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Mental Number Line, Number Line Estimation, and Mathematical Achievement: Their Interrelations in Grades 5 and 6

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As indicated by the distance effect and the spatial–numerical association of response codes (SNARC) effect, natural numbers are mentally represented on a number line. Purportedly, this number line underlies children's number sense, which supports the acquisition of more advanced mathematical competencies. In 3 studies with a total of 429 fifth and sixth graders, the authors compared the influences of each child's distance effect, SNARC effect, conceptual knowledge about decimal fractions, and numerical intelligence on mathematical school achievement. Additionally, they tested using decimal fractions whether number line estimation competence mediates the influence of the internal number line. In all, the results, found with path models, revealed that domain-specific conceptual knowledge, numerical intelligence, and number line estimation each were good predictors of achievement, while distance and SNARC effects were virtually unrelated to all other variables. Individual differences in the use of the internal number line, as assessed by these 2 effects, seem to be of little importance when it comes to the acquisition of the content of 5th- and 6th-grade mathematics lessons. The results instead highlight the importance of conceptual understanding and estimation competence.

Keywords: number line, distance effect, conceptual knowledge, numerical intelligence, mathematical achievement

Mathematical competence comprises a wide variety of different cognitive skills and processes. Some elementary mathematical skills such as the discrimination of numerosity or arithmetic computations with small sets of objects are already observed in infants (Starkey & Cooper, 1980; Wynn, 1992), which has been interpreted as evidence that the human brain is endowed with an innate number sense (Dehaene, 1997). In this context, *number sense* refers to the fundamental "ability to mentally represent and manipulate numerosities on a mental 'number line'" (Dehaene, 2001, p. 17). Numbers are regarded to be represented in an analogical format on this mental number line, allowing for an automatic and efficient processing of numerical quantities (cf. Newcombe, 2002). These mechanisms underlie mathematical intuitions that help "to quickly decide that 9 is larger than 5, that 3 falls in the middle of 2 and 4, or that 12 + 15 cannot equal 96, without much introspection as to how we perform these feats" (Dehaene, 2001, p. 16).

Thus, the mental number line can be regarded as a domain-specific foundation upon which the acquisition of more advanced mathematical concepts and procedures can be built. Case and Okamoto (1996) took this argument even further by stating that the mental number line "forms a sort of lens through which children view the world" and that it "constitutes a tool that [children] use to create new knowledge" (p. 8). However, empirical evidence of a link between the mental number line and higher order mathematical competence is sparse and inconclusive. With respect to the normal range of mathematical performance, virtually nothing is known about the relevance of the mental number line for the acquisition of more advanced mathematical concepts. In fact, educational research has primarily focused on the role of domain-specific conceptual knowledge and intelligence in the prediction of mathematical achievement in school. How the fundamental ability to represent and manipulate numerical information on the mental number line is related to mathematical knowledge and intelligence, and whether this basic capacity can explain individual differences in mathematical competences beyond these two well-established variables has not yet been investigated. This is the aim of the present study. In the following three sections, a brief outline about mental number line, conceptual knowledge, and intelligence as well as their relation to scholastic achievement is presented.

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The Mental Number Line

The term *mental number line*, or *internal number line*, refers to a language-independent analogical representation of numerical magnitude on which small numbers are represented on the left and large numbers are represented on the right (Dehaene, 1997). There are several lines of empirical evidence supporting the idea of an analog and one-dimensional representation of numbers (for reviews, cf. Dehaene, 1997; Hubbard, Piazza, Pinel, & Dehaene, 2005; Nieder, 2005). Probably the two most studied performance patterns in numerical cognition that can be accounted for by the assumption of a mental number line are the distance effect and the spatial–numerical association of response codes (SNARC) effect.

The *distance effect*, originally described by Moyer and Landauer (1967), is usually investigated by means of a simple number comparison task. Participants are asked to decide by pressing a button which of two visually presented Arabic numbers is larger (the left or the right). The speed of the response depends on the numerical distance between the numbers: Participants responded more quickly for number pairs with a large numerical distance (e.g., 2-9) compared with number pairs with a small numerical distance (e.g., 4-5). This finding conforms to the concept of the mental number line if it is assumed that the activation of one number spreads out to adjacent numerosities, therefore making discrimination more difficult for number pairs with a small than with a large distance (cf. Nieder, 2005). Subsequent research extended the number-comparison paradigm to two-digit stimuli revealing a logarithmic relation between the numerical distance and reaction time (Dehaene, Dupoux, & Mehler, 1990).

The SNARC effect represents another behavioral indicator of a mental number line. It was first reported by Dehaene et al. (1990), who instructed participants to judge by pressing a right or left button whether a visually presented two-digit number had a larger or smaller value than a reference number. They found that participants who pressed the left button for small numbers and the right button for large numbers displayed faster responses than those with the opposite magnitude-button mapping. In a subsequent study, Dehaene, Bossini, and Giraux (1993) asked participants to indicate by pressing right or left buttons whether one-digit numbers were odd or even. This so-called *parity judgment task*, too, elicited the SNARC effect: Participants again responded to smaller values faster with their left hand and to bigger values faster with their right hand. This association between numerical magnitude and space is independent of the notation (e.g., can be found for Arabic symbols as well as written words), the response type (e.g., button press vs. eye saccades), and handedness (cf. Hubbard et al., 2005).

Even though it is plausible to assume that the strength of both effects can serve as a marker for a person's reliance on the mental number line, individual differences in the distance effect and SNARC effect have largely remained unstudied. The few available findings, however, suggest that the strength of these effects is negatively associated with mathematical competence. For example, Dehaene et al. (1993) as well as Fischer and Rottmann (2005) observed weaker SNARC effects in students of mathematics, physics, or engineering (i.e., students being more proficient in mathematics) compared with literature and psychology students. In addition, conforming developmental trends have been reported for the distance effect. The size of the distance effect was repeatedly found to decrease with age (Holloway & Ansari, 2008; Sekuler & Mierkiewicz, 1977).

If the mental number line influences children's general math achievement, an important question to examine is what processes mediate this influence. Such mediating variables could be used to foster children's math learning. One variable could be children's competence in using external knowledge representations, for example, diagrams. The internal number line is a mental knowledge representation, while external knowledge representations are part of the outside world (e.g., printed on paper). However, like the internal number line, many external diagrams represent magnitudes in analogical form, for example, as position on an axis, as length of a bar, or as area of a pie chart (Zhang, 1996). The closest similarity exists between the internal number line and the external number line, since both are not only analogous but also one-dimensional and usually represent magnitudes in ascending order from left to right (Fias & Fischer, 2005).

Gattis (2001, 2002; Gattis & Holyoak, 1996) has characterized the process of diagram interpretation as structure mapping between an external and an internal knowledge representation in the sense of Gentner's (1983) structure mapping theory of analogy. Gentner suggested that the cognitive process underlying people's drawing of an analogy (e.g., between the solar system and an atom) consists mainly in the systematic mapping of conceptual relations in one domain (e.g., the planets revolve around the sun) onto relations in the other domain (e.g., the electrons revolve around the nucleus). Gattis pointed out that the process of interpreting diagrams is similar to this in that one has to map a system of visuospatial relations in the external diagram (e.g., the slope of a line graph) onto their mentally represented conceptual meanings (e.g., the rate of change of a variable).

Structural similarity is a precondition for analogical structure mapping (Gentner & Toupin, 1986). Therefore, the use of the external number line may be easier for children who tend to represent numbers on the structurally similar mental number line than for children who tend to represent numbers verbally or as digits. This effect could transfer to more complex diagrams, such as coordinate systems, especially since the two axes of coordinate systems basically are number lines.

The competent use of diagrams helps to efficiently communicate information (Larkin & Simon, 1987), solve complex problems (Novick, 2001), draw logical inferences (Stenning & Lemmon, 1999), and transfer knowledge between problems (Stern, Aprea, & Ebner, 2003). Thus, it is an important part of pupils' mathematics and sciences competence (Hardy, Schneider, Jonen, Stern, & Möller, 2005; National Council of Teachers of Mathematics, 2000; Shah & Hoeffner, 2002). For these reasons, pupils' diagram competence is very likely a mediator of the influence of the mental number line on general math achievement.

