Merging Information Under Constraints: A Logical Framework

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Abstract

The paper considers the problem of merging several belief bases in the presence of integrity constraints and proposes a logical characterization of operators having a majority behaviour or a consensual one. Then a representation theorem in terms of pre-orders on interpretations is given. The close connection between belief revision and merging operators is shown and it is shown that the proposal extends the pure merging case (i.e. without integrity constraints) studied in a previous work. Finally it is shown that Liberatore and Schaerf commutative revision operators can be seen as a special case of merging.

Keywords: Logic-based merging, belief revision, integrity constraints.

1 Introduction

In many computer science fields, one needs to synthesize a coherent belief from several sources. The problem is that, in general, these sources contradict each other. So the merging of belief sources is a non-trivial issue. The first work on that problem goes back at least to [8].

Merging multiple sources of information is particularly interesting for distributed databases, for multi-agent systems, and for distributed information systems in general. In the database field a key issue of incoming systems will be to be able to integrate multiple databases into a single database [34]. A lot of work has been done in the database area on the integration of schemas [5, 10, 37, 20, 15]. Concerning the handling of inconsistency due to conflicting data, there has been less effort than in the case of integration. In particular, we can find very few reports about the rationality of merging [32, 25, 22, 23].

Inconsistency problems also occur when one wants to combine several expert systems. Consider a set of belief bases coding the belief of several human experts. In order to build an expert system it is reasonable to try to combine all these belief bases in a single belief base that expresses the belief of the experts' group. This process allows one to discover new pieces of belief distributed among the sources. For example if an expert knows that a is true and another knows that $a \rightarrow b$ holds, then the 'synthesized' belief knows that b is true whereas none of the experts know it. This was called *implicit belief* in [17]. However, simply putting these belief bases together is wrong since there could certainly be contradictions between some experts.

Many merging methods have been proposed [8, 3, 4, 36, 12, 11]. In order to compare them one needs a general methodology to determine the properties of such techniques. We have proposed such a methodology in [23]. This paper is an extension of that work. The aim is on the one hand to propose general properties of merging operators in a logical framework. On the other hand we propose some families of merging operators which are studied in the light of these properties. This will allow one to decide if a particular merging method is better than another and to classify such methods.

We will consider information from different sources. This will be represented as a multi-set $\{\varphi_1, \ldots, \varphi_n\}$ where each φ_i represents the information of the source i. The choice of multi-sets for representing a group of sources of information is a key point in order to take into account the fact that the same piece of information could come from different sources (this is important, for instance, to address majority merging). We consider operators having two parameters: one of them is the multi-set of information from the different sources; the other one is a piece of information that codes the integrity constraints for the system (it can be physical constraints, unquestionable knowledge, etc.). The output of the operator has to be a piece of information that satisfies the integrity constraints.

This work has similarities with that of Alchourrón, Gärdenfors and Makinson, known as AGM belief revision theory [1, 16, 18]. They proposed a set of postulates that characterizes revision operators. Katsuno and Mendelzon proposed a representation theorem for revision operators in terms of pre-orders on possible worlds. Similarly, we propose a set of postulates that a merging operator has to satisfy in order to have good behaviour, and we also provide a representation theorem *à la* Katsuno and Mendelzon. Actually there are deeper connections between revision operators and merging operators: a merging operator is a generalization of a revision operator and in some cases a revision operator will generate a merging operator.

We define two subclasses of merging operators, namely majority merging and arbitration operators. The former striving to satisfy a maximum of protagonists, the latter trying to satisfy each protagonist to the best possible degree. In other words majority merging operators try to minimize global dissatisfaction whereas arbitration operators try to minimize individual dissatisfaction.

Some operators quite close to merging operators have already been formally studied. Revesz defined in [31, 32] model-fitting operators which can be considered as a generalization of revision for multiple belief bases. His postulates do not distinguish between majority and arbitration. Another essential difference between our approach and Revesz's one is our notion of *belief set* that is actually a multi-set of belief bases, and the notion of *equivalence* between two such belief sets.

Liberatore and Schaerf proposed postulates to characterize commutative revision operators (they also called these operators 'arbitration operators') [24, 25]. Their definition has a strong connection with revision operators, but the major drawback is that those operators merge only two belief bases. Furthermore in their setting the result of merging two belief bases will be logically stronger than the disjunction of the two belief bases. We consider that we cannot systematically ignore information which is not from these belief bases. To illustrate this situation consider the following example.

EXAMPLE 1.1

Suppose that we want to speculate on the stock exchange. We ask two financial experts about four shares A,B,C,D. We denote 1 if the share rises and 0 if it falls (we suppose that its value cannot be stable). These agents have the same expert level and so they are both equally reliable. The first one says that all the shares will rise: $\varphi_1 = \{(1, 1, 1, 1)\}$, the second one thinks that all the shares will fall: $\varphi_2 = \{(0, 0, 0, 0)\}$. Liberatore and Schaerf operators will merge these opinions and give the following result: $R = \{(0, 0, 0, 0), (1, 1, 1, 1)\}$. This means that either φ_1 is totally wrong or φ_2 is completely mistaken. But intuitively, if the two experts are equally reliable, there is no reason to think that one of them has failed more than the other: they both have to be at the same 'distance' from the truth. So they are certainly both wrong on two shares and the result has to be: $R' = \{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$. So two of the shares will rise and two will fall but we do not know which ones.

In our opinion Liberatore and Schaerf's operators have to be seen as selection operators and have to be used in applications which require the result to be one of the possibilities given by the protagonists. For example, if the result of the arbitration is a medical treatment, we cannot merge several therapies and so we have to use Liberatore and Schaerf operators. Conversely, if we try to merge indications provided by a set of sensors, the same sensor can be both valid[1] on a variable and wrong on another one. So, in some situations, it could be natural to take an 'average' of sensor indications. And, more generally, if we consider the problem of knowledge merging as some kind of negotiation between several agents, it is sensible to consider propositional letters (that are, *a priori*, logically independent) as the negotiation unit.

Liberatore and Schaerf's operators take, in a sense, the interpretation as unit of change. Our approach takes the propositional letter as such a unit, as Dalal says in [13]: *'Change in truth value of a single symbol can be considered as the smallest unit of change'*. We want to apply this idea to merging.

Another proposal in the literature is that of Lin and Mendelzon [28, 27]. They proposed a *theory merging by majority* operator which solves conflicts between belief bases by taking the majority into account. Their *theory merging operator* satisfies the properties characterizing our majority operators.

This paper is organized as follows. In Section 2 we give some definitions and some notation. In Section 3 we propose a logical definition of merging operators with integrity constraints, we define majority and arbitration operators and give a model-theoretic representation of those operators. In Section 4 we define three families of merging operators. In particular the $\triangle^{GMax}$ family of arbitration operators is, in the logical context, a new merging method. We illustrate the differences of behaviour of these three families on a concrete example. In Section 5 we underline the strong connection between merging operators and belief revision operators, showing that merging operators are a generalization of belief revision operators and we explore how to build merging operators from belief revision operators. In Section 6 we show the connections between this work and related work on merging. We first address the case where there is no integrity constraints and show that this work extends that of [22]. Then we compare this work with Liberatore and Schaerf's commutative revision operators, showing that those operators can be seen as a special case of merging operators. Next we discuss Lin and Mendelzon's proposal of theory majority merging operators, and we deal with Revesz's model-fitting operators that are very close to merging operators. Finally in Section 7 we give some conclusions and discuss open problems and future work.

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[1] Roughly speaking, an agent (a sensor, a database) is valid iff what it says is true in the real world. This notion was first introduced in the database field by A. Motro in [29]

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional letters. An interpretation is a function from $\mathcal{P}$ to $\{0,1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation I is a model of a formula if and only if it makes it true in the usual classical truth functional way. Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, i.e. $mod(\varphi) = \{I \in \mathcal{W} : I \models \varphi\}$. Let M be a set of interpretations, φ_M denotes a formula whose set of models is M. When $M = \{I\}$ we will use the notation φ_I for reading convenience.

A *belief base* K is a finite set of propositional formulae which can be seen as the formula φ which is the conjunction of the formulae of K.

Let $\varphi_1, \ldots, \varphi_n$ be n belief bases (not necessarily different). We call *belief set* the multi-set Ψ consisting of those n belief bases: $\Psi = \{\varphi_1, \ldots, \varphi_n\}$. We note $\bigwedge \Psi$ the conjunction of the belief bases of Ψ, i.e. $\bigwedge \Psi = \varphi_1 \wedge \cdots \wedge \varphi_n$. The union of multi-sets will be noted $\sqcup$. Belief bases will be denoted by lower case Greek letters and belief sets by upper case Greek letters.

REMARK 2.1
We consider that an inconsistent belief base gives no information for the merging process so we will suppose in the rest of the paper that the belief bases are consistent.

DEFINITION 2.2
A belief set Ψ is said to be consistent if and only if $\bigwedge \Psi$ is consistent. We will use $mod(\Psi)$ to denote $mod(\bigwedge \Psi)$ and write $I \models \Psi$ for $I \in mod(\Psi)$.

DEFINITION 2.3
Let Ψ_1, Ψ_2 be two belief sets. Ψ_1 and Ψ_2 are said to be equivalent, noted $\Psi_1 \leftrightarrow \Psi_2$, iff there exists a bijection f from $\Psi_1 = \{\varphi_1^1, \ldots, \varphi_n^1\}$ to $\Psi_2 = \{\varphi_1^2, \ldots, \varphi_m^2\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$. In particular if $\Psi_1 \leftrightarrow \Psi_2$ then $m = n$.

A pre-order $\leq$ over $\mathcal{W}$ is a reflexive and transitive relation on $\mathcal{W}$. A pre-order is total if $\forall I, J \in \mathcal{W}\ I \leq J$ or $J \leq I$. Let $\leq$ be a pre-order over $\mathcal{W}$, we define $<$ as follows: $I < J$ iff $I \leq J$ and $J \not\leq I$, and $\simeq$ as $I \simeq J$ iff $I \leq J$ and $J \leq I$. Let I be an interpretation we will write $I \in min(A, \leq)$ iff $I \in A$ and $\forall J \in A\ I \leq J$.

By abuse if φ is a belief base, and if $\Psi = \{\varphi\}$ is a belief set, then φ will denote also the belief set. In the same way $\Psi \sqcup \varphi$ will be used instead of $\Psi \sqcup \{\varphi\}$, and $\varphi_1 \sqcup \varphi_2$ instead of $\{\varphi_1\} \sqcup \{\varphi_2\}$. Let n and Ψ be a positive integer and a belief set respectively, we will denote Ψ^n the multi-set $\underbrace{\Psi \sqcup \ldots \sqcup \Psi}_{n}$.

The set of consistent belief bases will be denoted by $\mathcal{B}$. The set of belief sets will be denoted by $\mathcal{S}$. We will suppose from now on that integrity constraints are a finite set of formulae, i.e. a belief base. We will call μ this belief base. We are going to consider operators $\triangle : \mathcal{S} \times \mathcal{B} \longrightarrow \mathcal{B}$ in which the second argument is seen as the integrity constraints. We will write $\triangle_\mu(\Psi)$ instead of $\triangle(\Psi, \mu)$.

Let φ, μ, Ψ, and $\triangle$ be two belief bases, a belief set and an operator respectively. We define the sequence

$$\triangle_\mu^1(\Psi, \varphi) = \triangle_\mu(\Psi \sqcup \varphi)$$
$$\text{and} \quad \triangle_\mu^{n+1}(\Psi, \varphi) = \triangle_\mu(\triangle_\mu^n(\Psi, \varphi) \sqcup \varphi).$$

3 Merging with integrity constraints

In this section we give a logical definition of integrity constraints merging operators (IC merging operators from now on), more exactly we give a set of postulates the operators have to satisfy in order to have a good behaviour concerning the merging. We also define arbitration and majority merging operators. Then we provide a representation theorem for these three kinds of operators, by showing that a merging operator corresponds to a family of pre-orders on possible worlds.

DEFINITION 3.1
$\triangle$ is said to be an IC merging operator if and only if it satisfies the following postulates:

(IC0) $\triangle_\mu(\Psi) \vdash \mu$.

(IC1) If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent.

(IC2) If $\bigwedge \Psi$ is consistent with μ, then $\triangle_\mu(\Psi) = \bigwedge \Psi \wedge \mu$.

(IC3) If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \leftrightarrow \mu_2$, then $\triangle_{\mu_1}(\Psi_1) \leftrightarrow \triangle_{\mu_2}(\Psi_2)$.

(IC4) If $\varphi \vdash \mu$ and $\varphi' \vdash \mu$, then $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot \Rightarrow \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$.

(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \vdash \triangle_\mu(\Psi_1 \sqcup \Psi_2)$.

(IC6) If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$.

(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Psi)$.

(IC8) If $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \vdash \triangle_{\mu_1}(\Psi) \wedge \mu_2$.

Intuitively $\triangle_\mu(\Psi)$ is a belief base *close* to the belief set Ψ satisfying the integrity constraints μ. This idea is what the postulates try to capture. The meaning of the postulates is the following: *(IC0)* assures that the result of the merging satisfies the integrity constraints. *(IC1)* states that if the integrity constraints are consistent, then the result of the merging will be consistent. *(IC2)* states that if possible, the result of the merging is simply the conjunction of the belief bases with the integrity constraints. *(IC3)* is the principle of irrelevancy of syntax, i.e. if two belief sets are equivalent and two integrity constraints bases are logically equivalent then the belief bases result of the two merging will be logically equivalent. *(IC4)* is the fairness postulate, the point is that when we merge two belief bases, merging operators must not give preference to one of them. *(IC5)* expresses the following idea: if two groups Ψ_1 and Ψ_2 agree on some alternatives then these alternatives will be chosen if we join the two groups. *(IC5)* and *(IC6)* together state that if one could find two subgroups which agree on at least one alternative, then the result of the global merging will be exactly those alternatives the two groups agree on. *(IC7)* and *(IC8)* are a direct generalization of the *(R5-R6)* postulates for revision (cf. Section 5). They state that the notion of *closeness* is well-behaved (see [18] for a full justification).

In addition to these basic requirements, we have alternative postulates.

First we can demand an *iteration* property that give a more ‘topological’ behaviour to merging operators.

($\mathbf{IC_{it}}$) If $\varphi \vdash \mu$ then $\exists n \, \triangle_\mu^n(\Psi, \varphi) \vdash \varphi$. (iteration)

The intuitive idea of this property is that, since merging operators give, in a sense, the average belief of a belief set, if we always take the result of a merging and iterate with the same belief base, we have to reach this belief base after enough iterations.

Now we define two merging operator subclasses, namely majority merging operators and arbitration operators.

A majority merging operator is an IC merging operator that satisfies the following *majority* postulate:

(Maj) $\exists n\ \triangle_\mu (\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle_\mu(\Psi_2)$. (majority)

This postulate expresses the fact that if a subgroup appears quite enough in the whole group then it is the opinion of the subgroup that will prevail. In particular if an individual opinion has a large audience, it will be the opinion of the group.

An arbitration operator is an IC merging operator that satisfies the following postulate:

(Arb) $$\left.\begin{array}{l} \triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2) \\ \triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1).$$

(arbitration)

This postulate says that if a set of alternatives preferred among one set of integrity constraints μ_1 for a base φ_1 corresponds to the set of alternatives preferred among another set of integrity constraints μ_2 for a base φ_2, and if the alternatives that belong to a set of integrity constraints but not to the other are equally preferred for the whole group ($\varphi_1 \sqcup \varphi_2$), then the subset of preferred alternatives among the disjunction of integrity constraints will coincide with the preferred alternatives of each base among their respective integrity constraints. This property is much more intuitive when it is expressed in a model-theoretical way (cf. condition 8 of a fair syncretic assignment in Definition 3.4). It shows that this is the median possible choices that are preferred.

