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Merits and Limitations of Optimality Criteria Method for Structural Optimization

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Summary

The merits and limitations of the optimality criteria (OC) method for the minimum weight design of structures subjected to multiple load conditions under stress, displacement, and frequency constraints were investigated by examining several numerical examples. The examples were solved utilizing the OC Design Code that was developed for this purpose at NASA Lewis Research Center. This OC code incorporates OC methods available in the literature with generalizations for stress constraints, fully utilized design concepts, and hybrid methods that combine both techniques. It includes multiple choices for Lagrange multiplier and design variable update methods, design strategies for several constraint types, variable linking, displacement and integrated force method analyzers, and analytical and numerical sensitivities. On the basis of the examples solved, this method was found to be satisfactory for problems with few active constraints or with small numbers of design variables. The derivable OC method without stress constraints was found satisfactory even for large structural systems. For problems with large numbers of behavior constraints and design variables, the method appears to follow a subset of active constraints that can result in a heavier design. The computational efficiency of OC methods appears to be similar to some mathematical programming techniques.

I. Introduction

Three procedures are currently available for automated structural design: (1) the fully utilized design (FUD) (refs. 1 and 2), (2) the optimality criteria method (OC) (refs. 3 to 5), and (3) the mathematical programming techniques of operations research (MP) (refs. 6 to 9). The FUD, because of its simplicity, is popular in industry despite the availability of the other two design methodologies. FUD may, however, exhibit some limitations, especially for stiffness and dynamic constraints (ref. 6). To alleviate deficiencies associated with FUD, the structural design problem was formulated and solved during the 1960's as a nonlinear mathematical programming optimization problem. Structural mass was used as a typical objective and failure modes as the constraints (ref. 6). This last approach is conceptually sound and mathematically elegant, but its solution can be computationally prohibitive. At about the same time, a relatively simpler design technique, termed the optimality criteria method and based on a Lagrange multiplier approach, was introduced to solve certain types of structural optimization problems (refs. 3 to 5).

Theoretical aspects of the optimality criteria method have already been given (refs. 3 and 4). In this report the merits and limitations of this method, which may provide a practical design tool, are considered by examining several numerical examples. The examples are solved using the optimality criteria design code that was developed to assess the performance of this design technique.

The optimization section of the OC code contains numerous choices for optimality criteria update formulas, including most of the formulas in references 3 and 4, along with fully utilized design rules. The optimality criteria method for structural optimization was originally derived (refs. 3 and 4) for displacement buckling and frequency constraints. Stress and frequency constraints, however, are also included in the OC code, because such constraints are relevant to structural design. The analysis segment of the code includes a choice of three finite element analysis methods: (1) the displacement (or stiffness) method, (2) the integrated force method, and (3) a simplification of the integrated force method. The OC code has options to calculate design sensitivities of the behavior constraints either in closed form or using numerical differentiations. To compare results obtained by the optimality criteria methods with those of other structural optimizers, we integrated the OC code into the Comparative Evaluation Test Bed of Optimization and Analysis Routines for the Design of Structures (CometBoards), an optimal structural design test bed under development. CometBoards offers several choices for optimizers and analyzers.

The subject matter of this report is presented in the subsequent five sections. In section II, the theoretical aspects of the optimality criteria method are introduced. In section III, the structure of the OC code software is briefly discussed. In section IV, several numerical examples are provided. The issues associated with the optimality criteria method, as well as merits and limitations of the three analysis methods, are examined and illustrated in section V. The conclusions are given in section VI. Lists of symbols and of acronyms and initialisms are provided in appendixes A and B, respectively.

II. Theoretical Bases of Design Methods

The three automated design procedures—(1) optimization using mathematical programming techniques, (2) the optimality criteria method, and (3) fully utilized design concepts—are briefly examined and compared.

A. Mathematical Programming Techniques

In mathematical programming techniques, structural design is cast as the following optimization problem:

Find the n design variables $\vec{\chi}$ within prescribed upper and lower bounds $\left(\chi_i^{LB} \le \chi_i \le \chi_i^{UB},\ i = 1,2,\ldots,n\right)$ that make the scalar objective function $f(\vec{\chi})$ (here, weight) an extremum (minimum) subject to a set of m inequality constraints, usually defined as the failure modes of the design problem:

$$g_j(\vec{\chi}) \le 0 \qquad (j = 1,2,\ldots,m) \tag{1}$$

The constraints for structural design applications are typically nonlinear in the variables $\vec{\chi}$, thus it becomes a nonlinear programming problem. Note, here, that equality constraints could also be included.

This report considers stress, displacement, and frequency constraints (g_j) under multiple load conditions. For each load condition, the stress constraints are specified by

$$g_j = \left|\frac{\sigma_j}{\sigma_{j0}}\right| - 1.0 \le 0 \tag{2}$$

where σ_j is the design stress for the jth element and σ_{j0} is the permissible stress for the jth element. For each load condition, the displacement constraints are specified by

$$g_{js+j} = \left|\frac{u_j}{u_{j0}}\right| - 1.0 \le 0 \tag{3}$$

where u_j is the jth displacement component, u_{j0} is the displacement limitation for the jth displacement component, and js is the total number of stress constraints. Constraints on frequencies are specified by

$$g_f = \left(\frac{f_{n0}}{f_n}\right)^2 - 1.0 \le 0 \tag{4}$$

where f_n represents natural frequencies of the structure and f_{n0} the limitations on these frequencies.

The optimal design $\vec{\chi}^{\text{opt}}$ in a mathematical programming technique is obtained iteratively from an initial design $\vec{\chi}^0$ in, say, K design updates. At each iteration the design is updated by calculating two quantities: a direction $\vec{\varphi}$ and a step length α. The optimal design process, utilizing the direction and associated step length, can be symbolized as

$$\vec{\chi}^{\text{opt}} = \vec{\chi}^0 + \sum_{k=1}^{K} \alpha_k \vec{\phi}^k \tag{5}$$

where $\vec{\varphi}^k$ is the direction vector at the kth iteration and α_k is the step length along the direction vector.

The direction vector at the kth iteration is generated from the gradients of the objective function and the active constraint subset following one of the many available direction-generation algorithms (refs. 10 and 11). Along the direction vector $\vec{\varphi}^k$, a one-dimensional search is carried out to obtain the step length α_k, again utilizing one of several available procedures (refs. 12 and 13). The updated design is then checked against one or more stop criteria (ref. 1) until it converges. The details of the nonlinear mathematical programming techniques, well documented in the literature, are not elaborated here (refs. 10 to 15).