Conceptual Knowledge

In research on mathematics learning, conceptual knowledge is seen as knowledge of the core rules and principles as well as of their interrelations in a domain (Goldstone & Kersten, 2003; Hiebert, 1986; Rittle-Johnson, Siegler, & Alibali, 2001). Accordingly, it is assumed to be stored mentally in some form of relational representation, like schemas, hierarchies, or semantic networks (Byrnes & Wasik, 1991). Because conceptual knowledge is abstract in nature, consciously accessible, and embedded in larger knowledge structures, it can be verbalized and flexibly transformed through inference, elaboration, and reflection. It is therefore not bound up with specific problems but can, in principle, be generalized for a variety of problem types and external knowledge representations in a domain (e.g., Baroody, 2003).

As an example, consider Hiebert's (1992) analysis of what makes a conceptual understanding of decimal fractions. On the one hand, learners should know that fractions can quantify nonwhole quantities of something. This is useful in many real-life situations. On the other hand, learners have to understand the notational system of decimal fractions, including its base-10 structure and the fact that the decimal point separates whole units from 10ths units. Conceptual knowledge about decimal fractions, therefore, includes knowledge about relations between digits within a fraction, relations between different decimal fractions and their respective magnitudes, relations between fractions and whole numbers, and relations between fractions and every-day-life situations (see also Resnick et al., 1989).

Due to the rich and multifaceted nature of conceptual knowledge, it is advisable to measure it by a combination of tasks of different formats, such as asking people to evaluate the adequacies of different strategies for solving problems (Siegler & Crowley, 1994), compare and categorize objects or numbers (Resnick et al., 1989), represent relations in diagrams or sketches (Byrnes & Wasik, 1991), or give verbal explanations of conceptual relations (Rittle-Johnson & Alibali, 1999).

The influence of conceptual knowledge on students' mathematical competence is straightforward. Due to its relational nature, it enables the learner to see relations between different pieces of knowledge, leading to the activation of background knowledge. Thus, the student grasps the meaning and implications of specific pieces of knowledge. This can help him or her to construct new problem-solving strategies, to transfer strategies between related types of problems, to select among alternative strategies, to monitor strategy execution, and to check answer plausibility (Baroody, 2003; Rittle-Johnson et al., 2001). Due to these important functions, many educational researchers see well-integrated conceptual knowledge as one of the most important aims of school instruction (National Council of Teachers of Mathematics, 2000; Programme for International Student Assessment, 2006).

Intelligence

Intelligence, which may be defined as the "ability to reason, plan, solve problems, think abstractly, comprehend complex ideas, learn quickly and learn from experience" (Gottfredson, 1997, p. 13), is closely associated with learning and educational success (for reviews, cf. Gustafsson & Undheim, 1996; Jensen, 1998; Schmidt & Hunter, 1998). Measures of intelligence usually correlate at about .50 with school marks (i.e., grades given on report cards), performance in scholastic achievement tests, and years of education (Neisser et al., 1996). The relation between intelligence and educational success appears to be of a general nature as the predictive value of intelligence tests primarily derives from the g factor (general intelligence; cf. Brody, 1999; Jensen, 1998), reflecting the variance that is shared by different intelligence subscales (Spearman, 1904). In contrast to conceptual knowledge, the g factor is a domain-general construct. Subscales of intelligence tests usually measure the reasoning ability in a domain (e.g., numerical, verbal, or figural intelligence) but very little domain-specific content knowledge.

The origins of the substantial and robust correlation between intelligence and scholastic achievement, however, are largely unknown and heavily disputed. Some researchers have suggested reciprocal (e.g., Brody, 1997) causal relations; others have proposed unidirectional relationships, in which intelligence influences scholastic performance (e.g., Jensen, 2000) or vice versa (e.g., Blair, Gamson, Thorne, & Baker, 2005). Some even argued that the observed association is due to similarities in intelligence and achievement tests (e.g., Ceci, 1991; Flanagan, Andrews, & Genshaft, 1997). A very recent study suggesting that intelligence influences mathematical achievement was conducted by Watkins, Lei, and Canivez (2007). They applied a cross-lagged panel design to 289 students (mean age: 9 years) in which they assessed the participants' intelligence and achievement twice with a mean test–retest interval of about 3 years. Using structural equation modeling, they found that the optimal model reflected a causal precedence of intelligence on achievement. They interpreted their findings following Jensen (1998), who emphasized the importance of high general intelligence as a prerequisite for the successful acquisition of knowledge in school.

Aim of the Present Investigation

The mental number line is assumed to underlie number sense, that is, the fundamental ability to automatically and efficiently process numerical quantity information. Although there is no conclusive empirical evidence, it has repeatedly been claimed that individual differences in this basic capacity influence the acquisition of advanced mathematical concepts and, eventually, the achieved mathematical competence (Dehaene, 1997). In contrast, the relevance of mathematical conceptual knowledge and intelligence for mathematical achievement is well-documented. Our aim in the present investigation was to provide first insights into the relationship between the mental number line, conceptual knowledge, intelligence, and mathematical achievement in school.

While it has already been shown that the mental number line influences children's early abilities to understand and operate on whole numbers (Gilmore, McCarthy, & Spelke, 2007; Holloway & Ansari, in press), we focused on its influence on competencies, which are just a step further advanced. We investigated fifth and sixth graders' knowledge about decimal fractions, since this topic is taught shortly after whole-number arithmetic in many countries. Three empirical studies are reported in which the relations between the aforementioned variables were investigated using path analyses. In all three studies, reliance on the mental number line was operationalized as the strength of the distance and/or SNARC effect, conceptual knowledge was assessed in the domain of decimal fractions, and the mark in mathematics was taken as the measure for scholastic mathematical achievement. The reported studies are part of a larger research project on the relation between conceptual and procedural knowledge in the domain of mathematics (Schneider & Stern, 2005, 2008).

Based on the findings from previous investigations outlined in the introduction, we hypothesized that conceptual knowledge and intelligence are correlated with each other and that both display a significant association with mathematical achievement. The role of the mental number line, which was in the focus of the present investigation, is more unclear. Following the arguments by Dehaene (1997, 2001), as well as Case and Okamoto (1996), we expected a significant association between the strength of the distance effect and SNARC effect and mathematical achievement.

In order to reveal variables mediating the assumed link between number line and mathematical achievement, we administered two additional tasks. The first was the number line estimation task, because participants who more strongly rely on the internal representation of quantity on a mental number line should also perform better in tasks demanding a structurally similar external representation form. This task was given in all three studies. We used the second assessment, a test of graph understanding, in Study 3 only to examine the hypothesis that the influence of the mental number line transfers from external number line estimation to competence in working with more complex representations such as coordinate systems. Our path models directly mirror these theoretical assumptions: Whenever possible we specified regression paths from distance effect, SNARC effect, conceptual knowledge, and numerical intelligence to mathematical achievement. In addition to these direct paths, we modeled indirect paths with number line estimation accuracy as a mediator between the predictors and the math achievement.

Study 1

Research Questions

In Study 1, we administered a parsimonious design to provide first data on the relation between the mental number line, conceptual knowledge, and mathematical school achievement. Individual differences in reliance on the mental number line are operationalized by means of the distance effect size. If the mental number line is related to mathematical achievement, we expected that a significant correlation between the size of the distance effect and the mathematics mark should emerge. In addition to these three variables, the performance in an external number line test was included as a variable that potentially moderates the influence of knowledge and mental number line on the mathematics mark.

Method

Participants. The sample comprised 115 fifth graders from 11 schools in Berlin, Germany. They were volunteers and received monetary compensation. The sample mean age was 11.3 years (SD = 0.6, minimum = 10.2, maximum = 15.0). Of the children, 46.4% were girls. The children came to our research institute in small groups and worked individually on computers in a quiet room without seeing each other. The experimenter was present the entire time and available for questions.

