \ \text{ are comoving coordinates (constant on generators)} \ . \tag{2.15b}$$

If we define

$$Y \equiv \left[\frac{d\bar{t}}{d\bar{t}'}\right]_{\text{along generator}} = \left[\frac{\partial \bar{t}}{\partial \bar{t}'}\right]_{x^{a'}} , \tag{2.16a}$$

$$G \equiv \frac{1}{Y}\left[\frac{\partial Y}{\partial \bar{t}'}\right]_{x^{a'}} = \left[\frac{\partial Y}{\partial \bar{t}}\right]_{x^a} , \tag{2.16b}$$

$$W_a \equiv \left[\frac{\partial \bar{t}}{\partial x^{a'}}\right]_{\bar{t}'} = -Y\left[\frac{\partial \bar{t}'}{\partial x^a}\right]_{\bar{t}} , \tag{2.16c}$$

$$A_a \equiv \frac{1}{Y}\left[\frac{\partial W_a}{\partial \bar{t}'}\right]_{x^{b'}} = \frac{1}{Y}\left[\frac{\partial Y}{\partial x^{a'}}\right]_{\bar{t}'} = \left[\frac{\partial W_a}{\partial \bar{t}}\right]_{x^b} = \frac{1}{Y}\left[\frac{\partial Y}{\partial x^a} + G W_a\right] , \tag{2.16d}$$

then the relationships between the primed and unprimed bases are

$$\begin{aligned} &\mathbf{e}_{\bar{0}'} = \mathbf{l}' = Y\mathbf{l} = Y\mathbf{e}_{\bar{0}} , \\ &\omega^{\bar{0}'} = d\bar{t}' = Y^{-1}(\omega^{\bar{0}} - W_a\omega^a) = Y^{-1}(d\bar{t} - W_a dx^a) , \\ &\mathbf{e}_{a'} = \frac{\partial}{\partial x^{a'}} = \mathbf{e}_a + W_a \mathbf{l} = \frac{\partial}{\partial x^a} + W_a \frac{\partial}{\partial \bar{t}} , \\ &\omega^{a'} = dx^{a'} = \omega^a = dx^a . \end{aligned} \tag{2.17}$$

The behavior of the horizon fields under slicing transformations can be derived from these equations and the definitions (2.4)–(2.6):

$$\gamma'_{a'b'} = \gamma_{ab} , \tag{2.18a}$$

$$\theta'_H = Y\theta_H , \tag{2.18b}$$

$$\sigma'^{H}_{a'b'} = Y\sigma^H_{ab} , \tag{2.18c}$$

$$g'_H = Y g_H + G , \tag{2.18d}$$

$$\Omega'^{H}_{a'} = \Omega^H_a - [\sigma^{H\ b}_{a} + (\tfrac{1}{2}\theta_H - g_H)\delta^b_a]W_b + A_a , \tag{2.18e}$$

$$T_{\bar{0}'\bar{0}'} = Y^2 T_{\bar{0}\bar{0}} , \tag{2.18f}$$

$$T_{\bar{0}'a'} = Y(T_{\bar{0}a} + W_a T_{\bar{0}\bar{0}}) , \tag{2.18g}$$

$$C_{a'\bar{0}'b'\bar{0}'} = Y^2 C_{a\bar{0}b\bar{0}} . \tag{2.18h}$$

The first three equations show that γ_{ab}, θ_H, and σ^H_{ab} at any point on the horizon are independent of the manner in which $\mathscr{H}$ is sliced into sections $\mathscr{H}_{\bar{t}}$, aside from a change in scale (factor Y) due to the change in the ticking rate of time $\bar{t}$. (For this reason θ_H and σ^H_{ab}, rewritten in Newman-Penrose notation, have been called "optical scalars."[22]) The changes (2.18d) and (2.18e) in g_H and Ω^H_a are not so simple; but one can see their mathematical naturalness when one combines them with (2.18b) and (2.18c) into a single geometric equation for the change in the horizon's extrinsic curvature:

$$\overleftrightarrow{\mathbf{K}}' = Y\overleftrightarrow{\mathbf{K}} - \mathbf{l} \otimes dY . \tag{2.18i}$$

We shall reach a deeper understanding of these slicing transformation laws when we see them in the context of a 3 + 1 split of spacetime near the horizon (Appendix D).

Notice that by an appropriate choice of slicing one can make the hole's surface gravity g_H take on any value one wishes [Eq. (2.18d)], even zero. However, in practice we will tie our horizon slicing to a natural slicing of spacetime outside the horizon, thereby constraining g_H severely.

For the special case of a static, nonrotating black hole (which need not be axisymmetric[23]), spacetime outside the horizon possesses a unique, timelike Killing vector field $\mathbf{k}$ whose norm far from the hole is unity. In the limit as one approaches the horizon $\mathscr{H}$, $\mathbf{k}$ becomes null and tangent to the horizon generators. In this case it turns out (see Appendix B for a proof) that there exists a preferred horizon slicing $\bar{t}$ ("canonical slicing"), unique up to a time translation

$$\bar{t}' = \bar{t} + \text{const} , \tag{2.19}$$

for which

$$\mathbf{l} = (\partial/\partial\bar{t})_{x^a} = \mathbf{k} , \tag{2.20a}$$

$$\Omega^H_a = \sigma^H_{ab} = \theta_H = 0 , \tag{2.20b}$$

$$g_H = \text{const} , \tag{2.20c}$$

$$\partial\gamma_{ab}/\partial\bar{t} = 0 . \tag{2.20d}$$

As an example, for a Schwarzschild black hole with mass M and "horizon radius" $r_H = 2M$, the surface gravity and the 2-metric in spherical polar coordinates (θ,ϕ) are

$$\begin{aligned} &g_H = 1/4M , \quad \gamma_{\theta\theta} = r_H{}^2 , \\ &\gamma_{\phi\phi} = r_H{}^2\sin^2\theta , \quad \gamma_{\theta\phi} = 0 . \end{aligned} \tag{2.21}$$

For the special case of a stationary, rotating black hole (which must be axisymmetric[15]), spacetime possesses two independent, commuting Killing vector fields: $\mathbf{k}$ = ("generator of time translations") which is timelike with unit norm far from the hole, and ξ = ("generator of rotations about axis of symmetry"), which is spacelike with orbits that close on themselves after parameter length 2π. In this case it turns out (see Appendix B for proof) that there exists a preferred horizon slicing $\bar{t}$ ("canonical slicing"), unique up to a time translation

$$\bar{t}' = \bar{t} + \text{const} , \tag{2.22}$$

for which[24]

$$\xi \text{ lies in } \mathscr{H}_{\bar{t}} , \tag{2.23a}$$

$$\mathbf{l} = \mathbf{k} + \Omega_H \xi , \quad \Omega_H \equiv (\text{horizon angular velocity}) = \text{const} , \tag{2.23b}$$

$\mathbf{\Omega}_H$ is parallel to ξ with proportionality factor independent of $\bar{t}$, (2.23c)

$$\sigma^H_{ab} = \theta_H = 0 , \tag{2.23d}$$

$$g_H = \text{const} , \tag{2.23e}$$

$$\partial\gamma_{ab}/\partial\bar{t} = 0 . \tag{2.23f}$$

(Reluctantly we have adhered to past conventions and have used the scalar Ω_H to denote the horizon's angular velocity, and the vector $\mathbf{\Omega}_H$ or Ω_H^a to denote the horizon's Hajicek field; the reader should be wary and not confuse them with each other.) As an example, for a Kerr black hole with mass M, angular momentum J, rotation parameter $a \equiv J/M$, and "horizon radius" $r_H \equiv M + (M^2 - a^2)^{1/2}$, described in polar coordinates (θ, ϕ) with ranges $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$ and with $\xi = \partial/\partial\phi$, it turns out that

$$g_H = \frac{r_H - M}{2Mr_H}\,, \quad \Omega_H = \frac{a}{2Mr_H}\,, \tag{2.24a}$$

$$\Omega_\theta^H = 0\,, \quad \Omega_\phi^H = -\frac{a \sin^2\theta}{\rho_H{}^4}\,[\rho_H{}^2 r_H + M(r_H{}^2 - a^2\cos^2\theta)]\,, \tag{2.24b}$$

$$\gamma_{\theta\theta} = \rho_H{}^2 \equiv r_H{}^2 + a^2\cos^2\theta\,, \quad \gamma_{\phi\phi} = \frac{(2Mr_H)^2}{\rho_H{}^2}\sin^2\theta\,,$$
$$\gamma_{\theta\phi} = 0\,. \tag{2.24c}$$

It is straightforward to verify that the integral of $-\Omega_\phi^H/8\pi$ [Eq. (2.24b)] over the horizon's area $dA = (\gamma_{\theta\theta}\gamma_{\phi\phi})^{1/2} d\theta\, d\phi$ is equal to the hole's total angular momentum $J = Ma$, in accord with Eq. (2.7).

For fully dynamical holes none of the simplifications of static or stationary holes apply. We have, however, the possibility of specializing to slices with spatially constant g_H. We shall explore the constant-g_H slicing transformations and their generalization to slowly variable g_H in Appendix D.

III. THE 3 + 1 SPLIT OF SPACETIME NEAR THE HORIZON

We turn attention now from the hole's 3-dimensional, absolute event horizon $\mathcal{H}$ to 4-dimensional spacetime $\mathcal{S}$ outside and near the horizon. In Sec. III A we introduce a set of spacetime coordinates tied to our slicings $\bar{t}$ of the horizon. Then in Sec. III B we use these coordinates to perform the 3 + 1 split of spacetime $\mathcal{S}$ into "absolute space" $\mathcal{S}_t$ plus "universal time" t; and we define the hole's stretched horizon and relate its kinematic properties and evolution to those of the true horizon. Section III B focuses, for simplicity, on slicings for which the hole's surface gravity g_H is constant. In Sec. III C we repeat our study of the stretched horizon and its properties for slicings with slowly varying surface gravity; and in Appendix C we present, for the slowly varying case, a canonical form of the near-horizon spacetime metric—a form which meshes nicely with our 3 + 1 split, and we exhibit this metric's relationship to the Boyer-Lindquist form of the Kerr metric.

A. Carter coordinates

We introduce here a special class of coordinates for spacetime outside and near the horizon, which are the generalization to dynamical black holes of the coordinates used by Carter[25] in his proof that surface gravity is constant on stationary horizons.

We begin by introducing on the horizon $\mathcal{H}$ a specific time $\bar{t}$ and a specific set of spatial coordinates x^a which comove with the horizon's generators. At each point on $\mathcal{H}$ we introduce a future-directed ingoing null vector $\mathbf{n}$ which is orthogonal to the slice $\mathcal{H}_{\bar{t}}$ and has unit inner product with the horizon generator $\mathbf{l}$:

$$\mathbf{n}\cdot\mathbf{e}_a = 0\,, \quad \mathbf{n}\cdot\mathbf{l} = -1\,. \tag{3.1}$$

We then construct the unique congruence of ingoing null geodesics ("ingoing rays") which at the horizon $\mathcal{H}$ are tangent to $\mathbf{n}$ (Fig. 2); and we carry the coordinates $(\bar{t}, x^a)$ outward on these rays into the surrounding spacetime. As our fourth spacetime coordinate we use the affine parameter λ of the null-geodesic rays, with $\lambda = 0$ and $d/d\lambda = -\mathbf{n}$ on $\mathcal{H}$. Note that this construction corresponds to

$$-\left(\frac{\partial}{\partial\lambda}\right)_{x^a,\bar{t}} \text{ is tangent to ingoing null-geodesic ray}\,, \tag{3.2a}$$

$$\text{the horizon } \mathcal{H} \text{ is at } \lambda = 0\,, \tag{3.2b}$$

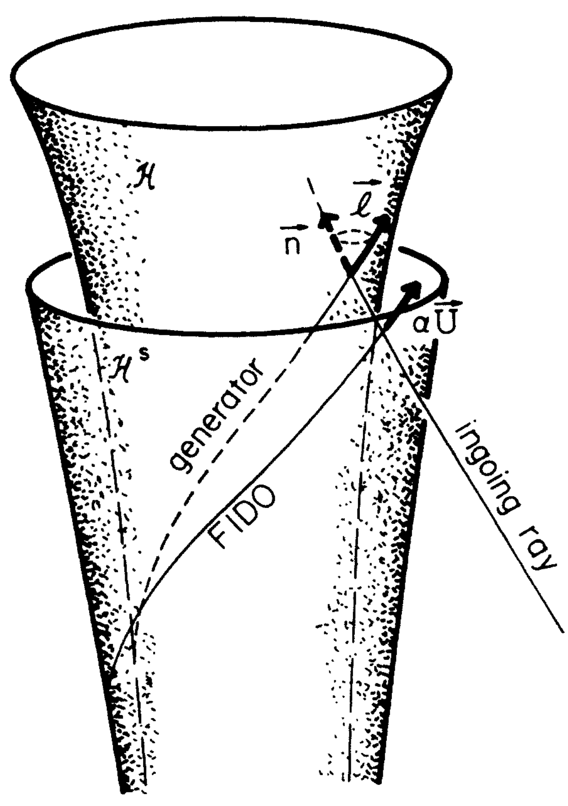

FIG. 2. Equivalence of horizon generators and fiducion world lines. In the horizon limit $\alpha\mathbf{U} \to \mathbf{l}$ along an ingoing ray.

$$\left[\frac{\partial}{\partial\lambda}\right]_{x^a,\bar{t}} = -\mathbf{n} \text{ at } \lambda=0 \,. \tag{3.2c}$$

In this coordinate system the metric coefficients $g_{\lambda\mu}$ (with $x^\mu=\bar{t},\lambda,x^a$) are independent of λ. This follows from the facts that $\partial/\partial\lambda$ and $\partial/\partial x^\mu$ commute, and that $\partial/\partial\lambda$ is null and satisfies the geodesic equation

$$0=\left[\frac{\partial}{\partial\lambda}\right]\cdot\left[\nabla_\lambda\frac{\partial}{\partial x^\mu}-\nabla_\mu\frac{\partial}{\partial\lambda}\right]=\left[\frac{\partial}{\partial\lambda}\right]\cdot\nabla_\lambda\frac{\partial}{\partial x^\mu}$$

$$=\nabla_\lambda g_{\lambda\mu}-\left[\frac{\partial}{\partial x^\mu}\right]\cdot\nabla_\lambda\frac{\partial}{\partial\lambda}$$

$$=\frac{\partial g_{\lambda\mu}}{\partial\lambda} \,, \tag{3.3}$$

where we have used the shorthand notation $\nabla_\lambda\equiv\nabla_{\partial/\partial\lambda}$, $\nabla_\mu\equiv\nabla_{\partial/\partial x^\mu}$. The λ-independent values of $g_{\lambda\mu}$ can be read off Eqs. (3.1) and (3.2c): $g_{\lambda\lambda}=g_{\lambda a}=0$, $g_{\lambda\bar{t}}=1$. We denote $g^{\lambda\lambda}$ by $\mathscr{I}$, g^{ab} by γ^{ab}, and the inverse of γ^{ab} by γ_{ab}. [Note that γ_{ab} will be the same as the horizon metric plus terms of $O(\lambda)$.] We further define $b^a=-g^{\lambda a}$ and $b_a=\gamma_{ab}b^b$. With these notations the covariant and contravariant metric coefficients are

$$g_{\mu\nu}=\begin{bmatrix} b^a b_a-\mathscr{I} & 1 & b_a \\ 1 & 0 & 0 \\ b_a & 0 & \gamma_{ab}\end{bmatrix} ,$$

$$g^{\mu\nu}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & \mathscr{I} & -b^a \\ 0 & -b^a & \gamma^{ab}\end{bmatrix} . \tag{3.4}$$

The horizon's extrinsic curvature [Eq. (2.8)] has components

$$K_H{}^{\bar{0}}{}_{\bar{A}}=-\langle d\bar{t},\nabla_{\bar{A}}l\rangle=-\Gamma^{\bar{0}}{}_{\bar{0}\bar{A}}=-\Gamma_{\lambda\bar{0}\bar{A}}=\tfrac{1}{2}g_{\bar{0}\bar{A},\lambda} \,;$$

which, together with $K_H{}^{\bar{0}}{}_{\bar{0}}=-g_H$, $K_H{}^{\bar{0}}{}_a=-\Omega_a^H$ and the vanishing of $b_a=g_{\bar{0}a}$ and $\mathscr{I}=-g_{\bar{0}\bar{0}}+b^a b_a$ on the horizon ($\lambda=0$), tells us that

$$\mathscr{I}=2g_H\lambda+O(\lambda^2) \,, \quad b_a=-2\Omega_a^H\lambda+O(\lambda^2) \,. \tag{3.5}$$

Thus, accurate to first order in λ the spacetime metric is

$$ds^2=-(2g_H\lambda)d\bar{t}^2+2d\bar{t}d\lambda$$
$$+\gamma_{ab}(dx^a-2\Omega_H^a\lambda d\bar{t})(dx^b-2\Omega_H^b\lambda d\bar{t})$$
$$+O(\lambda^2) \,. \tag{3.6}$$

In this spacetime metric the values of $g_H(\bar{t},x^a)$ and $\Omega_H^a(\bar{t},x^a)$ depend, of course, on the original choice $\bar{t}$ of horizon slicing. By performing a change of $\bar{t}$ slicing before constructing the coordinate system, one obtains a metric (3.6) whose g_H and Ω_H^a are changed in accord with the slicing transformations (2.18).

B. 3 + 1 split and stretched horizon for slicings with constant g_H

1. The 3+1 split, the stretched horizon, and FIDO's

We now specialize, for simplicity of analysis, to a slicing for which the surface gravity g_H is constant, i.e., is independent both of time $\bar{t}$ and spatial location x^a on the horizon $\mathscr{H}$.

The foundation for our 3 + 1 split of spacetime is a choice of "universal time" t and corresponding "absolute space" $\mathscr{S}_t$=(3-surface of constant t) outside the horizon $\mathscr{H}$. We must not choose $t=\bar{t}$ because the 3-surfaces of constant $\bar{t}$ are everywhere null. Instead, our choice for t must be one such that $\mathscr{S}_t$ is spacelike and the congruence of timelike curves orthogonal to $\mathscr{S}_t$ (the FIDO world lines) "snuggle up to the horizon" in the manner described in Sec. I A and Fig. 1.