B. Design Cast in a Lagrange Multiplier Formulation

The optimality criteria method is an alternative design tool for solving the optimization problem given by equations (1) to (4). This method can be considered to be a variant of the Lagrange multiplier approach, which has been specialized for structural design applications. We consider next the Lagrange multiplier method for solving the optimization problem.

The Lagrange multiplier approach adjoins the constraints to the objective function, and the Lagrangian is formed:

$$\mathcal{L}\left(\vec{\chi},\vec{\lambda}\right) = f(\vec{\chi}) + \sum_{\text{active set}} \lambda_j g_j^*(\vec{\chi}) \tag{6}$$

where the superscript asterisk indicates those constraints within the active set.

The passive constraints do not influence the design, and associated Lagrange multipliers are zero. The independent variables associated with the Lagrangian function (eq. (6)) are the design variables and multipliers associated with the active constraints. These two sets of unknowns, the design $\vec{\chi}$ and the multipliers $\vec{\lambda}$, in principle can be obtained from the stationary conditions of the Lagrangian, $\mathcal{L}(\vec{\chi},\vec{\lambda})$. The stationary condition of the Lagrangian with respect to the design variables yields

$$\nabla f(\vec{\chi}) + \sum_{\text{active set}} \lambda_j \nabla g_j^*(\vec{\chi}) = \vec{0} \tag{7}$$

Likewise, the stationary condition with respect to the multipliers yields the active constraints

$$g_j^*(\vec{\chi}) = 0 \quad \left(g_j^* \text{ within the active set}\right) \tag{8}$$

Simultaneous solution of equations (7) and (8) would yield the solution to the optimization problem. For structural design problems, this direct approach can be computationally intensive and may suffer from numerical instability.

The optimality criteria method bypasses direct Lagrangian solution. Instead, the intrinsic nature of the design problem has been exploited, and the process has produced several procedures. The derivation of an optimality criteria method for minimum weight optimization of structures in the context of displacement constraints is illustrated in appendix C.

Before the optimality criteria methods are presented, the relation between the number of design variables and the number of active constraints, which plays an important role in the optimization process, is examined. The three possible relationships between the number of design variables (NDV) and the number of active constraints (NAC) of the optimal solution are illustrated by considering a three-bar truss (fig. 1). The bar areas are taken to be the three design variables. The truss is subjected to two load conditions, which give rise to 6 stress, 4 displacement, and 1 frequency constraint, or a total of 11 inequality constraints.

C. Relationship Between Number of Active Constraints and Number of Design Variables

The three cases, (1) NAC greater than NDV, (2) NAC equal to NDV, and (3) NAC less than NDV, are examined separately. In the discussion of these relationships that follows, functional independence among the active constraints is assumed. Note, however, that functional dependence among the behavior constraints of structural systems can occur and may require special consideration as elaborated in reference 16.

1. When the number of active constraints is greater than the number of design variables (NAC > NDV).—Consider the design space of a problem with n design variables and $n + \nu$ active constraints at the optimum. The optimal solution, a point in that design space, can be specified as the intersection of any n of the $(n-1)$-dimensional surfaces defined by the active constraints, assuming that the active constraints are locally well behaved and functionally independent. In a more general sense, if there exists a subset, of any size, of active constraints the intersection of whose surfaces form the optimal solution point, then there exists a set of, at most, n of those surfaces whose intersection is sufficient to define the optimal point. The remaining ν constraints, which pass through the optimal point, may be termed "follower constraints." Such an occurrence is illustrated in figure 2(a).

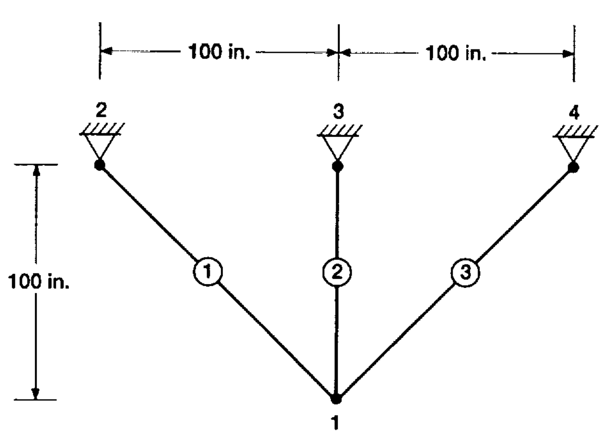


Figure 1.—Three-bar truss. (Elements are circled, nodes are not.)

For a three-bar truss with three design variables, consider an optimal design that has five active, functionally independent constraints, g_3, g_5, g_6, g_9, and g_{11}, as depicted in figure 2(a). Any set of three active constraints is sufficient to establish the optimal design (for example g_3, g_5, and g_6, as in fig. 2(b)). The two remaining active constraints (here, g_9 and g_{11}) are the follower constraints and need not be considered in optimization calculations. In summary, from geometrical considerations, the inclusion of a maximum of n active constraints in the optimization process is sufficient to generate the optimal design vector of dimension n. By considering only as many active constraints as there are design variables, we reduce the complexity of a given optimal design problem.

2. When the number of active constraints equals the number of design variables (NAC = NDV).—The solution of an optimal design problem in which there are n design variables and the same number of active constraints (i.e., NDV = NAC = n) is, by definition, a fully utilized design. The stationary conditions of the Lagrangian

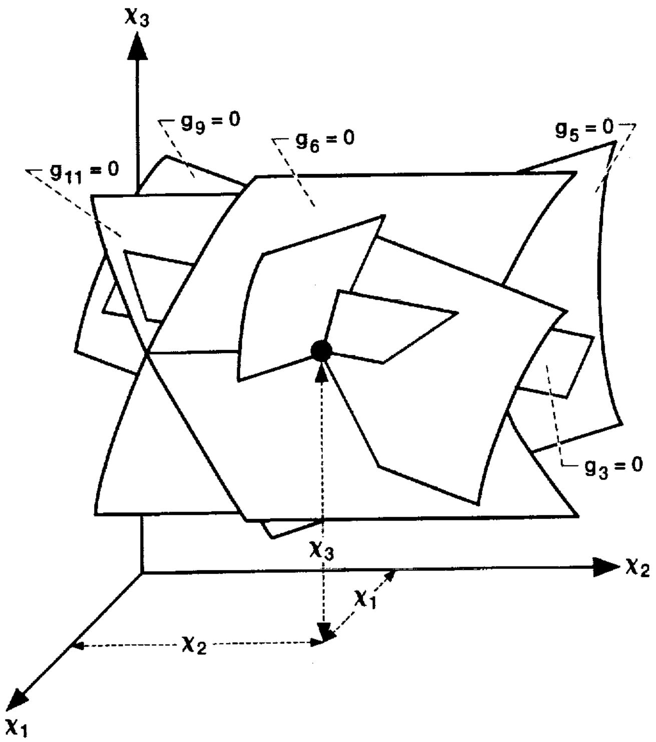
$$\mathcal{L}(\vec{\chi},\vec{\lambda}) = f(\vec{\chi}) + \sum_{j=1}^{n} \lambda_j g_j(\vec{\chi})$$

for this situation yield the following equations:

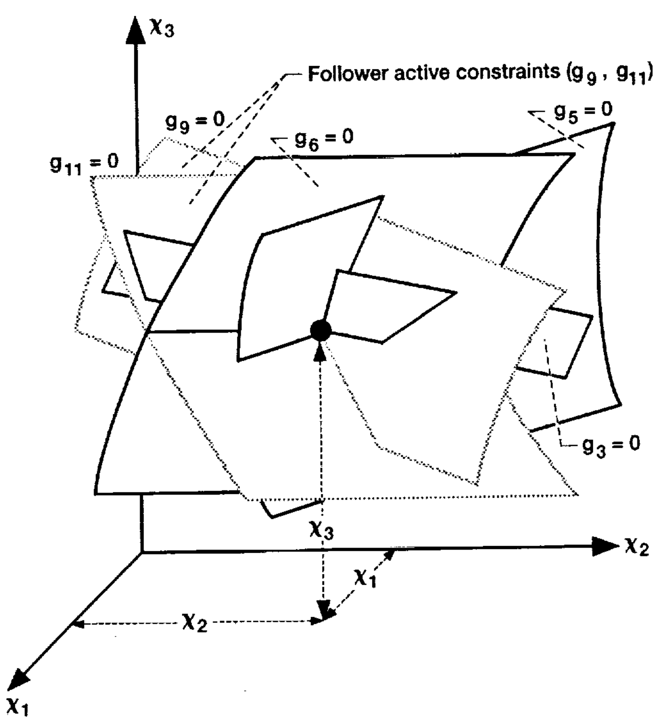
$$\nabla f(\vec{\chi}) + \sum_{j=1}^{n} \lambda_j \nabla g_j^*(\vec{\chi}) = \vec{0} \tag{9a}$$

$$g_j^*(\vec{\chi}) = 0 \qquad (j = 1, 2, \ldots, n) \tag{9b}$$

Equations (9a) and (9b) together represent $2n$ equations in $2n$ unknowns: n design variables $\{\chi_i\}$ and n Lagrange multipliers $\{\lambda_i\}$. The stationary condition of the Lagrangian with respect to the design variables (eq. (9a)) yields n equations in $2n$ unknowns (i.e., n Lagrange multipliers and n design variables). The stationary condition of the Lagrangian with respect to the multipliers reproduces the n constraint equations in n unknowns (eq. (9b)), which are independent of the Lagrange multipliers. The set of n constraint equations (eq.(9b)) alone is sufficient to generate the n design variables, if we assume that they are functionally independent. Once the design variables are known, equation (9a) can be used to calculate the Lagrange



(a) Five active constraints within the design space of a three-bar truss.



(b) Three active constraints, which are sufficient to determine the optimal point, and two follower constraints.

Figure 2.—Optimum design point.

multipliers. It is important to note that the objective function does not participate in this calculation of the design. In the case of the three-bar truss example, if exactly three constraints are active, then simultaneously solving the three constraint equations (eq. (9a)) would produce both the optimal and the fully utilized design. Equation (9a) can then be used to calculate Lagrange multipliers, if desired.

In summary, when an optimal design has functionally independent active constraints that equal or exceed the number of design variables, then the design represents both a fully utilized design and an optimal design.

3. ***When the number of active constraints is less than the number of design variables (NAC< NDV).***—If the number of active constraints in the optimal design is less than the number of design variables, the design is not fully utilized. Some features of this case can be illustrated by the three-bar truss. Let us assume that, at optimum, two constraints, g_1 and g_2, are active. The stationary conditions of the Lagrangian yield the following equations:

$$\nabla f(\vec{\chi}) + \lambda_1 \nabla g_1^*(\vec{\chi}) + \lambda_2 \nabla g_2^*(\vec{\chi}) = \vec{0} \tag{10a}$$

$$g_j^*(\vec{\chi}) = 0 \qquad (j = 1, 2, \ldots, n) \tag{10b}$$

Equations (10a) and (10b) represent five equations in five unknowns (three design variables, χ_1, χ_2, and χ_3, and two Lagrange multipliers, λ_1 and λ_2). The stationary condition of the Lagrangian with respect to the design variables (eq. (10a)) represents three equations in five unknowns (the two Lagrange multipliers and the three design variables). The two constraint equations given in equation (10b) represent two equations in three unknowns (the three design variables). Although these last two equations are independent of the Lagrange multipliers, they are insufficient in quantity to solve for the three design variables. Thus, all five equations (eqs. (10a) and (10b) are coupled in the design variables and the multipliers, and the entire set must be solved simultaneously to generate the solution. The gradient of the objective function and the Lagrange multipliers do participate in the calculation of the design variables because of the intrinsic coupling between equations (10a) and (10b). In other words, only when the number of active constraints is fewer than the number of design variables do both the constraints and the objective function participate in the optimization, this situation is the one most frequently encountered.

D. Optimality Criteria Method

The optimality criteria method provides several procedures to iteratively update both the design variables and the Lagrange multipliers. The update rules described below include modifications and generalizations of the original formulas given in references 3 and 4 for stress, displacement, and frequency constraints. The derivation of one rule is illustrated in appendix C.

The active constraint set is estimated through a constraint thickness parameter t_k that defines a finite interval $[-t_k, 0]$ within which all constraints are considered active. This constraint thickness becomes progressively tighter at each successive iteration, ultimately reducing to zero. The constraint thickness update rule used is

$$t_k = \tau^{k-1}\, t_0 \qquad (k = 1, 2, \ldots, K) \tag{11}$$

where t_k is the constraint thickness at the kth iteration, t_0 is the initial prescribed constraint thickness, $\tau(<1)$ is the factor by which the constraint thickness is reduced at each iteration, and K represents the number of iterations required to reach a solution. All violated constraints whose values are greater than zero are also included in the "active" constraint set.