Procedure. We measured distance effect, conceptual knowledge, number line estimation accuracy, number line estimation speed, and mathematics marks in a session that lasted about 70 min per child (Time 1). Five or six days later, during a second session, we measured the distance effect again (Time 2) to investigate its stability.

Statistical analysis. We investigated the covariance structure of our data by means of path analyses in the program MPlus (Muthén & Muthén, 1998–2006). We used the robust estimator MLR (Muthén & Muthén, 1998–2006, pp. 423–426), which allows for missing data and does not require a normal distribution of the data. In our theoretical model, distance effect and conceptual knowledge are predictors of both number line estimation accuracy and mathematical achievement. The influence of the distance effect on mathematical achievement can be direct as well as mediated by number line estimation accuracy. For reasons of parsimony and to ensure model identification, we included only paths that connect bivariately correlated variables in the path model.

Assessments. To assess the strength of each child's distance effect, we used 150 trials, the first 10 of which were practice trials and were not analyzed. The children got feedback on their solution correctness during the practice trials only. In each trial, a child saw a whole number between 10 and 99 in the middle of the computer screen. The child was asked to press a right-hand key on a standard computer keyboard with his or her right index finger if the number was larger than 55 and to press a left-hand key with his or her left index finger if the number was smaller than 55. The numbers were the same for all children and were selected by a pseudorandom algorithm. Reaction times were measured from stimulus onset until response. Incorrectly solved trials and outliers, defined as reaction times lying more than five standard deviations above or below the sample mean, were excluded from the analyses. For each trial, the absolute numerical distance between the number presented and the standard, 55, was computed. We then took the natural logarithm of this value. For example, the first stimulus was 64 for each child, which yields a distance of 9. The natural logarithm of 9 is 2.197. These logarithmic values were used as predictors of the reaction times in a linear regression conducted for each child. The standardized regression weight beta was taken as the measure of the distance effect for that child. Because there is a negative relationship between distance and reaction time, the regression weights should be mostly negative.

Four different types of tasks were used to assess children's conceptual knowledge about decimal fractions. In the first task, the children were asked to evaluate the correctness of eight verbally described strategies for finding the position of a decimal fraction on a number line by clicking one of two buttons on the screen. Only four of the strategies were correct. In the second task, the children were shown a decimal fraction together with four pie charts. A part of each pie chart was shaded, and the children were told to click on the pie chart where the proportion of the shaded area compared to the entire pie area corresponded to the decimal fraction. We computed the percentage of correct answers relative to the 20 trials used in total. In this and all following tasks, decimal fractions between zero and one in the German notation (i.e., with a comma instead of a decimal point) with one to three digits after the comma were used. In the third tasks, the students were shown 20 pairs of decimal fractions and were asked to click on the larger number of each pair. Again the percentage of correct answers was computed. In the fourth task, the children wrote down answers to four questions about the general properties of decimal fractions in a booklet. Their answers were coded independently by two trained raters as fully correct (2 points), partly correct (1 point), missing or wrong (0 points). For each child, we computed which percentage of the maximum number of points he or she achieved. Finally, we averaged the percentage values from the four different task types into an overall measure of the child's conceptual knowledge.

We assessed number line estimation accuracy with 20 trials of a task designed by Rittle-Johnson et al. (2001). In each trial, the children saw a decimal fraction and a number line ranging from zero to one on the screen. The number line had hatch marks as well as labels only at its both ends, with the label *0* below the first hatch mark and the label *1* below the second hatch mark. The decimal fractions had one to three digits after the comma. Their values were greater than zero and smaller than one. The children were asked to move a lever on the screen to the position on the number line indicated by the value of the decimal fraction. They were told that accuracy was more important than speed. In accordance with Rittle-Johnson et al. (2001), answers within ± 0.1 units around the correct position were scored as correct by the assessment program. We again computed the percentage of correct answers.

Number line estimation speed was measured in a similar manner, but with three differences: (a) The children gave their answers by freely clicking on the number line with the mouse, (b) they were asked to optimize speed as well as accuracy of their answers, and (c) the first three trials were not included in the analyses, because they might have been biased by children's general orientations in the task surface. The solution times were averaged over the remaining 17 trials.

Mathematical school achievement. The students' mathematical school achievement was measured in the form of their mathematics marks. The pupils were asked to write down the mark they had received according to their latest school report. In Germany, school marks range from 1 (best) to 6 (worst). Therefore, we expected to find inverse relations between students' marks and our measures of estimation and knowledge. In a recent metaanalysis (Kuncel, Credé, & Thomas, 2005), the correlation between self-reported and actual mathematics marks was found to be $r = .82$ for high school students. School marks have been shown to correlate at about $r = .60$ to .70 with the results of pupils' standardized tests on scholastic performance (cf. Tent, 2006).

Results

Eight of the 115 children had missing data on one or more variables, because either they did not complete all assessments or there were problems with the computer hardware. Solution rate on the distance effect tasks was .933 ($SD = .094$, minimum = .500). The descriptive statistics of our measures are given in Table 1. We found significant ($ps \leq .05$) individual distance effects for 84.1% of the sample at Time 1 and for 77.8% of the sample at Time 2. The means of the beta weights operationalizing the individual children's distance effect differed significantly from zero at Time 1, $t(108) = -25.2$, $p < .001$, and at Time 2, $t(106) = -21.9$, $p < .001$. The distance effect explained a mean variance proportion of $M = .126$ ($SD = .092$) of the reaction times per child at Time 1 and of $M = .100$ ($SD = .080$) at Time 2. The unstandardized regression coefficients (b) had a sample mean of $M = -117.5$ ($SD = 72.0$) at Time 1 and of $M = -117.8$ ($SD = 78.2$) at Time 2. Thus, as necessary for our later analyses, we found a general distance effect in our sample with large differences between persons. The correlation of the distance effects found at Time 1 and Time 2 ($r = .501$, $p \leq .001$) indicated that either the intrapersonal stability of the distance effect or the retest reliability was low. The scale reliabilities as estimated by Cronbach's alpha were .78 for number line estimation accuracy, .94 for number line estimation speed, and .74 for conceptual knowledge.

Table 2
Intercorrelations of the Measures in Study 1

Measure	Conceptual knowledge	Number line estimation accuracy	Mathematics mark
Distance effect (Time 1)	.001	.080	.000
Conceptual knowledge	—	.677***	−.458***
Number line estimation accuracy	—	—	−.469***

*** $p < .001$.

The intercorrelation matrix of the variables is given in Table 2. Conceptual knowledge, number line estimation accuracy, and mathematics mark were highly significantly correlated. The distance effect, however, was not significantly related to any of these variables. In additional explorative analyses, we found that the solution times in the number estimation task were not significantly correlated with any of our variables (all *r*s between –.15 and .15).

Based on our theoretical expectations and the intercorrelation matrix, we specified the path model shown in Figure 1, which has an excellent fit to the data, $\chi^2(3) = 1.405$, comparative fit index = 1.000, root-mean-square error of approximation = 0.000. The probability of finding the obtained or more extreme data under the assumption that the model holds for the population is $p = .704$. Table 3 shows the model parameters. All path coefficients are highly significant. The good model fit indicates that relations not specified in the model can indeed be neglected. The predictors explain a variance proportion of .255 for children's mathematics marks and of .467 for number line estimation accuracy. As expected, children with higher conceptual knowledge or higher estimation accuracy have lower (i.e., better) marks.