A suitable such choice, and the one we shall make, is

$$t=\bar{t}-\frac{1}{2g_H}\ln(2g_H\lambda)+O(\lambda) \,. \tag{3.7}$$

Since the FIDO world lines are orthogonal to $\mathscr{S}_t$ the FIDO 4-velocity $\mathbf{U}$, viewed as a 1-form $\widetilde{U}$, is given by

$$\widetilde{U}\equiv-\alpha\, dt \,, \tag{3.8}$$

where α is the "lapse function" for the 3 + 1 slicing and relates FIDO proper time τ to universal time t along a FIDO world line by

$$d\tau/dt=\alpha \,. \tag{3.9}$$

From $\mathbf{U}^2=-1$, expression (3.8) for $\widetilde{U}$, and the form (3.4), (3.5) of the metric it is straightforward to show that

$$\alpha=(2g_H\lambda)^{1/2}+O(\lambda^{3/2}) \,. \tag{3.10}$$

It is convenient for many purposes to use α rather than λ as our radial coordinate; for example, the 4-velocity $\mathbf{U}$ corresponding to (3.8) is written

$$\mathbf{U}=\left[\frac{1}{\alpha}+O(\alpha)\right]\frac{\partial}{\partial\bar{t}}+O(\alpha^3)\frac{\partial}{\partial\lambda}+\left[\frac{\Omega_H{}^a}{g_H}\alpha+O(\alpha^3)\right]\frac{\partial}{\partial x^a}$$

$$=\left[\frac{1}{\alpha}+O(\alpha)\right]\frac{\partial}{\partial\bar{t}}+O(\alpha^2)\frac{\partial}{\partial\alpha}+\left[\frac{\Omega_H{}^a}{g_H}\alpha+O(\alpha^3)\right]\frac{\partial}{\partial x^a} \,. \tag{3.11}$$

As discussed in Sec. I B, it is important in the membrane formalism to stretch the horizon so as to get rid of irrelevant layers of matter and fields which are relics of past history. We shall choose our instantaneous stretched horizon $\mathscr{H}^S_{t_0}$, at some initial moment of universal time t_0, to be at $\alpha=\alpha_{H_0}$ where $\alpha_{H_0} \ll 1$ is a constant; and by carrying this $\mathscr{H}^S_{t_0}$ forward in time along FIDO world lines we shall generate the 3-dimensional stretched horizon $\mathscr{H}^S$. We shall denote by $\alpha_H(x^a,t)$ the value of the lapse function α at location (x^a,t) on the stretched horizon $\mathscr{H}^S$. Thus, FIDO world lines are generators of $\mathscr{H}^S$; and, by Eqs. (3.9) and (3.11), the derivative of the lapse function with respect to universal time along these generators is $d\alpha_H/dt=O(\alpha_H{}^3)$, which means that, aside from fractional errors of $O(\alpha_H{}^2)$ which we shall typically neglect, the stretched horizon is a 3-surface of constant lapse function, $\alpha_H=\text{const}+O(\alpha_H{}^3)$. (If, after a long time, α_H evolves to become significantly spatially nonconstant, we shall readjust the location of the stretched horizon to make it constant again.)

Our spacetime is now sectioned by t into the 3-space $\mathscr{S}_t$ of constant universal time t ("absolute space") with coordinates x^a and λ (or α); and it is also sectioned by $\alpha=\alpha_H$ into the stretched horizon $\mathscr{H}^S$, coordinatized by x^a and t. The 2-dimensional spatial intersection $\mathscr{H}^S_t$ of $\mathscr{S}_t$ and $\mathscr{H}^S$ (the "instantaneous stretched horizon") is coordinatized by x^a.

Although t and α are fundamental in establishing the sectioning of spacetime, the spatial Carter coordinates x^a, which comove with the horizon generators, are not fundamental; and for astrophysical applications it often is more convenient to use noncomoving spatial coordinates $x^{a'}$ [e.g., the Boyer-Lindquist angular coordinates $\theta^\dagger$ and $\phi^\dagger$ of Eqs. (C7); Appendix C]. Such a change of x^a will cause at most a transformation among themselves of the basis vectors $\mathbf{e}_a$ lying in $\mathscr{H}^S_t$. The key equations derived in the remainder of this paper will be tensorial in $\mathscr{H}^S_t$ and hence will not refer specifically to Carter coordinates x^a; but many of our derivations of key equations will rely on Carter coordinates. In practice the Carter x^a are useful for deriving relationships, while other coordinates $x^{a'}$ are used in applications. Similarly, the full Carter-type spacetime coordinates $(\bar{t},\lambda,x^a)$ are useful in deriving relationships but $(t,\alpha,x^{a'})$ mesh more nicely with the 3 + 1 split and are more useful for astrophysical calculations; see Appendix C.

2. *Fiducions in the stretched horizon*

It will be conceptually useful to introduce in the stretched horizon $\mathscr{H}^S$ a family of timelike curves that are more tightly locked to the generators of the true horizon $\mathscr{H}$ than are the FIDO world lines, and to imagine that these generator-locked curves are the world lines of a family of fiducial particles ("fiducions"). Since the Carter spatial coordinates x^a in $\mathscr{H}^S$ are locked by null rays to the generators of $\mathscr{H}$, it is natural to choose the fiducions to be at rest in these coordinates. This will, in fact, be our choice for the case of a stationary or static black hole:

$$(dx^a/dt)_{\text{FN}}=0 \quad \text{for a stationary or static hole}, \tag{3.12a}$$

where x^a are Carter coordinates and "FN" means "fiducion." Notice [cf. Eq. (3.11)] that the FIDO's in $\mathscr{H}^S$ see the fiducions move with physical velocity

$$\begin{aligned} v^a_{\text{FN}} &= \left[\frac{dt}{d\tau}\right]\left[\left[\frac{dx^a}{dt}\right]_{\text{FN}} - \left[\frac{dx^a}{dt}\right]_{\text{FIDO}}\right] \\ &= -\left[\frac{\Omega^a_H}{g_H}\right]\alpha_H \ . \end{aligned} \tag{3.12b}$$

In the case of a dynamical hole it turns out to be best to lock the fiducions of $\mathscr{H}^S$ to the generators of $\mathscr{H}$ using a different family of null rays than were used in locking the Carter coordinates x^a. By an appropriate choice of locking rays, we obtain a fiducion physical velocity (relative to FIDO's), v^a_{FN}, which satisfies the following equation valid in any $\mathbf{e}_a$ basis (not necessarily Carter):

$$[(g_H-\tfrac{1}{2}\theta_H)\delta^a{}_b-\sigma^{Ha}{}_b]v^b_{\text{FN}}=-\alpha_H\Omega^a_H \quad \text{in general} . \tag{3.13}$$

(The required locking rays are outward, past-directed null geodesics such that the ray which starts at a point $\overline{\mathscr{P}}$ on a given generator of $\mathscr{H}$ emerges from $\overline{\mathscr{P}}$ orthogonal to a very special 2-flat $\mathscr{F}_{\overline{\mathscr{P}}}\subset\mathscr{H}$; this $\mathscr{F}_{\overline{\mathscr{P}}}$ is defined by the demand that, if ξ is a vector in $\mathscr{F}_{\overline{\mathscr{P}}}$ then $\nabla_\xi \mathbf{l}$ is also in $\mathscr{F}_{\overline{\mathscr{P}}}$. At any point $\overline{\mathscr{P}}$ on the horizon a slicing $\bar{t}$ can be found such that $\mathscr{F}_{\overline{\mathscr{P}}}$ is tangent to $\mathscr{H}_{\bar{t}}$, but in general a single slicing cannot be made which agrees with $\mathscr{F}_{\overline{\mathscr{P}}}$ at all points $\overline{\mathscr{P}}$ in a finite patch of the horizon. One can prove these statements using the slicing-transformation formalism of Sec. II C and Appendix D.) This choice of fiducions is actually possible and unique if and only if Eq. (3.13) can be inverted to give v^a_{FN}, i.e., if and only if

$$(g_H-\tfrac{1}{2}\theta_H)^2-\tfrac{1}{2}\sigma^H_a{}^b\sigma^H_b{}^a=0 \ . \tag{3.14}$$

Condition (3.13) is never violated for weakly perturbed holes, since they have $g_H \gg (\,|\theta_H|$ or $|\sigma^H_{ab}|\,)$. If a hole is so strongly perturbed from equilibrium that (3.14) is violated somewhere, then the focusing equation (2.12) for θ_H is likely to drive θ_H to infinity, producing caustics in the horizon which the membrane formalism in its present state of development is not prepared to handle. If, nevertheless, one wishes to apply the formalism to situations where (3.14) is violated, one can switch from the definition (3.13) of fiducions to (3.12a). The resulting formalism will be totally unchanged except for the conceptually attractive description, in Eq. (5.20c) below, of the physical origin of the horizon's momentum.

3. *Kinematics of the fiducions and FIDO's in the stretched horizon*

The FIDO's in the stretched horizon use as a natural set of spacetime basis vectors

$$\mathbf{e}_{\hat{0}}\equiv\mathbf{U}_{\text{FIDO}} \ , \quad \mathbf{e}_a \ , \quad \mathbf{N} \ . \tag{3.15}$$

Here $\mathbf{N}$ is the outward unit normal to $\mathscr{H}^S$, and the $\mathbf{e}_a$ are a pair of basis vectors lying in $\mathscr{H}^S_t$, which are obtained by smoothly transporting an arbitrary basis $\mathbf{e}_a$ in $\mathscr{H}_{\bar{t}}$ along the coordinate-locking rays of constant Carter $(\bar{t},x^a)$. (In mathematical derivations one might want to specialize to $\mathbf{e}_a = \partial/\partial x^a$.) Note that $\mathbf{N}$ and $\mathbf{e}_a$ lie in absolute space $\mathcal{S}_t$ while $\mathbf{e}_{\hat{0}}$ is orthogonal to it; $\mathbf{e}_{\hat{0}}$ and $\mathbf{e}_a$ lie in the stretched horizon $\mathcal{H}^S$ while $\mathbf{N}$ is orthogonal to it; $\mathbf{e}_a$ lie in the instantaneous, 2-dimensional stretched horizon $\mathcal{H}_t^S \equiv \mathcal{H}^S \cap \mathcal{S}_t$ while $\mathbf{N}$ and $\mathbf{e}_{\hat{0}}$ are orthogonal to it; and the spacetime metric in this basis is

$$g_{\hat{0}\hat{0}} = -1 \ , \quad g_{NN} = +1 \ , \quad g_{ab} = \gamma_{ab} \ ; \tag{3.16}$$

all others vanish. It will be important below to relate this basis to the true horizon's null basis $\boldsymbol{l}$, $\mathbf{n}$, $\mathbf{e}_a$. We shall describe the relationship of the two bases in terms of a limiting process $\rightarrow$ defined as "in the limit $\alpha_H \rightarrow 0$ with a chosen point of $\mathcal{H}^S$ moving inward along the coordinate-locking ray $(\bar{t}, x^a) = \text{const}$ until it reaches the corresponding point of $\mathcal{H}$." It is straightforward to verify from the definitions of $\mathbf{U}$, $\mathbf{N}$, and $\mathbf{e}_a$ that in this limit

$$\alpha_H \mathbf{e}_{\hat{0}} \rightarrow \boldsymbol{l} \ , \quad \alpha_H \mathbf{N} \rightarrow \boldsymbol{l} \ , \quad \frac{1}{\alpha_H}(\mathbf{e}_{\hat{0}} - \mathbf{N}) \rightarrow \mathbf{n} \ ,$$
$$(\mathbf{e}_a)_{\text{in } \mathcal{H}_t^S} \rightarrow (\mathbf{e}_a)_{\text{in } \mathcal{H}_{\bar{t}}} \ . \tag{3.17}$$

Note that in the limit $\rightarrow$ the world lines of the stretched-horizon fiducions and FIDO's both coalesce into coincidence with the horizon's generators

$$\alpha_H \mathbf{U}_{\text{FIDO}} \rightarrow \boldsymbol{l} \ , \quad \alpha_H \mathbf{U}_{\text{FN}} \rightarrow \boldsymbol{l} \ . \tag{3.18}$$

As a result, aside from fractional errors of order $\alpha_H{}^2$, the FIDO's and fiducions in $\mathcal{H}^S$ have the same expansion θ, shear $\vec{\sigma}$, and gravitational acceleration $\mathbf{g} \equiv -(4\text{-acceleration})$; and when renormalized with α_H to make them finite, these coalesce into the horizon's expansion, shear, and surface gravity:

$$\alpha_H \theta \equiv \alpha_H U^A{}_{|A} = \theta_H [1 + O(\alpha_H{}^2)] \rightarrow \theta_H \ , \tag{3.19a}$$

$$\alpha_H \sigma_{ab} \equiv \alpha_H (U_{(a|b)} - \tfrac{1}{2}\gamma_{ab}\theta) = \sigma_{ab}^H [1 + O(\alpha_H{}^2)] \rightarrow \sigma_{ab}^H \ , \tag{3.19b}$$

$$\begin{aligned} \alpha_H{}^2 \mathbf{g} &\equiv -\alpha_H{}^2 \nabla_{\mathbf{U}} \mathbf{U} \\ &= g_H(-\alpha_H \mathbf{N})[1 + O(\alpha_H{}^2)] + O(\alpha_H{}^4)\mathbf{e}_a \rightarrow -g_H \boldsymbol{l} \ , \end{aligned} \tag{3.19c}$$

$$\alpha_H g = g_H [1 + O(\alpha_H{}^2)] \rightarrow g_H \ . \tag{3.19d}$$

Here $g \equiv |\mathbf{g}|$; and the index notation of Table I is being used: $|$ denotes covariant derivative in $\mathcal{H}^S$ and the index A denotes components of a vector that lies in $\mathcal{H}^S$ [i.e., A ranges over $\hat{0}$, 2, 3 if the basis of Eq. (3.15) is used, or over 0, 2, 3 if the coordinate basis (t, x^a) is used]. Equations (3.19) are equally valid for the FIDO's and the fiducions.

The stretched-horizon equivalent of the Hajicek field involves the gradient of $\mathbf{N}$ in a spatial direction:

$$\mathbf{U} \cdot \nabla_a \mathbf{N} = U^\mu N_{\mu;a} = -\Omega_a^H [1 + O(\alpha_H{}^2)] \rightarrow -\Omega_a^H \ . \tag{3.20}$$

A rather different appearance can be given to the correspondence between quantities on $\mathcal{H}$ and on $\mathcal{H}^S$ if we introduce the extrinsic curvature $K^B{}_A$ of the stretched horizon:

$$\nabla_A \mathbf{N} = -K^B{}_A \mathbf{e}_B \ . \tag{3.21}$$

The field differentiated is the unit normal $\mathbf{N}$, not the FIDO 4-velocity, so [by contrast with the horizon's extrinsic curvature (2.8)] there is no immediate relationship to characteristic properties (acceleration, expansion, shear) of the FIDO and fiducion congruence. Nevertheless, because $\alpha \mathbf{N} \rightarrow \boldsymbol{l}$, one can show that

$$\alpha_H K^{\hat{0}}{}_{\hat{0}} \equiv \alpha_H \mathbf{U} \cdot \nabla_{\mathbf{U}} \mathbf{N} = K_H{}^{\bar{0}}{}_{\hat{0}} [1 + O(\alpha_H{}^2)] \rightarrow K_{H\bar{0}}^{\bar{0}} = -g_H \ , \tag{3.22a}$$

$$K^{\hat{0}}{}_a \equiv \mathbf{U} \cdot \nabla_a \mathbf{N} = K_{Ha}^{\bar{0}} [1 + O(\alpha_H{}^2)] \rightarrow K_{Ha}^{\bar{0}} = -\Omega_a^H \ , \tag{3.22b}$$

$$\begin{aligned} \alpha_H K_{ab} &\equiv -\alpha_H \mathbf{e}_a \cdot \nabla_b \mathbf{N} = K_{ab}^H [1 + O(\alpha_H{}^2)] \\ &\rightarrow K_{ab}^H = -(\sigma_{ab}^H + \tfrac{1}{2}\gamma_{ab}\theta_H) \ , \end{aligned} \tag{3.22c}$$

and that in comoving coordinates the time derivatives of the stretched-horizon metric coefficients are

$$\frac{\partial \gamma_{ab}}{\partial t} \rightarrow -2K_{ab}^H = 2(\sigma_{ab}^H + \tfrac{1}{2}\gamma_{ab}\theta_H) \tag{3.22d}$$

[cf. Eqs. (2.10)].

The FIDO's of our membrane formalism are not unique. Each different choice of the horizon-slicing time function $\bar{t}$ will produce, by the constructions of Secs. III A and III B, a different universal time t and a different family of FIDO's. This arbitrariness is greatly reduced when $\bar{t}$ is much larger (in units of g_H^{-1}) than the time at which the last generators join the horizon. See Appendix D for details.

C. 3 + 1 split and stretched horizon for slicings with slowly variable g_H

The choice made above of constant-g_H slicing gives the simplest introduction to the concept of the stretched horizon. It is not, however, ideally suited to problems of astrophysical interest. Astrophysical problems typically involve a stationary Schwarzschild or Kerr hole in the distant past, with a well-defined value of g_H [Eq. (2.21) or (2.24a)]. The hole undergoes perturbations due to stress-energy penetrating the horizon or due to the tidal fields of far-off bodies; and as a result the mass and angular momentum of the hole change, but on a time scale usually much longer than the characteristic time g_H^{-1} for the hole. A long time ($\gg g_H^{-1}$) after the perturbations act the hole again becomes stationary, but with a new (well-defined) value of g_H. To describe such a situation most aesthetically, we clearly need to make a slicing with a time-dependent g_H which agrees with the initial and final values of g_H and which "tracks" the hole's evolution during the era of perturbations in a physically reasonable way. [Less pleasing, but acceptable, would be a constant-g_H slicing during the evolution, followed by a slicing transformation with constant g_H' (Sec. II C and Appendix D) after the evolution is finished.]

Not only must a fully pleasing "tracking" g_H be time dependent, it must also be spatially variable (a function of x^a). Otherwise the final horizon slicing $\bar{t}$ will differ from the canonical slicing of Eqs. (2.22)–(2.24) by $\bar{t}=\bar{t}_{\rm canonical}+\Phi(x^a)$; and, as a result, the final Hajicek field will not have the canonical Kerr form [cf. Eqs. (2.24b) and (2.18e)].

It turns out that the membrane formalism gets into severe difficulties if one tries to evolve with a rapidly variable g_H; but for a slowly variable g_H the results of the last section are valid except for trivial changes and negligible errors. (For a discussion of the allowed slicing transformations in this case see Appendix D.)

By "slowly variable g_H" we mean a slicing $\bar{t}$ of the horizon $\mathscr{H}$ such that there exists a time scale t_* for which

$$g_H t_* \ggg 1 \;, \tag{3.23a}$$

$$\frac{|g_{H,\bar{t}}|}{g_H} \lesssim \frac{1}{t_*} \;, \qquad \frac{|g_{H,a}g_{H,b}\gamma^{ab}|^{1/2}}{g_H} \lesssim \frac{1}{t_*} \;. \tag{3.23b}$$

We shall refer to t_* as the "evolution time scale for g_H."