We will illustrate this in the following scenario.

EXAMPLE 3.2
Tom and David missed the soccer match yesterday between reds and yellows. So they do not know the result of the match. Tom listened in the morning that reds had a very good match. So he thinks that a win for reds is more plausible than a draw and that a draw is more reliable than a win for yellows. David was told that after that match yellows now have a good chance of winning the championship. From this information he infers that yellows won the match, and if not, at least drew. Confronting their points of view, Tom and David agree on the fact that the two teams are of the same strength, and that they had the same chances of winning the match. What *arbitration* demands is that with this information, Tom and David agree that a draw between the two teams is the more plausible result.

Another property, opposed to the majority postulate, is the *majority independence* postulate:

(MI) $\forall n\ \triangle_\mu (\Psi_1 \sqcup {\Psi_2}^n) \leftrightarrow \triangle_\mu(\Psi_1 \sqcup \Psi_2)$. (majority independence)

This very strong property states that the result of merging is fully independent of the popularity of the views. It simply takes into account each different view. A corollary of this property is that for operators satisfying (MI), belief sets that are indeed multi-sets can be seen as simple sets.

But this property is not compatible with those of an IC merging operator.

THEOREM 3.3
There is no IC merging operator satisfying (MI).

PROOF. This proof is due to P. Liberatore (personal communication). Let $\Psi_1 = \{\varphi, \neg\varphi\}$ and $\Psi_2 = \{\varphi\}$ be two belief sets. By (MI) we have that $\triangle_\top(\Psi_1 \sqcup \Psi_2) = \triangle_\top(\Psi_1)$. By $(IC4)$ we have also that $\triangle_\top(\Psi_1) \nvdash \varphi$ and $\triangle_\top(\Psi_1) \nvdash \neg\varphi$. Furthermore by $(IC2)$ we deduce $\triangle_\top(\Psi_2) = \varphi$. So $\triangle_\top(\Psi_1) \wedge \triangle_\top(\Psi_2)$ is consistent and by $(IC6)$ we have $\triangle_\top(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\top(\Psi_1) \wedge \triangle(\Psi_2)$, i.e. $\triangle_\top(\Psi_1) \vdash \triangle_\top(\Psi_1) \wedge \varphi$. Then $\triangle_\top(\Psi_1) \vdash \varphi$, which contradicts $(IC4)$. ∎

Nevertheless a weak form of the last property called *weak majority independence* is compatible with IC merging operators:

(WMI) $\forall\varphi' \, \exists\varphi \, \varphi' \nvdash \varphi \, \forall n \;\; \triangle_\mu(\varphi' \sqcup \varphi^n) \leftrightarrow \triangle_\mu(\varphi' \sqcup \varphi)$.
(weak majority independence)

This weak form of majority independence asks that there are cases where the result of merging is independent of the popularity of the views.

Now that we have a logical definition of IC merging operators, we will state a representation theorem that gives a more intuitive way to define IC merging operators. More precisely, we will show that to each IC merging operator corresponds a family of pre-orders on possible worlds.

First we have to introduce the notion of *syncretic assignment*. By this we mean an assignment uniting and blending together several pre-orders (preference relations).

DEFINITION 3.4
A *syncretic assignment* is a function mapping each belief set Ψ to a total pre-order $\leq_\Psi$ over interpretations such that for any belief sets Ψ, Ψ_1, Ψ_2 and for any belief bases φ, φ' the following conditions hold:

1. If $I \models \Psi$ and $J \models \Psi$, then $I \simeq_\Psi J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then $I <_\Psi J$.
3. If $\Psi_1 \leftrightarrow \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$.
4. $\forall I \models \varphi \, \exists J \models \varphi' \; J \leq_{\varphi \sqcup \varphi'} I$.
5. If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $I \leq_{\Psi_1 \sqcup \Psi_2} J$.
6. If $I <_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $I <_{\Psi_1 \sqcup \Psi_2} J$.

The first two conditions ensure that the models of the knowledge set (if any) are the more plausible interpretations for the pre-order associated to the knowledge set. The third condition states that two equivalent knowledge sets have the same associated pre-orders. These three conditions are very close to the ones existing in belief revision for faithful assignments [18]. The fourth condition states that, when merging two belief bases, for each model of the first one, there is a model of the second one that is at least as good as the first one. It ensures that the two knowledge bases are given the same consideration.

REMARK 3.5
Condition 4 is equivalent to the following condition:

4′. $\exists J \models \varphi' \, \forall I \models \varphi \; J \leq_{\varphi \sqcup \varphi'} I$.

PROOF. Condition 4′ implies condition 4 straightforwardly. To show that condition 4 implies condition $4'$ simply notice that $\leq_{\varphi \sqcup \varphi'}$ is a total pre-order, so if one chooses $J \in \min(mod(\varphi'), \leq_{\varphi \sqcup \varphi'})$ then by condition 4 and transitivity $\forall I \models \varphi\ J \leq_{\varphi \sqcup \varphi'} I$, that is condition $4'$ holds. ∎

The fifth condition says that if an interpretation I is at least as plausible as an interpretation J for a knowledge set Ψ_1 and if I is at least as plausible as J for a knowledge set Ψ_2, then if one joins the two knowledge set, then I will still be at least as plausible as J.

The sixth condition strengthen the previous condition by saying that an interpretation I is at least as plausible as an interpretation J for a knowledge set Ψ_1 and if I is strictly more plausible than J for a knowledge set Ψ_2, then if one joins the two knowledge set, then I will be strictly more plausible than J.

These two previous conditions corresponds to Pareto conditions in Social Choice Theory [2, 19].

We can also define two particular syncretic assignments with additional conditions.

DEFINITION 3.6
A *majority syncretic assignment* is a syncretic assignment which satisfies the following:

7. If $I <_{\Psi_2} J$, then $\exists n\ I <_{\Psi_1 \sqcup \Psi_2{}^n} J$.

A *fair syncretic assignment* is a syncretic assignment which satisfies the following:

8. $\left.\begin{array}{l} I <_{\varphi_1} J \\ I <_{\varphi_2} J' \\ J \simeq_{\varphi_1 \sqcup \varphi_2} J' \end{array}\right\} \Rightarrow I <_{\varphi_1 \sqcup \varphi_2} J.$

Condition 7 says that if an interpretation I is strictly more plausible than an interpretation J for a knowledge set Ψ_2, then there is a quorum n of repetitions of the knowledge set from which I will be more plausible than J for the larger knowledge set $\Psi_1 \sqcup \Psi_2{}^n$. This condition seems to be the weakest form of 'majority' condition one could state.

Condition 8 states that if an interpretation I is more plausible than an interpretation J for a belief base φ_1, if I is more plausible than J' for another base φ_2, and if J and J' are equally plausible for the knowledge set $\varphi_1 \sqcup \varphi_2$, then I has to be more plausible than J and J' for $\varphi_1 \sqcup \varphi_2$ (see Example 3.2 for an intuitive explanation).

Now we can state the following representation theorem for merging operators.

THEOREM 3.7
An operator $\triangle$ is an IC merging operator if and only if there exists a syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ such that

$$mod(\triangle_\mu(\Psi)) \quad = \quad \min(mod(\mu), \leq_\Psi). \tag{3.1}$$

When this equation holds we will say that the assignment represents the operator.

PROOF. *(Only if part)* Let $\triangle$ be an operator satisfying postulates (IC0-IC8). Let us define a syncretic assignment as follows: for each belief set Ψ we define a total pre-order $\leq_\Psi$ by putting $\forall I, J \in \mathcal{W}\ I \leq_\Psi J$ if and only if $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$.

First we show that $\leq_\Psi$ is a total pre-order:
Totality: $\forall I, J \in \mathcal{W}$, from (IC1) $\triangle_{\varphi_{\{I,J\}}}(\Psi) \neq \emptyset$ and from (IC0) $\triangle_{\varphi_{\{I,J\}}}(\Psi) \vdash \varphi_{\{I,J\}}$, so $I \leq_\Psi J$ or $J \leq_\Psi I$.

Reflexivity: From (IC0) and (IC1) we have that $\triangle_{\varphi_I}(\Psi) = \varphi_I$. So $I \leq_\Psi I$.
Transitivity: Assume that $I \leq_\Psi J$ and $J \leq_\Psi L$ and suppose towards a contradiction that $I \not\leq_\Psi L$. So by definition and from (IC0) and (IC1) $\triangle_{\varphi_{\{I,L\}}}(\Psi) \leftrightarrow \varphi_{\{L\}}$. By (IC7) we find that $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}} \vdash \triangle_{\varphi_{\{I,L\}}}(\Psi)$. We consider two cases:
Case 1: $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}}$ is consistent then $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}} \leftrightarrow \varphi_{\{L\}}$. Thus we have that $I \not\models \triangle_{\varphi_{\{I,J,L\}}}(\Psi)$. But by (IC1) $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \neq \emptyset$, so by (IC0) we have $mod(\triangle_{\varphi_{\{I,J,L\}}}(\Psi)) = \{J,L\}$ or $mod(\triangle_{\varphi_{\{I,J,L\}}}(\Psi)) = \{L\}$. In the first case by (IC7) and (IC8) we conclude that $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$ and so $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Contradiction. In the second case by (IC7) and (IC8) $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{J,L\}} \leftrightarrow \triangle_{\varphi_{\{J,L\}}}(\Psi)$ but $J \not\models \triangle_{\varphi_{\{I,J,L\}}}(\Psi)$ so $J \not\models \triangle_{\varphi_{\{J,L\}}}(\Psi)$. Contradiction.
Case 2: $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}}$ is not consistent, so $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) = \varphi_{\{J\}}$. Then $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,J\}} = \varphi_{\{J\}}$. By (IC7) and (IC8) it follows that $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_{\{J\}}$, that is by definition $J <_\Psi I$. Contradiction.

Now we show that $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. First for the inclusion $mod(\triangle_\mu(\Psi)) \subseteq \min(mod(\mu), \leq_\Psi)$ assume that $I \models \triangle_\mu(\Psi)$ and suppose towards a contradiction that I is not in $\min(mod(\mu), \leq_\Psi)$. So we can find a $J \models \mu$ s.t. $J <_\Psi I$, then $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Since $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}}$ is consistent from (IC7) and (IC8) we have $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$. But $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$ so $I \not\models \triangle_\mu(\Psi)$. Contradiction.

For the other inclusion $mod(\triangle_\mu(\Psi)) \supseteq \min(mod(\mu), \leq_\Psi)$, suppose that $I \in \min(mod(\mu), \leq_\Psi)$. We want to show that $I \models \triangle_\mu(\Psi)$. Since $I \in \min(mod(\mu), \leq_\Psi)$, $\forall J \models \mu\ I \leq_\Psi J$ and so $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Since $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}}$ is consistent from (IC7) and (IC8) we have $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$. But $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$ so $I \models \triangle_\mu(\Psi)$.

It remains to verify the conditions of the syncretic assignment:

1. If $I \models \Psi$ and $J \models \Psi$, then by (IC2) we have $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_{\{I,J\}}$, so $I \leq_\Psi J$ and $J \leq_\Psi I$ by definition and then $I \simeq_\Psi J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then by (IC2) $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_I$, so $I \leq_\Psi J$ and $J \not\leq_\Psi I$, i.e. $I <_\Psi J$.
3. Straightforward from (IC3)
4. We want to show that $\forall I \models \varphi\ \exists J \models \varphi'\ J \leq_{\varphi \sqcup \varphi'} I$. First we show that $\exists J \models \triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi'$. If not we have $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi' \vdash \bot$, from (IC0) and (IC1) we have that $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \vdash \varphi$, now by (IC4) we get that $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$. Contradiction.
   Let I be a model of φ and take J such that $J \models \triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi'$. We get from (IC7) and (IC8) that $J \models \triangle_{\varphi_{\{I,J\}}}(\varphi \sqcup \varphi')$. So $J \leq_{\varphi \sqcup \varphi'} I$.
5. If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$ then $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$. So from (IC5) $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and by definition $I \leq_{\Psi_1 \sqcup \Psi_2} J$.
6. Suppose that $I <_{\Psi_1} J$ and $I \leq_{\Psi_2} J$. We want to show that $I <_{\Psi_1 \sqcup \Psi_2} J$. By the hypothesis $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$. So from (IC5) and (IC6) $\triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2) = \varphi_I$. Then $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and by definition $I <_{\Psi_1 \sqcup \Psi_2} J$.

(If part) Let's consider a syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define an operator $\triangle$ by putting $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. We want to show that $\triangle$ satisfies (IC0-IC8).

(IC0) By definition $mod(\triangle_\mu(\Psi)) \subseteq mod(\mu)$.

(IC1) If μ is consistent, then $mod(\mu) \neq \emptyset$. As there is a finite number of interpretations, there is no infinite descending chains of inequalities, so $\min(mod(\mu), \leq_\Psi) \neq \emptyset$. Then $\triangle_\mu(\Psi)$

is consistent.

(IC2) Assume that $\bigwedge \Psi \wedge \mu$ is consistent. We want to show $\min(mod(\mu), \leq_\Psi) = mod(\bigwedge \Psi \wedge \mu)$. First note that if $I \models \Psi$ then from conditions 1 and 2, $I \in \min(\mathcal{W}, \leq_\Psi)$. So if $I \models \Psi \wedge \mu$ then $I \in \min(mod(\mu), \leq_\Psi)$. So $\min(mod(\mu), \leq_\Psi) \supseteq mod(\bigwedge \Psi \wedge \mu)$. For the other inclusion consider $I \in \min(mod(\mu), \leq_\Psi)$. Suppose towards a contradiction that $I \not\models \Psi \wedge \mu$. Since $I \not\models \Psi$, by condition 2 we have that $\forall J \models \Psi\ J <_\Psi I$. In particular $\forall J \models \Psi \wedge \mu\ J <_\Psi I$. So $I \notin \min(mod(\mu), \leq_\Psi)$. Contradiction.

(IC3) Direct from condition 3 and the definition of $\triangle$.

(IC4) Assume that $\varphi \vdash \mu$, $\varphi' \vdash \mu$, and $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot$, we want to show that $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$. Consider $I \models \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi$. Then $\forall I' \models \mu\ I \leq_{\varphi \sqcup \varphi'} I'$. But from condition 4 we have that $\exists J \models \varphi'$ such that $J \leq_{\varphi \sqcup \varphi'} I$. Then $\forall I' \models \mu\ J \leq_{\varphi \sqcup \varphi'} I'$. Then $J \models \triangle_\mu(\varphi \sqcup \varphi')$ and therefore $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$.

(IC5) If $I \models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ then $I \in \min(mod(\mu), \leq_{\Psi_1})$ and so $\forall J \models \mu\ I \leq_{\Psi_1} J$. We have in the same way $\forall J \models \mu\ I \leq_{\Psi_2} J$. So by condition 5 we have that $\forall J \models \mu\ I \leq_{\Psi_1 \sqcup \Psi_2} J$. So $I \in \min(mod(\mu), \leq_{\Psi_1 \sqcup \Psi_2})$. So by definition $I \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$.

(IC6) Assume that $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent. We want to show that $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ holds. Take $I \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$, so $\forall J \models \mu\ I \leq_{\Psi_1 \sqcup \Psi_2} J$. Suppose towards a contradiction that $I \not\models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$. So $I \not\models \triangle_\mu(\Psi_1)$ or $I \not\models \triangle_\mu(\Psi_2)$. Suppose that $I \not\models \triangle_\mu(\Psi_1)$ (the other case is symmetrical). As $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent $\exists J \models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$. So $J <_{\Psi_1} I$ and $J \leq_{\Psi_2} I$ so by condition 6 $J <_{\Psi_1 \sqcup \Psi_2} I$ and then $I \not\models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$. Contradiction.