Table 1
Number of Valid Cases, Means, Standard Deviations, Minimums, and Maximums of the Measures in Study 1

Measure	No. of valid cases	*M*	*SD*	Min.	Max.
Distance effect, Time 1 (β)	109	−0.331	0.137	−0.665	0.019
Distance effect, Time 2 (β)	107	−0.289	0.136	−0.612	0.068
Conceptual knowledge (%)	113	50.3	13.3	26.3	83.1
Number line estimation accuracy (%)	114	54.9	20.5	10.0	100.0
Mathematics mark (absolute)	114	2.46	0.78	1.00	5.00

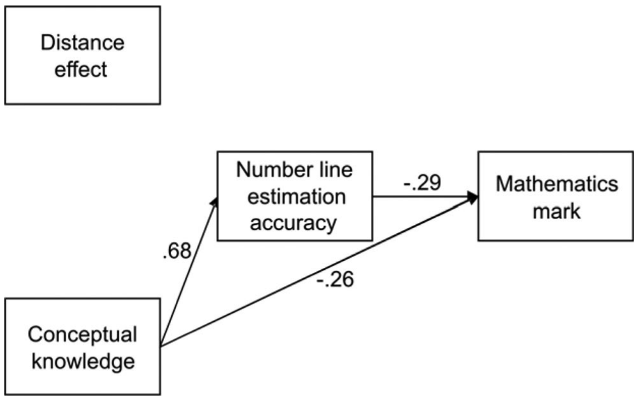

Figure 1. Path model of the relations between conceptual knowledge, distance effect, number line estimation accuracy, and math marks in Study 1.

Discussion

The results of this study corroborate the well-established relationship between conceptual knowledge and mathematical achievement. The bivariate correlation indicates that over 20% of the variance in the mathematics mark can be accounted for by our measure of conceptual knowledge. This finding appears highly notable considering that we assessed knowledge about decimal fractions that represents only a small part of the conceptual knowledge involved in school mathematics. The results of the path analysis in which the performance in the external number line task is modeled as a potential mediating variable suggest both a direct and indirect influence of conceptual knowledge on the mathematics mark.

The second hypothetical predictor of mathematical achievement, the distance effect, however, was uncorrelated with the mathematics mark (r = .000 [sic]) as well as our other measures. This null finding cannot be attributed to the absence of the distance effect in the sample or very low variability. Therefore, this study provides first evidence that the size of the distance effect might not be related to higher order mathematical knowledge and performance. Given the comparably low test–retest correlation of the distance effect sizes over 5–6 days, it might also be questioned whether the distance effect can be considered a trait variable.

Study 2

Research Questions

Study 1 provided no evidence of a relation between mental number line and mathematical achievement. However, mental number line was only measured in terms of the distance effect. In Study 2, we introduced the SNARC effect as an alternative measure for the mental number line. If both effects reflect individual differences in reliance on the mental number line, we expected that they should correlate. Moreover, the administration of both measures allowed us to investigate whether the distance effect or the SNARC effect represents a good predictor for variables of mathematical achievement. A larger number of trials were administered in the measurement of the distance effect in Study 2. We did this to find out whether the null correlation of the distance effect with the other performance measures in Study 1 might be due to a low reliability caused by using too few items.

Method

Participants. We used the same procedure as in Study 1, with 110 volunteering fifth graders from 13 different Berlin schools. The sample mean age was 11.1 years (SD = 0.5, minimum = 9.4, maximum = 12.3); 47.4% of the children were girls.

Procedure and assessments. All measures were identical to Study 1 with two exceptions: (a) We measured the SNARC effect, and (b) there were now 210 trials for the distance effect (of which the first 10 were designated practice trials and were not included in the analyses).

We measured the SNARC effect with 88 trials of the parity judgment task (cf. Dehaene et al., 1993). In each trial, the participants saw a number from zero to nine on the screen. In half of the trials (Block A), the students were told to press a left-hand button with their left index finger if the number was odd and to press a right-hand button with their right index finger if the number was even. In the other half of the trials (Block B), the children were instructed to press the left-right button for even numbers and the right-right button for odd numbers. The order of Block A and Block B was randomized per child, while the trial order within the blocks was always the same. The first four trials of each block, for which the children got feedback on the correctness of their solutions, were discarded as practice trials. As in Study 1, solution times of incorrectly solved tasks and times lying more than five standard deviations above or below the sample mean were excluded from the analyses. For each stimulus (0–9), we measured how fast each child responded with the left hand or with the right hand. The time measured for left-hand responses was subtracted from the time measured for right-hand responses. For example, 1 participant needed 660 ms to respond to the stimulus zero correctly with the left hand, but 715 ms to respond to the same stimulus correctly with the right hand, leading to a reaction time difference of 55 ms. The reaction time differences for the remaining nine stimulus numbers for this participant were obtained the same way.

Table 3
Model Estimated Path Coefficients in Study 1

Relation in the model	Unstandardized coefficient	SE	Standardized coefficient	p
Number line estimation accuracy on conceptual knowledge	1.045	0.098	.683	≤.001
Mathematics mark on conceptual knowledge	−0.015	0.005	−.258	.002
Mathematics mark on number line estimation accuracy	−0.011	0.004	−.293	.002

Table 4
Number of Valid Cases and Descriptive Characteristics of the Measures in Study 2

Measure	No. of valid cases	*M*	*SD*	Min.	Max.
Distance effect (β)	103	−0.284	0.124	−0.574	0.007
SNARC effect (β)	100	−0.170	0.313	−0.715	0.413
Conceptual knowledge (%)	107	50.4	14.7	26.9	81.9
Number line estimation accuracy (%)	107	52.4	22.2	5.0	100.0
Mathematics mark (absolute)	107	2.49	0.94	1.00	6.00

Note. SNARC = spatial–numerical association of response codes.

For each child, we then regressed the 10 reaction time differences on the 10 respective stimuli magnitudes (0–9). Since persons respond to small numbers faster with their left hand and to large numbers faster with their right hand, we expected mostly negative regression slopes. As recommended by Fias and Fischer (2005), the beta coefficient of this slope was taken as measure of the individual SNARC effect size.

Results

Table 4 shows the descriptive characteristics of all measures. Of the 110 children, 10 had missing data on one or more variables due to either hardware problems, not completing all assessments, or exclusion of error trials and reaction time outliers from the analyses. Solution rates were M = .880 (SD = .204) on the distance effect tasks and M = .922 (SD = .114) on the SNARC effect tasks. The distance effects of 7 children and the SNARC effects of 1 child were not computed, since the children solved less than 50% of the respective tasks correctly. The coefficients operationalizing the pupils' distance effects were significantly different from zero, $t(102) = –23.2$, $p < .001$. The distance effect explains a variance proportion of M = .099 (SD = .095) of the reaction times per child. The coefficients operationalizing the pupils' SNARC effects differed significantly from zero too, $t(99) = –5.4$, $p < .001$. The SNARC effect explains a variance proportion of M = .126 (SD = .128) of the reaction times per child. The unstandardized regression weights had a sample mean of –115.9 (SD = 87.5) for the distance effect and of –11.2 (SD = 29.0) for the SNARC effect. The average size of the SNARC effect conforms to previous studies on children of our age group (e.g., Bachot, Gevers, Fias, & Roeyers, 2005) and was somewhat larger than in previous studies on adults (Fischer & Rottmann, 2005).

The scale reliabilities as estimated by Cronbach's alpha were .83 for number line estimation accuracy and .71 for conceptual knowledge. The intercorrelations presented in Table 5 substantiate the finding of significant relations between conceptual knowledge, number line estimation accuracy, and mathematics mark in Study 1. Neither the distance effect nor the SNARC effect were significantly correlated with mathematics mark. There was only a small correlation between distance effect and number line estimation accuracy. In addition, distance effect and SNARC effect were significantly yet weakly correlated. Additional explorative analyses revealed that the solution times for the number line estimation tasks were significantly correlated with number line estimation accuracy (r = .200) but not with any other variable tested.

Based on these intercorrelations and our theoretical expectations, we specified the model shown in Figure 2. This model had an excellent fit to the data, $\chi^2(5) = 2.246$, $p = .814$, comparative fit index = 1.000, root-mean-square error of approximation = 0.000. The model parameters are reported in Table 6. The proportions of explained variance were .425 for number line estimation accuracy and .232 for mathematics mark.