We now assume that a horizon slicing $\bar{t}$ has been chosen for which g_H satisfies conditions (3.23); and from that slicing we construct Carter-type spacetime coordinates $(\bar{t},\lambda,x^a)$ in the manner of Sec. III A. We then face the issue of choosing a universal time function $t(\bar{t},\lambda,x^a)$ which meshes properly with the horizon slicing $\bar{t}$. The key meshing properties that we need, in order to make the stretched-horizon kinematics track the true-horizon kinematics in the manner of Eqs.(3.18)–(3.22), are

$$\alpha \mathbf{U}_{\rm FIDO} \to \mathbf{l} \;, \tag{3.24a}$$

$$\frac{1}{\alpha}\frac{d\alpha}{dt}=\frac{d\alpha}{d\tau}\equiv \alpha_{,\mu}U^{\mu}_{\rm FIDO}\to 0 \;, \tag{3.24b}$$

where α and $\mathbf{U}_{\rm FIDO}$ are defined by

$$\mathbf{U}_{\rm FIDO}\equiv -\alpha\nabla t \;, \quad |\mathbf{U}_{\rm FIDO}|\equiv 1 \;. \tag{3.24c}$$

[Condition (3.24c) says that the FIDO world lines are orthogonal to the slices $\mathscr{S}_t$ of constant time t, and that $\alpha=d\tau/dt$ is the FIDO lapse function; condition (3.24a) says that the FIDO congruence coalesces into the generator congruence as the stretched horizon approaches the true horizon; and condition (3.24b) says that the FIDO's near the stretched horizon move in surfaces of constant α.]

Unfortunately, it seems impossible to achieve conditions (3.24a) and (3.24b) simultaneously if the horizon's $\bar{t}$ slicing has variable g_H. On the other hand, if we make (as we shall) the same choice of universal time as in the constant-g_H case

$$t\equiv\bar{t}-\frac{1}{2g_H}\ln(2g_H\lambda)+O(\lambda) \;, \tag{3.25}$$

the conditions (3.24) can "almost" be achieved, in the following sense: The FIDO 4-velocity and lapse function as defined by (3.24c) will then be

$$\alpha\mathbf{U}_{\rm FIDO}=\frac{1}{1+O(\delta)}\left[\frac{\partial}{\partial\bar{t}}+2g_H\lambda O(\delta+g_H\lambda)\frac{\partial}{\partial\lambda}+2\mathbf{\Omega}_H\lambda+O(\delta)g_H\lambda\mathbf{e}_{\hat{a}}\right] , \tag{3.26a}$$

$$\alpha=\left[\frac{2g_H\lambda}{1+O(\delta)}\right]^{1/2}+O(\lambda^{3/2}) \;, \tag{3.26b}$$

where $\mathbf{e}_{\hat{a}}$ are unit basis vectors in the 2-space spanned by $\partial/\partial x^2$ and $\partial/\partial x^3$, and

$$\delta\equiv\frac{|\ln(g_H\lambda)|}{g_H t_*} \;, \tag{3.26c}$$

and they will satisfy

$$\frac{d\alpha}{d\tau}=O(\alpha^2 g_H)+O\left[\frac{\delta}{1+\delta}g_H\right] . \tag{3.26d}$$

[The notation in the second term of (3.26d) indicates that the time variation leads to $d\alpha/d\tau=O(g_H)$ if δ is not small; and in Eq. (3.26a) there are δ-independent, higher-order corrections which one can read off Eq. (3.11).] Note that in the $\to$ limit, as $\lambda\to 0$, δ becomes logarithmically infinite causing a violation of conditions (3.24a) and (3.24b). However, the blow up of δ becomes a problem only exceedingly close to the true horizon: As long as we restrict attention to $\lambda\gtrsim\lambda_{\rm MSH}$, where

$$\lambda_{\rm MSH}\equiv\frac{\ln(g_H t_*)}{g_H t_*}\frac{1}{2g_H} \;, \tag{3.27}$$

α^2 will be $\gtrsim\delta$, and the effects of δ will be negligible. Note that $\lambda_{\rm MSH}$ corresponds to

$$\alpha_{\rm MSH}{}^2\cong 2g_H\lambda_{\rm MSH}=\frac{\ln(g_H t_*)}{g_H t_*} \simeq 1\times 10^{-11} \text{ if hole and slicing have } M=10^8 M_\odot \;, \; g_H\simeq 1/4M \;, \; t_*=10^8\,\text{yr} \;. \tag{3.28}$$

We shall refer to the location $\alpha=\alpha_{\rm MSH}$ as the "minimally stretched horizon" and shall insist that the stretched horizon always be chosen at

$$\alpha_H\geq\alpha_{\rm MSH} \;. \tag{3.29}$$

Then a repetition of the constant-g_H analysis of Sec. III B for the case of slowly variable g_H shows that the stretched-horizon kinematics still closely approximate those of the true horizon; i.e., Eqs. (3.19)–(3.22) remain valid in the form

$$\alpha_H\theta=\theta_H[1+O(\alpha_H{}^2)] \;, \tag{3.30a}$$

$$\alpha_H\sigma_{ab}=\sigma^H_{ab}[1+O(\alpha_H{}^2)] \;, \tag{3.30b}$$

$$\alpha_H g = g_H[1+O(\alpha_H{}^2)]\ , \tag{3.30c}$$

$$\mathbf{U}\cdot\nabla_a \mathbf{N} = -\Omega_a^H[1+O(\alpha_H{}^2)]\ , \tag{3.30d}$$

$$\alpha_H K^{\hat 0}{}_{\hat 0} = K^{\bar 0}_{H\bar 0}[1+O(\alpha_H{}^2)] = -g_H[1+O(\alpha_H{}^2)]\ , \tag{3.30e}$$

$$\alpha_H K^{\hat 0}{}_a = K^{\bar 0}_{Ha}[1+O(\alpha_H{}^2)] = -\Omega_a^H[1+O(\alpha_H{}^2)]\ , \tag{3.30f}$$

$$\alpha_H K_{ab} = K^H_{ab}[1+O(\alpha_H{}^2)] = -(\sigma^H_{ab} + \tfrac{1}{2}\gamma_{ab}\theta^H)[1+O(\alpha_H{}^2)]\ , \tag{3.30g}$$

$$\frac{\partial \gamma_{ab}}{\partial t} = -2K^H_{ab}[1+O(\alpha_H{}^2)] = 2(\sigma^H_{ab} + \tfrac{1}{2}\gamma_{ab}\theta^H)[1+O(\alpha_H{}^2)] \quad \text{in comoving coordinates}\ . \tag{3.30h}$$

These relations will be a key foundation for the construction of gravitational aspects of the membrane paradigm in Sec. V.

IV. THE MEMBRANE PARADIGM FOR ELECTROMAGNETIC FIELDS

Now that the mathematical foundations for the membrane paradigm have been laid, we can proceed with developing the paradigm itself. As a first step, we digress from our study of horizon kinematics and focus attention, temporarily, on the structure and evolution of electromagnetic fields near a black-hole horizon. We do this because there are close similarities between the membrane treatments of electromagnetic fields and of gravitational fields; and we will want to highlight those similarities when translating into membrane language the horizon kinematics of Secs. II and III.

The membrane formalism for electromagnetic fields around a Schwarzschild or Kerr black hole has been developed and discussed in previous papers in this series.[1–3] The extension of that formalism to the more general context of a dynamical hole with slowly varying surface gravity follows directly from the ideas presented in Sec. III above, together with the general 3 + 1 split of Maxwell's equations as developed in Ref. 1.

In our 3 + 1 viewpoint the electromagnetic field is described by the electric field and magnetic field that a FIDO measures. In terms of the electromagnetic field tensor $F^{\mu\nu}$ and the FIDO 4-velocity U^μ these are

$$E^\mu \equiv F^{\mu\nu}U_\nu\ , \tag{4.1a}$$

$$B^\mu \equiv \tfrac{1}{2}\epsilon^{\nu\mu\alpha\beta}F_{\alpha\beta}U_\nu\ , \tag{4.1b}$$

where $\epsilon^{\nu\mu\alpha\beta}$ is the Levi-Civita tensor in spacetime. Since $\mathbf{E}\cdot\mathbf{U}=\mathbf{B}\cdot\mathbf{U}=0$, we can regard $\mathbf{E}$ and $\mathbf{B}$ as 3-vectors in absolute space $\mathscr{S}_t$. Similarly, we split the electromagnetic 4-current J^μ into the charge density ρ_e and current density $\mathbf{j}$ measured by a FIDO

$$\rho_e \equiv -J^\mu U_\mu\ , \quad j^\mu \equiv J^\mu - \rho_e U^\mu\ , \tag{4.1c}$$

which are a scalar and vector residing in absolute space $\mathscr{S}_t$. Maxwell's equations are then written and studied as 3-dimensional vector equations ($B^i{}_{|i}=0$, $E^i{}_{|i}=4\pi\rho_e$, etc.) in absolute 3-dimensional space; cf. Sec. 3.2 of Ref. 1.

Because the FIDO 4-velocity $\mathbf{U}$ is singular near the horizon [Eq. (3.18)], the FIDO-measured fields $\mathbf{E}$ and $\mathbf{B}$ are typically also singular. In order to study the behaviors of these fields near the horizon it is useful to introduce the tetrad of an observer with nonsingular motion near the horizon. The prototype of such an observer is one who falls freely and radially into a Schwarzschild hole from rest at spatial infinity. It is straightforward to show that this observer is seen by FIDO's to move inward with a Lorentz factor γ given by $\gamma=\alpha^{-1}=(1-2M/r)^{-1/2}$. In a more general spacetime we shall take our freely falling observer (FFO) to have a 4-velocity near the horizon of the form

$$\mathbf{U}_{\rm FFO} = A\partial/\partial\bar t - C\partial/\partial\lambda\ , \tag{4.2}$$

so that the FFO passes through the stretched horizon in the normal direction $-\mathbf{N}$, and through the true horizon with motion normal to $\mathscr{H}_{\bar t}$. The condition that this motion be well behaved at the horizon is that A and C be positive and of order unity as $\alpha\to 0$. Consequently, near the horizon this FFO will be observed by FIDO's to move with a Lorentz γ factor

$$\gamma = -\mathbf{U}_{\rm FFO}\cdot\mathbf{U}_{\rm FIDO} = C/\alpha \tag{4.3}$$

which, as in the Schwarzschild case, diverges as α^{-1}.

Because the FFO is physically well behaved, it must measure well-behaved electric and magnetic fields $\mathbf{E}_{\rm FFO}$ and $\mathbf{B}_{\rm FFO}$ as it falls through the horizon; and the singularity in the FIDO-measured $\mathbf{E}$ and $\mathbf{B}$ at the horizon are related to the divergence of the γ factor between the FIDO and the FFO motion. We can study the singular behavior of the FIDO-measured fields by decomposing them into pieces normal to and parallel to the stretched horizon. The decomposition into normal and parallel pieces is done using the projection tensor

$$\overleftrightarrow{\gamma} \equiv \gamma^{ab}\mathbf{e}_a\otimes\mathbf{e}_b \equiv {}^{(3)}\overleftrightarrow{\mathbf{g}} - \mathbf{N}\otimes\mathbf{N}\ , \tag{4.4}$$

where ${}^{(3)}\overleftrightarrow{\mathbf{g}}$ is the metric of absolute space, $(\mathbf{N},\mathbf{e}_a)$ are the absolute-space basis vectors of Eq. (3.15), and γ^{ab} is the metric in $\mathscr{H}^S_t$. More specifically, we define on $\mathscr{H}^S_t$

$$E_N \equiv \mathbf{E}\cdot\mathbf{N}\ , \quad B_N \equiv \mathbf{B}\cdot\mathbf{N}\ , \tag{4.5a}$$

$$\mathbf{E}_{||} \equiv \overleftrightarrow{\gamma}\cdot\mathbf{E}\ , \quad \mathbf{B}_{||} \equiv \overleftrightarrow{\gamma}\cdot\mathbf{B} \quad \text{at } \alpha=\alpha_H\ , \tag{4.5b}$$

and in terms of these the relationship of $\mathbf{E}$ and $\mathbf{B}$ to $\mathbf{E}_{\rm FFO}$ and $\mathbf{B}_{\rm FFO}$ is given by a standard Lorentz boost with velocity $\beta\mathbf{N}$ and γ factor $\gamma\equiv(1-\beta^2)^{-1/2}=O(\alpha_H{}^{-1})$:

$$E_N = E_N^{\rm FFO}\ , \quad B_N = B_N^{\rm FFO}\ , \tag{4.6a}$$

$$\mathbf{E}_{||} = \gamma(\mathbf{E}_{||}^{\rm FFO} + \beta\mathbf{N}\times\mathbf{B}_{||}^{\rm FFO})\ , \quad \mathbf{B}_{||} = \gamma(\mathbf{B}_{||}^{\rm FFO} - \beta\mathbf{N}\times\mathbf{E}_{||}^{\rm FFO})\ . \tag{4.6b}$$

Since $\mathbf{E}^{\rm FFO}$ and $\mathbf{B}^{\rm FFO}$ are nonsingular, Eqs. (4.6) show that the normal components of $\mathbf{E}$ and $\mathbf{B}$ are well behaved as $\alpha_H\to\alpha_{\rm MSH}$, but the transverse components diverge as α_H^{-1}.

It is straightforward to show, by analogy with the kinematic equations (3.30), that

$$\alpha_H E^a_{\|} = E^a_{ZD}[1+O(\alpha_H{}^2)] , \quad E_N = E^{\bar{0}}_{ZD}[1+O(\alpha_H{}^2)] , \tag{4.7a}$$

$$\alpha_H B^a_{\|} = B^a_{ZD}[1+O(\alpha_H{}^2)] , \quad B_N = B^{\bar{0}}_{ZD}[1+O(\alpha_H{}^2)] , \tag{4.7b}$$

where

$$E^{\bar{A}}_{ZD} \equiv F^{\bar{A}}{}_{\bar{0}} , \quad B^{\bar{A}}_{ZD} \equiv \tfrac{1}{2}\epsilon_{\bar{0}}{}^{\bar{A}\alpha\beta}F_{\alpha\beta} \text{ evaluated on } \mathscr{H} \tag{4.8}$$

are the (nonsingular) horizon electric and magnetic fields defined by Znajek[4] and Damour[5–7] and used in their horizon formalisms for black holes.

In the membrane formalism we abandon all attempts to study the electric and magnetic fields beneath the stretched horizon, and as a result we abandon any pretense at knowing *precisely* the fields $E^{\bar{A}}_{ZD}$ and $B^{\bar{A}}_{ZD}$ on the true horizon. Instead we content ourselves with an approximate knowledge of the horizon fields—a knowledge with fractional errors of $O(\alpha_H{}^2)$. In this spirit, we define "stretched-horizon fields" E_N, B_N [Eqs. (4.5a)] and

$$\mathbf{E}_H \equiv \alpha_H \mathbf{E}_{\|} , \quad \mathbf{B}_H \equiv \alpha_H \mathbf{B}_{\|} , \tag{4.9}$$

which approximate the Znajek-Damour fields (4.8) to within $O(\alpha_H{}^2)$ and which thus also obey the equations of Damour's horizon formalism to within errors of $O(\alpha_H{}^2)$; and we use Damour's horizon equations as stretched-horizon boundary conditions on the electromagnetic field of the external universe.

For comparison with the gravitational analysis of the next section, we shall sketch here a derivation of Damour's electromagnetic horizon equations in the context of the stretched horizon.

The inverse of the Lorentz boost equations (4.6) (i.e., $\mathbf{E}_{\rm FFO}$, $\mathbf{B}_{\rm FFO}$ in terms of $\mathbf{E}$, $\mathbf{B}$) gives

$$\mathbf{E}_{\|} - \mathbf{N}\times\mathbf{B}_{\|} = O(\alpha_H) , \quad \mathbf{B}_{\|} + \mathbf{N}\times\mathbf{E}_{\|} = O(\alpha_H) , \tag{4.10}$$

where we have used $\beta = 1 - O(\alpha_H{}^2)$. In terms of horizon fields these relations read

$$\mathbf{E}_H = \mathbf{N}\times\mathbf{B}_H + O(\alpha_H{}^2) , \quad \mathbf{B}_H = -\mathbf{N}\times\mathbf{E}_H + O(\alpha_H{}^2) . \tag{4.11}$$

These equations have the simple physical interpretation that to FIDO's at the stretched horizon, moving outward at nearly the speed of light, the parallel components of the electromagnetic field have the form of ingoing plane waves. Thus, the membrane is a perfect absorber of electromagnetic radiation.

In the membrane formalism the interaction of the horizon with external electromagnetic fields is understood by the artifice of attributing to the stretched horizon the electrical properties of a physical membrane. Specifically, the membrane is regarded as having a charge density σ_H which terminates the normal component of the electric field in accord with *Gauss's law:*

$$\sigma_H \equiv E_N / 4\pi . \tag{4.12}$$

The membrane is also regarded as electrically conductive: it possesses a horizon surface current density $\mathscr{J}_H$, a vector in $\mathscr{H}^S_t$ which measures the charge that flows across a unit length per unit universal time. This surface current density is defined by the demand that it terminate the tangential component of the magnetic field in accord with *Ampère's law:*

$$4\pi \mathscr{J}_H \times \mathbf{N} = \mathbf{B}_H . \tag{4.13}$$

The stretched-horizon Gauss law (4.12) and Ampère law (4.13) are merely definitions of the (fictitious but conceptually useful) surface densities of charge σ_H and current $\mathscr{J}_H$. The payoff of these definitions is the beautiful forms that they give to the stretched-horizon electromagnetic boundary conditions. In particular, (i) the "ingoing plane-wave condition" (4.11), by virtue of the horizon's Ampère law (4.13), is equivalent to *Ohm's law* for the stretched horizon:

$$\mathbf{E}_H = R_H \mathscr{J}_H , \tag{4.14a}$$

$$R_H \equiv 4\pi \simeq 377 \text{ ohms per square} ; \tag{4.14b}$$

i.e., the stretched horizon behaves as though it had a surface electrical resistivity R_H. Also, (ii) the normal component of the 3-dimensional Faraday law just above the stretched horizon [Eqs. (3.4c) of Ref. 1], by virtue of the horizon's Gauss and Ampère laws (4.12) and (4.13), is equivalent to the *law of charge conservation* for the stretched horizon

$$\partial\sigma_H/\partial t + \mathscr{J}_H{}^a{}_{\|a} + j^N_H = 0 . \tag{4.15}$$

Here

$$-j^N_H \equiv -\alpha_H \mathbf{j}\cdot\mathbf{N} \tag{4.16}$$

is the rate per unit universal time t and per unit area that charge (falling inward with the FIDO-measured speed of light) flows into the stretched horizon from the external universe. [Note that, by virtue of Eqs. (3.17) and (3.18), $\mathbf{j}\cdot\mathbf{N} = \mathbf{J}\cdot\mathbf{N}$ and $-\rho_e \equiv \mathbf{J}\cdot\mathbf{U}$ are equal and diverge as $1/\alpha_H$ at the stretched horizon; thus j^N_H is well behaved.] Equation (4.15) says that all charge flowing into the membrane from outside remains forever in the membrane; i.e., the stretched-horizon membrane is impermeable to the passage of charge.