1. ***Lagrange multiplier update formulas.***—Utilizing equations (eq. (7) and (8)), references 3 and 4 give several update rules for the Lagrange multipliers, two of which are included here. Three additional update methods are also discussed. Refer to appendix D for a summary of the update formulas.

a. The linear form: The linear form of the update formula for the Lagrange multipliers is obtained by manipulating the constraint equation and has the following form:

$$\lambda_{ja}^{k+1} = \lambda_{ja}^{k}\left[1.0 + \alpha^{k} p_0 (C_{ja} - C_{ja}^{*})\right] \tag{12}$$

where λ_{ja}^{k} is the value of the Lagrange multiplier associated with the jath "active" constraint at the kth iteration, C_{ja} is the actual value of the displacement at a particular node, or the stress in a particular element, and C_{ja}^{*} is the corresponding permissible value of that displacement or stress (i.e., $C_{ja} = (g_{ja} + 1)C_{ja}^{*}$). For frequency constraints, C_{ja} is taken to be the square of the inverse of the natural frequency, $1/f_n^2$, and C_{ja}^{*} is $1/f_{n0}^2$. Also in equation (12), p_0 is the initial value of an update parameter, and α is an acceleration parameter used to modify that update parameter. The value of p_0 is typically 0.5, and α is often taken as 1.0.

Only those Lagrange multipliers are updated which are associated with active constraints. Active constraints are defined on the basis of the thickness parameter given by equation (11) (i.e., λ_{ja}, where ja is such that $g_{ja} \geq -t_k$). However, all Lagrange multipliers may, in some cases, be updated, regardless of the corresponding constraint value. Initial values for the Lagrange multipliers are required to begin iterations.

b. The exponential form: The exponential form is also obtained from the constraint equation and has the following form:

$$\lambda_{ja}^{k+1} = \lambda_{ja}^{k}\left(\frac{C_{ja}}{C_{ja}^{*}}\right)^{\alpha^{k} p_0} \tag{13}$$

The optimality criteria formulas (eqs. (12) and (13)), do not use weight in calculating multipliers.

c. The Lagrange inverse forms: The multipliers in these methods are obtained from equation (7) by premultiplying it with the transpose of the gradient matrix $\nabla\vec{g}^{*}$ and then solving for the Lagrange multipliers:

$$\vec{\lambda}^{*} = \left[\left[\nabla\vec{g}^{*}\right]^{T}\left[\nabla\vec{g}^{*}\right]\right]^{-1}\left[\nabla\vec{g}^{*}\right]^{T}\nabla f \tag{14}$$

where $\vec{\lambda}^{*}$ is the vector of Lagrange multipliers associated with the "active" constraints, ∇f is the gradient of the objective function, and the superscript asterisk represents the fact that only active constraints are included in the sensitivity matrix. If the number of active constraints m exceeds the number of design variables n, then the number of columns m in $\nabla\vec{g}^{*}$ will be larger than the number of rows (n; $m > n$), making the $m \times m$ coefficient matrix $[[\nabla\vec{g}^{*}]^{T}[\nabla\vec{g}^{*}]]$ singular. As mentioned earlier, however, a maximum of n active constraints is sufficient to establish an optimal solution in an n-dimensional design space. Furthermore, except for functional dependence among active constraints, restricting attention to any n of the active constraints at optimum should yield the optimal solution. With this Lagrange inverse form of the multiplier update method, no more than n active constraints are considered at any given time.

Two more Lagrange multiplier update formulas can be derived—the unrestricted and the diagonalized inverse forms. In the unrestricted Lagrange inverse form, equation (14) is used, but the number of active constraints considered is allowed to exceed the number of design variables. This form is included to examine the difficulties that may arise from singularities whenever the number of active constraints exceeds the number of design variables.

The diagonalized Lagrange inverse form is obtained by simplifying equation (14) (i.e., only diagonal entries in the matrix $[\nabla\vec{g}^{*}]^{T}[\nabla\vec{g}^{*}]$ are taken). The coupling terms in the sensitivity matrix are neglected. This update rule becomes

$$\lambda_{ja} = \frac{\left[\nabla g_{ja}\right]^{T}\nabla f}{\left[\nabla g_{ja}\right]^{T}\nabla g_{ja}} \tag{15}$$

2. ***Design variable update formulas.***—The optimality criteria update formulas utilize a vector $\vec{D}$ of dimension n. The vector $\vec{D}$ is calculated from the stationary condition (eq. (9)) by scaling, component by component, with respect to the gradient of the objective function and has the following form:

$$D_i = -\frac{\sum_{ja} \lambda_{ja}\left(\nabla g_{ja}\right)_i}{\nabla f_i} \qquad (i = 1, 2, \ldots, n) \tag{16}$$

where $\sum \lambda_{ja}\left(\nabla g_{ja}\right)_i$ represents the ith component of the summation of the product of the gradients of the active constraints and the Lagrange multipliers, and ∇f_i is the ith component of the gradient of the objective function. The stationary condition of the Lagrangian with respect to the design variables (eq. (9)), in terms of the vector $\vec{D}$, can be represented as

$$D_i = 1.0 \qquad (i = 1, 2, \ldots, n) \tag{17}$$

Design updates that are based on equation (17) (refs. 3 and 4) are given next. Refer to appendix D for a summary of the design update formulas.

a. The exponential form: The exponential form of the design variable update method is as follows:

$$\chi_i^{k+1} = \chi_i^k D_i^{1/(\beta^k q_0)} \tag{18}$$

where q_0 and β are acceleration parameters. Typical values of q_0 and β are 2.0 and 1.0, respectively.

b. The linearized form: The linearized form of the design variable update method can be considered to be a truncated Taylor expansion of the exponential form (eq. (18)) and is given by

$$\chi_i^{k+1} = \chi_i^k \left[1.0 + \frac{1}{\beta^k q_0}\left(D_i - 1.0\right)\right] \tag{19}$$

c. The reciprocal form: A Taylor expansion of the inverse of the exponential form (eq. (18)) yields the reciprocal form:

$$\chi_i^{k+1} = \frac{\chi_i^k}{1.0 - \frac{1}{\beta^k q_0}\left(D_i - 1.0\right)} \tag{20}$$

d. The melange form: These rather complicated formulas to update the design variables are an attempt to combine influences of several factors simultaneously: active constraint types, the rescaling vector $\vec{D}$, and factors to stabilize convergence. The design variable is updated in two steps. The initial step, represented by $k + \frac{1}{2}$, is

$$\chi_i^{k+1/2} = a\chi_i^k + \frac{b}{njad}\sum_{jad=1}^{njad} \chi_i^k\left(1.0 + g_{jad}\right) + \frac{c}{njaf}\sum_{jaf=1}^{njaf} \chi_i^k\left(1.0 + g_{jaf}\right) + \mathrm{d}\,\chi_i^k D_i \tag{21a}$$

The final step is

$$\chi_i^{k+1} = \begin{cases} \frac{1}{2}\left[\chi_i^{k+1/2} + \chi_i^k\left(1.0 + g_{is}\right)\right] & \text{if } \chi_i \in \left\{\chi_i \middle| g_{is} \geq -t_k\right\} \\ \chi_i^{k+1/2} & \text{if } \chi_i \notin \left\{\chi_i \middle| g_{is} \geq -t_k\right\} \end{cases} \tag{21b}$$

In equation (21a), the coefficients, a, b, c, and d depend on active constraint types (stress, displacement, or frequency), as well as certain prescribed parameters. The coefficients, however, always sum to unity. Here, g_{jad} and g_{jaf} are constraint values for specific "active" displacement and frequency constraints, respectively; njad and njaf are the number of each of these types of constraints in the "active" set; and $\chi_i^{k+1/2}$ is an intermediate design value that may be modified within a given iteration. Only active constraints are considered in this update formula.