Discussion

In line with our expectations, Study 2 has shown that distance effect and SNARC effect are significantly correlated with each other. Even though this finding may point to a common basis for both behavioral effects, it should be emphasized that the size of the correlation was very small, accounting for only about 6% of the variance. Similar to Study 1, Study 2 also found no significant association between distance effect or SNARC effect and mathematics mark. There was only a weak correlation between the distance effect and external number line accuracy, indicating that the ability to represent and manipulate magnitudes on the mental number line may partly be reflected in the competence to solve estimation tasks with external number lines, which, in turn, influences mathematical achievement. Besides this weak effect, the

Table 5
Intercorrelations of the Five Measures in Study 2

Measure	SNARC effect	Conceptual knowledge	Number line estimation accuracy	Mathematics mark
Distance effect	.245*	−.108	−.202*	.120
SNARC effect	—	−.018	−.106	.082
Conceptual knowledge	—	—	.645***	−.431***
Number line estimation accuracy	—	—	—	−.448***

Note. SNARC = spatial–numerical association of response codes.
* $p < .05$. *** $p < .001$.

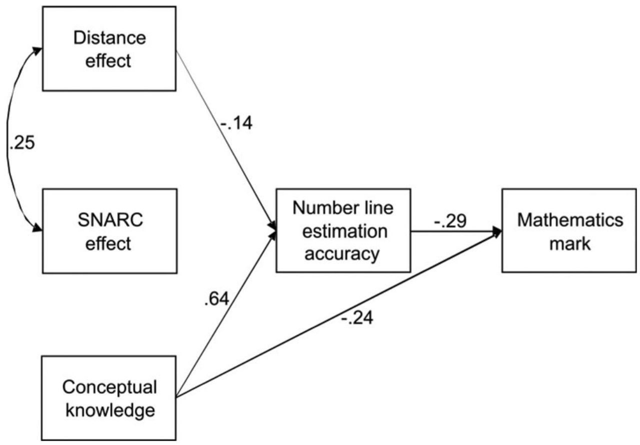

Figure 2. Path model of the relations between conceptual knowledge, distance effect, spatial–numerical association of response codes (SNARC) effect, number line estimation accuracy, and math marks in Study 2.

results of the path analysis again showed a strong direct and indirect influence of conceptual knowledge on the mathematics mark.

Study 3

Research Questions

Studies 1 and 2 revealed that conceptual knowledge is strongly related to mathematical achievement at school, while distance effect and SNARC effect are not. The still unanswered questions, however, are the following: How do these variables relate to the individual's numerical intelligence, and how much incremental variance of students' mathematical achievement can be explained by the intelligence measure? Therefore, the participants' numerical intelligence was introduced as additional predictor in Study 3. In light of considerable evidence that both knowledge and intelligence impact on scholastic performance, we expected an independent effect of intelligence on the mathematics mark.

Furthermore, Studies 1 and 2 have shown that the impact of conceptual knowledge on mathematical performance might be mediated by competence in using an external representation of numbers. However, this held true only for the accuracy but not the speed of task performance. Therefore, and for reasons of parsimony, we only included the external number line accuracy in Study 3.

Since competence in using external numerical representations may also transfer to more complex representations such as coordinate systems, an additional graph test was administered. We explored how external number line performance and graph test performance relate to each other and whether graph competence represents a further mediating variable between mental number line and higher order mathematical achievement.

Method

Participants. Participants were 204 volunteers from the fifth and sixth grades of 14 Berlin schools, who were monetarily compensated. The sample mean age was 11.3 years (SD = 0.07, minimum = 9.3, maximum = 13.8). Of the children, 47.3% were fifth graders and 51.2% were girls.

Procedure. As previously explained, this study is part of a larger project. For logistical reasons, the measures analyzed in Study 3 were assessed at the third point (Time 3) of a longitudinal one-group design. The first two measurement points (Time 1 and Time 2) were 1 day apart, while approximately 4 month lay between Time 2 and Time 3. We assessed children's conceptual knowledge about decimal fractions and their accuracy on the number line estimation task at all three measurement points. At Time 3, we additionally measured the distance effect, the SNARC effect, a graph test, and intelligence using the same method as in Study 1 and Study 2.

Between Time 1 and Time 2, all of the children played the catch-the-monster game, which was invented by Rittle-Johnson et al. (2001). The children saw a decimal fraction and a number line ranging from zero to one. Only the start point and the end point of the number line were labeled. The children were told that a monster was hiding at the position on the line indicated by the decimal fraction and that they could catch the monster by clicking there. After a child had entered his or her answer, the monster appeared at the correct position of the decimal fraction, thus providing feedback. In Rittle-Johnson et al.'s study, this game successfully increased children's conceptual knowledge about decimal fractions and their solution rates on the number line estimation task. The intervention is of importance for the present study only in so far as the children tested at Time 3 had more knowledge about decimal fractions than the children in Study 1 and Study 2.

Table 6
Estimated Path Coefficients of the Model in Study 2

Relation in the model	Unstandardized coefficient	SE	Standardized coefficient	p
Regression parameters				
Number line estimation accuracy on distance effect	−25.037	14.435	−.142	.041
Number line estimation accuracy on conceptual knowledge	0.952	0.104	.637	<.001
Mathematics mark on conceptual knowledge	−0.015	0.007	−.243	.017
Mathematics mark on number line estimation accuracy	−0.012	0.004	−.290	.003
Covariance parameters				
Distance effect with SNARC effect	0.009	0.004	.247	.005

Note. SNARC = spatial–numerical association of response codes.

Statistical analysis and assessments. At Time 3, the same measures as in Study 2 were administered plus two additional tests: an intelligence test and a graph test. In our theoretical model, distance effect, SNARC effect, conceptual knowledge, and numerical intelligence predict children's mathematics marks. The influences of the distance effect and the SNARC effect are both direct and mediated by a path leading over number line estimation accuracy and graph competence.

The students completed the Kognitiver Fähigkeitstest (Heller & Perleth, 2000), that is, the German version of Thorndike's Cognitive Abilities Test (e.g., Lohman et al., 2001), as a measure of their intelligence. A standardized short form of the revised version, adequate for our age group, was used. It contains a verbal, a numerical, and a figural subscale. Only the numerical subscale, comprising set comparison tasks and number series tasks, was used in our analyses. The test is 90 min long and has an internal consistency of .95 and correlations of .37–.51 with students' mathematics marks.

The second new assessment was a newly constructed graph test. The children saw six coordinate systems with one or two linear graphs in them, respectively. A sentence describing the represented situation was written at the top of each coordinate system. Four different interpretations of the specific processes represented by the line graphs were offered under each coordinate system. For a pair of two intersecting distance-time graphs representing the bike rides of two girls, Anna and Beth, one interpretation could, for example, be, "Where Anna's graph is on top of Beth's graph, Anna is riding faster than Beth." About half of the interpretations were correct. The children had to mark each of the 24 interpretations as either agreeing with the graphs or not agreeing with the graphs. The percentage of correct answers was computed per child. The graphs represented (a) distance traveled per time, (b) fuel needed per distance, (c) fat per kilogram cheese, and (d) liters of one kind of lemonade per liters of a second kind of lemonade, when mixing them. Some of the correct interpretations related to the meaning of the widths or heights of the graphs, while others related to the meaning of the slope as representing the proportion of y-change and x-change (Shah & Hoeffner, 2002). The wrong interpretations included children's typical graph misconceptions found in previous studies (Leinhardt, Zaslavsky, & Stein, 1990; Mevarech & Kramarsky, 1997). The test was iteratively optimized in an unpublished pilot study with 40 pupils.