(IC7) Let's take $I \models \triangle_{\mu_1}(\Psi) \wedge \mu_2$. We have $\forall J \models \mu_1\ I \leq_\Psi J$. So $\forall J \models \mu_1 \wedge \mu_2\ I \leq_\Psi J$, so $I \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$.

(IC8) Assume that $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, so $\exists J \models \triangle_{\mu_1}(\Psi) \wedge \mu_2$. Consider $I \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$ and suppose that $I \not\models \triangle_{\mu_1}(\Psi)$. So $J <_\Psi I$. But $J \models \mu_1 \wedge \mu_2$ then $I \notin \min(mod(\mu_1 \wedge \mu_2), \leq_\Psi)$. Thus $I \not\models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$. Contradiction.

■

An analysis of the proof of Theorem 3.7 reveals that postulate (IC6) is used only in the proof of condition 6 and that condition 6 is used only in the proof of postulate (IC6). Similarly (IC4) corresponds to condition 4 on the assignment. This simple observation gives us the following corollary (of the previous proof).

COROLLARY 3.8
An operator satisfies (IC0–IC5),(IC7) and (IC8) if and only if it can be represented by an assignment satisfying conditions 1–5.
An operator satisfies (IC0–IC3),(IC5–IC8) if and only if it can be represented by an assignment satisfying conditions 1–3, 5 and 6.

Next we will give another variant of Theorem 3.7 by weakening postulate (IC6) and its corresponding condition on the assignment.

(IC6') If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \vee \triangle_\mu(\Psi_2)$

This property states that if an alternative is taken by a group, then if we split the group in two subgroups (which agree on something), at least one of the these subgroups will take the same alternative. This property correspond to the following condition that is obviously weaker than condition 6.

6′. If $I <_{\Psi_1} J$ and $I <_{\Psi_2} J$, then $I <_{\Psi_1 \sqcup \Psi_2} J$.

DEFINITION 3.9
We will call IC quasi-merging operator an operator satisfying (IC0–IC5), (IC6′), (IC7) and (IC8), and quasi-syncretic assignment an assignment satisfying conditions 1–5 and 6′.

THEOREM 3.10
An operator $\triangle$ is an IC quasi-merging operator if and only if it can be represented by a quasi-syncretic assignment.

PROOF. *(Only if part)* Let $\triangle$ be an operator satisfying postulates (IC0–IC5), (IC6′), (IC7) and (IC8). Define an assignment as in the proof of Theorem 3.7.

By Corollary 3.8 this assignment representing $\triangle$ satisfies conditions 1–5. It remains to prove condition 6′. Suppose that $I <_{\Psi_1} J$ and $I <_{\Psi_2} J$. We want to show that $I <_{\Psi_1 \sqcup \Psi_2} J$. By the hypothesis $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \vee \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$. So from (IC6′) $\triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2) = \varphi_I$. Then $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and by definition $I <_{\Psi_1 \sqcup \Psi_2} J$.
(If part) Let us consider an assignment satisfying conditions 1–5 and 6′ that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define the operator $\triangle$ by the equation (3.1). By Corollary 3.8 we know that $\triangle$ satisfies (IC0–IC5),(IC7) and (IC8). It remains to prove (IC6′). We want to show that if $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \vee \triangle_\mu(\Psi_2)$ holds. Assume that $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent and take $I \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$, so $\forall J \models \mu\ I \leq_{\Psi_1 \sqcup \Psi_2} J$. Condition 6′ is equivalent to If $I \leq_{\Psi_1 \sqcup \Psi_2} J$, then $I \leq_{\Psi_1} J$ or $I \leq_{\Psi_2} J$. Now suppose towards a contradiction that $I \not\models \triangle_\mu(\Psi_1) \vee \triangle_\mu(\Psi_2)$, that is $I \not\models \triangle_\mu(\Psi_1)$ and $I \not\models \triangle_\mu(\Psi_2)$. This can be rewritten as $\exists J_1 \in \mu\ J_1 <_{\Psi_1} I$ and $\exists J_2 \in \mu\ J_2 <_{\Psi_2} I$. But $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent so $\exists J_3 \in \mu$ such that $\forall I' \in \mu\ J_3 \leq_{\Psi_1} I'$ and $J_3 \leq_{\Psi_2} I'$. In particular we have that $J_3 \leq_{\Psi_1} J_1$ and $J_3 \leq_{\Psi_2} J_2$. By transitivity we find $J_3 <_{\Psi_1} I$ and $J_3 <_{\Psi_2} I$, and by condition 6′ we conclude $J_3 <_{\Psi_1 \sqcup \Psi_2} I$. So $I \not\models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$. Contradiction. ∎

We have also representation theorems for majority merging operators and arbitration operators.

THEOREM 3.11
An operator $\triangle$ is an IC majority merging operator if and only if it can be represented by a majority syncretic assignment.

PROOF. *(Only if part)* Let $\triangle$ be an operator satisfying postulates (IC0–IC8) and (Maj). Define an assignment as in the proof of Theorem 3.7.

By Theorem 3.7 this is a syncretic assignment representing $\triangle$. It remains to prove condition 7. Assume that $I <_{\Psi_2} J$. Then $\triangle_{\varphi_{\{I,J\}}}(\Psi_2) = \varphi_I$. From (Maj) we get that $\exists n$ such that $\triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$, so $\exists n\, \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup {\Psi_2}^n) = \varphi_I$, i.e. $\exists n\ I <_{\Psi_1 \sqcup {\Psi_2}^n} J$.
(If part) Let's consider a majority syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define the operator $\triangle$ by letting $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. By Theorem 3.7 we know that $\triangle$ satisfies (IC0–IC8). It remains to prove (Maj). From conditions 6 and 7 we get easily the following condition:

$$I <_{\Psi_2} J \ \Rightarrow\ \exists n_0\ \forall n \geq n_0\ \ I <_{\Psi_1 \sqcup {\Psi_2}^n} J.$$

Since for each Ψ, $\leq_\Psi$ is total this condition is equivalent to

$$\forall n_0\ \exists n \geq n_0\ I \leq_{\Psi_1 \sqcup {\Psi_2}^n} J \ \ \Rightarrow\ \ I \leq_{\Psi_2} J. \qquad (*)$$

Now, suppose towards a contradiction that $\forall n \, \triangle_\mu \, (\Psi_1 \sqcup {\Psi_2}^n) \nvdash \triangle_\mu(\Psi_2)$. From this hypothesis we get that $\forall n \, \exists I \models \mu \, \forall J \models \mu \; I \leq_{\Psi_1 \sqcup {\Psi_2}^n} J$ and $\exists J' \models \mu \; J' <_{\Psi_2} I$. Since the number of possible worlds is finite, by a combinatorial argument (pigeon hole principle) there exists I such that $I \leq_{\Psi_1 \sqcup {\Psi_2}^n} J$ for any $J \models \mu$ and an infinity of integers n and such that $\exists J' \models \mu \; J' <_{\Psi_2} I$. This obviously entails the premisses of condition $(*)$, so we have $I \leq_{\Psi_2} J$ for any $J \models \mu$ which is obviously in contradiction with the fact that $\exists J' \models \mu \; J' <_{\Psi_2} I$. ■

REMARK 3.12
Notice that in the previous proof only conditions 6 and 7 on the syncretic assignment are used to prove the postulate (Maj).

THEOREM 3.13
An operator $\triangle$ is an IC arbitration operator if and only if it can be represented by a fair syncretic assignment.

PROOF. *(Only if part)* Let $\triangle$ be an operator satisfying postulates (IC0-IC8) and (Arb). Define an assignment as in the proof of Theorem 3.7.

By Theorem 3.7 this assignment is a syncretic assignment, so it remains to show that condition 8 holds. Assume that both $J <_{\varphi_1} I$, $J <_{\varphi_2} J'$ and $I \simeq_{\varphi_1 \sqcup \varphi_2} J'$ hold. First if $I = J'$ then $J <_{\varphi_1 \sqcup \varphi_2} I$ follows from condition 6. Now suppose $I \neq J'$. By hypothesis $\triangle_{\varphi_{\{I,J\}}}(\varphi_1) \leftrightarrow \triangle_{\varphi_{\{J,J'\}}}(\varphi_2) \leftrightarrow \varphi_J$ and $\triangle_{\varphi_{\{I,J'\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_{\{I,J'\}}$. By the assumption $I \neq J'$, we have that both of $\varphi_{\{I,J\}} \wedge \neg\varphi_{\{I,J'\}}$ and $\varphi_{\{I,J'\}} \wedge \neg\varphi_{\{I,J\}}$ are consistent. Then by (Arb) we get that $\triangle_{\varphi_{\{I,J,J'\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_J$. And by (IC7) and (IC8) we conclude that $\triangle_{\varphi_{\{I,J\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_J$, that is $J <_{\varphi_1 \sqcup \varphi_2} I$.
(If part) Let's consider a fair majority syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define $\triangle$ by putting $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. We know by Theorem 3.7 that $\triangle$ satisfies (IC0-IC8), then it is enough to prove (Arb).

Assume that $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$, $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$, $\mu_1 \wedge \neg\mu_2 \nvdash \bot$ and $\mu_2 \wedge \neg\mu_1 \nvdash \bot$ hold. We want to show that $\triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$.
First we prove that $\triangle_{\mu_1}(\varphi_1) \vdash \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Consider $I \models \triangle_{\mu_1}(\varphi_1)$ and suppose towards a contradiction that $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Then $\exists J \models \mu_1 \vee \mu_2 \; J <_{\varphi_1 \sqcup \varphi_2} I$.

We consider three cases: $J \models \mu_1 \wedge \mu_2$, $J \models \mu_1 \wedge \neg\mu_2$ or $J \models \neg\mu_1 \wedge \mu_2$.
Case 1: $J \models \mu_1 \wedge \mu_2$. Since $I \models \triangle_{\mu_1}(\varphi_1)$, $I \leq_{\varphi_1} J$. By hypothesis $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$. So $I \models \triangle_{\mu_2}(\varphi_2)$ and then $I \leq_{\varphi_2} J$. Then by condition 5 we have that $I \leq_{\varphi_1 \sqcup \varphi_2} J$. Contradiction.
Case 2: $J \models \mu_1 \wedge \neg\mu_2$ (the *Case 3*, $J \models \neg\mu_1 \wedge \mu_2$, is symmetrical). Since $J \not\models \mu_2$ and $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$ we have $J \not\models \triangle_{\mu_1}(\varphi_1)$, so $I <_{\varphi_1} J$. By hypothesis we can find a $J' \models \mu_2 \wedge \neg\mu_1$ and with an analogous argument $I <_{\varphi_2} J'$. We also know that $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$, this implies $J \simeq_{\varphi_1 \sqcup \varphi_2} J'$. And then by condition 8 we get that $I <_{\varphi_1 \sqcup \varphi_2} J$. Contradiction.

Now we prove $\triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \vdash \triangle_{\mu_1}(\varphi_1)$. Assume that $I \models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$ and suppose towards a contradiction that $I \not\models \triangle_{\mu_1}(\varphi_1)$. There are three cases:
Case 1: $I \models \mu_1 \wedge \mu_2$ then $\exists J \models \triangle_{\mu_1}(\varphi_1)$, so $J <_{\varphi_1} I$. And, as $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$, $J <_{\varphi_2} I$. So by condition 8 we have that $J <_{\varphi_1 \sqcup \varphi_2} I$, so $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Contradiction.
Case 2: $I \models \mu_1 \wedge \neg\mu_2$ (the *Case 3*, where $I \models \neg\mu_1 \wedge \mu_2$, is symmetrical). By hypothesis we know that $\exists I' \models \neg\mu_1 \wedge \mu_2$. Since $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$ $\exists J \models \triangle_{\mu_1}(\varphi_1)$ such that $J <_{\varphi_1} I$ and $J <_{\varphi_2} I'$. We obtain also from $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$ that $I \simeq_{\varphi_1 \sqcup \varphi_2} I'$, so by condition 8 we get that $J <_{\varphi_1 \sqcup \varphi_2} I$. So $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Contradiction. ■

REMARK 3.14
Notice that in the previous proof only conditions 5 and 8 on the assignment are used to prove the postulate (Arb).

4 Examples of operators

In this section we will define three families of operators and show that the first family gives majority merging operators, the second family gives quasi-merging operators and the third family gives arbitration operators.

We will suppose here that we have a distance between interpretations (possible worlds), that is a function $d : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) such that:
$d(I, J) = d(J, I)$
$d(I, J) = 0$ iff $I = J$.

From now on we define the distance between an interpretation I and a belief base φ in the following way:

$$d(I, \varphi) = \min_{J \models \varphi} d(I, J).$$

We also define the distance between two belief bases φ and φ' induced by the above distance. This definition is not required for the definition of the operators but will be useful in the proofs:

$$d(\varphi, \varphi') = \min_{I \models \varphi, J \models \varphi'} d(I, J).$$

4.1 Σ operators

We define here a family of operators that will actually be majority merging operators. This definition is similar to Borda rule in the framework of social choice theory [19, 2]. Indeed, Borda rule tries to minimize the sum of ranks and our operator tries to minimize the sum of distances.

DEFINITION 4.1
Let Ψ be a belief set, let I be an interpretation and let d be a distance between interpretations. We define the Σ-distance between an interpretation and a belief set as

$$d_\Sigma(I, \Psi) = \sum_{\varphi \in \Psi} d(I, \varphi).$$

Then we have the following pre-order:

$$I \leq^\Sigma_\Psi J \text{ iff } d_\Sigma(I, \Psi) \leq d_\Sigma(J, \Psi).$$

The operator $\triangle^\Sigma$ is defined by

$$mod(\triangle^\Sigma_\mu(\Psi)) = \min(mod(\mu), \leq^\Sigma_\Psi).$$

The $\triangle^\Sigma$ operators rely on the definition of the distance between an interpretation and a belief set defined as the sum of the distances between this interpretation and the belief bases of the belief set. The result of $\triangle^\Sigma$ operators can be considered as the 'election' of the most popular possible choices among the integrity constraints.

Lin and Mendelzon give a $\triangle^\Sigma$ operator (when the chosen distance is the Dalal distance [13]) as an example of what they called operators of *theory merging by majority* in [28]. And independently, Revesz gives the same operator as an example of weighted model fitting in [31].

This operator is indeed a majority merging operator as stated in the theorem below.

THEOREM 4.2
$\triangle^{\Sigma}$ is a majority merging operator.

PROOF. First of all notice the following fact, the proof of which is straightforward by definition:

$$d_{\Sigma}(I, \Psi_1 \sqcup \Psi_2) = d_{\Sigma}(I, \Psi_1) + d_{\Sigma}(I, \Psi_2). \tag{4.1}$$

We prove that the assignment $\Psi \mapsto \leq_{\Psi}^{\Sigma}$ is a majority syncretic assignment. Then by Theorem 3.11 we conclude that $\triangle_{\Sigma}$ satisfies $(IC0 - IC8)$ and (Maj).