If one asks about the implications of the membrane paradigm for the region of absolute space immediately below the stretched horizon, one encounters a certain awkwardness. The paradigm would have E^N and $\mathbf{B}_H$ terminate at the membrane and thus vanish below it, but would leave B^N and $\mathbf{E}_H$ continuous and nonzero. But the vanishing of $\mathbf{B}_H$ in the interior implies (incorrectly), via Maxwell's laws, the vanishing of $\mathbf{E}_H$.

This awkwardness can be resolved in a number of ways. Perhaps the best is to confess that the membrane paradigm makes no pretense whatsoever to give any account of the region below the stretched horizon. It gives neither a correct account of the layer-upon-layer of fine-scale structure contained there, nor even an account of a fictitious interior that at least satisfies Maxwell's equations. Rather, it confines attention solely to the membrane's exterior and uses the membrane's properties (laws of Gauss, Ampère, Ohm, and charge conservation) to give a heuristically powerful description of the exterior's boundary conditions.

If, despite this confession, one seeks to treat the region below the membrane in a Maxwell-equation-correct way (while admitting that one's description has nothing to do with the real-world layers of structure there), perhaps the best way to do so is that developed by Znajek.[4] Znajek's version of the formalism (which was developed independently of and simultaneously with Damour's) endows the horizon not only with electric charge and current, but also with magnetic charge and current; and these charges and currents live in a thin boundary layer rather than being confined to a precisely two-dimensional membrane. In this variant Maxwell's equations are valid above, inside, and below the boundary layer; and below they drive all electric and magnetic fields to zero. This nice feature is bought, however, at a price which we have chosen not to pay: that of introducing on the horizon magnetic surface charges and currents which fail to mesh nicely with one's elementary physics experience.

Interior fields are irrelevant to the paradigm, and their nature constitutes an irrelevant point of principle except for one consequence: In a real conducting membrane Ohm's law relates current flow to the electric field in the material of the membrane. If there is ambiguity about $\mathbf{E}_H$ inside the membrane, this has no meaning; we must thus take "Ohm's law" in Eq. (4.14) to relate $\mathcal{J}_H$ to $\mathbf{E}_H$ just above the membrane. This does not detract in any way from the usefulness of the membrane viewpoint and in fact has been ignored or overlooked in previous papers in this series. It is made explicit here in order to help clarify related issues in the gravitational formalism of the following section (see end of Appendix E).

V. THE MEMBRANE PARADIGM FOR GRAVITATIONAL FIELDS

A. The 3 + 1 split of gravitational fields

We now turn attention from electromagnetic fields around a black hole to gravitational fields.

The Weyl tensor $C_{\mu\nu\lambda\sigma}$ is a gravitational analog of the electromagnetic field tensor; and just as the 3 + 1 split of spacetime induces a split of $F_{\mu\nu}$ into the 3-dimensional electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, so also it induces a split of $C_{\mu\nu\lambda\sigma}$ into a "gravitoelectric" tidal field $\overleftrightarrow{\mathscr{E}}$, and a "gravitomagnetic" tidal field $\overleftrightarrow{\mathscr{B}}$. These tidal fields are second-rank tensors defined in terms of $C_{\mu\nu\lambda\sigma}$ and the FIDO 4-velocity U^μ by[26]

$$\mathscr{E}_{\alpha\beta}\equiv C_{\alpha\mu\beta\nu}U^\mu U^\nu \ , \tag{5.1a}$$

$$\mathscr{B}_{\alpha\beta}\equiv \tfrac{1}{2}\epsilon_{\mu\alpha\rho\sigma}C^{\rho\sigma}{}_{\beta\nu}U^\mu U^\nu \tag{5.1b}$$

[cf. Eqs. (4.1)]. From the symmetries of the Weyl tensor it is straightforward to verify that (i) $\mathscr{E}_{\alpha\beta}$ and $\mathscr{B}_{\alpha\beta}$ are orthogonal to $\mathbf{U}$ and thus can be regarded as 3-tensors $\mathscr{E}_{ij}$, $\mathscr{B}_{ij}$ that reside in absolute space $\mathscr{S}_t$, (ii) $\overleftrightarrow{\mathscr{E}}$ and $\overleftrightarrow{\mathscr{B}}$ are symmetric and trace free,

$$\mathscr{E}_{jk}=\mathscr{E}_{kj}\ ,\quad \mathscr{E}^j{}_j=0\ ,\quad \mathscr{B}_{jk}=\mathscr{B}_{kj}\ ,\quad \mathscr{B}^j{}_j=0\ , \tag{5.2}$$

and thus have five independent components each, and (iii) the purely spatial components of $C_{\mu\nu\sigma\lambda}$ (i.e., the projection into $\mathscr{S}_t$) can be expressed as

$$C_{ipjq}=g_{ij}\mathscr{E}_{pq}+g_{pq}\mathscr{E}_{ij}-g_{iq}\mathscr{E}_{pj}-g_{pj}\mathscr{E}_{iq}\ , \tag{5.3}$$

where g_{ij} is the 3-metric in absolute space. It follows that all of the information in the Weyl tensor is contained in $\overleftrightarrow{\mathscr{E}}$ and $\overleftrightarrow{\mathscr{B}}$. The names gravitoelectric and gravitomagnetic are justified by the fact that, as measured by FIDO's $\overleftrightarrow{\mathscr{E}}$ governs the velocity-independent part, and $\overleftrightarrow{\mathscr{B}}$ the velocity-dependent part, of geodesic deviation.

Just as the study of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ near the stretched horizon is facilitated by decomposing them into normal and tangential parts [Eqs. (4.5)], so our near-horizon study of $\mathscr{E}_{jk}$ and $\mathscr{B}_{jk}$ will be facilitated by a similar decomposition. For the tidal fields the decomposition produces three parts: a "normal-normal" component

$$\mathscr{E}_{NN}\equiv\mathbf{N}\cdot\overleftrightarrow{\mathscr{E}}\cdot\mathbf{N}=N^i\mathscr{E}_{ij}N^j \tag{5.4a}$$

and similarly for $\overleftrightarrow{\mathscr{B}}$; a "normal-transverse" part (two components)

$$\boldsymbol{\mathscr{E}}^T\equiv\overleftrightarrow{\gamma}\cdot\overleftrightarrow{\mathscr{E}}\cdot\mathbf{N}\ ,\quad \text{i.e.,}\quad \mathscr{E}^T{}_a\equiv\mathscr{E}_{aj}N^j \tag{5.4b}$$

and similarly for $\overleftrightarrow{\mathscr{B}}$; and a "transverse-traceless" tensor in the transverse plane (two independent components)

$$\overleftrightarrow{\mathscr{E}}^{\rm TT}=\overleftrightarrow{\gamma}\cdot\overleftrightarrow{\mathscr{E}}\cdot\overleftrightarrow{\gamma}+\tfrac{1}{2}\overleftrightarrow{\gamma}\mathscr{E}_{NN}\ ,\quad \text{i.e.,}\quad \mathscr{E}^{\rm TT}_{ab}\equiv\mathscr{E}_{ab}+\tfrac{1}{2}\gamma_{ab}\mathscr{E}_{NN} \tag{5.4c}$$

and similarly for $\overleftrightarrow{\mathscr{B}}$. Note that tracelessness of $\mathscr{E}^{\rm TT}_{ab}$ follows from

$$\mathrm{Tr}(\overleftrightarrow{\mathscr{E}}^{\rm TT})=\gamma^{ab}\mathscr{E}_{ab}+\mathscr{E}_{NN}=g^{ij}\mathscr{E}_{ij}=0\ . \tag{5.5}$$

This decomposition of $\overleftrightarrow{\mathscr{E}}$ and $\overleftrightarrow{\mathscr{B}}$ is intimately related to the Newman-Penrose description of the Weyl tensor (cf. Appendix A). It should be noted in particular that it is $\overleftrightarrow{\mathscr{E}}^{\rm TT}$ and $\overleftrightarrow{\mathscr{B}}^{\rm TT}$ that carry gravitational-wave energy in the $\pm\mathbf{N}$ direction and that are encoded in the Newman-Penrose Weyl scalars Ψ_0 and Ψ_4.

Following the prescription of Sec. IV, we require that the Weyl tensor be nonsingular as observed by FFO's. The relationship to FIDO measurements is given by the Lorentz boost with velocity $\beta\mathbf{N}$ and $\gamma=(1-\beta^2)^{-1/2}\propto\alpha_H{}^{-1}$ from the FFO frame to the FIDO frame:

$$\mathscr{E}_{NN}=\mathscr{E}^{\rm FFO}_{NN}\ ,\quad \mathscr{B}_{NN}=\mathscr{B}^{\rm FFO}_{NN}\ , \tag{5.6a}$$

$$\boldsymbol{\mathscr{E}}^T=\gamma(\boldsymbol{\mathscr{E}}^T_{\rm FFO}+\beta\mathbf{N}\times\boldsymbol{\mathscr{B}}^T_{\rm FFO})\ ,\qquad \boldsymbol{\mathscr{B}}^T=\gamma(\boldsymbol{\mathscr{B}}^T_{\rm FFO}-\beta\mathbf{N}\times\boldsymbol{\mathscr{E}}^T_{\rm FFO})\ , \tag{5.6b}$$

$$\overleftrightarrow{\mathscr{E}}^{\rm TT}=\gamma^2[(1+\beta^2)\overleftrightarrow{\mathscr{E}}^{\rm TT}_{\rm FFO}+2\beta\mathbf{N}\times\overleftrightarrow{\mathscr{B}}^{\rm TT}_{\rm FFO}]\ , \tag{5.6c}$$

$$\overleftrightarrow{\mathscr{B}}^{\rm TT}=\gamma^2[(1+\beta^2)\overleftrightarrow{\mathscr{B}}^{\rm TT}_{\rm FFO}-2\beta\mathbf{N}\times\overleftrightarrow{\mathscr{E}}^{\rm TT}_{\rm FFO}]\ . \tag{5.6d}$$

Here we are using the notation

$$(\mathbf{N}\times\overleftrightarrow{\mathscr{E}})^{ij}\equiv\epsilon^{ikp}N_k\mathscr{E}_p{}^j\ , \tag{5.7}$$

where ϵ^{ikp} is the 3-dimensional Levi-Civita tensor in absolute space $\mathscr{S}_t$. It follows from Eqs. (5.6) that physically acceptable tidal fields (i.e., fields for which the FFO components are finite) must satisfy (i) $\mathscr{E}_{NN}$ finite, (ii) $\mathscr{E}^T$ and $\mathscr{B}^T$ of $O(\alpha_H{}^{-1})$, and (iii) $\overleftrightarrow{\mathscr{E}}^{TT}$ and $\overleftrightarrow{\mathscr{B}}^{TT}$ of $O(\alpha_H{}^{-2})$. This motivates us to define, by analogy with Eqs. (4.9) for $\mathbf{E}_H$ and $\mathbf{B}_H$, a set of "stretched-horizon tidal fields"

$$\mathscr{E}^H \equiv \alpha_H \mathscr{E}^T , \quad \mathscr{B}^H \equiv \alpha_H \mathscr{B}^T ,$$
$$\overleftrightarrow{\mathscr{E}}^H \equiv \alpha_H{}^2 \overleftrightarrow{\mathscr{E}}^{TT} , \quad \overleftrightarrow{\mathscr{B}}^H \equiv \alpha_H{}^2 \overleftrightarrow{\mathscr{B}}^{TT} . \tag{5.8}$$

The inverses of Eqs. (5.6b) give $\mathscr{E}^T_{\rm FFO}$, $\mathscr{B}^T_{\rm FFO}$ in terms of $\mathscr{E}^T$, $\mathscr{B}^T$; and the condition that the former pair be finite imposes constraints on the latter. In terms of the horizon fields these constraints are

$$\mathscr{E}_H = \mathbf{N} \times \mathscr{B}_H + O(\alpha_H{}^2) , \quad \mathscr{B}_H = -\mathbf{N} \times \mathscr{E}_H + O(\alpha_H{}^2) . \tag{5.9}$$

The same considerations applied to Eqs. (5.6c) and (5.6d) give

$$\overleftrightarrow{\mathscr{E}}_H = \mathbf{N} \times \overleftrightarrow{\mathscr{B}}_H + O(\alpha_H{}^4) , \quad \overleftrightarrow{\mathscr{B}}_H = -\mathbf{N} \times \overleftrightarrow{\mathscr{E}}_H + O(\alpha_H{}^4) . \tag{5.10}$$

Equations (5.10) have the physical interpretation, in analogy with that of Eqs. (4.11), that to FIDO's at the stretched horizon the transverse-traceless tidal fields have the form of ingoing plane gravitational waves. Thus, the horizon acts as a perfect absorber of gravitational radiation. Similarly, Eq. (5.9) is identical to the ingoing-wave electromagnetic boundary condition (4.10), but it applies to the "subradiative" tidal fields $\mathscr{E}_H$ and $\mathscr{B}_H$. The relationship of the stretched-horizon tidal fields $\overleftrightarrow{\mathscr{E}}_H$, $\overleftrightarrow{\mathscr{B}}_H$, $\mathscr{E}_H$, $\mathscr{B}_H$, $\mathscr{E}_{NN}$, and $\mathscr{B}_{NN}$ to the Newman-Penrose fields Ψ_0, Ψ_1, and Ψ_2 is spelled out in Appendix A [Eqs. (A7)–(A9)].

Tidal gravity in general relativity is fully characterized by the Riemann curvature tensor, of which the Weyl tensor is only one part. The other part is the Ricci curvature tensor $R_{\alpha\beta}$, which by virtue of the Einstein field equations is equivalent to the stress-energy tensor $T_{\alpha\beta}$. The $3+1$ split of spacetime induces a split of $T_{\alpha\beta}$ into an energy density ϵ, a momentum density (energy flux) S^i, and a stress T^{ij}, which are, respectively, a scalar, a vector, and a second-rank symmetric tensor in absolute space. In terms of $T^{\alpha\beta}$ and the FIDO 4-velocity $\mathbf{U}$ these are given by

$$\epsilon \equiv T^{\alpha\beta} U_\alpha U_\beta , \quad S^\alpha \equiv -T^{\alpha\beta} U_\beta - \epsilon U^\alpha ,$$
$$(T^{\alpha\beta})_{\rm stress} \equiv T^{\alpha\beta} - U^\alpha S^\beta - U^\beta S^\alpha - \epsilon U^\alpha U^\beta . \tag{5.11}$$

At the stretched horizon the components of S^i and T^{ij} are further decomposed into normal ($\mathbf{N}$) and transverse ($\mathbf{e}_a$) components. The Lorentz boost from the FFO to the FIDO frame produces the following relationship between the FFO-measured and FIDO-measured fields:

$$\epsilon = \gamma^2 (\epsilon_{\rm FFO} - 2\beta S^N_{\rm FFO} + \beta^2 T^{NN}_{\rm FFO}) , \tag{5.12a}$$

$$S^N = \gamma^2 [-\beta \epsilon_{\rm FFO} + (1+\beta^2) S^N_{\rm FFO} - \beta T^{NN}_{\rm FFO}] , \tag{5.12b}$$

$$T^{NN} = \gamma^2 (\beta^2 \epsilon_{\rm FFO} - 2\beta S^N_{\rm FFO} + T^{NN}_{\rm FFO}) , \tag{5.12c}$$

$$S^a = \gamma (S^a_{\rm FFO} - \beta T^{aN}_{\rm FFO}) , \tag{5.12d}$$

$$T^{aN} = \gamma (-\beta S^a_{\rm FFO} + T^{aN}_{\rm FFO}) , \tag{5.12e}$$

$$T^{ab} = T^{ab}_{\rm FFO} . \tag{5.12f}$$

From these transformations with $\beta = 1 - O(\alpha_H{}^2)$, $\gamma = O(\alpha_H{}^{-1})$, and from the finiteness of the FFO fields we infer that ϵ, $-S^N$, and T^{NN} are divergent as $\alpha_H{}^{-2}$ and are equal to within fractional errors of $O(\alpha_H{}^2)$; that S^a and $-T^{aN}$ are divergent as $\alpha_H{}^{-1}$ and are equal to within fractional errors of $O(\alpha_H{}^2)$; and that T^{ab} is finite. This motivates us to define, by analogy with the normal charge flux j^N_H of Eq. (4.16),

$$\mathscr{F}^H \equiv -\alpha_H{}^2 S^N \equiv (\text{red-shifted energy entering the stretched horizon per unit universal time}) , \tag{5.13a}$$

$$\mathscr{G}^H_a \equiv -\alpha_H T_a{}^N \equiv (a \text{ component of momentum entering the stretched horizon per unit universal time}) . \tag{5.13b}$$

For the relationship of these quantities to Newman-Penrose quantities see Eqs. (A10) and (A11) of Appendix A. Notice that the relations

$$\epsilon[1+O(\alpha_H{}^2)] = T^{NN}[1+O(\alpha_H{}^2)] = -S^N = \mathscr{F}^H \alpha_H{}^{-2} , \tag{5.14a}$$

$$S_a[1+O(\alpha_H{}^2)] = -T_a{}^N = \mathscr{G}^H_a \alpha_H{}^{-1} \tag{5.14b}$$

have the physical interpretation that the FIDO's see the dominant part of the stress energy as that of a medium crossing the stretched horizon almost (but not quite) precisely inward and at almost the speed of light.

B. Mechanical properties of the membrane

The electromagnetic paradigm has the following central features: (i) The membrane at $\mathscr{H}^S_t$ is endowed with electrical properties, specifically charge and current density; (ii) all external current flows terminate at the membrane in accord with the membrane's law of charge conservation; (iii) the normal $\mathbf{E}$ field and transverse $\mathbf{B}$ field are terminated at the membrane in accord with the membrane's Gauss and Ampère laws; (iv) current in the membrane is viewed as driven by the transverse $\mathbf{E}$ field just above the membrane in accord with the membrane's Ohm's law. These features give a physical and intuitively appealing way of understanding the electrical properties of the hole in terms of electrical properties of a membrane.