Results

Table 7 shows the descriptive characteristics of all measures. Of the 204 children, 31 had missing data on one or more variables due to either hardware problems, noncompletion of all assessments, or data cleaning. Solution rates were $M = .942$ ($SD = .048$, minimum = .700) on the distance effect tasks and $M = .939$ ($SD = .068$, minimum = .510) on the SNARC effect tasks. The coefficients operationalizing the pupils' distance effects were significantly different from zero, $t(102) = -38.3$, $p < .001$. The distance effect explains a variance proportion of $M = 0.129$ ($SD = 0.085$) of the reaction times per child. The coefficients operationalizing the pupils' SNARC effects differed significantly from zero too, $t(172) = -7.5$, $p < .001$. The SNARC effect explains a variance proportion of $M = 0.098$ ($SD = 0.133$) of the reaction times per child. The unstandardized regression weights had a sample mean of –108.9 ($SD = 53.2$) for the distance effect and of –12.5 ($SD = 25.7$) for the SNARC effect.

As expected, the means for conceptual knowledge and number line estimation accuracy were higher than in Studies 1 and 2 due to the influence of the intervention. Nevertheless there is sufficient between-persons variance and no ceiling effect. The Kognitiver Fähigkeitstest total score, which was normed to $M = 50$ ($SD = 10$), was $M = 53.8$ ($SD = 8.3$, minimum = 26, maximum = 76) in our sample, indicating, together with the mean mathematics mark of 2.47, that our sample covers a wide competence range. The correlation between the Kognitiver Fähigkeitstest total score and mathematics mark was $r = -.598$, suggesting a high validity of children's self-reported mathematics mark as a competence measure. The negative sign indicates that children with higher intelligence got better (i.e., lower) marks.

The scale reliabilities as estimated by Cronbach's alpha were .85 for number line estimation accuracy, .77 for conceptual knowledge, and .49 for the graph test. Since the graph test items tap the understanding of different graph components (height, width, and slope) in different content areas, the low internal consistency likely reflects the heterogeneity of the items.

The intercorrelations in Table 8 show highly significant relations between conceptual knowledge, numerical intelligence, number line estimation accuracy, graph test, and mathematics mark. Neither the distance effect nor the SNARC effect were correlated with these variables except for a rather small but significant correlation between distance effect and number line estimation accuracy ($p = .019$).

Based on these intercorrelations and our theoretical expectations, we specified the model shown in Figure 3. Most indices show a good fit of the model to the data, $\chi^2(11) = 17.269$, $p = .100$, comparative fit index = 0.980, root-mean-square error of approximation = 0.053. The model parameters are reported in

Table 7
Number of Valid Cases and Descriptive Characteristics of the Measures in Study 3

Measure	No. of valid cases	M	SD	Min.	Max.
Distance effect (β)	203	−0.336	0.125	−0.634	−0.027
SNARC effect (β)	173	−0.156	0.273	−0.909	0.623
Conceptual knowledge (%)	204	66.1	14.9	26.9	95.6
Number line estimation accuracy (%)	204	86.2	17.5	20.0	100.0
Mathematics mark (absolute)	200	2.47	0.91	1.00	5.00
KFT, numerical subscale	195	56.7	9.1	24.0	82.0
Graph test (%)	194	54.5	13.3	20.8	87.5

Note. KFT = Kognitiver Fähigkeitstest.

Table 8
Full Intercorrelation Matrix of the Measures Used in Study 3

Measure	SNARC effect	Conceptual knowledge	KFT, numerical subscale	Number line estimation accuracy	Graph test	Mathematics mark
Distance effect	−.027	−.113	−.074	−.164*	−.005	.094
SNARC effect	—	−.060	−.094	−.094	−.104	.123
Conceptual knowledge	—	—	.429***	.731***	.509***	−.569***
KFT, numerical subscale	—	—	—	.415***	.305***	−.463***
Number line estimation accuracy	—	—	—	—	.323***	−.545***
Graph test	—	—	—	—	—	−.431***

Note. SNARC = spatial–numerical association of response codes; KFT = Kognitiver Fähigkeitstest.
* $p < .05$. *** $p < .001$.

Table 9. The explained variance proportions were .547 for number line estimation accuracy, .271 for graph competence, and .398 for mathematics mark.

Discussion

Several results corroborate the findings from Studies 1 and 2. There is a strong relation between conceptual knowledge and mathematical achievement. The path analysis of Study 3 suggests that this influence may be mediated by the competence in using simple visuospatial knowledge representations (i.e., external number lines) as well as more complex ones (i.e., graphs in Cartesian coordinate systems) that are structurally similar to the internal number line. In addition, the influence of external number line performance on the mathematics mark seems to be mediated by the graph competence.

Also similar to the results from Studies 1 and 2, neither measure of reliance on the mental number line (distance effect and SNARC effect) displayed a direct relation with more complex mathematical performance. Although accuracy in the number line estimation task seems to be influenced by the distance effect, the size of this effect is negligibly small. In contrast to Study 2, distance and SNARC effect were not significantly correlated.

In line with our expectations, the inclusion of numerical intelligence in Study 3 could explain an incremental portion of the variance of mathematical achievement. The results from the path analysis suggest that its effect might be both direct and mediated, similar to conceptual knowledge. The effect size for numerical intelligence, however, was smaller than that for conceptual knowledge.

General Discussion

During the past 2 decades, researchers have claimed that the mental number line is not only important for the representation of numerical quantities but also for mathematical competencies in general (Case & Okamoto, 1996). Dehaene (2001, p. 16) postulated "that higher-level cultural developments in arithmetic emerge through the establishment of linkages between this core analogical representation (the 'number line') and other verbal and visual representations of number notations." While this seems to be the case for elementary arithmetic (Gilmore et al., 2007; Holloway & Ansari, in press), our results show that the limits of this claim are reached when it comes to the content of fifth- and sixth-grade mathematics classes. Using tasks from the domain of decimal fractions, we showed that in this age group, the influence of the

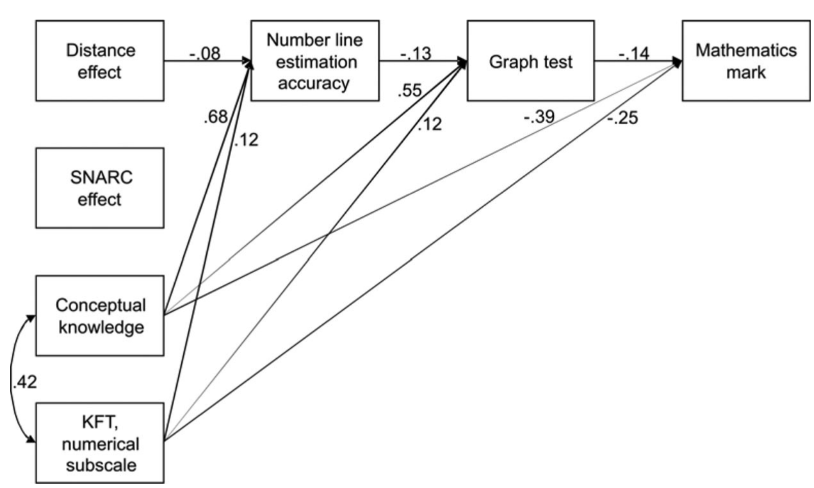

Figure 3. Path model of the relations between conceptual knowledge, numerical intelligence, distance effect, spatial–numerical association of response codes (SNARC) effect, number line estimation accuracy, graph competence, and math marks in Study 3. KFT = Kognitiver Fähigkeitstest.

Table 9
Model Estimated Path Coefficients in Study 3

Relation in the model	Unstandardized coefficient	*SE*	Standardized coefficient	*p*
Regression parameters				
Number line estimation accuracy on distance effect	−11.106	6.759	−.080	.050
Number line estimation accuracy on conceptual knowledge	0.789	0.066	.676	≤.001
Number line estimation accuracy on KFT, numerical subscale	0.230	0.106	.120	.015
Graph competence on conceptual knowledge	0.485	0.078	.547	≤.001
Graph competence on numerical intelligence	0.178	0.094	.123	.028
Graph competence on number line estimation accuracy	−0.099	0.063	−.131	0.058
Mathematics mark on conceptual knowledge	−0.024	0.005	−.393	≤.001
Mathematics mark on numerical intelligence	−0.025	0.007	−.247	≤.001
Mathematics mark on graph competence	−0.010	0.004	−.144	.013
Covariance parameters				
Conceptual knowledge with numerical intelligence	57.070	10.639	.424	≤.001

Note. KFT = Kognitiver Fähigkeitstest.

mental number line on school achievement is negligible when compared to the influences of conceptual knowledge and numerical intelligence.