Let us verify the conditions of a majority syncretic assignment:

1. If $I \models \Psi$ and $J \models \Psi$, then $d_{\Sigma}(I, \Psi) = 0$ and $d_{\Sigma}(J, \Psi) = 0$, so $I \simeq_{\Psi} J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then $d_{\Sigma}(I, \Psi) = 0$ and $d_{\Sigma}(J, \Psi) > 0$, so $I <_{\Psi} J$.
3. Straightforward.
4. We want to show that $\forall I \models \varphi \; \exists J \models \varphi' \; J \leq_{\varphi \sqcup \varphi'} I$. We have that $d(I, \varphi) = 0$ and $d(I, \varphi') = \min_{J \models \varphi'} d(I, J)$, so choose $J \models \varphi'$ such that $d(I, J) = d(I, \varphi')$. Then $d(J, \varphi) = \min_{I' \models \varphi} d(J, I') \leq d(J, I)$, and $d(J, \varphi') = 0$. So $d_{\Sigma}(J, \varphi \sqcup \varphi') = d_{\Sigma}(J, \varphi) \leq d_{\Sigma}(I, \varphi') = d_{\Sigma}(I, \varphi \sqcup \varphi')$. So by definition $J \leq_{\varphi \sqcup \varphi'} I$.
5. If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $d_{\Sigma}(I, \Psi_1) \leq d_{\Sigma}(J, \Psi_1)$ and $d_{\Sigma}(I, \Psi_2) \leq d_{\Sigma}(J, \Psi_2)$, so by equation 4.1 $d_{\Sigma}(I, \Psi_1 \sqcup \Psi_2) \leq d_{\Sigma}(J, \Psi_1 \sqcup \Psi_2)$.
6. Follows from equation (4.1) as previous property.
7. If $I <_{\Psi_2} J$, then $d_{\Sigma}(I, \Psi_2) < d_{\Sigma}(J, \Psi_2)$. We want to show that $\exists n \; I <_{\Psi_1 \sqcup \Psi_2{}^n} J$, that is

$$\exists n \; d_{\Sigma}(I, \Psi_1) + n * d_{\Sigma}(I, \Psi_2) < d_{\Sigma}(J, \Psi_1) + n * d_{\Sigma}(J, \Psi_2)$$

so simply choose $n > \frac{d_{\Sigma}(I, \Psi_1) - d_{\Sigma}(J, \Psi_1)}{d_{\Sigma}(J, \Psi_2) - d_{\Sigma}(I, \Psi_2)}$.

∎

THEOREM 4.3
If the distance $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{N}$ satisfies the triangular inequality, i.e. $d(I, J) \leq d(I, J') + d(J', J)$ then $\triangle^{\Sigma}$ satisfies (IC$_{it}$).

PROOF. First assume that $\varphi \vdash \mu$ and $\varphi' \vdash \mu$. We will prove that $\exists n \; \triangle_{\mu}^{\Sigma\, n}(\varphi', \varphi) \vdash \varphi$. Let a be the distance between φ and φ', i.e. $d(\varphi, \varphi') = a$. Take $I \models \varphi$ and $J \models \varphi'$ such that $dist(I, J) = a$. By using the triangular inequality, it is easy to see that $a = \min\{dist_{\Sigma}(I', \varphi \sqcup \varphi') : I' \models \mu\}$ thus $I \models \triangle_{\mu}^{\Sigma}(\varphi \sqcup \varphi')$ and then, by (IC2), $\triangle_{\mu}^{\Sigma}(\triangle_{\mu}^{\Sigma}(\varphi' \sqcup \varphi) \sqcup \varphi) \vdash \varphi$. Therefore $\exists n \; \triangle_{\mu}^{\Sigma\, n}(\varphi', \varphi) \vdash \varphi$. From this, by putting $\varphi' = \triangle_{\mu}^{\Sigma}(\Psi \sqcup \varphi)$, (IC$_{it}$) follows. ∎

We will now illustrate the behaviour of majority operators through the following example showing the $\triangle^{\Sigma}$ operator at work. We will also use the same example with the two other operators we define in this section. In particular, it will serve to highlight the difference of behaviour between majority and arbitration operators.

EXAMPLE 4.4
We will choose as distance for the operators the Dalal distance [13]. The Dalal distance between two interpretations is the number of propositional letters on which the two interpretations differ, for example the Dalal distance between $(1, 0, 0)$ and $(1, 1, 0)$ is 1 since the two interpretations differ only on the second letter.

At a meeting of a block of flat co-owners, the chairman proposes for the coming year the construction of a swimming pool, of a tennis court and a private car park. But if two of these three items are built, the rent will increase significantly. We will denote by S, T, P respectively the construction of the swimming pool, the tennis court and the private car park. We will denote I the rent increase.

The chairman outlines that building two items or more will have an important impact on the rent: $\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$

There are four co-owners $\Psi = \{\varphi_1 \sqcup \varphi_2 \sqcup \varphi_3 \sqcup \varphi_4\}$. Two of the co-owners want to build the three items and do not care about the rent increase: $\varphi_1 = \varphi_2 = S \wedge T \wedge P$. The third one thinks that building any item will caused at some time an increase of the rent and wants to pay the lowest rent so he is opposed to any construction: $\varphi_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$. The last one thinks that the block really needs a tennis court and a private car park but does not want a high rent increase: $\varphi_4 = T \wedge P \wedge \neg I$.

The propositional letters S, T, P, I will be considered in that order for the valuations:

$mod(\mu) = \mathcal{W} \setminus \{(0,1,1,0),(1,0,1,0),(1,1,0,0),(1,1,1,0)\}$
$mod(\varphi_1) = \{(1,1,1,1),(1,1,1,0)\}$ $\quad$ $mod(\varphi_3) = \{(0,0,0,0)\}$
$mod(\varphi_2) = \{(1,1,1,1),(1,1,1,0)\}$ $\quad$ $mod(\varphi_4) = \{(1,1,1,0),(0,1,1,0)\}$

We sum up the calculations in Table 1, for each interpretation we give the distances between this interpretation and the four belief bases and the distance between this interpretation and the belief set according to the $\triangle^{\Sigma}$ operator. The lines shadowed correspond to the interpretations rejected by the integrity constraints. Thus the result has to be found among the interpretations that are not shadowed.

TABLE 1. $\triangle^{\Sigma}$ operator

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist_{\Sigma}}$
$(0,0,0,0)$	3	3	0	2	8
$(0,0,0,1)$	3	3	1	3	10
$(0,0,1,0)$	2	2	1	1	6
$(0,0,1,1)$	2	2	2	2	8
$(0,1,0,0)$	2	2	1	1	6
$(0,1,0,1)$	2	2	2	2	8
$(0,1,1,0)$	1	1	2	0	4
$(0,1,1,1)$	1	1	3	1	6
$(1,0,0,0)$	2	2	1	2	7
$(1,0,0,1)$	2	2	2	3	9
$(1,0,1,0)$	1	1	2	1	5
$(1,0,1,1)$	1	1	3	2	7
$(1,1,0,0)$	1	1	2	1	5
$(1,1,0,1)$	1	1	3	2	7
$(1,1,1,0)$	0	0	3	0	3
$(1,1,1,1)$	0	0	4	1	**5**

If one takes the decision according to the majority wishes then with the $\triangle^{\Sigma}$ operator we have 5 as minimum distance, so $mod(\triangle^{\Sigma}_{\mu}(\Psi)) = \{(1,1,1,1)\}$, and the decision that satisfies the majority in the group is to build the three items and to increase the rent.

4.2 *Max operators*

We will define in this section the $\triangle^{Max}$ operators. These operators are very close to the minimax rule used in decision theory [33]. The minimax rule tries to minimize the worst cases and similarly our operator $\triangle^{Max}$ tries to minimize the more remote distances. But as we will see they are too rough as merging operators and then they do not satisfy all the postulates. Nevertheless $\triangle^{Max}$ operators are quasi-merging operators. The behaviour of this family can be seen as an approximation to the behaviour of arbitration operators.

DEFINITION 4.5
Let Ψ be a belief set, let I be an interpretation and let d be a distance between interpretations. We define the Max-distance between an interpretation and a belief set as:

$$d_{Max}(I, \Psi) = \max_{\varphi \in \Psi} d(I, \varphi).$$

Then we define the following pre-order:

$$I \leq_{\Psi}^{Max} J \text{ iff } d_{Max}(I, \Psi) \leq d_{Max}(J, \Psi).$$

The operator $\triangle^{Max}$ is defined by:

$$mod(\triangle_{\mu}^{Max}(\Psi)) = \min(mod(\mu), \leq_{\Psi}^{Max}).$$

The idea of this operator is to find the closest possible worlds to the overall belief set. So it seems to be a good arbitration operator but it does not satisfy all the postulates. Revesz gives a $\triangle^{Max}$ operator (with the Dalal distance [13] as chosen distance) as an example of model fitting operators in [31].

THEOREM 4.6
$\triangle^{Max}$ is a quasi-merging operator. Furthermore it satisfies (Arb), (IC$_{it}$) and (MI). In particular $\triangle^{Max}$ cannot satisfy (IC6) and (Maj).

PROOF. First note that we have the following fact, the proof of which is straightforward:

REMARK 4.7
$d_{Max}(I, \Psi_1 \sqcup \Psi_2) = \max\{d_{Max}(I, \Psi_1), d_{Max}(I, \Psi_2)\}$

We show that the assignment $\Psi \mapsto \leq_{\Psi}^{Max}$ is a quasi-syncretic assignment. So by Theorem 3.10 $\triangle^{Max}$ is an IC quasi-merging operator.

1. If $I \models \Psi$ and $J \models \Psi$, then $d_{Max}(I, \Psi) = 0$ and $d_{Max}(J, \Psi) = 0$, so $I \simeq_{\Psi} J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then $d_{Max}(I, \Psi) = 0$ and $d_{Max}(J, \Psi) > 0$, so $I <_{\Psi} J$.
3. Straightforward.
4. We want to show that $\forall I \models \varphi\ \exists J \models \varphi'\ J \leq_{\varphi \sqcup \varphi'} I$. We have that $d(I, \varphi) = 0$ and $d(I, \varphi') = \min_{J \models \varphi'} d(I, J)$, so choose $J \models \varphi'$ such that $d(I, J) = d(I, \varphi')$.
   Then $d(J, \varphi) = \min_{I' \models \varphi} d(J, I') \leq d(J, I)$, and $d(J, \varphi') = 0$. So $d_{Max}(J, \varphi \sqcup \varphi') = d_{Max}(J, \varphi) \leq d_{Max}(I, \varphi') = d_{Max}(I, \varphi \sqcup \varphi')$. So by definition $J \leq_{\varphi \sqcup \varphi'} I$.
5. If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $d_{Max}(I, \Psi_1) \leq d_{Max}(J, \Psi_1)$ and $d_{Max}(I, \Psi_2) \leq d_{Max}(J, \Psi_2)$, so by Remark 4.7 $d_{Max}(I, \Psi_1 \sqcup \Psi_2) \leq d_{Max}(J, \Psi_1 \sqcup \Psi_2)$.

6′. If $I <_{\Psi_1} J$ and $I <_{\Psi_2} J$, then $d_{Max}(I, \Psi_1) < d_{Max}(J, \Psi_1)$ and $d_{Max}(I, \Psi_2) < d_{Max}(J, \Psi_2)$, so by Remark 4.7 $d_{Max}(I, \Psi_1 \sqcup \Psi_2) < d_{Max}(J, \Psi_1 \sqcup \Psi_2)$.

In order to prove that (Arb) holds it is enough to show that condition 8 holds because of Remark 3.14. Thus suppose that $I <_{\varphi_1} J$, $I <_{\varphi_2} J'$ and $J \simeq_{\varphi_1 \sqcup \varphi_2} J'$. We want to show that $I <_{\varphi_1 \sqcup \varphi_2} J$. By remark 4.7 we have $d_{Max}(I, K_1 \sqcup \varphi_2) = \max\{d_{Max}(I, \varphi_1), d_{Max}(I, \varphi_2)\}$. By the hypotheses $\max\{d_{Max}(I, \varphi_1), d_{Max}(I, \varphi_2)\} < \max\{d_{Max}(J, \varphi_1), d_{Max}(J', \varphi_2)\}$. Since $J \simeq_{\varphi_1 \sqcup \varphi_2} J'$, $\max\{d_{Max}(J, \varphi_1), d_{Max}(J, \varphi_2)\} = \max\{d_{Max}(J', \varphi_1), d_{Max}(J', \varphi_2)\}$. From this, it is easy to see that $\max\{d_{Max}(J, \varphi_1),\ d_{Max}(J', \varphi_2)\} \leq \max\{d_{Max}(J, \varphi_1),\ d_{Max}(J, \varphi_2)\}$. And by transitivity $\max\{d_{Max}(I, \varphi_1),\ d_{Max}(I, \varphi_2)\} < \max\{d_{Max}(J, \varphi_1),\ d_{Max}(J, \varphi_2)\}$, i.e. $I <_{\varphi_1 \sqcup \varphi_2} J$.

Now we prove that (IC_{it}) holds. Assume that $\varphi \vdash \mu$ and $\varphi' \vdash \mu$. It is enough to show that $\exists n \, \triangle_\mu^{Max^n}(\varphi', \varphi) \vdash \varphi$ because from this (IC_{it}) follows by putting $\varphi' = \triangle_\mu^{Max}(\Psi \sqcup \varphi)$. Let a be the distance between φ and φ'. We proceed by induction on a.

If $a = 0$, then by (IC2), $\triangle_\mu^{Max}(\varphi' \sqcup \varphi) = \varphi \wedge \varphi'$ and by (IC2) again $\triangle_\mu^{Max}(\triangle_\mu^{Max}(\varphi' \sqcup \varphi) \sqcup \varphi) \vdash \varphi$. Suppose that $a \geq 1$ and for any i if $i < a$ we have that $d(\varphi, \varphi') = i$ implies $\exists n \, \triangle_\mu^{Max^n}(\varphi', \varphi) \vdash \varphi$. We want to show that if $d(\varphi, \varphi') = a$ then $\exists n \, \triangle_\mu^{Max^n}(\varphi', \varphi) \vdash \varphi$. Since $d(\varphi, \varphi') = a$ we can take I, I' such that $I \models \varphi$, $I' \models \varphi'$ and $d(I, I') = a$. If $\min\{d_{Max}(J, \varphi \sqcup \varphi') : J \models \mu\} = a$, then $I \models \triangle_\mu^{Max}(\varphi' \sqcup \varphi)$ and $\triangle_\mu^{Max}(\triangle_\mu^{Max}(\varphi' \sqcup \varphi) \sqcup \varphi) \vdash \varphi$. Otherwise, $\min\{d_{Max}(I', \varphi \sqcup \varphi') : I' \models \mu\} = i_0 < a$, then $mod(\triangle_\mu^{Max}(\varphi' \sqcup \varphi)) = \{J \models \mu : d_{Max}(J, \varphi \sqcup \varphi') = i_0\}$. Let's call φ'' this belief base and notice that $\varphi'' \vdash \mu$, so put a' the distance between φ and φ''. Notice that $a' \leq i_0 < a$ so by the induction hypothesis $\exists n \, \triangle_\mu^{Max^n}(\varphi'', \varphi) \vdash \varphi$. Then $\exists n \, \triangle_\mu^{Max^n}(\varphi', \varphi) \vdash \varphi$.

The proof that $\triangle^{Max}$ satisfies (MI) follows easily from Remark 4.7. From this and Theorem 3.3 follows that $\triangle^{Max}$ does not satisfy (IC6).

Finally notice that $\triangle_\mu^{Max}(\varphi_1 \sqcup \varphi_2^n) = \triangle_\mu^{Max}(\varphi_1 \sqcup \varphi_2)$ from this and (IC4) is easy to see that (Maj) fails. ■

Let's see what the $\triangle^{Max}$ operator gives on the block of flats example (Example 4.4).

EXAMPLE 4.8
We recall that we use the Dalal distance in this example and that S, T, P, I denotes respectively the construction of the swimming pool, the tennis court, the private car park and the rent increase.

With $\triangle^{Max}$ as merging criterion we have 2 as minimum distance between possible choices and the belief set (cf. Table 2) so the corresponding interpretations are selected: $mod(\triangle_\mu^{Max}(\Psi)) = \{(0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (1,0,0,0)\}$. Then the decision that best fit the group wishes is not to increase the rent and to build one of the three items, or to increase the rent and build the tennis court or the private car park.

4.3 GMax operators

We define in this section a new kind of merging operators, namely the $\triangle^{GMax}$ family. The aim is to capture the 'arbitration' behaviour of the $\triangle^{Max}$ family but without losing the properties of an IC merging operator. The idea behind these operators has been used in social choice theory [30], where they are called *leximin* functions (see also [14] for an example in decision theory).