We wish now to develop a paradigm for gravitational interactions of the hole, and therefore for mechanical properties of the hole, in terms of mechanical properties of a membrane. In the electromagnetic case the electromagnetic field, its sources, and its evolution could be considered distinct from the dynamics of the horizon itself. For the gravitational paradigm that is no longer the case; the dynamical properties of the membrane must describe the dynamics of the horizon itself, and the paradigm will consequently be rather more intricate than the electromagnetic paradigm. One part of the paradigm will be a way of understanding the radiative boundary conditions in Eq. (5.10), but the membrane paradigm must go further; to represent horizon dynamics the paradigm must lead to a physical representation of the dynamical equations (2.12)–(2.14) for the horizon quantities θ^H, σ^H_{ab}, and Ω^H.

1. The membrane's stress-energy tensor and laws of energy and momentum conservation

The analog of the membrane's surface densities of charge and current will be membrane surface densities of energy, momentum, and stress, embodied in a surface stress-energy tensor $S^A{}_B$. The equations of motion of $S^A{}_B$ (the analog of Maxwell's equations for the electromagnetic field) are

$$S_A{}^B{}_{|B}+[T_A{}^N]=0\,, \tag{5.15a}$$

where $[T_A{}^N]$ is the discontinuity in the tangential-normal component of the external stress-energy tensor [MTW Eq. (21.170)].

By analogy with the electromagnetic case we would like to choose $S^A{}_B$ so that no energy or momentum flows through the membrane [cf. Eq. (4.15)] and therefore so that

$$[T_A{}^N]=(T_A{}^N)_{\text{just above membrane}}\,. \tag{5.15b}$$

Following the analogy further, we would like to require that certain components of $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ be terminated at the membrane by $S^A{}_B$ and that, with the addition of an analog of Ohm's law, the radiative boundary conditions (5.10) follow. Unlike the electromagnetic case we have the further requirement that Eqs. (5.15) represent the dynamical evolution of the horizon.

It turns out that all these requirements *can* be met in a mathematically consistent way: The Einstein field equations dictate that the surface stress-energy tensor $S^A{}_B$ generate a discontinuity $[K^A{}_B]$ in the extrinsic curvature of our membrane at $\mathscr{H}^S$ according to

$$[K^A{}_B]-\delta^A{}_B[K^C{}_C]=8\pi S^A{}_B \tag{5.16a}$$

[Israel's[27] junction condition; see MTW Eq. (21.168b)]. The requirements of the paradigm will turn out to follow if we take the extrinsic curvature $K^A{}_B$ to vanish on the bottom face of the membrane, so that

$$[K^A{}_B]=(K^A{}_B)_{\text{on top face of membrane}}\,. \tag{5.16b}$$

By combining Eqs. (5.16), (3.30), and (3.19) we see that, aside from fractional errors of $O(\alpha_H{}^2)$ which we ignore,

$$S^{\hat{0}\hat{0}}=-S^{\hat{0}}{}_{\hat{0}}=\frac{1}{8\pi}K^a{}_a=-\frac{1}{8\pi}\theta\,, \tag{5.17a}$$

$$S^{\hat{0}}{}_a=\frac{1}{8\pi}K^{\hat{0}}{}_a=-\frac{1}{8\pi}\Omega^H_a\,, \tag{5.17b}$$

$$S^a{}_b=\frac{1}{8\pi}[K^a{}_b-\delta^a{}_b(K^{\hat{0}}{}_{\hat{0}}+K^c{}_c)]$$
$$=-\frac{\sigma^a{}_b}{8\pi}+\delta^a{}_b\left[\frac{g}{8\pi}+\frac{\theta}{16\pi}\right]. \tag{5.17c}$$

Here $\sigma^a{}_b$, θ, and g can be regarded equally well [to within negligible fractional corrections of $O(\alpha_H{}^2)$] as the shear, expansion, and acceleration of the stretched-horizon fiducions or FIDO's.

Equations (5.17) have a remarkable interpretation: The stretched-horizon fiducions can be regarded as a 2-dimensional viscous fluid with surface energy density Σ, surface pressure P, shear and bulk viscosities η and ζ, and velocity $v^a_{\rm FN}$, all as measured by the FIDO's, given by

$$\Sigma\equiv-\frac{\theta}{8\pi}\,,\quad P\equiv\frac{g}{8\pi}\,,\quad \eta=\frac{1}{16\pi}\,,\quad \zeta=-\frac{1}{16\pi}\,, \tag{5.18}$$

$$v^a_{\rm FN}\equiv[\text{expression (3.13)}]=O(\alpha_H)\,. \tag{5.19}$$

In terms of these quantities the stress-energy tensor as measured by FIDO's [Eq. (5.17)], is given by the standard expressions for a viscous fluid with low velocity [low because $v^a_{\rm FN}=O(\alpha_H)$]:

$$S^{\hat{0}\hat{0}}=\Sigma\,, \tag{5.20a}$$

$$S^a{}_b=(P-\zeta\theta)\delta^a{}_b-2\eta\sigma^a{}_b\,, \tag{5.20b}$$

$$S^{\hat{0}}{}_b\equiv\Pi_b=[(\Sigma+P-\zeta\theta)\delta^a{}_b-2\eta\sigma^a{}_b]v_{{\rm FN}a}\,. \tag{5.20c}$$

[Note that our definition (3.13) of fiducion velocity was carefully chosen so as to make Eq. (5.20c) come out right, i.e., to make the fiducions comove with the fluid elements.] The fact that the bulk viscosity ζ is negative is intimately connected with the acausal, teleological nature of a black-hole horizon [cf. Eq. (5.34) below].

Because the fiducions' shear $\sigma^a{}_b$, expansion θ, and acceleration g are divergently large, i.e., $\propto 1/\alpha_H$, the fluid's Σ, P, $S^{\hat{0}\hat{0}}$, and $S^a{}_b$ are also divergently large; but $S^{\hat{0}}{}_a=\Pi_a$, being the product of an $O(\alpha_H{}^{-1})$ stress energy with an $O(\alpha_H)$ velocity [Eq. (5.20c)], is finite. Throughout the membrane paradigm we seek to avoid all $O(\alpha_H{}^{-1})$ divergences by renormalizing the divergent quantities. In this spirit, and with the aid of Eqs. (3.30), we henceforth abandon Σ, P, η, ζ, $S^A{}_B$, and work, instead, in terms of renormalized "horizon" (H) quantities

$$\Sigma_H \equiv \alpha_H \Sigma = (\text{red-shifted energy per unit area}) = -\frac{1}{8\pi}\theta_H \ , \tag{5.21a}$$

$$P_H \equiv \alpha_H P = (\text{horizon surface pressure})$$
$$= (\text{momentum crossing a unit length per unit universal time}) = \frac{1}{8\pi} g_H \ , \tag{5.21b}$$

$$\eta_H \equiv \eta = (\text{coefficient of shear viscosity}) = \frac{1}{16\pi} \ , \tag{5.21c}$$

$$\zeta_H \equiv \zeta = (\text{coefficient of bulk viscosity}) = -\frac{1}{16\pi} \ , \tag{5.21d}$$

$$\Pi_H^a \equiv \Pi^a = (\text{momentum per unit area}) = -\frac{1}{8\pi}\Omega_H^a \ , \tag{5.21e}$$

$$S_H^{ab} \equiv \alpha_H S^{ab} = (\text{horizon stress tensor})$$
$$= (a \text{ component of momentum crossing a unit length with normal in } b \text{ direction, per unit universal time})$$
$$= (P_H - \zeta_H \theta_H)\gamma^{ab} - 2\eta_H \sigma_H^{ab} \ . \tag{5.21f}$$

The physical interpretation of the changes in these quantities under a slicing transformation is discussed at the end of Appendix D.

Note that the renormalization factor α_H in (5.21a) can be regarded as converting from locally measured energy to red-shifted energy, while the factors α_H in (5.21b) and (5.21f) convert from per unit FIDO-measured time to per unit universal time. Note further, in this spirit, that Π^a regarded as a momentum density is finite and requires no renormalization; but that its relation to energy flux is

$$(\text{red-shifted energy crossing a unit length normal to } a \text{ direction per unit universal time}) = {\alpha_H}^2 \Pi^a \simeq 0 \ . \tag{5.21g}$$

Thus, our membrane cannot support a tangential flux of red-shifted energy, even when it has a tangential gradient in its surface temperature (Sec. V C below); i.e., our membrane has vanishing thermal conductivity.

Although the fiducion velocity $v_{\rm FN}^a$ is useful in helping us to understand the physical origin of the fiducions' momentum, we shall not keep it as a key concept in the paradigm. Instead we will work directly with the momentum density Π_H^a.

Turn attention, now, to the membrane's law of energy-momentum conservation ${S_A}^B{}_{|B} + {T_A}^N = 0$ [Eq. (5.15), which follows from Israel's junction condition (5.16) and the Einstein field equations; cf. Sec. 21.13 of MTW]. When split separately into energy conservation $(A = \hat{0})$ and momentum conservation $(A = a)$ and reexpressed in terms of the renormalized horizon quantities (5.21) and (5.13), this law becomes (see Appendix E)

$$D_t \Sigma_H + \theta_H \Sigma_H = -P_H \theta_H + \zeta_H {\theta_H}^2 + 2\eta_H \sigma_{ab}^H \sigma_H^{ab} + \mathcal{F}_H \ , \tag{5.22}$$

$$D_t \Pi_a^H + ({\sigma^H_a}^b + \tfrac{1}{2}\theta_H {\delta_a}^b)\Pi_b^H + \theta_H \Pi_a^H = -P_{H,a} + \zeta_H \theta_{H,a} + 2\eta_H {\sigma_a^{Hb}}_{||b} + \mathcal{G}_a^H \ . \tag{5.23}$$

Here D_t is the covariant derivative with respect to universal time moving with the fiducions [or, equivalently to within negligible errors of $O({\alpha_H}^2)$, moving with the FIDO's]:

$$D_t \Sigma_H \equiv \Sigma_{H,\mu}(\alpha_H U^\mu) \ , \tag{5.24a}$$

$$D_t \Pi_\lambda^H \equiv \Pi_{\mu;\nu}^H(\alpha_H U^\nu){\gamma^\mu}_\lambda \ ; \tag{5.24b}$$

cf. Eq. (2.11).

The membrane's law of energy conservation (5.22) has the standard form for a viscous fluid. The $\theta_H \Sigma_H$ term accounts for the decrease in energy density due to fluid expansion $\theta_H = (1/\text{area})(d\ \text{area}/dt)$ with total energy conserved; $-P_H \theta_H$ is the energy loss due to work done by the fluid's pressure (analog of $P\,dV$ work for a 3-dimensional fluid); $\zeta_H {\theta_H}^2$ and $2\eta_H \sigma_{ab}^H \sigma_H^{ab}$ are the energy increases due to viscous dissipation; and $\mathcal{F}_H$ is the rate at which energy flows into the membrane from the outside universe.

Similarly, the law of momentum conservation (5.23) has the standard form for a viscous fluid; it is, in fact, the standard Navier-Stokes equation for a low-velocity fluid [recall: $|\mathbf{v}_{\rm FN}| = O(\alpha_H) \ll 1$] as viewed in a reference frame (that of the FIDO's) which is expanding and shearing. The terms $({\sigma^H_a}^b + \frac{1}{2}\theta_H {\delta_a}^b)\Pi_b^H$ on the left-hand side, which may be unfamiliar to the reader, account for the expansion and shear of the FIDO's reference frame, with respect to which our 3 + 1 split has been made; $\theta_H \Pi_a^H$ accounts for the decrease of momentum density due to fluid expansion with total momentum conserved; $-P_{H,a}$ is the force due to pressure gradients; $\zeta_H \theta_{H,a}$ and $2\eta {{\sigma^H}_a}^b{}_{||b}$ are the viscous forces; and $\mathcal{G}_a^H$ is the force due to momentum being deposited in the membrane from the external universe.

The membrane's law of energy conservation (5.22) and Navier-Stokes equation (5.23) are stretched-horizon equivalents of the true-horizon focusing equation (2.12) and Hajicek equation (2.14). One can verify this by inserting expressions (5.21) into (5.22) and (5.23), and by noting that because $\alpha_H \mathbf{U} = \partial/\partial \bar{t} + O({\alpha_H}^2)$ [Eq. (3.11)],

$$D_t \equiv \nabla_{\alpha_H \mathbf{U}} = \nabla_{\partial/\partial \bar{t}} + O({\alpha_H}^2) = D_{\bar{t}} + O({\alpha_H}^2) \ . \tag{5.25}$$

In fact, the interpretation of a black-hole horizon as endowed with a viscous-fluid stress tensor was originally developed by Damour[6,7] largely on the basis of the resemblance of the focusing and Hajicek equations to viscous-fluid equations.

2. *Termination of tidal fields, and tidal-force equation*

Not only does the stretched horizon's membrane terminate the external universe's energy flow, momentum flow, and extrinsic curvature, it also terminates some, but not all of the tidal fields.

In particular, the Gauss-Codazzi equations[28]

$$R_{NABC}=K_{AC\,|\,B}-K_{AB\,|\,C}\ , \tag{5.26a}$$

$$R_{ABCD}={}^{(3)}R_{ABCD}-K_{AC}K_{BD}+K_{AD}K_{BC} \tag{5.26b}$$

for the Riemann curvature $R_{\alpha\beta\gamma\delta}$ of spacetime in terms of the stretched horizon's 3-dimensional Riemann curvature ${}^{(3)}R_{ABCD}$ and extrinsic curvature K_{AB}, together with the termination of the extrinsic curvature [Israel's junction condition (5.16)] imply that R_{NABC} is terminated, but R_{ABCD} typically is not. It is straightforward to show that R_{NABC} decomposes into the spacetime Ricci tensor, which terminates by virtue of the Einstein field equations and the horizon laws of energy and momentum conservation (5.15), plus $\mathcal{B}^{\rm TT}_{ab}=\mathcal{B}^{H}_{ab}/\alpha_H{}^2$, $\mathcal{E}_{aN}=\mathcal{E}^{H}_{a}/\alpha_H$, and $\mathcal{B}_{NN}$; and that these tidal fields must therefore terminate:

$$\mathcal{B}^{H}_{ab}\ ,\mathcal{E}^{H}_{a}\ ,\mathcal{B}_{NN}\ \text{are terminated by the membrane's surface layer of stress energy}\ . \tag{5.27a}$$

On the other hand, R_{ABCD} decomposes into Ricci plus $\mathcal{E}^{\rm TT}_{ab}=\mathcal{E}^{H}_{ab}/\alpha_H{}^2$, $\mathcal{B}_{aN}=\mathcal{B}^{H}_{a}/\alpha_H$, and $\mathcal{E}_{NN}$; and since R_{ABCD} typically does not terminate [cf. Eq. (5.26b)],

$$\mathcal{E}^{H}_{ab}\ ,\mathcal{B}^{H}_{a}\ ,\mathcal{E}_{NN}\ \text{are typically not terminated}\ . \tag{5.27b}$$

Note the similarity to electromagnetism: The Gauss-Codazzi equation (5.26a) with (5.16) is a gravitational analog of the horizon's Ampère and Gauss laws (4.13) and (4.12). Just as Ampère and Gauss dictate the termination of $\mathbf{B}_H$ and E_N by the membrane's surface current and charge, so Gauss-Codazzi dictates the termination of $\overleftrightarrow{\mathcal{B}}_H$, $\mathcal{E}_H$, and $\mathcal{B}_{NN}$ by the membrane's surface stress energy. It is straightforward to derive from a "2 + 1 + 1" decomposition of the Gauss-Codazzi equation (5.26a) plus Israel's junction condition (5.16) for K_{AB}, the following explicit forms of the Gauss-Codazzi tidal-field termination laws (see Appendix E):

$$(\mathbf{N}\times\overleftrightarrow{\mathcal{B}}_H)_{ab}=8\pi[-2\eta_H D_t\sigma^{H}_{ab}+(\Sigma_H+P_H)\sigma^{H}_{ab}]\ , \tag{5.28a}$$

$$\mathcal{E}^{H}_{a}=8\pi[(2\eta_H\sigma_H{}^{b}{}_{a}+\tfrac{1}{2}\Sigma_H\delta^{b}{}_{a})_{||b}+(\sigma_H{}^{b}{}_{a}-\tfrac{1}{2}\theta_H\delta^{b}{}_{a})\Pi^{H}_{b}+\tfrac{1}{2}\mathcal{G}^{H}_{a}]\ , \tag{5.28b}$$

$$\mathcal{B}_{NN}=(\text{a quantity not simply expressible in terms of membrane properties})\ . \tag{5.28c}$$

We have now met the gravitational analogs of three of the four electromagnetic membrane laws: Gauss, Ampère, and charge conservation. The analog of the fourth electromagnetic law, Ohm's law $\mathbf{E}_H=R_H\boldsymbol{\mathcal{J}}_H$, is the tidal-force equation (2.13). Since $D_{\bar t}=D_t+O(\alpha_H{}^2)$ [Eq. (5.25)] and $C_{a\bar 0 b\bar 0}=\mathcal{E}^{H}_{ab}$ [Eqs. (5.1a), (5.8), and (3.26a)], the tidal-force equation says

$$D_t\sigma^{H}_{ab}+(\theta_H-g_H)\sigma^{H}_{ab}=-\mathcal{E}^{H}_{ab}\ . \tag{5.29}$$

Just as Ohm's law does not follow directly from any of the Maxwell equations on the timelike stretched horizon $\mathcal{H}^S$, so this tidal-force equation does not follow directly from any of the Gauss-Codazzi equations on $\mathcal{H}^S$. And just as Ohm and Ampère together guarantee that the ingoing plane-wave boundary condition $\mathbf{E}_H=\mathbf{N}\times\mathbf{B}_H$ is satisfied, so the tidal-force equation (5.29) and the Gauss-Codazzi termination law (5.28a) together guarantee that the ingoing plane-wave boundary condition $\overleftrightarrow{\mathcal{E}}_H=\mathbf{N}\times\overleftrightarrow{\mathcal{B}}_H$ [Eq. (5.10)] is satisfied. (The reader can construct for himself or herself an analog of the tidal-force equation (5.29) which, together with the Gauss-Codazzi termination law (5.28b), will guarantee that the "subradiative" boundary condition $\boldsymbol{\mathcal{E}}_H=\mathbf{N}\times\boldsymbol{\mathcal{B}}_H$ [Eq. (5.9)] is satisfied. We have not found that equation to be particularly useful, so we do not bother to write it down.)