The second term in equation (21a) shows a sum over the displacement constraints included in the "active" set. If there are none, this term is dropped. The factor, $1 + g$, is the scaling factor that will move the design just inside the feasible domain with respect to any one displacement constraint. Note that the largest such factor is the one used in the fully utilized design methodology described later. Although stress constraints are closely linked to the cross-sectional area of the element corresponding to each constraint, displacements are global responses and can be significantly influenced by any or all design variables. The third term shows a sum over all frequency constraints included in the "active" set and also represents a global response. The last term uses the rescaling vector $\vec{D}$, with an exponent of one, and the first term retains the old (prior iterate) value of the design variable.

Any design variable associated with an element whose corresponding stress constraint is included in the "active" set is further modified by taking the arithmetic mean of the value obtained in equation (21a) with the value of that design variable that would be obtained by a fully stressed (fully utilized) design technique (see eq. (21b)). Here, g_{is} is the value of the stress constraint corresponding to the ith element. Note that only one element is influenced by any one active stress constraint. All design variables not associated with any such element retain their values given in equation (21a).

*3. **Hybrid design variable update methods.***—The hybrid methods represent an augmentation of the optimality criteria design update rules to the fully utilized design concept. These three methods were devised in an attempt to retain the benefits of the fully stressed design for stresses with the optimality criteria method for displacements and frequencies. For these methods, the rescaling vector $\vec{D}$, given by equation (16), is calculated for active displacement and active frequency constraints only. To initiate the design variable update, each method uses a corresponding optimality criteria update formula (eq. (18), (19), or (20)) to obtain an intermediate design. Next, an alternative intermediate design is obtained with a fully stressed design for the stress constraints.

$$\chi_i^{k+1/2} = \chi_i^k \left(1.0 + g_{is}\right) \tag{22}$$

Whenever more than one stress constraint is associated with any given design variable (e.g., with multiple load conditions, or linking of design variables), the largest of the scaling factors $(1 + g)$ is used. Finally, the two intermediate designs are compared, one design variable at a time, and the largest is chosen. These three design variable update methods are termed (1) the hybrid exponential form, which incorporates the exponential form of the optimality criteria design update formula (eq. (18)); (2) the hybrid linearized form, which uses the OC linearized form (eq. (19)); and (3) the hybrid reciprocal form, which uses the OC reciprocal form (eq. (20)).

E. Fully Utilized Design Approach

The fully utilized design is obtained in two steps. First, a fully stressed design, designated $\vec{\chi}^{\sigma,\text{opt}}$, is obtained by using the stress ratio technique. The design is then rescaled for violated displacement and/or frequency constraints, if any. The two steps are—

*1. **Fully stressed design.***—For a fully stressed design, the design variables are updated for stress constraints only, as

$$\chi_i^{\sigma,k+1} = \chi_i^{\sigma,k} \boldsymbol{R}_{\sigma i} \quad (\boldsymbol{i} = 1, 2, \ldots, \boldsymbol{n}) \tag{23}$$

where $\chi_i^{\sigma,k}$ is the ith component of the fully stressed design at the kth iteration, and the scaling factor $\boldsymbol{R}_{\sigma i}$ for the ith design variable is

$$\boldsymbol{R}_{\sigma i} = \frac{\max\left(\sigma_{1i}, \sigma_{2i}, \ldots, \sigma_{Li}\right)}{\sigma_{i0}} \tag{24}$$

where σ_{1i}, σ_{2i}, ..., σ_{Li} represent the stress values for each element associated with χ_i under each load condition. Note that linking of design variables can cause more than one element to be associated with a given design variable for a given load condition (see section III.D). The fully stressed design $\vec{\chi}^{\sigma,\text{opt}}$ is known to converge in a few cycles, irrespective of the number of design variables (ref. 1).

*2. **Fully utilized design.***—The fully utilized design for simultaneous stress, displacement, and frequency constraints is obtained in one additional step by scaling the fully stressed design to satisfy the maximum violated displacement and frequency constraint, if any:

$$\vec{\chi}^{\text{opt}} = \vec{\chi}^{\sigma,\text{opt}}\left(1 + g_{\max}\right) \tag{25}$$

where $g_{\max}$ is the value of the largest violated displacement or frequency constraint.

III. Optimality Criteria Computer Code

The OC computer code is composed of three modules: (1) the optimization module, (2) the analysis module, and (3) the interface module. The code was developed on the basis of existing interfaces to analysis routines and was incorporated into CometBoards, an optimal structural design test bed.

A. Interface Module

The interface module reads the problem specification, such as the geometry, material properties, design limitations, and optimization parameters from free-format keyword-based data files.

Interactive input of some of the optimization parameters is an additional optional feature. This module also initializes variables, calls the optimizer, and prints out the final results.

B. Optimization Module

The optimization module of the OC code was designed to provide considerable flexibility, such as multiple choices of both Lagrange multiplier and design variable update methods and parameters, as well as various strategies for approaching problems with several constraint types.

The core iteration loop of this module proceeds as follows. The thickness for each constraint type (stress, displacement, and frequency) is modified by a constant factor, producing iteration-dependent thicknesses as given by equation (11). From these constraint thicknesses, a set of active constraints is identified and a reduced set of "active" constraints is determined. The reduced set of active constraints is obtained by considering the user-specified (1) minimum and maximum percentage of active constraints identified for each constraint type, (2) minimum and maximum number of active constraints to be used (by constraint type), and (3) overall minimums and maximums. Thus, fewer, or more, constraints than the number of active constraints identified from the constraint thickness criteria may be included in the reduced set for subsequent calculations.

For the reduced set of constraints, the user-specified pair of OC update methods (one for the Lagrange multipliers and another for the design variables) is invoked, using iteration-dependent acceleration parameters. Furthermore, if more than one pair of methods is chosen, a weighted average of the resulting updates is taken. This average is based on the weights also supplied by the user in the input data. The rescaling vector $\vec{D}$ is calculated after the updates to the Lagrange multipliers but prior to updates to the design variables.