TABLE 2. $\triangle^{Max}$ operator

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist_{Max}}$
$(0,0,0,0)$	3	3	0	2	3
$(0,0,0,1)$	3	3	1	3	3
$(0,0,1,0)$	2	2	1	1	**2**
$(0,0,1,1)$	2	2	2	2	**2**
$(0,1,0,0)$	2	2	1	1	**2**
$(0,1,0,1)$	2	2	2	2	**2**
$(0,1,1,0)$	1	1	2	0	2
$(0,1,1,1)$	1	1	3	1	3
$(1,0,0,0)$	2	2	1	2	**2**
$(1,0,0,1)$	2	2	2	3	3
$(1,0,1,0)$	1	1	2	1	2
$(1,0,1,1)$	1	1	3	2	3
$(1,1,0,0)$	1	1	2	1	2
$(1,1,0,1)$	1	1	3	2	3
$(1,1,1,0)$	0	0	3	0	3
$(1,1,1,1)$	0	0	4	1	4

DEFINITION 4.9
Let Ψ be a belief set and let d be a distance between interpretations. Suppose $\Psi = \{\varphi_1 \dots \varphi_n\}$. For each interpretation I we build the list $(d_1^I \dots d_n^I)$ of distances between this interpretation and the n belief bases in Ψ, i.e. $d_j^I = d(I, \varphi_j)$. Let L_I^Ψ be the list obtained from $(d_1^I \dots d_n^I)$ by sorting it in descending order. We will denote $d_{GMax}(I, \Psi)$ the list L_I^Ψ. Let $\leq_{lex}$ be the lexicographical order between sequences of integers (of the same length). We define the following total pre-order:

$$I \leq_\Psi^{GMax} J \text{ iff } L_I^\Psi \leq_{lex} L_J^\Psi$$

and the operator $\triangle^{GMax}$ is defined by:

$$Mod(\triangle_\mu^{GMax}(\Psi)) = \min(mod(\mu), \leq_\Psi^{GMax}).$$

By definition it is easy to show that the $\triangle^{GMax}$ operator is a refinement of the $\triangle^{Max}$ operator.

REMARK 4.10
$\triangle_\mu^{GMax}(\Psi) \vdash \triangle_\mu^{Max}(\Psi)$.

We have to state some results in order to prove in an easy manner that the operator $\triangle^{GMax}$ has good properties.

DEFINITION 4.11
Let L_1 and L_2 be two lists of numbers sorted in descending order. We define $L_1 \odot L_2$ the list obtained by sorting in descending order the concatenation of L_1 with L_2.

LEMMA 4.12
Let L_1, L_1', L_2, L_2' be four lists of integers sorted in descending order such that $card(L_1) = card(L_1')$ and $card(L_2) = card(L_2')$. If $L_1 \leq_{lex} L_1'$ and $L_2 \leq_{lex} L_2'$ then $L_1 \odot L_2 \leq_{lex} L_1' \odot L_2'$.

PROOF. Suppose that $L_1 \leq_{lex} L'_1$ and $L_2 \leq_{lex} L'_2$. It is easy to see that the two following inequalities hold: $L_1 \odot L_2 \leq_{lex} L'_1 \odot L_2$ and $L'_1 \odot L_2 \leq_{lex} L'_1 \odot L'_2$. So by transitivity $L_1 \odot L_2 \leq_{lex} L'_1 \odot L'_2$. ∎

LEMMA 4.13
Let L_1, L'_1, L_2, L'_2 be four lists of integers sorted in descending order such that $card(L_1) = card(L'_1)$ and $card(L_2) = card(L'_2)$. If $L_1 \leq_{lex} L'_1$ and $L_2 <_{lex} L'_2$ then $L_1 \odot L_2 <_{lex} L'_1 \odot L'_2$.

PROOF. With the assumptions it is easy to see that $L_1 \odot L_2 \leq_{lex} L'_1 \odot L_2$ and $L'_1 \odot L_2 <_{lex} L'_1 \odot L'_2$. We conclude by transitivity of $\leq_{lex}$. ∎

Now we can show that the $\triangle^{GMax}$ operators are arbitration operators.

THEOREM 4.14
$\triangle^{GMax}$ is an arbitration operator that satisfies (IC$_{it}$).

PROOF. In order to show $\triangle^{GMax}$ satisfies postulates (IC0–IC8) and (Arb) we use the representation theorem and we show that the assignment $\Psi \mapsto \leq_{\Psi}^{GMax}$ is a fair syncretic assignment.

1. If $I \models \Psi$ and $J \models \Psi$, then $\forall \varphi_i \in \Psi \ I \models \varphi_i$ and $J \models \varphi_i$, then $L_I = (0, \ldots, 0)$ and $L_J = (0, \ldots, 0)$, so $I \simeq_{\Psi} J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then $L_I = (0, \ldots, 0)$ and $L_J \neq (0, \ldots, 0)$, so $I <_{\Psi} J$.
3. If $\Psi_1 \leftrightarrow \Psi_2$, then it is obvious that $\leq_{\Psi_1} = \leq_{\Psi_2}$.
4. We want to show that $\forall I \models \varphi \ \exists J \models \varphi' \ J \leq_{\varphi \sqcup \varphi'} I$. We have that $d(I, \varphi) = 0$ and $d(I, \varphi') = \min_{J \models \varphi'} d(I, J)$, so choose $J \models \varphi'$ such that $d(I, J) = d(I, \varphi')$. Then $d(J, \varphi) = \min_{I' \models \varphi} d(J, I') \leq d(J, I)$, and $d(J, \varphi') = 0$. So $d_{GMax}(J, \varphi \sqcup \varphi') \leq_{lex} d_{GMax}(I, \varphi \sqcup \varphi')$. So by definition $J \leq_{\varphi \sqcup \varphi'} I$.
5. If $L_I^{\Psi_1} \leq_{lex} L_J^{\Psi_1}$ and $L_I^{\Psi_2} \leq_{lex} L_J^{\Psi_2}$. By Lemma 4.12, we have $L_I^{\Psi_1 \sqcup \Psi_2} \leq_{lex} L_J^{\Psi_1 \sqcup \Psi_2}$.
6. If $L_J^{\Psi_1} <_{lex} L_I^{\Psi_1}$ and $L_J^{\Psi_2} \leq_{lex} L_I^{\Psi_2}$, by Lemma 4.13 follows $L_J^{\Psi_1 \sqcup \Psi_2} <_{lex} L_I^{\Psi_1 \sqcup \Psi_2}$.
8. Suppose that $I <_{\varphi_1}^{GMax} J$, $I <_{\varphi_2}^{GMax} J'$ and $J \simeq_{\varphi_1 \sqcup \varphi_2}^{GMax} J'$. We want to show that $I <_{\varphi_1 \sqcup \varphi_2}^{GMax} J$. From $L_I^{\varphi_1} <_{lex} L_J^{\varphi_1}$ we get $d(I, \varphi_1) < d(J, \varphi_1)$ and in the same way $d(I, \varphi_2) < d(J', \varphi_2)$. From $L_J^{\varphi_1 \sqcup \varphi_2} \simeq_{lex} L_{J'}^{\varphi_1 \sqcup \varphi_2}$ it is easy to see that either $d(J, \varphi_1) = d(J', \varphi_1)$ and $d(J, \varphi_2) = d(J', \varphi_2)$, or $d(J, \varphi_1) = d(J', \varphi_2)$ and $d(J, \varphi_2) = d(J', \varphi_1)$. In the first case we get $L_I^{\varphi_2} <_{lex} L_J^{\varphi_2}$ and since $L_I^{\varphi_1} <_{lex} L_J^{\varphi_1}$ we have by Lemma 4.13 that $L_I^{\varphi_1 \sqcup \varphi_2} <_{lex} L_J^{\varphi_1 \sqcup \varphi_2}$. In the second case we get $d(I, \varphi_2) < d(J, \varphi_1)$ and since $d(I, \varphi_1) < d(J, \varphi_1)$, we obtain easily $L_I^{\varphi_1 \sqcup \varphi_2} <_{lex} L_J^{\varphi_1 \sqcup \varphi_2}$.

Finally the proof that postulate (IC$_{it}$) holds for $\triangle^{GMax}$ is similar to the one for $\triangle^{Max}$. ∎

Let us return to the block of flats example (Example 4.4).

EXAMPLE 4.15
Table 3 sums up the calculation for the $\triangle^{GMax}$ operator (using the Dalal distance).

As we have seen in the previous section the $\triangle^{Max}$ operator is not an IC merging operator because it is not sharp enough. The $\triangle^{GMax}$ operator narrows the $\triangle^{Max}$ choices and then we have $mod(\triangle_{\mu}^{GMax}(\Psi)) = \{(0,0,1,0), (0,1,0,0)\}$, so the decision that best fits the group and that is allowed by the integrity constraints is to build either the tennis court or the private car park, without increasing the rent.

TABLE 3. $\triangle^{GMax}$ operator

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist_{GMax}}$
$(0,0,0,0)$	3	3	0	2	(3,3,2,0)
$(0,0,0,1)$	3	3	1	3	(3,3,3,1)
$(0,0,1,0)$	2	2	1	1	**(2,2,1,1)**
$(0,0,1,1)$	2	2	2	2	(2,2,2,2)
$(0,1,0,0)$	2	2	1	1	**(2,2,1,1)**
$(0,1,0,1)$	2	2	2	2	(2,2,2,2)
$(0,1,1,0)$	1	1	2	0	(2,1,1,0)
$(0,1,1,1)$	1	1	3	1	(3,1,1,1)
$(1,0,0,0)$	2	2	1	2	(2,2,2,1)
$(1,0,0,1)$	2	2	2	3	(3,2,2,2)
$(1,0,1,0)$	1	1	2	1	(2,1,1,1)
$(1,0,1,1)$	1	1	3	2	(3,2,1,1)
$(1,1,0,0)$	1	1	2	1	(2,1,1,1)
$(1,1,0,1)$	1	1	3	2	(3,2,1,1)
$(1,1,1,0)$	0	0	3	0	(3,0,0,0)
$(1,1,1,1)$	0	0	4	1	(4,1,0,0)

The majority 'vote', *à la* $\triangle^{\Sigma}$, seems to be more 'democratic' than the others methods. But in this case it works only if φ_3 agrees to conform to the majority wishes that are strongly opposed to his own. In this case φ_3 could decide to quit the co-owners committee, and the work will perhaps not carry on because of lack of money. So if a decision, as in this example or in a peace agreement or in a price agreement in a competitive market, requires the approval of all the members, an arbitration method like $\triangle^{GMax}$ seems more appropriate.

5 Connection with belief revision

We show in this section that IC merging operators are a generalization of revision operators [1, 16, 18] to multiple belief bases and how to build an IC merging operator from a revision operator.

First we recall the Katsuno and Mendelzon [18] postulates for belief revision operators: let φ be a belief base and μ be a new information. $\varphi \circ \mu$ denotes the belief base result of the revision of φ by μ. The operator $\circ$ is a revision operator if it satisfies the following postulates:

(R1) $\varphi \circ \mu$ implies μ.

(R2) If $\varphi \wedge \mu$ is consistent then $\varphi \circ \mu \leftrightarrow \varphi \wedge \mu$.

(R3) If μ is consistent then $\varphi \circ \mu$ is consistent.

(R4) If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \circ \mu_1 \leftrightarrow \varphi_2 \circ \mu_2$.

(R5) $(\varphi \circ \mu) \wedge \phi$ implies $\varphi \circ (\mu \wedge \phi)$.

(R6) If $(\varphi \circ \mu) \wedge \phi$ is consistent then $\varphi \circ (\mu \wedge \phi)$ implies $(\varphi \circ \mu) \wedge \phi$.

When one works with a finite propositional language Katsuno and Mendelzon postulates are equivalent to AGM ones [1, 16], see [18] for a full justification. In this paper they give also a representation theorem for revision operators.

DEFINITION 5.1
A faithful assignment is a function mapping each belief base φ to a pre-order $\leq_\varphi$ over interpretations such that:

1. If $I \models \varphi$ and $J \models \varphi$, then $I \simeq_\varphi J$.
2. If $I \models \varphi$ and $J \not\models \varphi$, then $I <_\varphi J$.
3. If $\varphi \leftrightarrow \varphi'$, then $\leq_\varphi = \leq_{\varphi'}$.

THEOREM 5.2
An operator $\circ$ is a revision operator if and only if there exists a faithful assignment that maps each belief base φ to a total pre-order $\leq_\varphi$ such that

$$mod(\varphi \circ \mu) = \min(mod(\mu), \leq_\varphi).$$

5.1 From IC merging to revision

Intuitively revision operators select in a formula (the new evidence) the closest information to a ground information (the old belief base). Identically, IC merging operators select in a formula (the integrity constraints) the closest information to a ground information (a set of belief bases).

So following this idea it is easy to make a correspondence between IC merging operators and belief revision operators.

THEOREM 5.3
If $\triangle$ is an IC merging operator (it satisfies (IC0–IC8)), then the operator $\circ$, defined as $\varphi \circ \mu = \triangle_\mu(\varphi)$, is an AGM revision operator (it satisfies (R1–R6)).

PROOF. The proof is straightforward. (R1) comes from (IC0), (R2) comes from (IC2), (R3) comes from (IC1), (R4) comes from (IC3), (R5) comes from (IC7), and (R6) comes from (IC8). ∎

Conversely, we can wonder if we can build a merging operator from a given revision operator.

5.2 From revision to IC merging

We have seen in the previous section that each IC merging operator defines a revision operator. We can try to connect more deeply these two kinds of operators, so the point is now to determine if each revision operator $\circ$ defines an IC merging operator Δ.

First, it is important to note that, thanks to representation theorem (Theorem 3.7), we can identify a belief set Ψ with a *preference relation* between valuations, that is, the pre-order $\leq_\Psi$ associated to belief set Ψ via the syncretic assignment. In particular, each belief base (formula) belonging to a belief set can be seen as a singleton belief set and therefore as a preference relation between valuations. Because of this, the problem of defining a merging operator from a revision operator can be reduced to the problem of merging agent (individual) preferences — given by the revision operator via the representation Theorem 5.2 — into the group (global) preferences. Thus, a way to build a merging operator from a revision operator is to choose the method to combine *individual* preferences in a *global* preference. As we will see, we can choose a method *à la* $\triangle^{\Sigma}$ or $\triangle^{GMax}$, among others.

We will examine here the properties of the pairs ⟨revision operator, merging method⟩.

We propose the following definition of a merging operator from a given revision operator $\circ$.

DEFINITION 5.4

- Considers the faithful assignment corresponding to the revision operator $\circ$.
- Define $f^{\circ}_{\varphi}(I) = n$ where n is the level where the interpretation I appears in the $\leq_{\varphi}$ pre-order. More formally n is the length of the longest chain of strict inequalities $I_0 < \ldots < I_n$ with $I_0 \models \varphi$ and $I_n = I$.
- Define $f^{\circ}_{\Psi}(I)$ with the merging method chosen (for example $f^{\circ}_{\Psi}(I) = \sum_{\varphi \in \Psi}(f^{\circ}_{\varphi}(I))$ if Σ is the chosen method).
- Define $I \leq_{\Psi} J$ iff $f^{\circ}_{\Psi}(I) \leq f^{\circ}_{\Psi}(J)$.
- Finally $mod(\triangle^{\circ}_{\mu}(\Psi)) = \min(mod(\mu), \leq_{\Psi})$.

Now the question we try to address is the following one: given a merging method, what are the properties required of the operator $\circ$ in order to get good properties for the operator $\triangle^{\circ}$?