The tidal-force equation (5.29) has a simple physical interpretation: It says that the tidal field $\mathcal{E}^{H}_{ab}$ drives changes in the fiducions' shear. Such tidally induced shears are familiar from Newtonian physics. Consider, for example, a 2-dimensional inviscid pressureless fluid driven by a Newtonian gravitational potential Φ, for which the force equation in Cartesian coordinates reads

$$D_t v_a\equiv v_{a,t}+v^b v_{a,b}=-\Phi_{,a}\ , \tag{5.30a}$$

and the shear is $\sigma_{ab}=v_{(a,b)}-\frac{1}{2}\delta_{ab}v^c{}_{,c}$. By computing the symmetric trace-free (STF) part of the spatial gradient of this equation and noting that in two dimensions $(v_{a,c}v^c{}_{,b})^{\rm STF}=\theta\sigma_{ab}$, we obtain

$$D_t\sigma_{ab}+\theta\sigma_{ab}=-\mathcal{E}^{\rm TT}_{ab} \tag{5.30b}$$

where

$$\mathcal{E}^{\rm TT}_{ab}\equiv\Phi_{,ab}-\tfrac{1}{2}\delta_{ab}\Phi_{,c}{}^{c} \tag{5.30c}$$

is the Newtonian limit of general relativity's tidal-field $\mathcal{E}^{\rm TT}_{ab}=\mathcal{E}^{H}_{ab}/\alpha_H{}^2$.

Notice that the Newtonian tidal-force equation (5.30b) is identical in form to that for our fiducions, Eq. (5.29), with one exception: the absence of a surface-gravity term. This similarity may seem surprising, since the inviscid, pressureless fluid in Eq. (5.30a) is influenced only by tidal forces, while our fiducion fluid might be expected to have its shear driven by pressure and viscous forces. Indeed, if the shear in question were that of the fiducion velocity *relative to the* FIDO's, $\sigma^{\rm FN}_{ab}=v^{\rm FN}_{(a||b)}-\frac{1}{2}\gamma_{ab}v^{{\rm FN}c}{}_{||c}$, its evolution could be computed by taking the symmetric trace-free part of the spatial gradient of the Navier-Stokes equation (5.23) and (5.20c); and the result would contain pressure and viscous shear-driving terms. However, the horizon shear σ^{H}_{ab} in Eq. (5.29) and elsewhere is *not* that of the fiducions relative to the FIDO's, but rather that of fiducions relative to a local nondeforming frame. If the

FIDO's motion were shear free, as it is far from the hole, then the fiducions' motion relative to FIDO's would show the same shear as their motion relative to a local nondeforming frame; i.e., we would have $\alpha_H\sigma_{ab}^{\rm FN}=\alpha_H\sigma_{ab}=\sigma_{ab}^H$. However, the irresistible dragging of inertial frames by the hole's rotation nearly locks the FIDO's motion to that of the fiducions and generators, thereby giving the FIDO's motion the same shear σ_{ab}^H as the fiducions' motion and as the generators' motion, aside from fractional differences of $O(\alpha_H{}^2)$. In other words, frame dragging causes the $\alpha_H\sigma_{ab}^{\rm FN}$, whose evolution is computable from the horizon Navier-Stokes equation, to be a tiny, $O(\alpha_H{}^2)$ correction to σ_{ab}^H; and it thereby decouples the horizon's tidal-force equation (5.29) from its Navier-Stokes equation (5.23), forcing the tidal-force equation like Ohm's law to be posed independently of the other horizon equations. Moreover, since the nearly identical fiducion and FIDO motions are locked to the motions of massless free particles (the horizon generators), it is evident that the horizon's tidal-force equation (5.29) should be free of pressure and viscous forces and, in fact, should be similar to the inviscid, pressureless Newtonian equation (5.30b).

There is but one difference in form between the horizon's tidal-force equation (5.29) and the inviscid, pressureless Newtonian equation (5.30b): the presence of a surface gravity term in (5.29). This surface gravity term has a simple origin: it results from our use on the horizon of a time parameter $\bar{t}$ which is not analogous to Newtonian time, and correspondingly our use on the stretched horizon of a non-Newtonian-like $t=\bar{t}+\text{const}$ [Eq. (3.25)]. The analog of Newtonian time is, in fact, any affine parameter along the horizon's generators; and by making a slicing transformation that changes $\bar{t}$ to affine time, we cause g_H to become zero, thereby annulling the surface-gravity term and making the horizon's tidal-force equation assume an identical form to that of an inviscid, pressureless Newtonian fluid (5.30b).

C. Thermodynamics of the membrane and discussion of its evolution

Hawking[29] has shown that a stationary black hole with surface gravity g_H and surface area A_H behaves as though its horizon were endowed with a surface temperature T_H and entropy S_H given by

$$T_H=\frac{\hbar}{2\pi k}g_H\ ,\quad S_H=\frac{k}{4\hbar}A_H\ . \tag{5.31}$$

Here $\hbar$ and k are Planck's constant and Boltzmann's constant. Correspondingly, we shall regard a small bundle of fiducions occupying an area ΔA on the stretched horizon as endowed with a temperature T_H and entropy ΔS given by

$$T_H=\frac{\hbar}{2\pi k}g_H\ ,\quad \Delta S=\frac{k}{4\hbar}\Delta A\ . \tag{5.32}$$

It is a remarkable fact that the horizon's law of energy conservation (5.22), when rewritten using equations (5.21), (5.32), and $\theta_H=D_t(\ln\Delta A)$, becomes the "dissipation equation"

$$T_H\left[D_t\Delta S-\frac{1}{g_H}D_t{}^2\Delta S\right]=(\zeta_H\theta_H{}^2+2\eta_H\sigma_{ab}^H\sigma_H^{ab}+\mathcal{F}_H)\Delta A\ . \tag{5.33}$$

The right-hand side of this equation is familiar from ordinary fluid mechanics: the terms $\zeta_H\theta_H{}^2$ and $2\eta_H\sigma_{ab}^H\sigma_H^{ab}$ produce viscous dissipation (viscous entropy increase), and $\mathcal{F}_H$ is the agent that feeds entropy into the horizon from the external universe. The term $(-T_H/g_H)\partial^2\Delta S/\partial t^2$ on the left-hand side, however, is absent in ordinary fluid mechanics; it is peculiar to black-hole horizons. In special circumstances at special locations on the horizon it may be locally important—e.g., when a compact lump of mass falls into the hole. However, in locations where the local horizon evolution occurs on the same slow time scale t_* as we have required for the hole's surface gravity, this peculiar term gives fractional contributions of $O(1/g_Ht_*)$ and thus can be ignored.[30]

In order to solve the evolution equations for an ordinary fluid, one must pose initial conditions at some initial moment of time and then integrate forward in time. Not so for the horizon of a black hole, and similarly not so for the stretched horizon. Because of the "teleological" definition of the horizon as the boundary of the region that cannot send outgoing null rays to future null infinity,[15] for the stretched horizon one must pose final conditions on some quantities and then integrate backward in time to find out the initial conditions. The final conditions are the demands[15,16] (which follow from the definition of a horizon) that in its final, stationary state the hole must have

$$\sigma_{ab}^H=\theta_H=\Sigma_H=\partial\Delta S/\partial t=0\quad\text{at } t\to\infty\ . \tag{5.34}$$

(This, of course, ignores the effects of Hawking radiation,[29] which are negligible for macroscopic holes and thus have been omitted from the above equations.) These teleological boundary conditions, together with the law of energy conservation (5.22) or equivalently the dissipation equation (5.33) imply that[15,16]

$$\theta_H\geq 0\ ,\quad D_t\Delta S\geq 0\ ,\quad \Sigma_H\leq 0\ \text{ always}\ . \tag{5.35}$$

In words, (i) the horizon can expand but can never contract, (ii) the entropy of a bundle of fiducions can increase but can never decrease, and (iii) [cf. Eq. (5.22) with $P_H=g_H/8\pi$ and $\theta_H=-8\pi\Sigma_H$] at a time $\Delta t\sim g_H{}^{-1}$ before energy is deposited on the stretched horizon by the external universe, the horizon's pressure P_H causes it to begin to expand, the work done in that expansion drives its energy density Σ_H negative, and then the deposited energy $\mathcal{F}_H$ drives Σ_H back to its steady-state zero value.

Our presentation of the membrane formalism is now complete, with one exception: We have not given a prescription for computing the evolution of the hole's mass. That prescription has been known for a long time for any evolution that takes the hole (slowly or rapidly) from one nearly stationary, axisymmetric configuration to another slightly different one; it is the hole's first law of thermodynamics:[31]

$$dM=T_HdS^H+\Omega_HdJ\ . \tag{5.36}$$

Here T_H is the spatially constant temperature of the horizon, computed from g_H via Eq. (5.32); dS^H is the change of the hole's total entropy, computed by integrating the dissipation equation (5.33) over the stretched horizon and over time; Ω_H is the angular velocity of the stretched horizon's fiducions as measured by distant observers [Eq. (2.23b); not to be confused with the Hajicek vector Ω_H^a]; and dJ is the change of angular momentum, computed from the Navier-Stokes equation (5.23) for $D_t\Pi_\phi^H$ (where ϕ denotes a component along the axial Killing vector $\xi \equiv \partial/\partial\phi$), and from the surface integral

$$J=\int_{\mathscr{H}_t^S}\Pi_\phi^H dA \tag{5.37}$$

for J [cf. Eqs. (2.7) and (5.21e)].

VI. CONCLUSION

This paper has focused on the derivation of the membrane formalism for black holes. Because the derivation makes extensive use of four-dimensional spacetime arguments, it obscures the spirit of the membrane paradigm.

In the membrane paradigm one abandons four-dimensional language, and works instead entirely in the three-dimensional physical language of absolute space and universal time. Similarly, one abandons the true horizon of the hole and deals instead with its surrogate, the stretched horizon, which one regards as a membrane with a variety of simple physical properties. The laws of structure and evolution of the stretched-horizon membrane then become boundary conditions on the physics of the external universe—boundary conditions that permit complex astrophysical interactions to be studied with the aid of elementary physical intuition.

The laws of structure and evolution of the stretched-horizon membrane include electromagnetic laws and gravitational laws. The electromagnetic laws are the definitions of the membrane's electric and magnetic fields in terms of the fields measured by FIDO's there [Eqs. (4.5) and (4.9)], Gauss's law (4.12), which defines the membrane's surface charge density so as to annihilate the normal electric field, Ampère's law (4.13), which defines the membrane's surface current density so as to annihilate the tangential magnetic field, Ohm's law (4.14), which describes how the tangential electric field drives the membrane's surface current, and the law of charge conservation (4.15), which describes how the membrane acquires and conserves all charge that enters it from the external universe.

The stretched horizon's gravitational laws are the definitions of its tidal fields in terms of the fields measured by FIDO's there [Eqs. (5.4) and (5.8)], the law of metric change (3.30h), which describes how the kinematic properties (expansion and shear) of the membrane's fiducions are related to the time evolution of the membrane's metric, the law of vanishing extrinsic curvature on the bottom face of the membrane and the resulting expressions (5.20), (5.21) for the membrane's surface stress-energy tensor in terms of the fiducions' kinematic properties, the Gauss-Codazzi laws (5.27), (5.28) by which the horizon's stress-energy tensor annihilates half of the tidal fields, the tidal-force equation (5.29), which describes how the gravitoelectric tidal field produces fiducion shear, the law of energy conservation (5.22), which describes how the membrane acquires and conserves all energy that enters it from the external universe, the law of momentum conservation (5.23) (Navier-Stokes equation), which describes how the membrane acquires and conserves all tangential momentum that enters it from the external universe, Hawking's laws (5.31), (5.32), expressing the membrane's surface temperature and entropy in terms of its surface gravity and area, the dissipation equation (5.33), which describes how the membrane's entropy increases due to the fiducions' viscous dissipation and due to energy flowing into the membrane from the external universe, the teleological boundary conditions (5.34) on the fiducions' kinematic properties and entropy changes, the expression (5.37) for the hole's total angular momentum in terms of an integral over the membrane's surface density of momentum, and the hole's first law of thermodynamics (5.36) for changes of its mass in terms of changes of its entropy and angular momentum.

In applying these laws, the user of the membrane formalism must select (wisely) the location of the stretched horizon α_H and the constant or slowly varying surface gravity g_H. The laws will then fully determine the evolution of the stretched horizon in response to the external universe's driving forces—i.e. the electric, magnetic, and tidal fields at the stretched horizon, and the flow of charge, energy, and momentum into the stretched horizon.

Examples of such applications, as given elsewhere, are a variety of idealized electromagnetic model problems in Refs. 3, 9, and 10, a variety of idealized gravitational model problems in Refs. 32 and 9, and models for power generation in quasars and active galactic nuclei in Refs. 2, 13, and 9.

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APPENDIX A: THE NEWMAN-PENROSE FORMALISM ON THE HORIZON

We outline here the explicit connection between the description of the horizon given in this paper, especially in Sec. II, and the Newman-Penrose formalism for the horizon, which is often convenient for calculations. In this outline we shall refer to the classic 1962 paper by Newman and Penrose[18] and shall cite equations from that paper by NP. The equations in that paper, unfortunately, are based on the metric signature $+---$. To make the comparison with our formalism tractable we shall introduce a "rationalized" NP formalism based on our choice of signature $(-+++)$ and the conventions for the Riemann, Weyl, and Ricci tensors as given in MTW. The signs in the "rationalized" NP formalism are given by the following rules. (i) The null tetrad $\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, $\mathbf{m}^*$ have all inner products vanishing except

$$\mathbf{l}\cdot\mathbf{n}=-1\ ,\quad \mathbf{m}\cdot\mathbf{m}^*=1\ . \tag{A1}$$

(ii) The definitions of the spin coefficients, of the Weyl projections Ψ_i, and of the Ricci projections Φ_{ij} and Λ are taken to be precisely those given by NP Eqs. (4.1a) and (4.3). (iii) In NP Eqs. (4.2), the equations needed here, the only signs that must be reversed are the sign of Λ and the signs of terms quadratic in the spin coefficients.

On the horizon we choose $\mathbf{l}$ as given by Eq. (2.1), we take $\mathbf{m}$, $\mathbf{m}^*$ to lie in the horizon section $\mathscr{H}_{\bar{t}}$ (i.e., the space spanned by $\mathbf{e}_a$), and $\mathbf{n}$ then coincides with the $\mathbf{n}$ of Eq. (3.1). For simplicity we choose the $\mathbf{m}$, $\mathbf{m}^*$ legs to propagate along the generators according to

$$\mathbf{m}\cdot\nabla_{\mathbf{l}}\mathbf{m}^*=\mathbf{m}^*\cdot\nabla_{\mathbf{l}}\mathbf{m}=0\ . \tag{A2}$$

This can always be done by an appropriate rotation of $\mathbf{m}$, $\mathbf{m}^*$ at each point of the horizon. With this tetrad the spin coefficients have, *on the horizon*, the following simplifications and relations to our notation:

$$\begin{aligned}&\tilde{\rho}=\tilde{\rho}^*=\tfrac{1}{2}\theta^H\ ,\quad \tilde{\epsilon}=\tilde{\epsilon}^*=-\tfrac{1}{2}g^H\ ,\\ &\tilde{\kappa}=0\ ,\quad \tilde{\pi}=\tilde{\alpha}+\tilde{\beta}^*=-\mathbf{m}^*\cdot\mathbf{\Omega}^H\ ,\\ &\tilde{\mu}=\tilde{\mu}^*\ ,\quad \tilde{\sigma}=m^a m^b\sigma^H_{ab}\ .\end{aligned} \tag{A3}$$

Here an asterisk denotes complex conjugation and a tilde has been used over spin coefficients to avoid confusion. The reality of $\tilde{\rho}$ (i.e., $\tilde{\rho}=\tilde{\rho}^*$) follows from the fact that the generators (trajectories of $\mathbf{l}$) are hypersurface orthogonal; the reality of $\tilde{\mu}$ (i.e., $\tilde{\mu}=\tilde{\mu}^*$) follows from the fact that the commutator $[\mathbf{m},\mathbf{m}^*]$ lies in the $\mathbf{m}\wedge\mathbf{m}^*$ 2-flat; the equation $\tilde{\pi}=\tilde{\alpha}+\tilde{\beta}^*$ follows from the fact that horizon sections are Lie transported along $\mathbf{l}$ [Eq. (2.1)]; all other relations in Eq. (A3) follow directly from the definitions of σ^H_{ab}, θ^H, g^H, $\mathbf{\Omega}^H$, the definitions in NP Eq. (4.1a), and the special properties of the tetrad.

Equation NP (4.2a) with "rationalized" signs reads

$$D\tilde{\rho}=-(\tilde{\rho}^2+\tilde{\sigma}\tilde{\sigma}^*)-2\tilde{\epsilon}\tilde{\rho}+\Phi_{00}\ . \tag{A4}$$

With the relations in Eq. (A3) and with $\Phi_{00}=-\frac{1}{2}R_{ll}=-4\pi T_{\bar{0}\bar{0}}$ [from NP Eq. (4.3b) and the Einstein field equations] this becomes the focusing equation (2.12). Similarly NP Eq. (4.2b),

$$D\tilde{\sigma}=-2\tilde{\rho}\tilde{\sigma}-2\tilde{\epsilon}\tilde{\sigma}+\Psi_0\ , \tag{A5}$$

becomes the tidal-force equation (2.13), and the sum of NP Eqs. (4.2e) and (4.2k) and the complex conjugate of NP Eq. (4.2d),

$$\begin{aligned}D\tilde{\pi}^*+\delta\tilde{\rho}-2\delta\tilde{\epsilon}=&(\delta^*+2\tilde{\alpha}-2\tilde{\beta}^*)\tilde{\sigma}-\tilde{\sigma}\tilde{\pi}\\&-3\tilde{\rho}\tilde{\pi}^*+2\Phi_{01}\ ,\end{aligned} \tag{A6}$$

becomes the Hajicek equation (2.14).

In connection with the description in Sec. V of the Weyl tensor and stress-energy tensor, we note that Ψ_0, the NP field which satisfies the Teukolsky equation for perturbations of the Kerr geometry, is related to the gravitoelectric and gravitomagnetic tidal fields on the stretched horizon [Eqs. (5.8)] by

$$\begin{aligned}\Psi_0&\equiv -m^a m^b C_{lalb}\\ &=-m^a m^b\alpha_H{}^2\mathscr{E}_{ab}[1+O(\alpha_H{}^2)]\\ &=-m^a m^b\mathscr{E}^H_{ab}[1+O(\alpha_H{}^2)]\\ &=-m^a m^b(\mathbf{N}\times\overleftrightarrow{\mathscr{B}}^H)_{ab}[1+O(\alpha_H{}^2)]\ ,\end{aligned} \tag{A7}$$

and similarly

$$\begin{aligned}\Psi_1&=\mathbf{m}\cdot\mathscr{E}_H[1+O(\alpha_H{}^2)]\\ &=\mathbf{m}\cdot(\mathbf{N}\times\mathscr{B}_H)[1+O(\alpha_H{}^2)]\ ,\end{aligned} \tag{A8}$$

$$\Psi_2=-\tfrac{1}{2}(\mathscr{E}_{NN}\pm i\mathscr{B}_{NN})[1+O(\alpha_H{}^2)]\ , \tag{A9}$$

$$\Phi_{00}=-4\pi\mathscr{F}_H[1+O(\alpha_H{}^2)]\ , \tag{A10}$$

$$\Phi_{01}=\Phi_{10}{}^*=4\pi\mathbf{m}\cdot\mathscr{S}_H[1+O(\alpha_H{}^2)]\ . \tag{A11}$$

Here the sign of the imaginary part of Ψ_2 depends on the choice of $\mathbf{m}$; setting $\mathbf{m}_{\text{new}}=\mathbf{m}^*_{\text{old}}$ reverses the sign.