At the end of each iteration, the design is normally rescaled to ensure feasibility, unless otherwise specified by the user. This core iteration process continues either until convergence is achieved or the maximum number of iterations is reached.

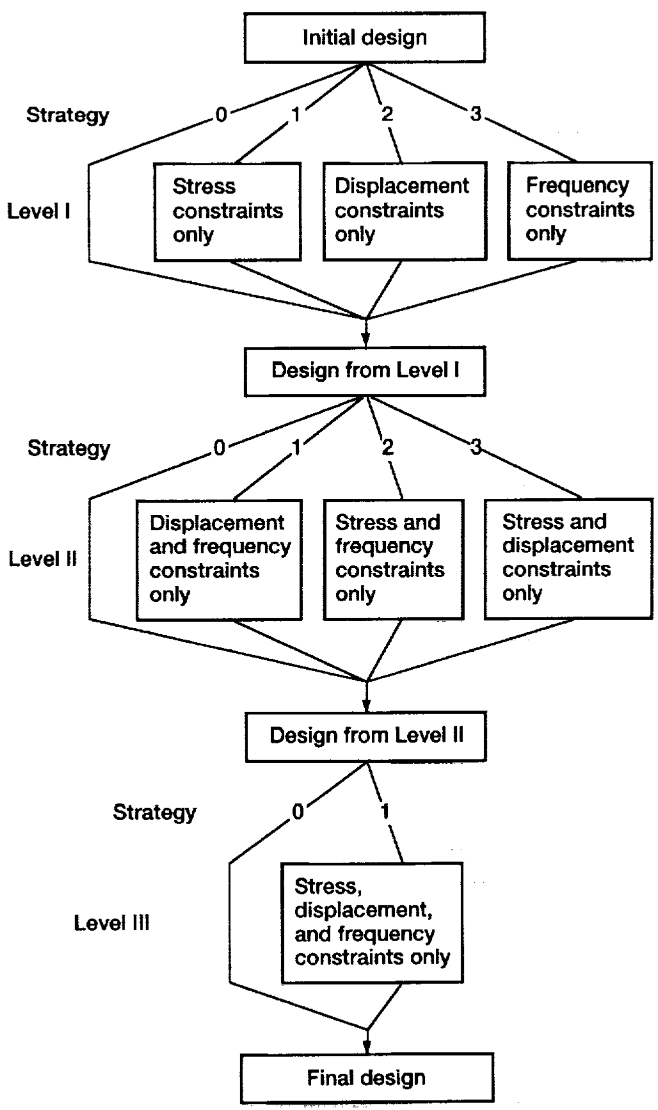


Figure 3.—Optimality criteria code levels and constraint inclusion strategies. (Strategy 0 allows the level to be skipped).

The optimization procedure has three levels, and each level has several choices of strategies, as depicted in figure 3. The strategy at each level is specified by an input parameter.

• Level I considers optimization for one constraint type, such as stress constraints (Strategy I), displacement constraints (Strategy II), or frequency constraints (Strategy III).

• Level II allows simultaneous consideration of two types of constraints: displacement and frequency (Strategy I), stress and frequency (Strategy II), or stress and displacement (Strategy III).

• Level III allows the simultaneous consideration of all three constraint types (i.e., stress, displacement, and frequency).

Any levels may be skipped. For example, one could choose to solve a problem that contained all three types of constraints by skipping the first two levels and considering all constraints simultaneously at Level III. Alternatively, one could use Level I, Strategy I for stress constraints, then utilize these results as an initial design for Level II, Strategy III to include both stress and displacement constraints. Finally, these results could be used to initiate Level III, thus including the frequency constraints. This second alternative is generally recommended.

Besides these three sets of iteration loops, the initial and final design are rescaled to ensure design feasibility. The final design may also be rescaled to ensure that at least one constraint becomes active. In addition, options are available to specify convergence criteria, the maximum iteration limits, and the initial values of the Lagrange multipliers. Furthermore, users can remove restrictions normally placed on the Lagrange multipliers and the design variables during each iteration by (1) allowing infeasibility of the design at most intermediate stages, (2) allowing the Lagrange multipliers to become negative, and (3) allowing the rescaling vector $\vec{D}$ to include negative values.

C. Analysis Modules

The behavior constraints and objective function are generated in the analysis module. The objective function is given simply by the weight of the structure. The structure is analyzed to calculate the behavior constraints following a finite element technique. A choice of three analysis methods are currently available: (1) the displacement (or stiffness) method, (2) the integrated force method, and (3) a simplified integrated force method.

1. Displacement method.—The stress, displacement, and frequency constraints for the structure under single or multiple load conditions are obtained with the displacement (stiffness) method as implemented in the ANALYZ/DANLYZ code developed at Wright-Patterson Air Force Base (ref. 17). It generates displacement constraints at specified nodes and directions and/or specified frequency constraints, along with all stress constraints. The design sensitivity matrix for stress, displacement, and frequency constraints is obtained analytically. However, finite difference sensitivity calculations are also available as an option.

The information passed on to the optimization module from the analysis module consists of the value of the objective function, the gradient of the objective function, the constraint array, and the gradient (or sensitivity matrix) of the constraints. Analysis and sensitivity calculations via the displacement (or stiffness) method have become routine procedures, so no further discussion of this method is presented in this report (see refs. 18 and 19).

It should be noted here that certain apparent anomalies are observed with certain constraint gradient information provided by the ANALYZ/DANLYZ implementation of the analysis method. This is not deemed sufficient, however, to preclude the use of this code for comparing optimizers.

2. Integrated force method. — The force method analysis generates and passes along the same basic information (the objective function, its gradient, the constraints, and their sensitivities) to the optimization module. Because the integrated force method is relatively new, a review of the basic equations for the analysis and sensitivity calculations is presented (see ref. 20).