For example if we choose $f^{\circ}_{\Psi}(I) = \sum_{\varphi \in \Psi}(f^{\circ}_{\varphi}(I))$ as merging method we get the following result, without any additional property required of $\circ$:

THEOREM 5.5

If a merging operator $\triangle^{\circ}$ is defined from a revision operator $\circ$ and from the merging method $f^{\circ}_{\Psi}(I) = \sum_{\varphi \in \Psi}(f^{\circ}_{\varphi}(I))$, according to Definition 5.4, then the operator $\triangle^{\circ}$ satisfies (IC0–IC3), (IC5–IC8) and (Maj).

PROOF. Let's verify the conditions on the assignment, then we conclude by Corollary 3.8 and Remark 3.12:

1. If $I \models \Psi$ and $J \models \Psi$, then $f^{\circ}_{\Psi}(I) = 0$ and $f^{\circ}_{\Psi}(J) = 0$, so $I \simeq_{\Psi} J$.
2. If $I \models \Psi$ and $J \not\models \Psi$, then $f^{\circ}_{\Psi}(I) = 0$ and $f^{\circ}_{\Psi}(J)) > 0$, so $I <_{\Psi} J$.
3. Straightforward.
5. If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $f^{\circ}_{\Psi_1}(I) \leq f^{\circ}_{\Psi_1}(J)$ and $f^{\circ}_{\Psi_2}(I) \leq f^{\circ}_{\Psi_2}(J)$, so trivially $f^{\circ}_{\Psi_1 \sqcup \Psi_2}(I) \leq f^{\circ}_{\Psi_1 \sqcup \Psi_2}(J)$.
6. Straightforward (similar to 5).
7. If $I <_{\Psi_2} J$, then $f^{\circ}_{\Psi_2}(I) < f^{\circ}_{\Psi_2}(J)$. We want to show that

$$\exists n \; f^{\circ}_{\Psi_1}(I) + n * f^{\circ}_{\Psi_2}(I) < f^{\circ}_{\Psi_1}(J) + n * f^{\circ}_{\Psi_2}(J)$$

so simply choose $n > \frac{f^{\circ}_{\Psi_1}(I) - f^{\circ}_{\Psi_1}(J)}{f^{\circ}_{\Psi_2}(J) - f^{\circ}_{\Psi_2}(I)}$.

■

Unfortunately condition 4 of the syncretic assignment is not always satisfied as we will see at the end of this section. We extend the definition of $f^{\circ}_{\varphi}(\cdot)$ to belief bases by putting $f^{\circ}_{\varphi}(\varphi') = \min\{f^{\circ}_{\varphi}(I) : I \models \varphi'\}$ and we consider the following condition of *symmetry*:

(Sym) $f^{\circ}_{\varphi}(\varphi') = f^{\circ}_{\varphi'}(\varphi)$ for any belief bases φ and φ'.

This condition characterizes the revision operators generating IC merging operators. More precisely we have the following:

THEOREM 5.6
If a merging operator $\triangle^\circ$ is defined from a revision operator $\circ$ according to Definition 5.4 using the merging method $f^\circ_\Psi(I) = \sum_{\varphi \in \Psi} f^\circ_\varphi(I)$, then the operator $\triangle^\circ$ is an IC majority merging operator if and only if the condition (Sym) holds.

PROOF. From Theorems 5.5 and 3.11 it remains simply to show that condition 4 of the assignment corresponds to condition (Sym).

First we show that condition 4 implies $f^\circ_\varphi(\varphi') = f^\circ_{\varphi'}(\varphi)$. Condition 4 is equivalent to $\forall I \models \varphi\, \exists J \models \varphi'\ f^\circ_\varphi(J) + f^\circ_{\varphi'}(J) \leq f^\circ_\varphi(I) + f^\circ_{\varphi'}(I)$. But if $I \models \varphi$ and $J \models \varphi'$ we have $f^\circ_\varphi(I) = 0$ and $f^\circ_{\varphi'}(J) = 0$, so condition 4 is equivalent to $\forall I \models \varphi\, \exists J \models \varphi'\ f^\circ_\varphi(J) \leq f^\circ_{\varphi'}(I)$. From this we deduce easily that $f^\circ_\varphi(\varphi') \leq f^\circ_{\varphi'}(\varphi)$ and as the role of φ and φ' in condition 4 is symmetrical we also obtain $f^\circ_{\varphi'}(\varphi) \leq f^\circ_\varphi(\varphi')$. Therefore $f^\circ_\varphi(\varphi') = f^\circ_{\varphi'}(\varphi)$.

Conversely, suppose that $f^\circ_\varphi(\varphi') = f^\circ_{\varphi'}(\varphi)$ holds. Towards a contradiction suppose that condition 4 does not hold, that is $\exists I \models \varphi\, \forall J \models \varphi' J >_{\varphi \sqcup \varphi'} I$. Then $\exists I \models \varphi\, \forall J \models \varphi' f^\circ_{\varphi \sqcup \varphi'}(J) > f^\circ_{\varphi \sqcup \varphi'}(I)$. From this, since $f^\circ_{\varphi'}(J) = f^\circ_\varphi(I) = 0$, we have $\exists I \models \varphi\, \forall J \models \varphi' f^\circ_\varphi(J) > f^\circ_{\varphi'}(I)$. Therefore $f^\circ_\varphi(\varphi') > f^\circ_{\varphi'}(\varphi)$. Contradiction. ∎

Actually this condition of symmetry works for other merging methods. In particular if we modify the Definition 5.4 as follows

- Define $f^\circ_\Psi(I)$ as the list of $f^\circ_\varphi(I)$ sorted in descending order, i.e. $f^\circ_\Psi(I) = (f^\circ_{\varphi_1}(I), \ldots, f^\circ_{\varphi_n}(I))$ where $\Psi = \{\varphi_1 \ldots, \varphi_n\}$ and $f^\circ_{\varphi_i}(I) \geq f^\circ_{\varphi_{i+1}}(I)$ for $1 \leq i \leq n$.
- Define $I \leq_\Psi J$ iff $f^\circ_\Psi(I) \leq_{lex} f^\circ_\Psi(J)$.

The operator $\triangle^\circ$ defined in this way will be termed the $Gmax$ operator associated to $\circ$. For this kind of operator we can prove in a similar way to Theorem 5.6 the following result.

THEOREM 5.7
Let $\triangle^\circ$ be the Gmax operator associated to $\circ$. Then the operator $\triangle^\circ$ satisfies (IC0–IC3), (IC5–IC8) and (Arb). Furthermore the operator $\triangle^\circ$ is an IC arbitration operator if and only if the condition (Sym) holds.

Actually the condition (Sym) is satisfied if and only if the revision is defined from a distance. More precisely a revision operator $\circ$ is said to be defined from a distance d iff the following conditions hold:

- d is a distance, that is d is a function $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{N}$ that satisfies: $d(I, J) = d(J, I)$ and $d(I, I) = 0$ iff $I = J$.
- Let φ be a belief base and I be an interpretation: $d(I, \varphi) = \min\{d(I, J) : J \models \varphi\}$.
- $I \leq_\varphi J$ iff $d(I, \varphi) \leq d(J, \varphi)$.
- $mod(\varphi \circ \mu) = \min(mod(\mu), \leq_\varphi)$.

Notice that if $\circ$ is defined from a distance, the Σ and the $Gmax$ methods will give respectively the operators $\triangle^\Sigma$ and $\triangle^{GMax}$ defined in Section 4.

PROPOSITION 5.8
Let $\circ$ be a revision operator. Then the condition (Sym) holds iff $\circ$ is defined from a distance.

PROOF. The *if part* is straightforward. For the *only if part* define the following distance: $d(I,J) = f^{\circ}_{\varphi_I}(J)$. Let $\varphi \mapsto \leq_\varphi$ the assignment representing $\circ$. We want to prove that $I \leq_\varphi J$ iff $d(I,\varphi) \leq d(J,\varphi)$. Notice that by definition $d(I,\varphi) \leq d(J,\varphi)$ if and only if $\min\{f^{\circ}_{\varphi_{I'}}(I) : I' \models \varphi\} \leq \min\{f^{\circ}_{\varphi_{I'}}(J) : I' \models \varphi\}$ which by (Sym) is equivalent to $\min\{f^{\circ}_{\varphi_I}(I') : I' \models \varphi\} \leq \min\{f^{\circ}_{\varphi_J}(I') : I' \models \varphi\}$. This is exactly $f^{\circ}_{\varphi_I}(\varphi) \leq f^{\circ}_{\varphi_J}(\varphi)$ that is equivalent, using (Sym) once again, to $f^{\circ}_{\varphi}(I) \leq f^{\circ}_{\varphi}(J)$, which is exactly $I \leq_\varphi J$. ∎

As a corollary of Theorems 5.6, 5.7 and Proposition 5.8 we have the following result.

THEOREM 5.9
A merging operator $\triangle$ defined from a revision operator $\circ$ and the Σ or the $Gmax$ methods is an IC merging operator if and only if $\circ$ is defined from a distance.

In particular, as it is well known that there are revision operators which are not defined from a distance, the operators $\triangle^{\circ}$ associated to them are not fair, i.e. they do not satisfy (IC4).

6 Connections with related work

In this section we study the connections between this work and related work. We first investigate the case where there are no integrity constraints for the merging and show that this work extends that of [22]. Then we deal with the relationship between our operators and Liberatore and Schaerf's ones [24, 25], showing that those operators can be seen as a special case of IC merging operators. And finally, we briefly address Lin and Mendelzon's theory majority merging operators [28, 26], and Revesz's model-fitting operators [31, 32].

Concerning the connection with important work on schema integration, we will simply say a few words. The aim of this community is merging information in which the representation is heterogeneous. They do not address the problem of merging contradictory information. Even work about integration of deductive databases (see [35, 38]) does not consider the problem of inconsistency. Our work on merging inconsistent information uses the fact that the different sources have homogeneous representation. This can be used as a final stage in a complete process of merging heterogeneous information.

6.1 Pure merging

A logical definition of merging operators in the case where there is no integrity constraints was proposed in [22]. From now on, we will call those operators *pure merging* operators, but often simply refer to this case as merging, majority merging and arbitration operators (without the IC). Although the characterization of pure merging operators was simpler, the representation theorem was not fully satisfactory because in this case we have a very coerced definition of the pre-orders associated to a given belief set. An interesting point is to study the behaviour of the postulates of IC merging operators when there is no integrity constraints, which is simulated by putting $\mu = \top$. Actually we will see that the properties of this specialization are compatible with the characterization given in [22].

The definition proposed in that paper was the following:

DEFINITION 6.1
Let $\triangle$ be an operator mapping a belief set Ψ to a belief base $\triangle(\Psi)$. $\triangle$ is said to be a pure merging operator if and only if it satisfies the following postulates:

(A1) $\triangle(\Psi)$ is consistent.
(A2) If $\bigwedge \Psi$ is consistent, then $\triangle(\Psi) = \bigwedge \Psi$.
(A3) If $\Psi_1 \leftrightarrow \Psi_2$, then $\triangle(\Psi_1) \leftrightarrow \triangle(\Psi_2)$.
(A4) If $\varphi \wedge \varphi'$ is not consistent, then $\triangle(\varphi \sqcup \varphi') \nvdash \varphi$.
(A5) $\triangle(\Psi_1) \wedge \triangle(\Psi_2) \vdash \triangle(\Psi_1 \sqcup \Psi_2)$.
(A6) If $\triangle(\Psi_1) \wedge \triangle(\Psi_2)$ is consistent, then $\triangle(\Psi_1 \sqcup \Psi_2) \vdash \triangle(\Psi_1) \wedge \triangle(\Psi_2)$.

Moreover, a pure merging operator is said to be a pure majority operator if it satisfies the following postulate:

(M7) $\forall \varphi \, \exists n \;\; \triangle\, (\Psi \sqcup \varphi^n) \vdash \varphi$.

And finally, a pure merging operator is said to be a pure arbitration operator if it satisfies the following postulate:

(WMI) $\forall \varphi' \; \exists \varphi \, \varphi' \nvdash \varphi \, \forall n \;\; \triangle\, (\varphi' \sqcup \varphi^n) = \triangle(\varphi' \sqcup \varphi)$.

The characterization of IC merging operators is a generalization of this characterization, as we will see easily below. Actually, it is enough to study the shape of the postulates of Section 3 when $\mu = \top$. We will note $\triangle_\top(\Psi) = \triangle(\Psi)$.

(IC0$_\top$) $\triangle(\Psi) \vdash \top$.
(IC1$_\top$) $\triangle(\Psi)$ is consistent.
(IC2$_\top$) If Ψ is consistent, then $\triangle(\Psi) = \bigwedge \Psi$.
(IC3$_\top$) If $\Psi_1 \leftrightarrow \Psi_2$, then $\triangle(\Psi_1) \leftrightarrow \triangle(\Psi_2)$.
(IC4$_\top$) $\triangle(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot \Rightarrow \triangle(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$.
(IC5$_\top$) $\triangle(\Psi_1) \wedge \triangle(\Psi_2) \vdash \triangle(\Psi_1 \sqcup \Psi_2)$.
(IC6$_\top$) If $\triangle(\Psi_1) \wedge \triangle(\Psi_2)$ is consistent, then $\triangle(\Psi_1 \sqcup \Psi_2) \vdash \triangle(\Psi_1) \wedge \triangle(\Psi_2)$.
(IC7$_\top$) $\triangle(\Psi) \wedge \top \vdash \triangle(\Psi)$.
(IC8$_\top$) If $\triangle(\Psi) \wedge \top$ is consistent, then $\triangle(\Psi) \vdash \triangle(\Psi)$.
(Maj$_\top$) $\exists n \;\; \triangle\, (\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle(\Psi_2)$.
(Arb$_\top$) $\left.\begin{array}{l} \triangle(\varphi_1) \leftrightarrow \triangle(\varphi_2) \\ \triangle_\bot(\varphi_1 \sqcup \varphi_2) \leftrightarrow \bot \\ \top \nvdash \top \\ \top \nvdash \top \end{array}\right\} \Rightarrow \triangle(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle(\varphi_1)$.
(WMI$_\top$) $\forall \varphi' \; \exists \varphi \, \varphi' \nvdash \varphi \, \forall n \;\; \triangle\, (\varphi' \sqcup \varphi^n) = \triangle(\varphi' \sqcup \varphi)$.

(IC0$_\top$), (IC7$_\top$) and (IC8$_\top$) are trivially true. (Arb$_\top$) is also trivially true because the premiss of the implication is always false. Actually, the meaning of (Arb) is not expressible without integrity constraints. It is easy to see that the other postulates are nearly the same that those given in [22]. The main differences is that postulate (IC4$_\top$) is stronger than (A4) and that postulate (Maj$_\top$) is stronger than (M7).

Concerning (WMI), let us remark that this rule expresses only a kind of non-majority rule and thus is not a direct characterization of arbitration, whereas (Arb) defines in a more positive manner the arbitration behaviour.

The previous observations lead to the following result:

PROPOSITION 6.2
If $\triangle$ is an IC merging operator (i.e. it satisfies (IC0–IC8)), then $\triangle_\top$ is a merging operator (i.e. it satisfies (A1–A6)).

Furthermore if $\triangle$ is an IC majority merging operator, then $\triangle_\top$ is a majority merging operator.

6.2 Liberatore and Schaerf commutative revision operators

Liberatore and Schaerf [24, 25] have defined merging operators that they call arbitration operators or commutative revision operators. In this section we establish some links between our operators and Liberatore and Schaerf's. In the sequel we will call commutative revision the Liberatore and Schaerf's operators since the authors used the two names and since arbitration operators have already been defined in our framework.

They consider operators $\diamond$ mapping a couple of belief bases into a belief base. Next we give their definition of commutative revision operators.

An operator $\diamond$ is said to be a commutative revision operator if the following postulates hold:

(LS1) $\varphi \diamond \mu \leftrightarrow \mu \diamond \varphi$.

(LS2) $\varphi \wedge \mu$ implies $\varphi \diamond \mu$.