APPENDIX B: STATIC AND STATIONARY HORIZONS

We justify here the claims made in Sec. II about preferred choices of slicing for static and stationary horizons.

For a stationary black hole there is a unique Killing vector $\mathbf{k}$ with normalization $\mathbf{k}\cdot\mathbf{k}=-1$ at spatial infinity, which generates "time translations." If the horizon is static, $\mathbf{k}$ must be tangent to the horizon generators[33] and we normalize $\bar{t}$ such that $\mathbf{l}=\mathbf{k}$. If the hole is rotating, it must be axisymmetric[15] and we denote $\boldsymbol{\xi}$ as the axial Killing vector and fix its norm by requiring $\boldsymbol{\xi}=\partial/\partial\phi$ with ϕ having the usual range 0 to 2π. In this case $\mathbf{l}$ can be taken to be $\mathbf{k}+\Omega_H\boldsymbol{\xi}$ where Ω_H, the horizon's angular velocity, is constant by the "rigidity theorem."[34]

With $\mathbf{l}$ uniquely fixed in this manner, and with the requirement $\mathbf{l}=\partial/\partial\bar{t}$, the slicing function $\bar{t}$ is fixed up to transformations of the form

$$\bar{t}\,'=\bar{t}+f(x^a)\ . \tag{B1}$$

The surface gravity g_H is then fixed [Eq. (2.4)] and, by the "zeroth law of black-hole mechanics,"[25] is constant over the horizon. Since $\mathbf{l}=\partial/\partial\bar{t}$ generates an isometry on the horizon, the metric γ_{ab} must be independent of $\bar{t}$ in comoving coordinates and [cf. Eq. (2.10b)] both θ^H and σ^H_{ab} must vanish.

The arbitrariness inherent in Eq. (B1) can now be exploited to constrain the Hajicek field as claimed in Eqs. (2.20b) and (2.23c). To do this we introduce comoving spatial coordinates $x^a=\theta,\phi$ with ϕ, in the rotating case, the ignorable cyclic coordinate. In terms of the spacetime Carter coordinates of Eq. (3.4), Carter[25] shows that under very general conditions there exists a function $G(\lambda,x^a)$ well behaved at the horizon such that, in the static case

$$(\mathscr{J}-b^a b_a)\frac{\partial G}{\partial\theta}=b_\theta\ , \tag{B2a}$$

$$(\mathcal{J}-b^a b_a)\frac{\partial G}{\partial \phi}=b_\phi \ . \tag{B2b}$$

In the horizon limit [Eqs. (3.5)] these relations give

$$g_H\frac{\partial G}{\partial \theta}=-\Omega_\theta^H \ , \ g_H\frac{\partial G}{\partial \phi}=-\Omega_\phi^H \ . \tag{B3}$$

With the specialization to stationary horizons ($\sigma_{ab}^H=\theta^H=0$) and for the restricted slicing transformations of Eq. (B1) the Hajicek field transforms as [cf. Eq. (2.18e)]

$$\Omega_{a'}^{H'}=\Omega_a^H-g_H(\partial f/\partial x^a) \ . \tag{B4}$$

By choosing the transformation function f to be minus Carter's G function (at $\lambda=0$) we see that, in the static case, the slicing may be chosen so that $\mathbf{\Omega}^H$ vanishes, and this slicing is unique up to time translation.

If the hole is rotating Eq. (B2b) is no longer valid; it is replaced by the requirement $\partial G/\partial\phi=0$. Equation (B2a), however, still applies and shows that with the choice $f(\theta)=-G(\lambda=0,\theta)$ the θ component of $\mathbf{\Omega}^H$ is set to zero, and $\mathbf{\Omega}^H$ is parallel to ξ for this slicing, which is unique up to time translation.

APPENDIX C: SPACETIME METRIC NEAR THE HORIZON

In this appendix we develop a canonical form for the spacetime metric near the horizon of a slowly evolving black hole. This canonical form uses universal time t as its time coordinate, lapse function α as its radial coordinate, and transverse coordinates $x^{2'}$ and $x^{3'}$ which differ from Carter coordinates x^a by a term chosen to suppress the radial-transverse metric coefficient $g_{\alpha a'}$:

$$t=\bar{t}-\frac{1}{2g_H}\ln(2g_H\lambda) \ , \tag{C1a}$$

$$\alpha=(2g_H\lambda)^{1/2}+O(\lambda^{3/2}) \ , \tag{C1b}$$

$$x^{a'}=x^a-\frac{\Omega_H{}^a}{g_H}\lambda=x^a-\frac{1}{2}\frac{\Omega_H{}^a}{g_H{}^2}\alpha^2+O(\alpha^4) \tag{C1c}$$

[cf. Eqs. (3.7) and (3.10)].

The specific slow-evolution requirements that we shall need are (i) the constraints on g_H discussed in Sec. III C [Eqs. (3.23) plus $\lambda\gtrsim\lambda_{\rm MSH}$ so $\alpha^2\gtrsim\delta$], and (ii)

$$|\,\Omega_H{}^a{}_{,\bar{t}}\Omega_H{}^b{}_{,\bar{t}}\gamma_{ab}\,|^{1/2}\lesssim\frac{1}{t_*}\max(g_H,|\,\mathbf{\Omega}_H\,|) \ . \tag{C2}$$

By inserting the coordinate transformation (C1) into the Carter metric (3.6) and invoking these slow-evolution conditions, we find the following *canonical form* for the spacetime metric at $\delta\lesssim\alpha^2\ll 1$:

$$ds^2=-\alpha^2dt^2+\frac{d\alpha^2}{g_H{}^2}+\gamma_{a'b'}\left[dx^{a'}-\frac{\Omega_H{}^a}{g_H}\alpha^2dt\right]\left[dx^{b'}-\frac{\Omega_H{}^b}{g_H}\alpha^2dt\right]+(\text{higher-order corrections}) \ , \tag{C3}$$

where

$$\gamma_{a'b'}=\gamma_{cd}\frac{\partial x^c}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{b'}}=\gamma_{ab}+\tfrac{1}{2}g_H{}^{-2}\alpha^2(\gamma_{cb}\Omega^c{}_{,a}+\gamma_{ac}\Omega^c{}_{,b})+O(\alpha^4) \ , \tag{C4}$$

and the magnitudes of the corrections are

$$\Delta g_{\alpha\alpha}=O(\alpha^2) \ , \ \Delta g_{\alpha t}\sim\Delta g_{\alpha a'}=O(\alpha^3) \ , \quad \Delta g_{tt}\sim\Delta g_{ta'}\sim\Delta g_{a'b'}=O(\alpha^4) \ . \tag{C5}$$

Note that the canonical transverse spatial coordinates $x^{a'}$, like those of Carter x^a, are tied to the horizon generators. It is often convenient in applications of the membrane formalism to tie one's spatial coordinates to the hole's asymptotic rest frame at infinity. The resulting "infinity-tied" coordinates are obtainable from $x^{a'}$ by a time-dependent (rotating) coordinate transformation. As an example consider a Kerr black hole, begin in standard Boyer-Lindquist coordinates $(r,\theta^\dagger,\phi^\dagger)$ which are tied to infinity, and choose Boyer-Lindquist time t to play the role of universal time. Then the spacetime metric is

$$ds^2=-\alpha^2dt^2+g_{jk}(dx^j+\beta^jdt)(dx^k+\beta^kdt) \ ; \tag{C6}$$

the metric of absolute space is

$$ds^2=g_{jk}dx^jdx^k=(\rho^2/\Delta)dr^2+\rho^2d\theta^{\dagger 2}+(\Sigma\sin\theta^\dagger/\rho)^2d\phi^{\dagger 2} \ ; \tag{C7a}$$

and α and β^j are given by

$$\alpha=\rho\Delta^{1/2}/\Sigma \ , \tag{C7b}$$

$$\beta^r=\beta^{\theta^\dagger}=0 \ , \ \beta^{\phi^\dagger}=-\omega\equiv-2Mar/\Sigma^2 \ , \tag{C7c}$$

where

$$\Delta\equiv r^2+a^2-2Mr \ , \ \rho^2\equiv r^2+a^2\cos^2\theta^\dagger \ , \quad \Sigma^2\equiv(r^2+a^2)^2-a^2\Delta\sin^2\theta^\dagger \ . \tag{C7d}$$

The canonical, horizon-tied angular coordinates are related to the above infinity-tied coordinates by

$$\theta'=\theta^\dagger-\frac{\rho_H{}^2{}_{,\theta^\dagger}}{\rho_H{}^4}\frac{\alpha^2}{4g_H{}^2} \ , \ \phi'=\phi^\dagger-\Omega_Ht \ , \tag{C8a}$$

where Ω_H and g_H are the hole's angular velocity and surface gravity, and ρ_H is the value of ρ on the horizon:

$$\Omega_H\equiv\lim_{\alpha\to 0}\omega=a/2Mr_H \ , \ r_H\equiv M+(M^2-a^2)^{1/2} \ , \tag{C8b}$$

$$g_H=(r_H-M)/2Mr_H \ , \ \rho_H{}^2=r_H{}^2+a^2\cos^2\theta^\dagger \ . \tag{C8c}$$

By inserting the change of coordinates (C8a) into the spacetime metric (C6) and expanding in powers of α in the neighborhood of the horizon, we obtain the canonical form (C3) for the near-horizon metric, with the Hajicek field $\Omega_H{}^{a'}$ given by the index-raised version of Eq. (2.24b)

$$\Omega_H^{\theta'}=0\ ,\quad \Omega_H^{\phi'}=\lim_{\alpha\to 0}\frac{g_H}{\alpha^2}(\omega-\Omega_H)=-\frac{a}{(2Mr_H)^2\rho_H{}^2}[\rho_H{}^2 r_H+M(r_H{}^2-a^2\cos^2\theta')]\ , \tag{C8d}$$

and with the horizon metric $\gamma_{a'b'}$ given by Eq. (2.24c) plus $O(\alpha^2)$ corrections. [Note that in Eqs. (C.8b)–(C.8d), which refer to the horizon $\alpha=0$, we can set $\theta^{\dagger}=\theta'$; cf. Eq. (C.8a).]

APPENDIX D: SLICING TRANSFORMATIONS

In this appendix we discuss the effects of slicing transformations (Sec. II C) on universal time, on the FIDO congruence, and on properties of the stretched horizon; i.e., we study the arbitrariness inherent in these quantities. Throughout this appendix, as in Sec. II C, we use comoving coordinates x^a on the horizon.

1. Slicings with constant surface gravity and with $\partial^2\bar t'/\partial\bar t^2=0$

We begin by confining attention to slicing functions $\bar t$ and $\bar t'$ which give constant surface gravity, and hence to solutions of Eq. (2.18d) for which both g_H and $g_H{}'$ are constants.

The simplest solution to Eq. (2.18d) is that for which [cf. Eqs. (2.16)]

$$Y=g_H{}'/g_H=\text{const}\ ,\quad G=0 \tag{D1}$$

so that

$$g_H(\bar t-\bar t_0)=g_H{}'(\bar t'-\bar t'_0)\ ,\quad \bar t_0=\bar t_0(x^a)\ ,\quad \bar t'_0=\bar t'_0(x^a)\ , \tag{D2a}$$

$$W_a=\bar t_{0,a}-Y\bar t'_{0,a}\ ,\quad A_a=0\ . \tag{D2b}$$

By the construction of Sec. III A one can show that the radial Carter coordinates λ' and λ of the two slicings are related by

$$\lambda'=Y\lambda+O(\lambda^2) \tag{D3a}$$

corresponding to

$$\alpha'=Y\alpha+O(\alpha^3)\ . \tag{D3b}$$

From this result and (D2a) it is straightforward to find the relation of universal times t and t' in the two systems,

$$g_H(t-\bar t_0)=g'_H(t'-\bar t'_0)+\ln(g'_H/g_H)+O(\alpha^2)\ , \tag{D4}$$

and to show that the FIDO 4-velocities $\mathbf{U}_{\rm FIDO}\equiv-\alpha\nabla t$ and $\mathbf{U}'_{\rm FIDO}=-\alpha'\nabla t'$ are related (aside from corrections of order α^2) by

$$\mathbf{U}_{\rm FIDO}=\mathbf{U}'_{\rm FIDO}-\alpha\mathbf{W}\ , \tag{D5a}$$

where

$$\mathbf{W}\equiv W^a(\partial/\partial x^a)_{\alpha,t}=W^a(\partial/\partial x^{a'})_{\alpha',t}+O(\alpha')\mathbf{U}'\ ,\qquad W^a\equiv\gamma^{ab}W_b\ . \tag{D5b}$$

Correspondingly, the primed FIDO's see the unprimed FIDO's move with physical velocity $-\alpha\mathbf{W}$ tangential to the stretched horizon; the unprimed FIDO's see the primed FIDO's move with physical velocity $\alpha\mathbf{W}$; and the stretched horizons for the two slicings can be chosen to be the same with [cf. (D3)]

$$\alpha_H=(g_H/g'_H)\alpha'_H \text{ on } \mathcal{H}^S=\mathcal{H}^{S'}\ . \tag{D6}$$

2. Slicings with constant surface gravity and with $\partial^2\bar t'/\partial\bar t^2\neq 0$

For g_H and g'_H constant the general slicing transformation [i.e., the general solution to Eq. (2.18d)] follows from a simple observation: If on a generator $\bar t$ is a time parameter for which g_H is constant, then $e^{g_H\bar t}$ is an affine parameter. The general relationship between $\bar t$ and $\bar t'$ can then be inferred from the fact that any two affine parameters are linear functions of each other with constant coefficients:

$$Ae^{g_H\bar t}+B=Ce^{g'_H\bar t'}+D\ . \tag{D7}$$

Although the coefficients are constant on generators, they may differ from generator to generator, i.e., A, B, C, and D are functions of x^a. We will constrain $\bar t$ and $\bar t'$ both to increase to the future so that A and C must have the same sign, which we will take to be positive. If $B=D$, Eq. (D7) reduces to (D2a). For patches of the horizon with $D>B$ we can write Eq. (D7) as

$$e^{g_H(\bar t-\bar t_0)}=e^{g'_H(\bar t'-\bar t'_0)}+1\ ,\quad \bar t_0=\bar t_0(x^a)\ ,\quad \bar t'_0=\bar t'_0(x^a)\ , \tag{D8}$$

(with $\bar t_0=-g_H^{-1}\ln[A/(D-B)]$ and $\bar t'_0=-g'^{-1}_H\ln[C/(D-B)]$). The case of $B>D$ is equivalent to reversing the primed and unprimed quantities, so there is no loss of generality in confining attention to (D8).

The explicit relation of $\bar t$ and $\bar t'$ in (D8) is given by

$$\bar t-\bar t_0=T(\bar t'-\bar t'_0)\equiv g_H^{-1}\ln(e^{g'_H(\bar t'-\bar t'_0)}+1)\ , \tag{D9a}$$

from which we calculate the horizon functions of Eqs. (2.16)

$$Y=Y(\bar t-\bar t'_0)\equiv\frac{g'_H/g_H}{1+e^{-g'_H(\bar t'-\bar t'_0)}}\ ,$$

$$G=G(\bar t'-\bar t'_0)\equiv\frac{g'_H}{e^{g'_H(\bar t'-\bar t'_0)}+1}\ , \tag{D9b}$$

$$W_a=\bar t_{0,a}-Y\bar t'_{0,a}\ ,\quad A_a=-G\bar t'_{0,a}\ .$$

It should be noted that at times $\bar t'-\bar t'_0\gg g'^{-1}_H$ the transformation (D9a) asymptotes to that of the last section [Eq. (D2a)] and the horizon functions in (D9b) approach those in (D1) and (D2b).

From the constructions of Secs. III A and III B one can derive, aside from corrections of $O(\alpha^2)$, the following relations between these two slicings' universal times t and t', and between their lapse functions α and α':

$$t=\bar{t}_0+T(t'+g_H'^{-1}\ln\alpha'-\bar{t}_0')-g_H^{-1}\ln\alpha \,, \tag{D10a}$$

$$\alpha=\left[\frac{g_H}{g_H' Y}\right]^{1/2}\alpha' \,, \quad \text{where } Y=Y(t'+g_H'^{-1}\ln\alpha'-\bar{t}_0') \,. \tag{D10b}$$

The functions T and Y appearing here are those of Eqs. (D9b). From these relations we infer that the 4-velocities of the two FIDO's at the same location in spacetime are related by

$$\mathbf{U}_{\rm FIDO}=\gamma(\mathbf{U}'_{\rm FIDO}+v_N\mathbf{N}'+\alpha'\mathbf{w}) \,, \tag{D11a}$$

where

$$v_N=\frac{\chi/2}{1-\chi/2}\,, \quad \gamma=\frac{1-\chi/2}{\sqrt{1-\chi}}\,,$$

$$\chi\equiv\frac{1}{1+e^{g_H'(\bar{t}'-\bar{t}_0')}}=\frac{1}{1+\alpha' e^{g_H'(t'-\bar{t}_0')}}$$

$$=\frac{1}{\alpha e^{g_H(t-\bar{t}_0)}}\,, \tag{D11b}$$

$$\mathbf{w}\equiv(\gamma^{ab}/Y)(v_N\bar{t}_{0,b}-W_b)\mathbf{e}_{a'} \,.$$

Note that, as $g_H(\bar{t}'-\bar{t}_0')$ ranges from $\ll -1$ to $\gg +1$, $g_H(\bar{t}-\bar{t}_0)$ ranges from 0 to $\gg +1$, Y ranges from 0 to g_H'/g_H, v_N ranges from 1 to 0, and γ ranges from ∞ to 1. From this, the physical situation as seen by the unprimed FIDO's should be clear.