This finding conforms to the observation that with increasing grade levels natural numbers become less important, while abstract relations and rules become more important in math learning and teaching. However, such abstract conceptual knowledge cannot directly be transferred from the teacher to the learner, because learners interpret new information in the light of their prior knowledge. Hence, the facilitation of conceptual knowledge and the reduction of misconceptions can only be effective if mathematics teachers take the content of learners' prior knowledge into account (Mack, 1990; Nesher, 1987). Facing this rationale, recent efforts to broaden the scope of conceptual change approaches from science to mathematics learning seem especially worthwhile (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004). Currently a lot more is known about the facilitation of conceptual change in the domain of science learning than in mathematics learning (diSessa, 2006). The search for commonalities and differences between these two fields is thus a useful direction for future research (Vosniadou & Verschaffel, 2004).

Our dependent variable, the mathematics mark, reflects a number of different competencies taught in fifth and sixth grades in an undifferentiated way. Therefore, we complemented it with a more specific competence measure, number line estimation accuracy. Using decimal fractions as stimuli, we found virtually no relation between children's ability to use the external number line and their reliance on the internal number line, as assessed by the distance effect and SNARC effect. In contrast to our measures of the mental number line, an estimation task with the external number line predicted mathematical school achievement over and above conceptual knowledge. This finding has two implications. First, it shows that the relation between the internal and the external number line is only indirect. Children's estimation patterns on the external number line should not be interpreted as direct evidence of their use of the internal number line, as is sometimes done in the literature. The internal and the external number line cannot be equated. The former is a hypothetical construct postulated to explain certain behavioral effects, and its neural bases are by no means visually similar to external number lines (e.g., Feigenson, Dehaene, & Spelke, 2004; Nieder, 2005). The cognitive processes by which activation patterns on the internal number line and positions on the external number line are translated into each other are an important area for future research. The second implication of our results is the importance of external number line estimation competencies for children's mathematical school achievement in addition to conceptual understanding and intelligence. This finding replicates the results of Siegler and Booth (2004), who found a correlation between a battery of estimation tasks and a standardized math achievement test even after controlling for intelligence and age.

Our main finding—that the reliance on the internal number line has no substantial influence on number line estimation competence and mathematical school achievement in general—comes as a surprise. Nevertheless, in light of the three studies with independent samples, we have no reason to doubt its validity. The relations we found between all other variables besides the distance effect and the SNARC effect are plausible and without exception replicate well-established results of previous studies. We also found highly significant distance effects and SNARC effects, with effect sizes that, too, conform to previous findings and vary substantially between participants.

In Study 3, the distance effect and the SNARC effect did not correlate with each other, and in Study 2, they only weakly correlated. This challenges the notion that these behavioral effects share a common basis, that is, the mental number line. In addition, Study 1 revealed a comparatively low correlation of the distance effects measured twice within an interval of a few days. Thus, it can be questioned whether the distance effect indeed reflects a trait variable indicating the individual reliance on the mental number line. This may also be true for the SNARC effect, which "does not seem to tap into a fixed component of long-term representation of numbers" (Fias & Fischer, 2005, p. 49). For instance, the spatial reference frame on which numbers are allocated can easily be changed. When asked to think of digits as times on an analog clock, participants preferentially responded to small numbers with their right hand instead of their left hand (Bächtold, Baumüller, & Brugger, 1998). Further, cross-cultural comparisons show that the direction of the SNARC effect depends on the given culture's predominant direction for reading and writing texts (Gevers & Lammertyn, 2005).

In the present research, we regarded the mental number line as the core of a potentially innate number sense as proposed by Dehaene (1997). However, it must be emphasized that many definitions of number sense exist, and they vary greatly in defining the skills encompassed by the construct. Berch (2005) listed 30 different features used to describe or define number sense in the literature. The definitions of number sense fall into two types: Number sense is either viewed as a lower order, innate, perceptual sense of quantity, similar to the notion put forward by Dehaene (1997), or it is seen as an "acquired 'conceptual sense making' of mathematics" (Berch, 2005, p. 334). Irrespective of which view might best capture the foundations and precursors of mathematical competence, more research is needed into the relation between number sense and mathematical school achievement. For example, Jordan, Kaplan, Locuniac, and Ramineni (2007) showed that number sense measured at the beginning of kindergarten correlates at $r = .70$ with children's scores on standardized math achievement tests at the end of their 1st year in school. However, the assessment of number sense and the math achievement test in this study might have partly assessed the same competencies as they both included calculation problems. Clearly, future research is in need of standardized and well-validated tests of number sense based on precise definitions of the construct (see also Reys & Yang, 1998).

Paradigms for the in-depth investigation of internal knowledge representations, such as the mental number line, have mainly been developed by cognitive researchers, while behavioral indicators of the number sense and mathematical school achievement are in the focus of math educators and educational psychologists. The mutual utility of more basic cognitive science research and more applied pedagogical research is sometimes questioned. We see the field of research on relations between the mental number line, number sense, and math achievement as a very positive example of how these different disciplines can benefit from each other (cf. Siegler, 2003).

References

Bachot, J., Gevers, W., Fias, W., & Roeyers, H. (2005). Number sense in children with visuospatial disabilities: Orientation of the mental number line. *Psychology Science, 47,* 172–183.

Bächtold, D., Baumüller, M., & Brugger, P. (1998). Stimulus–response compatibility in representational space. *Neuropsychologia, 36,* 731–735.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1–33). Mahwah, NJ: Erlbaum.

Berch, D. B. (2005). Making sense of number sense: Implications for children with mathematical disabilities. *Journal of Learning Disorders, 38,* 333–339.

Blair, C., Gamson, D., Thorne, S., & Baker, D. (2005). Rising mean IQ: Cognitive demand of mathematics education for young children, population exposure to formal schooling, and the neurobiology of the prefrontal cortex. *Intelligence, 33,* 93–106.

Brody, N. (1997). Intelligence, schooling, and society. *American Psychologist, 52,* 1046–1050.

Brody, N. (1999). What is intelligence? *International Review of Psychiatry, 11,* 19–25.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology, 27,* 777–786.

Case, R., & Okamoto, Y. (1996). *The role of central conceptual structures in the development of children's thought.* New York: Blackwell.

Ceci, S. J. (1991). How much does schooling influence general intelligence and its cognitive components? A reassessment of the evidence. *Developmental Psychology, 27,* 703–722.

Dehaene, S. (1997). *The number sense.* New York: Oxford University Press.

Dehaene, S. (2001). Précis of "the number sense." *Mind and Language, 16,* 16–32.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General, 122,* 371–396.

Dehaene, S., Dupoux, E., & Mehler, J. (1990). Is numerical comparison digital? Analogical and symbolic effects in two-digit number comparison. *Journal of Experimental Psychology: Human Perception and Performance, 16,* 626–641.

diSessa, A. A. (2006). A history of conceptual change research. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 265–280). Cambridge, England: Cambridge University Press.

Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences, 8,* 307–314.

Fias, W., & Fischer, M. H. (2005). Spatial representations of numbers. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 43–54). New York: Psychology Press.

Fischer, M. H., & Rottmann, J. (2005). Do negative numbers have a place on the mental number line? *Psychology Science, 47,* 22–32.

Flanagan, D. P., Andrews, T. J., & Genshaft, J. L. (1997). The functional utility of intelligence tests with special education populations. In D. P. Flanagan, J. L. Genshaft, & P. L. Harrison (Eds.), *Contemporary intellectual assessment: Theories, tests, and issues* (pp. 457–483). New York: Guilford Press.