(LS3) If $\varphi \wedge \mu$ is consistent then $\varphi \diamond \mu$ implies $\varphi \wedge \mu$.

(LS4) $\varphi \diamond \mu$ is inconsistent iff both φ and μ are inconsistent.

(LS5) If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \diamond \mu_1 \leftrightarrow \varphi_2 \diamond \mu_2$.

(LS6) $\varphi \diamond (\mu \vee \theta) = \begin{cases} \varphi \diamond \mu \text{ or} \\ \varphi \diamond \theta \text{ or} \\ (\varphi \diamond \mu) \vee (\varphi \diamond \theta). \end{cases}$

(LS7) $\varphi \diamond \mu$ implies $\varphi \vee \mu$.

(LS8) If φ is satisfiable then $\varphi \wedge (\varphi \diamond \mu)$ is also satisfiable.

Now suppose that $\triangle$ is an IC merging operator. We would like to define a commutative revision operator from $\triangle$. Because of postulates (LS1), (LS2) and (LS7) there are two natural choices for the operator $\diamond_\triangle$ which we want to define. One option is by putting $\varphi_1 \diamond_\triangle \varphi_2 = \triangle_{\varphi_1 \vee \varphi_2}(\varphi_1 \sqcup \varphi_2)$. The other one is by putting $\varphi_1 \diamond_\triangle \varphi_2 = (\varphi_1 \circ \varphi_2) \vee (\varphi_2 \circ \varphi_1)$, where $\circ$ is the revision operator associated to $\triangle$ as in Theorem 5.3. Although the two definitions coincide as we will see below, we prefer the first formulation because it can be generalized to more than two belief bases in an obvious way.

DEFINITION 6.3
If $\triangle$ is an IC merging operator we define a commutative revision operator $\diamond_\triangle$ by $\varphi \diamond_\triangle \mu = \triangle_{\varphi \vee \mu}(\varphi \sqcup \mu)$. We will say that $\diamond_\triangle$ is the commutative revision operator associated with $\triangle$.

The following theorem shows that the two ways of defining a commutative operator from an IC merging operator agree.

THEOREM 6.4
If $\triangle$ is an IC merging operator then it satisfies

$$\triangle_{\varphi\vee\mu}(\varphi \sqcup \mu) \leftrightarrow \triangle_{\varphi}(\mu) \vee \triangle_{\mu}(\varphi).$$

Recall that $\varphi \circ \mu = \triangle_{\mu}(\varphi)$ is the AGM revision operator associated with $\triangle$. Thus the previous equivalence can be written in the following way:

$$\triangle_{\varphi\vee\mu}(\varphi \sqcup \mu) \leftrightarrow \mu \circ \varphi \vee \varphi \circ \mu.$$

PROOF. Let $\varphi \mapsto \leq_{\varphi}$ be a syncretic assignment representing $\triangle$. If $\varphi \wedge \mu$ is consistent the result follows trivially. Now suppose that $\varphi \wedge \mu$ is inconsistent. We have

$$\min(\varphi \vee \mu, \leq_{\varphi\sqcup\mu}) = \begin{cases} \min(\varphi, \leq_{\varphi\sqcup\mu}) \text{ or} \\ \min(\mu, \leq_{\varphi\sqcup\mu}) \text{ or} \\ \min(\varphi, \leq_{\varphi\sqcup\mu}) \cup \min(\mu, \leq_{\varphi\sqcup\mu}). \end{cases}$$

But the first two cases are not possible from condition 4 (fairness) and the fact that $\varphi \wedge \mu$ is inconsistent. So only the last case is possible so $\min(\varphi \vee \mu, \leq_{\varphi\sqcup\mu}) = \min(\varphi, \leq_{\varphi\sqcup\mu})$ $\vee \min(\mu, \leq_{\varphi\sqcup\mu})$, which can be rewritten $\min(\varphi \vee \mu, \leq_{\varphi\sqcup\mu}) = \min(\varphi, \leq_{\mu}) \cup \min(\mu, \leq_{\varphi})$ which is, by definition, $\triangle_{\varphi\vee\mu}(\varphi \sqcup \mu) \leftrightarrow \triangle_{\varphi}(\mu) \vee \triangle_{\mu}(\varphi)$. ∎

In order to prove that the operators $\diamond_{\triangle}$ defined in such a way are commutative revision operators we need some additional properties for the operator $\triangle$. This will be seen easily after setting a new formulation of the representation theorem for commutative revision operators.

THEOREM 6.5
If $\triangle$ is an IC merging operator, then the operator $\diamond_{\triangle}$ associated with it satisfies (LS1–LS5), (LS7) and (LS8).

PROOF. (LS1) direct by definition. (LS2) direct from (IC2). (LS3) direct from (IC2). (LS4) direct from (IC0) and (IC1). (LS5) direct from (IC3). (LS7) direct by definition. (LS8) direct from (IC4) and (IC2). ∎

In order to prove that the commutative operators associated to IC merging operators are indeed commutative revision operators in the sense of Liberatore and Schaerf, we need some additional properties derived from an analysis of their representation theorem.

We begin by recalling that their representation theorem is formulated in terms of families of pre-orders over sets of interpretations.

A mapping $F \mapsto \leq_F$ from subsets of interpretations into total pre-orders over sets of interpretation is said to be a *good assignment*, if the following properties hold:

L1 if $A \leq_F B$ and $B \leq_F C$ then $A \leq_F C$.
L2 if $A \subseteq B$ then $B \leq_F A$.
L3 $A \leq_F A \cup B$ or $B \leq_F A \cup B$.
L4 $B \leq_F C$ for every C iff $F \cap B \neq \emptyset$.
L5 $A \leq_{C\cup D} B \Leftrightarrow \begin{cases} C \leq_{A\cup B} D \text{ and } A \leq_C B \quad \text{or} \\ D \leq_{A\cup B} C \text{ and } A \leq_D B. \end{cases}$

Following [25], if A is a set of interpretations, $\hat{A}$ will denote the set $\{\{I\}|I \in A\}$. Liberatore and Schaerf's representation theorem is stated next.

THEOREM 6.6
An operator $\diamond$ is a commutative revision operator if and only if there is a good assignment $F \mapsto \leq_F$ such that

$$mod(\varphi \diamond \mu) = \{I | \{I\} \in \min(\widehat{mod(\varphi)}, \leq_{mod(\mu)}) \cup \min(\widehat{mod(\mu)}, \leq_{mod(\varphi)})\}.$$

An analysis of the proof of this theorem shows that any commutative revision operator is defined from a revision operator having some additional properties. What is not totally straightforward is to establish which are exactly these properties in terms of the revision operator. We do that next. Then it will be easy to characterize IC merging operators which generate commutative revision operators.

Let $\varphi \mapsto \leq_\varphi$ be a faithful assignment. We define the *lifting* of this as a map $F \mapsto \leq_F$ where F is a set of interpretations and $\leq_F$ is a total pre-order over sets of interpretations which satisfies

$$A \leq_F B \text{ iff } \exists I \in A \, \forall J \in B \; I \leq_\varphi J$$

where φ is a formula such that $mod(\varphi) = F$.

The following result is implicit in the work of Liberatore and Schaerf. That is in fact the kernel of their proof of the representation theorem. We think that it is quite important to state it explicitly.

OBSERVATION 6.7
$\diamond$ is a commutative revision operator iff there is a unique revision operator $\circ$ such that
(i) the lifting of the faithful assignment representing $\circ$ is a good assignment and
(ii) $\mu \diamond \theta = (\mu \circ \theta) \vee (\theta \circ \mu)$

Now we are ready to give our characterization of revision operators inducing commutative revision operators.

THEOREM 6.8
An operator $\diamond$ defined from a revision operator $\circ$ by $\varphi \diamond \mu = \varphi \circ \mu \vee \mu \circ \varphi$ satisfies (LS1–LS8) if and only if $\circ$ satisfy the following condition:

$$(\mu \vee \theta) \circ \varphi = \begin{cases} \mu \circ \varphi & \text{if } \varphi \circ (\mu \vee \theta) \vdash \neg\theta \\ \theta \circ \varphi & \text{if } \varphi \circ (\mu \vee \theta) \vdash \neg\mu \\ (\mu \circ \varphi) \vee (\theta \circ \varphi) & \text{otherwise.} \end{cases} \tag{6.1}$$

PROOF. First notice that by using the representation theorem for belief revision [18] we get easily that condition (6.1) corresponds to the following condition on the faithful assignment representing $\circ$:

$$\min(mod(\varphi), \leq_{\mu \vee \theta}) = \begin{cases} \min(mod(\varphi), \leq_\mu) & \text{if } \mu <_\varphi \theta \\ \min(mod(\varphi), \leq_\theta) & \text{if } \theta <_\varphi \mu \\ \min(mod(\varphi), \leq_\mu) \cup \min(mod(\varphi), \leq_\theta) & \text{otherwise} \end{cases} \tag{6.2}$$

where $\mu \leq_\varphi \theta$ is defined, as the lifting, in the natural way:

$$\mu \leq_\varphi \theta \text{ iff } \exists I \models \mu \, \forall J \models \theta \; I \leq_\varphi J.$$

(If part) Assume that $\circ$ is a revision operator satisfying condition (6.1) and let $\diamond$ be defined by putting $\varphi \diamond \mu = (\varphi \circ \mu) \vee (\mu \circ \varphi)$. We want to show that $\diamond$ is a commutative revision

operator. Let us consider $\varphi \mapsto \leq_\varphi$ the faithful assignment representing $\circ$ and let $F \mapsto \leq_F$ the lifting associated with it. By Theorem 6.6 it is enough to prove that $F \mapsto \leq_F$ is a good assignment.

The verification that the assignment $F \mapsto \leq_F$ satisfies L1–L4 is straightforward. It remains to verify L5. We will in fact show the following (that is clearly equivalent to L5):

$$A \leq_{C\cup D} B \Leftrightarrow \begin{cases} (i) & C <_{A\cup B} D \text{ and } A \leq_C B & \text{or} \\ (ii) & D <_{A\cup B} C \text{ and } A \leq_D B & \text{or} \\ (iii) & D \simeq_{A\cup B} C \text{ and } (A \leq_C B \text{ or } A \leq_D B). \end{cases}$$

Let φ, μ, θ be formulas such that $mod(\varphi) = A \cup B$, $mod(\mu) = C$ and $mod(\theta) = D$. From right to left we consider the three cases:
(i) holds. For reductio suppose that $A \leq_{C\cup D} B$ does not hold, then $B <_{C\cup D} A$. From this we have $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A = \emptyset$. From $C <_{A\cup B} D$ alias $\mu <_\varphi \theta$ and condition (6.2) we get that $\min(A \cup B, \leq_{\mu\vee\theta}) = \min(A \cup B, \leq_\mu)$. From $A \leq_C B$ we get $\min(A \cup B, \leq_\mu) \cap A \neq \emptyset$. So $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A \neq \emptyset$. Contradiction.
(ii) holds. The proof is similar to the one of the case (i).
(iii) holds. Suppose that $A \leq_{C\cup D} B$ does not hold, that is $A >_{C\cup D} B$. From $D \simeq_{A\cup B} C$ alias $\mu \simeq_\varphi \theta$ and property (6.2), we get

$$(\star) \qquad \min(A \cup B, \leq_{\mu\vee\theta}) = \min(A \cup B, \leq_\mu) \cup \min(A \cup B, \leq_\theta).$$

By hypothesis we know that $A \leq_C B$ or $A \leq_D B$, suppose w.l.g that $A \leq_C B$, then $\min(A \cup B, \leq_\mu) \cap A \neq \emptyset$. From this and equation $(\star)$ we get $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A \neq \emptyset$. But $A >_{C\cup D} B$ implies that $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A = \emptyset$. Contradiction.

From left to right we consider also three cases: either $C <_{A\cup B} D$ or $D <_{A\cup B} C$ or $C \simeq_{A\cup B} D$.
Case $C <_{A\cup B} D$: we want to show that $A \leq_C B$. Suppose towards a contradiction that it is not the case, i.e. $B <_C A$. From $A \leq_{C\cup D} B$ we have that $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A \neq \emptyset$. But from $C <_{A\cup B} D$ we have $\min(A \cup B, \leq_{\mu\vee\theta}) = \min(A \cup B, \leq_C)$. Now from $B <_C A$ we have $\min(A \cup B, \leq_C) \cap A = \emptyset$. Contradiction.
Case $D <_{A\cup B} C$: This case is similar to the first one.
Case $C \simeq_{A\cup B} D$: Suppose that $A \leq_C B$ or $A \leq_D B$ does not hold, that is $A >_C B$ and $A >_D B$, then $\min(A \cup B, \leq_\mu) \cap A = \emptyset$ and $\min(A \cup B, \leq_\theta) \cap A = \emptyset$. From $C \simeq_{A\cup B} D$ we have $\min(A \cup B, \leq_{\mu\vee\theta}) = \min(A \cup B, \leq_\mu) \cup \min(A \cup B, \leq_\theta)$. From $A \leq_{C\cup D} B$ we have that $\min(A \cup B, \leq_{\mu\vee\theta}) \cap A \neq \emptyset$. But from the supposition we have $(\min(A \cup B, \leq_\mu) \cup \min(A \cup B, \leq_\theta)) \cap A = \emptyset$. Contradiction.

(Only if) Assume that we dispose of a commutative revision operator $\diamond$ satisfying (LS1–LS8), which is defined from a revision operator $\circ$ by $\varphi \diamond \mu = (\varphi \circ \mu) \vee (\mu \circ \varphi)$. We want to show that $\circ$ satisfies condition (6.1). From (LS6) we get by definition that

$$(\varphi \circ (\mu \vee \theta)) \vee ((\mu \vee \theta) \circ \varphi) = \begin{cases} (\varphi \circ \mu) \vee (\mu \circ \varphi) & \text{or} \\ (\varphi \circ \theta) \vee (\theta \circ \varphi) & \text{or} \\ (\varphi \circ \mu) \vee (\mu \circ \varphi) \vee (\varphi \circ \theta) \vee (\theta \circ \varphi). \end{cases} \tag{6.3}$$

Remember the following classical property of revision operator [16]:

$$\varphi \circ (\mu \vee \theta) = \begin{cases} \varphi \circ \mu & \text{or} \\ \varphi \circ \theta & \text{or} \\ (\varphi \circ \mu) \vee (\varphi \circ \theta). \end{cases} \tag{6.4}$$

We consider the three cases of condition (6.1).

Case 1: Assume that $\varphi \circ (\mu \vee \theta) \vdash \neg\theta$. In this case the two last alternatives of (6.4) are clearly impossible, thus necessarily we have $\varphi \circ (\mu \vee \theta) = \varphi \circ \mu$. The assumption also implies that $\varphi \wedge \theta \vdash \bot$ and from this and the assumption it is easy to see that the two last cases of (6.3) are impossible, thus necessarily we have

$$(\varphi \circ (\mu \vee \theta)) \vee ((\mu \vee \theta) \circ \varphi) = (\varphi \circ \mu) \vee (\mu \circ \varphi). \tag{6.5}$$

Now suppose that $\varphi \wedge \mu \vdash \bot$. From this, $\varphi \wedge \theta \vdash \bot$ and the equation (6.5) we obtain $(\mu \vee \theta) \circ \varphi = \mu \circ \varphi$. Otherwise, suppose that $\varphi \wedge \mu \nvdash \bot$. Then, by the revision properties and the fact that $\varphi \wedge \theta \vdash \bot$, we have $(\mu \vee \theta) \circ \varphi = \mu \wedge \varphi = \mu \circ \varphi$. That is the first case of condition (6.1) holds.

Case 2: Assume $\varphi \circ (\mu \vee \theta) \vdash \neg\mu$. In this case we follow the reasoning of case 1 but changing the role of μ and θ. Then we obtain $(\mu \vee \theta) \circ \varphi = \theta \circ \varphi$. That is the second case of condition (6.1) holds.