Prior to time $\bar{t}\equiv t+g_H^{-1}\ln\alpha=\bar{t}_0$ only the unprimed FIDO's are in the vicinity of the horizon. At $\bar{t}=\bar{t}_0$ the primed FIDO's descend toward the horizon at very nearly the speed of light, blasting their rockets with near infinite acceleration $g'=g_H'/\alpha'$ to slow their descent; and at time $\bar{t}-\bar{t}_0\gg g_H^{-1}$ they come to rest slightly above the horizon, at locations $\alpha=(g_H/g_H')\alpha'$. Superimposed on these (initially) high-speed inward motions are infinitesimal-speed $[O(\alpha')]$ motions parallel to the horizon.

Because of their relative inward motions, the two families of FIDO's possess different stretched horizons at early times; but at $\bar{t}-\bar{t}_0\gg g_H^{-1}$ (corresponding to $\bar{t}'-\bar{t}_0'\gg g_H'^{-1}$), when their radial motions coincide, their stretched horizons also coincide.

Of all choices of FIDO's which one might make, the obviously correct one is that choice in which the FIDO's are attached to the horizon, i.e., not descending, throughout the horizon's history—or, more precisely, throughout the history of that patch of horizon to which the FIDO's belong. We can make this choice by requiring that the affine parameter $A(x^a)e^{g_H\bar{t}}$ [cf. Eq. (D7)] vanish (i.e., $\bar{t}=-\infty$) on each generator at the point at which the generator attaches to the horizon. With this constraint the slicing transformations (D7) are limited to those with $B=D=0$, i.e., to those of Sec. 1 [Eq. (D2)] with the FIDO's fixed except for their (relatively unimportant) small $[O(\alpha)]$ motions parallel to their common stretched horizon.

3. Slicings with slowly variable surface gravity g_H

We next consider the case of a surface gravity g_H with slow temporal and spatial variations, the choice made in Sec. III C for the description of the evolution of a typical astrophysical hole. For this case we assume that the ambiguity in the $\bar{t}=-\infty$ point on each generator has been resolved as discussed at the end of Sec. 2, thereby specializing to slicing transformations analogous to those of Sec. 1. However, since g_H is now slowly variable, we must broaden the class of admissible transformations discussed in Sec. 1 by using Eqs. (2.15) and (2.16) with Y slowly variable:

$$\bar{t}=\bar{t}(\bar{t}',x^a)\,, \tag{D12a}$$

$$Y\equiv\bar{t}_{,\bar{t}'} \quad \text{variable in space and time on scales } \gtrsim t_* \,, \tag{D12b}$$

$$W_a\equiv\bar{t}_{,a} \quad \text{variable in time on scales } \gtrsim t_* \text{ but rapidly variable in space}\,, \tag{D12c}$$

$$G\equiv Y^{-1}Y_{,\bar{t}'} \quad \text{implying } |G|\lesssim 1/t_* \tag{D12d}$$

$$A_a\equiv Y^{-1}Y_{,a'}=W_{a,\bar{t}'} \quad \text{implying } |\mathbf{A}|\lesssim 1/t_* \,. \tag{D12e}$$

When inserted into the kinematic transformation laws (2.18) the "acceleration functions" G and A_a have no significant effect: They produce fractional changes of g_H and $\mathbf{\Omega}^H$ that are generally $\lesssim 1/g_H t_*$, and that in turn cause no significant long-term secular effects in the evolution equations (2.12)–(2.14) and (2.15). By contrast, the weak spatial and temporal dependence in Y produces corresponding dependences in g_H' which (i) can be used to make g_H' track the hole's evolving mass and angular momentum in the standard Kerr manner, and (ii) through the $g_{H,a}$ term in the Hajicek equation (2.14) can be used to influence the evolution of the Hajicek field so as to keep it in canonical or near-canonical Kerr form.

The fact that the acceleration functions G and A_a have negligible effects in the slicing transformations permits us to approximate them as zero:

$$G\cong 0\,, \quad A_a\cong 0\,. \tag{D12f}$$

Then the universal times t and t' are related by Eqs. (D12a), (3.25), and (3.26b):

$$t+g_H^{-1}\ln\alpha=\bar{t}(\bar{t}',x^{a'})\,, \quad \bar{t}'\equiv t'+g_H'^{-1}\ln\alpha' \,, \tag{D13a}$$

where [by Eqs. (2.18d) and (D12b)]

$$\left(\frac{\partial t}{\partial t'}\right)_{\alpha',x^{a'}}=\left(\frac{\partial \bar{t}}{\partial \bar{t}'}\right)_{x^{a'}}=Y=\frac{g'_H}{g_H}\ , \quad (D13b)$$

and the lapse functions and four-velocities of the primed and unprimed FIDO's are related by Eqs. (D3b) and (D5a)

$$\alpha'=Y\alpha=(g'_H/g_H)\alpha\ , \quad (D13c)$$

$$\mathbf{U}_{\rm FIDO}=\mathbf{U}'_{\rm FIDO}-\alpha\mathbf{W}\ . \quad (D13d)$$

[Throughout Eqs. (D13) we neglect negligible fractional errors of $O(1/g_H t_*)$.] Thus, as in the constant-g_H case so also in this slowly-variable-g_H case: (i) the primed and unprimed FIDO's share a common stretched horizon $\mathscr{H}^S$; and (ii) the primed FIDO's see the unprimed FIDO's move with physical velocity $-\alpha\mathbf{W}$ tangential to the stretched horizon.

4. Effect of slicing transformations with slowly variable g_H on the stress-energy tensor of the stretched horizon

Since the surface stress-energy tensor $S^A{}_B$ of Sec. V is an intrinsic property of the stretched horizon, it will be the same for the primed and unprimed FIDO's of Sec. 3 of this appendix. But the two families of FIDO's move differently in the stretched horizon and hence will slice $S^A{}_B$ into different temporal and spatial pieces: From Eqs. (2.18) and (5.21) we see that

$$\Sigma'_H=Y\Sigma_H\ ;\ S^{H'}{}_{a'}{}^{b'}=YS^{Hb}_a\ , \quad (D14a)$$

$$\Pi^{H'}_{a'}=\Pi^H_a-S^H{}_a{}^bW_b-\Sigma_H W_a\ . \quad (D14b)$$

The changes in Σ_H and $S^H{}_a{}^b$ are the obvious ones which go along with their physical interpretations as red-shifted energy and as momentum flow per unit length per unit universal time: the change $Y=\partial t/\partial t'$ in the ticking rate of universal time, and the accompanying change in the red-shift factor, produce the factors of Y in Σ'_H and $S'_{Ha}{}^b$. Similarly, the change in Π^H_a is the obvious one that accompanies a Galilean change of reference frame. [To understand why it is a Galilean change that is relevant and why there are no velocity-change effects in Σ_H or $S^H{}_a{}^b$, reexamine Eqs. (D14) in terms of the unrenormalized, FIDO-measured energy $\Sigma=\Sigma_H/\alpha_H$, momentum $\Pi_a=\Pi^H_a$, stress $S_a{}^b=S^H{}_a{}^b/\alpha_H$, and relative velocity $v'_a=\alpha_H W_a$. The transformation must be Lorentz in terms of these physically measured quantities; but since $|\mathbf{v}'|=O(\alpha_H)\ll 1$, it is actually the Galilean limit of a Lorentz transformation; and since $\Pi_a=O(1)$ but $\Sigma=O(\alpha_H{}^{-1})$, $S_a{}^b=O(\alpha_H{}^{-1})$, this Galilean transformation affects $\mathbf{\Pi}$ but has only a negligible effect on Σ and $S_a{}^b$.]

APPENDIX E: SOME DETAILS OF THE GRAVITATIONAL PARADIGM

We present here some of the details of the calculations leading to the gravitational paradigm equations in Sec. V. The basis vectors for the stretched horizon $\mathscr{H}^S$ are $\mathbf{e}_A$ ($A=0,2,3$) consisting of the fiducion 4-velocity $\mathbf{e}_{\hat{0}}=\mathbf{U}$ and an *arbitrary* pair $\mathbf{e}_a$ ($a=2,3$) that are orthogonal to $\mathbf{U}$ and thus span the section $\mathscr{H}^S_t$. The connection coefficients for the stretched horizon's 3-geometry are computed from

$$\Gamma^A{}_{BC}=\langle\omega^A,\nabla_C\mathbf{e}_B\rangle\ . \quad (E1)$$

Here ω^A is the basis dual (in the 3-geometry) to $\mathbf{e}_A$ and ∇ is covariant differentiation with respect to the 4-geometry (equivalent to covariant differentiation with respect to the 3-geometry since ω^A projects into $\mathscr{H}^S$). We denote by $\mathbf{N}$ the unit outward normal ($\mathbf{e}_A\cdot\mathbf{N}=\langle\omega^A,\mathbf{N}\rangle=0$). In this basis we have, for example

$$\begin{aligned}\Gamma^b{}_{\hat{0}a}&=\langle\omega^b,\nabla_a\mathbf{U}\rangle=\langle\omega^b,[(\sigma_a{}^c+\tfrac{1}{2}\delta_a{}^c\theta)\mathbf{e}_c+U^N{}_{;a}\mathbf{N}]\rangle\\&=\sigma_a{}^b+\tfrac{1}{2}\delta_a{}^b\theta\\&=\alpha_H{}^{-1}(\sigma^{Hb}_a+\tfrac{1}{2}\delta_a{}^b\theta^H)\ . \end{aligned}\quad (E2)$$

Here we have used the fact that the rotation of the congruence vanishes since $\mathbf{U}$ is hypersurface orthogonal; $\overleftrightarrow{\sigma}$ and θ are the standard shear and expansion for a 2-dimensional congruence, while $\overleftrightarrow{\sigma}^H$ and θ^H are these kinematic quantities, reexpressed on a per-unit-universal time basis [Eqs. (3.30)]. By similar calculations we obtain for the other connection coefficients for small α_H:

$$\Gamma^{\hat{0}}{}_{\hat{0}A}=0\ ,\ \ \Gamma^a{}_{\hat{0}\hat{0}}=\gamma^{ab}\Gamma^{\hat{0}}{}_{b\hat{0}}=O(\alpha_H{}^2)\ ,$$

$$\Gamma^{\hat{0}}{}_{ab}=\gamma_{ac}\Gamma^c{}_{\hat{0}b}=\alpha_H{}^{-1}(\tfrac{1}{2}\gamma_{ab}\theta^H+\sigma^H_{ab})\ , \quad (E3)$$

$$\Gamma^a{}_{b\hat{0}}=(\text{a quantity depending on the unspecified time evolution of } \mathbf{e}_b)\ ,$$

together with $\Gamma^a{}_{bc}$, which are the same as those computed for the 2-geometry of $\mathscr{H}^S{}_t$ and depend on the unspecified details of the basis $\mathbf{e}_a$. [In Eqs. (E2), (E3), and throughout this appendix we ignore fractional corrections of $O(\alpha_H{}^2)$ or smaller.]

With these connection coefficients, with

$$\alpha_H S^{\hat{0}\hat{0}}=\Sigma^H\ ,\ \ S^{\hat{0}}{}_b=\Pi^H{}_b\ ,$$

$$\alpha_H S_{ab}=(P^H-\zeta^H\theta^H)\gamma_{ab}-2\eta_H\sigma^H_{ab}\ , \quad (E4)$$

and with Eqs. (5.11) and (5.14), the law of energy conservation (5.22) and Navier-Stokes equation (5.23) follow directly from the law of energy-momentum conservation (5.15).

The derivation of the tidal-field termination law (5.28a) starts with Eq. (5.1b) in the form

$$\mathscr{B}_{ab}=\tfrac{1}{2}\epsilon_{ajk}C^{jk}{}_{b\hat{0}}=\epsilon_{acN}C^{cN}{}_{b\hat{0}}\ , \quad (E5)$$

rewritten in terms of the Riemann and Ricci tensors

$$\mathscr{B}_{ab}=\epsilon_a{}^{cN}(R_{cNb\hat{0}}-\tfrac{1}{2}\gamma_{bc}R_{N\hat{0}})\ . \quad (E6)$$

The Riemann tensor is then written via the Gauss-Codazzi equation (5.26a) in terms of the extrinsic curvature K_{AB} as

$$R_{cNb\hat{0}}=K_{cb|\hat{0}}-K_{c\hat{0}|b}\ . \quad (E7)$$

The conditions that $\mathscr{B}_{ab}$, $R_{N\hat{0}}$, and K_{AB} terminate at the membrane, together with the Einstein field equation $R_{N\hat{0}}=8\pi T_{N\hat{0}}$ and Israel's junction conditions (5.16), bring (E6) and (E7) into the form

$$\mathscr{B}_{ab}=8\pi\epsilon_a{}^{cN}(S_{cb\,|\,\hat{0}}-\tfrac{1}{2}\gamma_{cb}S^A{}_{A\,|\,\hat{0}}-S_{c\hat{0}\,|\,b}-\tfrac{1}{2}\gamma_{bc}T_{N\hat{0}}) , \tag{E8}$$

where $\mathscr{B}_{ab}$ is now the tidal field immediately above the membrane. The right-hand side is of order $\alpha_H{}^{-2}$ and (aside from corrections higher order in α_H) is traceless, and therefore represents $\alpha_H{}^{-2}\mathscr{B}^H_{ab}$. By taking the cross product with **N** we therefore obtain

$$\begin{aligned}(\overleftrightarrow{N}\times\overleftrightarrow{\mathscr{B}}^H)_{ab}&=\epsilon_{aN}{}^c\mathscr{B}^H_{cb}\\&=8\pi\alpha_H{}^2(S_{ab\,|\,\hat{0}}-\tfrac{1}{2}\gamma_{cb}S^A{}_{A\,|\,\hat{0}}\\&\quad-S_{a\hat{0}\,|\,b}-\tfrac{1}{2}\gamma_{ab}T_{N\hat{0}}) .\end{aligned} \tag{E9}$$

When Eqs. (E3), (E4), (5.13a), and the law of energy conservation (5.22) are used, the tidal-field termination law (5.28a) follows.

The tidal-field termination law (5.28b) can be derived in a similar manner:

$$\begin{aligned}\mathscr{E}^H_a=\alpha_H\mathscr{E}_{aN}&=(R_{N\hat{0}a\hat{0}}+\tfrac{1}{2}R_{aN})\\&=\alpha_H(K_{\hat{0}\hat{0}\,|\,a}-K_{\hat{0}a\,|\,\hat{0}}+\tfrac{1}{2}R_{aN})\\&=8\pi\alpha_H(S_{\hat{0}\hat{0}\,|\,a}+\tfrac{1}{2}S^A{}_{A\,|\,a}-S_{\hat{0}a\,|\,\hat{0}}+\tfrac{1}{2}T_{aN})\\&=8\pi[D_t\Pi^H{}_a+(P^H+\tfrac{1}{2}\Sigma^H-\zeta^H\theta^H)_{,a}\\&\quad+\theta^H\Pi^H_a+2\Pi^H{}_b\sigma^b{}_a-\tfrac{1}{2}\mathscr{G}^H_a] .\end{aligned} \tag{E10}$$

Here the third equality, which shows that $\mathscr{E}^H$ terminates, follows from the Gauss-Codazzi equation (5.26a), the fourth from Israel's junction conditions, and the fifth from Eqs. (E3), (E4), and (5.13b). By combining (E10) with the Navier-Stokes equation (5.23) we obtain (5.28b).

The termination of $\mathscr{B}_{NN}$ is proved following the above pattern. From Eqs. (5.1b), (5.26a), and (5.16) we have

$$\begin{aligned}\mathscr{B}_{NN}=\tfrac{1}{2}\epsilon_{\hat{0}Nab}C^{ab}{}_{N\hat{0}}&=\tfrac{1}{2}\epsilon_N{}^{ab}R_{N\hat{0}ab}\\&=\epsilon_N{}^{ab}K_{\hat{0}b\,|\,a}=\epsilon_N{}^{ab}S_{\hat{0}b\,|\,a} ,\end{aligned} \tag{E11}$$

which shows that $\mathscr{B}_{NN}$ terminates at the membrane. When one tries to proceed further and express the right-hand side of (E11) in terms of Σ^H, P^H, θ^H, σ^H_{ab}, and Π^H_a [as was done in (E9) and (E10)], one finds that it depends on $O(\alpha_H{}^2)$ corrections to these properties—corrections not incorporated into our membrane formalism. More specifically, it depends on the symmetric, trace-free part of $n_{a;b}$ where **n** is the ingoing null vector of Eq. (3.1); equivalently, it depends on the Newman-Penrose spin coefficient λ, which has not been included in our membrane formalism.

In the electromagnetic paradigm there is awkwardness about the tangential **E** field interior to the membrane (see the discussion at the end of Sec. IV). This awkwardness is present also for the gravitoelectric field $\mathscr{E}_{ab}$. The relationship of the Weyl, Riemann, and Ricci tensors and the Ricci scalar gives

$$\mathscr{E}_{ab}=R_{a\hat{0}b\hat{0}}+\tfrac{1}{2}(R_{ab}-\gamma_{ab}R_{\hat{0}\hat{0}})-\tfrac{1}{6}\gamma_{ab}R , \tag{E12}$$

and hence

$$\begin{aligned}\mathscr{E}^H_{ab}&=\alpha_H{}^2(\mathscr{E}_{ab}-\tfrac{1}{2}\gamma_{ab}\mathscr{E}^c{}_c)\\&=\alpha_H{}^2(R_{a\hat{0}b\hat{0}}-\tfrac{1}{2}\gamma_{ab}R^c{}_{\hat{0}c\hat{0}})\\&\quad+\tfrac{1}{2}\alpha_H{}^2(R_{ab}-\tfrac{1}{2}\gamma_{ab}R^c{}_c) .\end{aligned} \tag{E13}$$

The Riemann terms are related to the Riemann tensor ${}^{(3)}R_{ABCD}$ and extrinsic curvature K_{AB} of $\mathscr{H}^S$ by the Gauss-Codazzi equation (5.26b):

$$\alpha_H{}^2R_{a\hat{0}b\hat{0}}=\alpha_H{}^2({}^{(3)}R_{a\hat{0}b\hat{0}}+K_{a\hat{0}}K_{b\hat{0}}-K_{\hat{0}\hat{0}}K_{ab}) . \tag{E14}$$

These terms are finite as $\alpha_H\to 0$ (though they are discontinuous, at any finite α_H, across the membrane). The Ricci terms $\alpha_H{}^2R_{ab}$ have contributions from the surface stress tensor S_{ab} (of order $\alpha_H{}^{-1}$), which is infinite at any α_H, in the surface itself. The contribution of these terms to $\mathscr{E}^H_{ab}$ is of order α_H, but infinite in the surface. Thus, as with the electric fields, we cannot ask what the gravitoelectric tidal field is *in* the surface; rather, the tidal force equation (5.29) must be taken as relating the stretched horizon's shear to the value of $\mathscr{E}^H_{ab}$ just above the membrane.

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