The integrated force method (IFM), considers all the internal forces $\vec{F}$ to be the simultaneous unknowns. The force equilibrium equations $[B]\vec{F} = \vec{P}$ and the strain compatibility conditions $[C][G]\vec{F} = \delta\vec{R}$ are concatenated to obtain the governing equations of the method:

$$\begin{bmatrix} [B] \\ \hdashline [C][G] \end{bmatrix} \vec{F} = \begin{bmatrix} \vec{P} \\ \hdashline \delta\vec{R} \end{bmatrix} \tag{26a}$$

or

$$[S]\vec{F} = \vec{P}^* \tag{26b}$$

where $[B]$ is the $m \times n$ equilibrium matrix, $[C]$ is the $r \times n$ compatibility matrix, $[G]$ is the $n \times n$ concatenated flexibility matrix that links deformations β to forces $\vec{F}$ as ($\vec{\beta} = [G]\vec{F}$), $\vec{P}$ is the m component load vector, $\delta\vec{R}$ is the r component effective initial deformation vector ($\delta\vec{R} = -[C]\vec{\beta}_0$), $\vec{\beta}_0$ is the n component initial deformation vector, $[S]$ is the $n \times n$ governing matrix, and $m + r = n$. The matrices $[B]$, $[C]$, $[G]$, and $[S]$ are banded and have full row ranks of m, r, n, and n, respectively.

The solution of equation (26a) or (26b) yields the n forces $\vec{F}$. The m displacements $\vec{X}$ are obtained from the forces as

$$\vec{X} = [J]\left([G]\vec{F} + \vec{\beta}_0\right) \tag{27}$$

Here, $[J]$ is the $m \times n$ deformation coefficient matrix defined as ($[J] = m$ rows of $[\,[S]^{-1}]^T$).

The basic equation of IFM for frequency analysis of an undamped structure is given by

$$\left[[S] - \omega^2\left[M^*\right]\right]\vec{f}_m = \vec{0} \tag{28}$$

where

$$\left[M^*\right] = \begin{bmatrix} [M][J][G] \\ \hdashline [0] \end{bmatrix} \tag{29}$$

here, $[M]$ is the mass matrix, ω is the circular frequency, and $\vec{f}_m$ is termed the "force mode shape" of the eigenvalue problem.

Note that in equation (26a), matrices $[B]$ and $[C]$ are independent of the design variables and need not be regenerated after the first analysis of a particular problem. Only the block diagonal flexibility matrix $[G]$, which contains the design variables, has to be regenerated for reanalysis.

The equations used for the sensitivity calculations are given for trusses. Here, the areas $\vec{A}$ are taken as the design variables.

The sensitivity matrix, of dimension $n \times n$, for stress constraints is

$$[\nabla\sigma] = [D]^T\left[\frac{1}{A}\right] - \left[\frac{F}{A^2}\right] \tag{30}$$

where

$$[\nabla\sigma] \equiv \left[\nabla\vec{\sigma}_1, \nabla\vec{\sigma}_2, \ldots, \nabla\vec{\sigma}_n\right]$$

$$\nabla\vec{\sigma}_i^T = \left(\frac{\partial\sigma_i}{\partial A_1}, \frac{\partial\sigma_i}{\partial A_2}, \ldots, \frac{\partial\sigma_i}{\partial A_n}\right)^T$$

$$[D] = [S]^{-1}\begin{bmatrix} [0] \\ \hdashline [C][\bar{\bar{G}}] \end{bmatrix}[\bar{F}]$$

and $[1/A]$, $[F/A^2]$, $[\bar{\bar{G}}]$, and $[\bar{F}]$ are diagonal matrices of dimension $n \times n$, whose diagonal elements are $1/A_i$, F_i/A_i^2, $\ell_i/(A_i^2 E_i)$, and F_i, respectively. Here, ℓ_i and E_i are the lengths and moduli of elasticity of each element, and F_i is the ith component of the force vector $\vec{F}$.

The gradient matrix, of dimension $n \times m$, for displacement constraints is

$$[\nabla X] = \left[[J][G][D] + [J]\left[S_{dg}\right]\right]^T \tag{31}$$

The elements of the diagonal matrix $[S_{dg}]$ are given by

$$\left(S_{dg}\right)_{ii} = \frac{-g_{ii}F_i}{A_i} \tag{31a}$$

where g_{ii} is the value of the ith diagonal element of the flexibility matrix $[G]$.

The gradient matrix for frequency constraints may be obtained by following the approach of Rudisill (ref. 19 and 20). A typical element of this sensitivity matrix is given by

$$\frac{\partial \omega}{\partial A_i} = \frac{1}{2\omega}\left(\frac{\vec{f_\ell}^T\left(\left[\overline{S_i}\right] - \omega^2\left[\frac{[M][J]\left[\overline{G_i}\right] + [M]\left[\overline{J_i}\right][G]}{[0]}\right]\right)\vec{f}}{\vec{f_\ell}^T\left[M^*\right]\vec{f_r}}\right) \tag{32}$$

where

$$\left[\overline{S_i}\right] = \frac{\partial}{\partial A_i}[S] = -\overline{g}_{ii}\left[\frac{[0]}{\left[\vec{0}, \vec{0}, \ldots, \vec{C_i}, \ldots, \vec{0}\right]}\right]$$

$$\left[\overline{G_i}\right] = \frac{\partial}{\partial A_i}[G] = -\overline{g}_{ii}\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$\left[\overline{J_i}\right] = \frac{\partial}{\partial A_i}[J]$$

$$\overline{g}_{ii} = \frac{\ell_i}{A_i^2 E_i}$$

$\vec{C_i}$ is the ith column of $[C]$, $\vec{f_\ell}$ and $\vec{f_r}$ are the left and right eigenvectors of the eigenvalue problem, respectively, and $\overline{J_i}$ may be obtained through its relationship with $[S]$.

*3. **Simplified integrated force method.***—The sensitivity analysis equations of the integrated force method are simplified, and alternative versions that are numerically less expensive than the original formulas are obtained. As suggested by the illustration in appendix C, replacement of the expression $[J][G][D]$ in equation (31) by a null matrix, simplifies the displacement sensitivity, which becomes

$$[\nabla X]^a = \left[S_{dg}\right][J]^T \tag{33}$$

where the superscript a represents an approximation to the sensitivity matrix.

The stress gradient given by equation (30) can be approximated by dropping the first term, which represents the effect of compatibility in the sensitivity calculations. The approximate expression for the stress sensitivity has the following form:

$$[\nabla\sigma]^a = -\left[\frac{F}{A^2}\right] \tag{34}$$

The effects of the simplified sensitivity formulas are discussed in section V.

D. Linking of Design Variables

Design variables in a given problem can be reduced by linking them into groups. Specifically, the design variables are divided into a small number of groups. Linking factors can be assigned to elements of the group.