Gattis, M. (2001). Reading pictures: Constraints on mapping conceptual and spatial schemas. In M. Gattis (Ed.), *Spatial schemas and abstract thought* (pp. 223–245). Cambridge, MA: MIT Press.

Gattis, M. (2002). Structure mapping in spatial reasoning. *Cognitive Development, 17,* 1157–1183.

Gattis, M., & Holyoak, K. J. (1996). Mapping conceptual to spatial relations in visual reasoning. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22,* 231–239.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7,* 155–170.

Gentner, D., & Toupin, C. (1986). Systematicity and surface similarity in the development of analogy. *Cognitive Science, 10,* 277–300.

Gevers, W., & Lammertyn, J. (2005). The hunt for SNARC. *Psychology Science, 47,* 10–21.

Gilmore, C., McCarthy, S. E., & Spelke, E. (2007, May 31). Symbolic arithmetic without instruction. *Nature, 447,* 589–591.

Goldstone, R. L., & Kersten, A. (2003). Concepts and categorization. In I. B. Weiner (Ed.), *Handbook of psychology* (Vol. 4, pp. 599–621). Hoboken, NJ: Wiley.

Gottfredson, L. S. (1997). Intelligence and social policy. *Intelligence, 24,* 1–12.

Gustafsson, J.-E., & Undheim, J. O. (1996). Individual differences in cognitive functions. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational Psychology* (pp. 186–242). New York: Macmillan.

Hardy, I., Schneider, M., Jonen, A., Stern, E., & Möller, K. (2005). Fostering diagrammatic reasoning in science education. *Swiss Journal of Psychology, 64,* 207–217.

Heller, K. A., & Perleth, C. (2000). Kognitiver Fähigkeitstest für 4. bis 12. Klassen, Revision [Cognitive ability test for class levels 4 to 12, revised version]. Göttingen, Germany: Hogrefe.

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of mathematics for mathematics teaching* (pp. 283–322). Hillsdale, NJ: Erlbaum.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Erlbaum.

Holloway, I. D., & Ansari, D. (2008). Domain-specific and domain-general changes in children's development of number comparison. *Developmental Science, 11,* 644–649.

Holloway, I. D., & Ansari, D. (in press). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology.*

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience, 6,* 435–448.

Jensen, A. R. (1998). *The g factor: The science of mental ability.* Westport, CT: Praeger.

Jensen, A. R. (2000, August). *The g factor and the design of education*. Paper presented at the 108th Annual Convention of the American Psychological Association, Washington, DC.

Jordan, N. C., Kaplan, D., Locuniak, M. N., & Ramineni, C. (2007). Predicting first-grade math achievement from developmental number sense trajectories. *Learning Disabilities Research & Practice, 22,* 36–46.

Kuncel, N. R., Credé, M., & Thomas, L. L. (2005). The validity of self-reported grade point averages, class ranks, and test scores: A meta-analysis and review of the literature. *Review of Educational Research, 75,* 63–82.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11,* 65–99.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60,* 1–64.

Lohman, D. F., Thorndike, R. L., Hagen, E. P., Smith, P., Fernandes, C., & Strand, S. (2001). *Cognitive Abilities Test* (3rd ed.). London, England: NferNelson.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21,* 16–32.

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the process of change. *Learning and Instruction, 14,* 519–534.

Mevarech, Z. R., & Kramarsky, B. (1997). From verbal descriptions to graphic representations: Stability and change in students' alternative conceptions. *Educational Studies in Mathematics, 32,* 229–263.

Moyer, R. S., & Landauer, T. K. (1967, September 30). Time required for judgments of numerical inequality. *Nature, 215,* 1519–1520.

Muthén, L. K., & Muthén, B. O. (1998–2006). *MPlus user's guide* (4th ed.) [Computer software and manual]. Los Angeles, CA: Authors.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics: Standards 2000.* Reston, VA: Author.

Neisser, U., Boodoo, G., Bouchard, T. J., Boykin, A. W., Brody, N., Ceci, S. J., et al. (1996). Intelligence: Knowns and unknowns. *American Psychologist, 51,* 77–101.

Nesher, P. (1987). Towards an instructional theory: The role of students' misconceptions. *For the Learning of Mathematics, 7,* 33–40.

Newcombe, N. S. (2002). The nativist-empiricist controversy in the context of recent research on spatial and quantitative development. *Psychological Science, 13,* 395–401.

Nieder, A. (2005). Counting on neurons: The neurobiology of numerical competence. *Nature Reviews Neuroscience, 6,* 1–14.

Novick, L. R. (2001). Spatial diagrams: Key instruments in the toolbox for thought. In D. L. Medin (Ed.), *The psychology of learning and motivation* (Vol. 40, pp. 1242–1256). San Diego, CA: Academic Press.

Programme for International Student Assessment. (2006). *Assessing scientific, reading and mathematical literacy: A framework for PISA 2006.* Paris: Organisation for Economic Co-operation and Development.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20,* 8–27.

Reys, R. E., & Yang, D.-C. (1998). Relationship between computational performance and number sense among sixth- and eighth-grade students in Taiwan. *Journal for Research in Mathematics Education, 29,* 225–237.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91,* 175–189.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93,* 346–362.

Schmidt, F. L., & Hunter, J. E. (1998). The validity and utility of selection methods in personnel psychology: Practical and theoretical implications of 85 years of research findings. *Psychological Bulletin, 124,* 262–274.

Schneider, M., & Stern, E. (2005). Conceptual and procedural knowledge of a mathematics problem: Their measurement and their causal interrelations. In B. G. Bara, L. Barsalou, & M. Bucciarelli (Eds.), *Proceedings of the 27th Annual Meeting of the Cognitive Science Society* (pp. 341–347). Mahwah, NJ: Erlbaum.

Schneider, M., & Stern, E. (2008). *The developmental relations between conceptual and procedural knowledge: a multimethod approach.* Manuscript submitted for publication.

Sekuler, R., & Mierkiewicz, D. (1977). Children's judgments of numerical inequality. *Child Development, 48,* 630–633.

Shah, P., & Hoeffner, J. (2002). Review of graph comprehension research: Implications for instruction. *Educational Psychology Review, 14,* 47–69.

Siegler, R. S. (2003). Implications of cognitive science research for mathematics education. In J. Kilpatrick, W. B. Martin, & D. E. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 219–233). Reston, VA: National Council of Teachers of Mathematics.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development, 75,* 428–444.

Siegler, R. S., & Crowley, K. (1994). Constraints on learning in nonprivileged domains. *Cognitive Psychology, 27,* 194–226.

Spearman, C. (1904). "General intelligence," objectively determined and measured. *American Journal of Psychology, 15,* 201–293.

Starkey, P., & Cooper, R. G. (1980, November 28). Perception of numbers by human infants. *Science, 210,* 1033–1035.

Stenning, K., & Lemmon, O. (1999). Aligning logical and psychological perspectives on diagrammatic reasoning. *Artificial Intelligence Review, 13,* 1–34.

Stern, E., Aprea, C., & Ebner, H. G. (2003). Improving cross-content transfer in text processing by means of active graphical representation. *Learning and Instruction, 13,* 191–203.

Tent, L. (2006). Zensuren [Marks]. In D. H. Rost (Ed.), *Handwörterbuch Pädagogische Psychologie* [Handbook of Educational Psychology] (pp. 873–879). Weinheim, Germany: Psychologie Verlags Union.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14,* 453–467.

Vosniadou, S., & Verschaffel, L. (2004). The problem of conceptual change in mathematics. *Learning and Instruction, 14,* 445–451.

Watkins, M. W., Lei, P.-W., & Canivez, G. L. (2007). Psychometric intelligence and achievement: A cross-lagged panel analysis. *Intelligence, 35,* 59–68.

Wynn, K. (1992, August 27). Addition and subtraction by human infants. *Nature, 358,* 749–750.

Zhang, J. (1996). A representational analysis of relational information displays. *International Journal of Human-Computer Studies, 45,* 59–74.

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