Case 3: In this case is easy to see that

$$\varphi \circ (\mu \vee \theta) = (\varphi \circ \mu) \vee (\varphi \circ \theta). \tag{6.6}$$

We consider two sub-cases. First assume that $\varphi \wedge (\mu \vee \theta) \nvdash \bot$. Then $\varphi \circ (\mu \vee \theta) = (\varphi \wedge \mu) \vee (\varphi \wedge \theta)$ with $\varphi \wedge \mu \nvdash \bot$ and $\varphi \wedge \theta \nvdash \bot$. From this it follows $(\mu \vee \theta) \circ \varphi = (\mu \wedge \varphi) \vee (\theta \wedge \varphi) = (\mu \circ \varphi) \vee (\theta \circ \varphi)$.

Now suppose that $\varphi \wedge (\mu \vee \theta) \vdash \bot$. From this and the following instance of (LS6)

$$(\varphi \circ (\mu \vee \theta)) \vee ((\mu \vee \theta) \circ \varphi) = \begin{cases} (\varphi \circ \mu) \vee (\mu \circ \varphi) & \text{or} \\ (\varphi \circ \theta) \vee (\theta \circ \varphi) & \text{or} \\ (\varphi \circ \mu) \vee (\varphi \circ \theta) \vee (\mu \circ \varphi) \vee (\theta \circ \varphi) & \end{cases} \tag{6.7}$$

it is easy to see that $mod((\mu \vee \theta) \circ \varphi) \subseteq mod(\mu \circ \varphi) \cup mod(\theta \circ \varphi)$. In order to see the converse inclusion take $I \in mod(\mu \circ \varphi)$ (the other case is analogous). Then $\varphi_I \leq_\mu \varphi$. From this and the fact that $\mu \simeq_\varphi \theta$ (which is a consequence of (6.6)), we have by L5 $\varphi_I \leq_{\mu \vee \theta} \varphi$, that is I is minimal in $mod(\varphi)$ for $\leq_{\mu \vee \theta}$. ∎

REMARK 6.9

Actually, it is not hard to see that if $\circ$ is defined from a distance, then property (6.1) is verified. In that case the operator $\diamond$ associated by $\varphi \diamond \mu = (\varphi \circ \mu) \vee (\mu \circ \varphi)$ is indeed a commutative revision operator.

From Theorem 6.8 it is easy to state a representation theorem for commutative revision operators.

DEFINITION 6.10

A commutative faithful assignment is a function mapping each belief base φ to a total pre-order $\leq_\varphi$ over interpretations such that:

1. If $I \models \varphi$ and $J \models \varphi$, then $I \simeq_\varphi J$.
2. If $I \models \varphi$ and $J \not\models \varphi$, then $I <_\varphi J$.
3. If $\varphi \leftrightarrow \varphi'$, then $\leq_\varphi = \leq_{\varphi'}$.
4. $\min(mod(\varphi), \leq_{\mu \vee \theta}) = \begin{cases} \min(mod(\varphi), \leq_\mu) & \text{if } \mu <_\varphi \theta \\ \min(mod(\varphi), \leq_\theta) & \text{if } \theta <_\varphi \mu \\ \min(mod(\varphi), \leq_\mu) \cup \min(mod(\varphi), \leq_\theta) & \text{otherwise} \end{cases}$

where $\mu \leq_\varphi \theta$ is defined in the natural way:

$$\mu \leq_\varphi \theta \text{ iff } \exists I \models \mu \, \forall J \models \theta \; I \leq_\varphi J.$$

THEOREM 6.11
An operator $\diamond$ satisfies (LS1–LS8) if and only if there exists a commutative faithful assignment that maps each belief base φ to a total pre-order $\leq_\varphi$ such that:

$$mod(\varphi \diamond \varphi') = \min(mod(\varphi), \leq_{\varphi'}) \cup \min(mod(\varphi'), \leq_\varphi).$$

PROOF. The *if part* is a consequence of Theorem 6.8. For the *only if part* notice that from a commutative revision operator $\diamond$ we can define a revision operator $\circ$ and from Theorem 6.8 conclude that this operator satisfies condition 4 of the commutative faithful assignment. ∎

We have seen in Theorem 6.8 how to characterize the revision operators inducing commutative revision operators. Then it is obvious that commutative revision operators defined from IC merging operators do not automatically satisfy (LS6). Actually, from Theorem 6.4 one can easily see that the property corresponding to equation 6.2 is the following

$$\triangle_\varphi(\mu \vee \theta) = \begin{cases} \triangle_\varphi(\mu) & \text{if } \triangle_{\mu \vee \theta}(\varphi) \vdash \neg\theta \\ \triangle_\varphi(\theta) & \text{if } \triangle_{\mu \vee \theta}(\varphi) \vdash \neg\mu \\ \triangle_\varphi(\mu) \vee \triangle_\varphi(\theta) & \text{otherwise.} \end{cases} \tag{6.8}$$

Thus by Theorems 5.3 and 6.4 the following theorem can simply be seen as a reformulation of Theorem 6.8.

THEOREM 6.12
If $\triangle$ is an IC merging operator (it satisfies (IC0–IC8)), then the operator $\diamond$ defined as $\varphi \diamond \mu = \triangle_{\varphi \vee \mu}(\varphi \sqcup \mu)$ satisfies (LS1–LS8) if and only if $\triangle$ satisfies property (6.8).

As we know $\triangle^\Sigma$ and $\triangle^{GMax}$ families are defined from a distance, so in particular the revision operators associated to them are also defined from a distance. Therefore, by Remark 6.9 operators $\diamond$ defined from the $\triangle^\Sigma$ and $\triangle^{GMax}$ families satisfy all the Liberatore and Schaerf's postulates.

It is interesting to note that property (6.8) implies the following property.

LEMMA 6.13
Property (6.8) implies $\triangle_\mu(\varphi) = \triangle_\mu(\triangle_\varphi(\mu))$.

PROOF. Consider $\varphi' = \min(mod(\varphi), \leq_\mu)$, that is $\varphi' = \triangle_\varphi(\mu)$. And define φ'' as $mod(\varphi'') = \{I \models \varphi \text{ and } I \not\models \varphi'\}$. From property (6.8) we get $\min(mod(\mu), \leq_{\varphi' \vee \varphi''}) = min(mod(\mu), \leq_{\varphi'})$, since by hypothesis $\varphi' <_\mu \varphi''$. So we have $\min(mod(\mu), \leq_\varphi) = min(mod(\mu), \leq_{\varphi'})$. Thus $\triangle_\mu(\varphi) = \triangle_\mu(\varphi')$, and from the definition of φ' this is $\triangle_\mu(\varphi) = \triangle_\mu(\triangle_\varphi(\mu))$. ∎

This property is maybe not very natural but it is very expressive in the framework of belief revision, since the corresponding revision operator satisfies

$$\varphi \circ \mu = (\mu \circ \varphi) \circ \mu. \tag{6.9}$$

That is to say that the result of the revision of φ by μ depends only on the models of φ that are the closest to μ.

A serious drawback of commutative revision definition is that it does not allow one to merge more than two belief bases since it is not associative (see [24, 25]), but the idea that the result of the merging has to imply the disjunction of the belief bases can be very useful in a lot of applications. So in order to generalize Liberatore and Schaerf operators to n belief bases, we could then define the merging of a belief set $\{\varphi_1 \sqcup \ldots \sqcup \varphi_n\}$ as

$$\triangle_{\varphi_1 \vee \ldots \vee \varphi_n}(\varphi_1 \sqcup \ldots \sqcup \varphi_n).$$

6.3 *Lin and Mendelzon majority merging operator*

Lin and Mendelzon have defined a kind of merging operator, denoted $\blacktriangle$, that they call a majority merging operator [28, 26].

The postulates given by Lin and Mendelzon for these operators are:

(LM1) $\blacktriangle(\Psi)$ is consistent.

(LM2) If $\bigwedge \Psi$ is consistent then $\blacktriangle(\Psi) = \bigwedge \Psi$.

(LM3) If $\Psi \leftrightarrow \Psi'$, then $\blacktriangle(\Psi) \leftrightarrow \blacktriangle(\Psi')$.

(LM4) For a literal sentence l, if $|\{\varphi_i \in \Psi : \varphi_i \models l\}| > |\{\varphi_i \in \Psi : \varphi_i \models \neg l\}| + |\{\varphi_i \in \Psi : \varphi_i \triangleright \neg l\}|$ then $\blacktriangle(\Psi)$ implies l.

$\varphi_i \triangleright \neg l$ means that the belief base φ_i *partially supports* $\neg l$, that is, there exists a β, which mentions no atom appearing in $\neg l$, such that $\varphi_i \models \neg l \vee \beta$ but $\varphi_i \not\models \neg l$ and $\varphi_i \not\models \beta$. Whereas $\varphi_i \models \neg l$ is called a *full support* of $\neg l$. Then the postulate (LM4) simply expresses the idea of a vote for or against l, that is l is 'elected' if there is more explicit supports to l than explicit and implicit supports to $\neg l$. It is necessary to take into account implicit supports in order to avoid some incoherent answers (see [28] for full justification).

Lin and Mendelzon provide an example of operator satisfying these properties. It is an $\triangle^{\Sigma}$ operator with the Dalal's distance as chosen distance. They give in particular an interesting definition of this operator when the belief bases are expressed in Disjunctive Normal Form.

We can note that postulates (LM1), (LM2) and (LM3) correspond respectively to postulates (A1), (A2) and (A3) for pure merging. Note also that if an operator satisfies (LM4) then it satisfies a weak version of (M7), i.e. at the level of literals. So a Lin and Mendelzon operator is a sort of weak pure majority merging operator, provided that it has some good properties (i.e. it satisfies (A4–A6)).

6.4 *Revesz model-fitting operators*

Revesz proposes in [31, 32] a kind of operators he called model-fitting operators that are very close to IC merging operators. One important difference between his approach and ours is the notion of belief set and the notion of equivalence of belief sets. He defines equivalence as follows:

DEFINITION 6.14
Let Ψ_1, Ψ_2 be two belief sets. $\Psi_1 \Rightarrow \Psi_2$ iff $\forall \varphi_2 \in \Psi_2\ \exists \varphi_1 \in \Psi_1$ s.t. $\varphi_2 \leftrightarrow \varphi_1$. $\Psi_1 \Leftrightarrow \Psi_2$ iff $\Psi_1 \Rightarrow \Psi_2$ and $\Psi_2 \Rightarrow \Psi_1$.

In particular for Revesz all belief set can be reduced to a set of formulas. That is false in our approach.

Let Ψ be a belief set and μ be a belief base, $\Psi \triangleright \mu$ denotes the belief base result of the model-fitting of Ψ by μ. The postulates he gives for these operators are:

(M1) $\Psi \triangleright \mu$ implies μ.

(M2) If $\Psi \wedge \mu$ is consistent, then $\Psi \triangleright \mu \leftrightarrow \Psi \wedge \mu$.

(M3) If μ is consistent, then $\Psi \triangleright \mu$ is consistent.

(M4) If $\Psi_1 \Leftrightarrow \Psi_2$ and $\mu \leftrightarrow \phi$, then $\Psi_1 \triangleright \mu \leftrightarrow \Psi_2 \triangleright \phi$.

(M5) $(\Psi \triangleright \mu) \wedge \phi$ implies $\Psi \triangleright (\mu \wedge \phi)$.

(M6) If $(\Psi \triangleright \mu) \wedge \phi$ is consistent then $\Psi \triangleright (\mu \wedge \phi)$ implies $(\Psi \triangleright \mu) \wedge \phi$.

(M7) $(\Psi_1 \triangleright \mu) \wedge (\Psi_2 \triangleright \mu)$ implies $(\Psi_1 \cup \Psi_2 \triangleright \mu)$.

Revesz provides also a representation theorem for these operators, see [31, 32] for more details about this theorem and the justifications of the postulates. Note the following:

THEOREM 6.15
Postulate (M4) is equivalent to postulates (IC3) and (MI).

Then it is straightforward that Revesz's postulates correspond directly to the following set of postulates: (IC0–IC3),(IC5),(IC7),(IC8) and (MI). Remember that the set of postulates (IC0–IC8) and (MI) is inconsistent (Theorem 3.3), so Revesz postulates are inconsistent with IC merging postulates. We have also show in Section 4.2 that the set of postulates (IC0–IC5),(IC7),(IC8),(MI) is consistent, model-fitting operators are a subset of this set of postulates (all but (IC4)). In particular note that $\triangle^{Max}$ family operators are model-fitting operators.

7 Conclusion

In this paper we have presented a logical framework for belief base merging in the presence of integrity constraints when there is no preference over the belief bases. We stated a set of properties an IC merging operator should satisfy in order to have a rational behaviour. This set of properties can then be used to classify particular merging methods. In particular, we have made a distinction between arbitration and majority operators, arbitration operators striving to minimize individual dissatisfaction and majority operators trying to minimize global dissatisfaction.

We have provided a model-theoretic characterization for IC merging operators. This characterization is much more natural than the one in [22], due to the presence of integrity constraints. Especially, we have defined three families of merging operators that illustrate the logical characterization. We defined, in particular, the $\triangle^{GMax}$ family that is, in the logical context, a new merging method. The difference of behaviour between these three families is shown on a concrete example.

We have studied the connections of this work with previous ones. First we show that this work consistently extends the work of [22] that merges belief bases in the pure case, i.e. without integrity constraints. We have also shown the strong connection between belief revision operators and IC merging operators. In particular our IC merging operators are a generalisation of AGM revision operators. An interesting result is that we can use revision operators for building merging operators if and only if the revision operator is defined from a distance. Concerning the Liberatore and Schaerf's commutative revision operators we have shown that they can be seen as a special case of IC merging operators. We have made a brief comparison between our proposal and the Lin and Mendelzon's one for theory merging by majority and the Revesz's one for model-fitting operators.

In many situations, in a committee, all the protagonists do not have the same weight on the final decision, so one generally needs to weight each belief base to reflect this. The idea behind weights is that the higher weight a belief base has, the more important it is. If the belief bases reflect the view of several people, weights could represent, for example, the cardinality of each group. We want to characterize logically the use of these weights. Majority operators are close to this idea of weighted operators since they allow one to take cardinalities into account. But a more subtle treatment of weights in merging is still to be done, in particular the notion of weighted arbitration operators is missing.

Another point of interest is to study practical merging operators, in particular syntactical merging operators, i.e. operators that do not satisfy (IC3), in order to find which is the best compromise between logical properties and calculability. In [21] merging operators that adopt a coherence approach to theory merging are studied. These operators are based on a union of all the belief bases and on the selection of some maximal subsets due to a given order (not necessarily the inclusion). Once the union of the belief bases is settled, the problem becomes one of finding a coherent belief base from an inconsistent one [9, 6, 7]. A lot of work on belief base merging adopts this coherence approach [3, 4]. An important drawback of coherence merging operators is that the source of each belief is lost in the merging process. So the problem is to take into account the source of each piece of information in order to allow subtler behaviours for merging operators, for example define majority or arbitration operators.

We would like to conclude with two open questions. The first one concerns distances. Notice that all examples of IC merging operators we have given are built from a distance. The question that naturally arise is the following: Are all the IC merging operators built from a distance?

The second question concerns the relationships between the postulates (Maj) and (Arb). It is not hard to see that if we consider only one propositional variable the operators $\triangle^{\Sigma}$ and $\triangle^{GMax}$ are equal. Indeed, we know that (Maj) and (Arb) together are consistent with IC merging postulates. Actually, if we fix the cardinality of the language, we can build a generalization of the $\triangle^{\Sigma}$ operators [2] that is both an arbitration and majority operator. So the question is to find if there is another property that captures the arbitration behaviour and such that the class of operators satisfying this property is disjoint of the class of IC majority merging operators.

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