Consider, for example, the ring structure depicted in figure 4. The 60 element areas are divided into 25 linked groups (as depicted in table 23);

Group 1 is $\{\chi_{49}, \chi_{50}, \ldots, \chi_{60}\} = \{1.0\chi_1^L, 1.0\chi_1^L, \ldots, 1.0\chi_1^L\}$

Group 2 is $\{\chi_1, \chi_{13},\} = \{1.0\chi_2^L, 1.0\chi_2^L\}$

Group 3 is $\{\chi_2, \chi_{14}\} = \{1.0\chi_3^L, 1.0\chi_3^L\}$

Group 25 is $\{\chi_{36}, \chi_{48}\} = \{1.0\chi_{25}^L, 1.0\chi_{25}^L\}$

where χ_j^L represents the jth linked design variable. The linking factors in this example are 1.0; these factors, however, can be different.

The efficiency of this technique is tested in section IV and discussed in section V.

E. CometBoards

The OC code was incorporated into the Comparative Evaluation Test Bed of Optimization and Analysis Routines for the Design of Structures (CometBoards). This test bed is available on the VM/CMS computer system, as well as on the Posix-based Cray and Convex computers and Iris workstations. The command-level user interface is similar on all four platforms. The command syntax for invocation of the OC

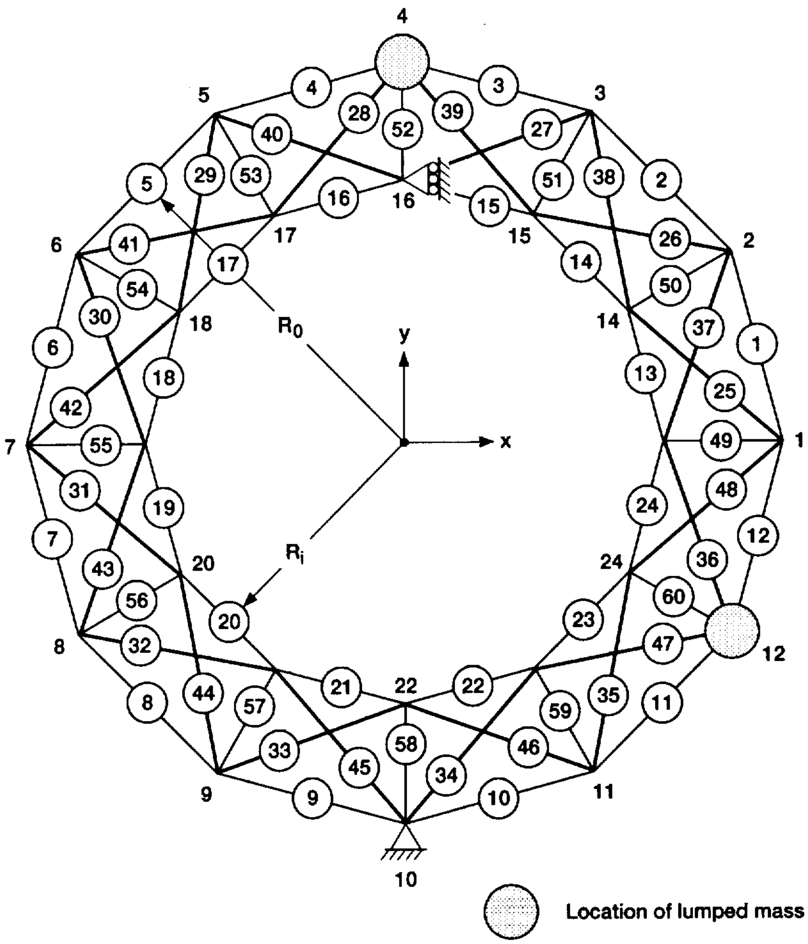


Figure 4.—Sixty-bar trussed ring. (Elements are circled, nodes are not.)

code on VM/CMS follows. Bold type should be input exactly as shown. Italic type highlights parameters where the user has a choice of options.

optimize oc *analyzer>fileid*

where

analyzer is **disp** for the displacement method analyzer
ifm for the integrated force method analyzer
sifmsd for the simplified integrated force method analyzer

fileid is the file identification for the output file on VM CMS (the default is for output to come to the screen)

The user on VM/CMS is prompted for the fileids of three input data files. For example, to run the optimality criteria method optimizer with the displacement analyzer on a user-specified problem with stress, displacement, and frequency constraints, and to place the output into the following file—prob1sdf output a —the user would type

optimize oc disp>prob1sdf output a

Then, following each of three prompts, the user would enter each of three fileids containing input data, such as

prob1sdf anldat a
prob1sdf idsdat a
prob1sdf ocdat a

The anldat file would include the specification of the finite element model. The idsdat file would contain design input data needed for constraint generation. The ocdat file would include parameters for the optimizer, including which optimality criteria update formulas are to be used.

IV. Numerical Examples

Several numerical examples for minimum weight design under single and multiple load conditions with stress, displacement and/or frequency constraints are solved using the OC code. The results obtained are compared with a mathematical programming technique referred to as a Sequence of Unconstrained Minimizations Technique (SUMT), which is available in the CometBoards test bed. Solutions obtained by SUMT are qualified by other optimization methods. Results from these example problems are briefly presented in this section.

Seventeen independent combinations of update formulas within the OC code were used to solve the example problems. These 17 procedures include optimality criteria methods, modified OC methods, a fully utilized design, and hybrid methods, which combine some of the OC methods with a fully utilized design. The salient features of the 17 methods are described next. An OC method, method 6 for example, updates the Lagrangian multipliers by using the exponential form, whereas it updates the design by using the reciprocal form.

The first six methods are standard optimality criteria methods given in references 3 and 4. Each method combines one update formula for the calculation of the Lagrange multipliers and one formula for updates to the design variables. These combinations are given in the following chart.

Method	Lagrange multiplier update formula	Design variable update formula
1	Linear form	Exponential form
2	Linear form	Linearized form
3	Linear form	Reciprocal form
4	Exponential form	Exponential form
5	Exponential form	Linearized form
6	Exponential form	Reciprocal form

The next seven methods are modified OC methods. Their combinations of update formulas are given in the following chart.

Method	Lagrange multiplier update formula	Design variable update formula
7	Diagonalized Lagrange inverse	Exponential form
8	Diagonalized Lagrange inverse	Linearized form
9	Diagonalized Lagrange inverse	Reciprocal form
10	Unrestricted Lagrange inverse	Exponential form
11	Unrestricted Lagrange inverse	Linearized form
12	Unrestricted Lagrange inverse	Reciprocal form
13	Lagrange inverse form	Melange form

The next three methods are hybrid methods, each using a hybrid design variable update formula as described in section II. The following chart depicts the update formula combinations for each of these methods.

Method	Lagrange multiplier update formula	Design variable update formula
14	Lagrange inverse form	Hybrid exponential form
15	Lagrange inverse form	Hybrid linearized form
16	Lagrange inverse form	Hybrid reciprocal form

Method 17 is the fully utilized design for stress, displacement, and frequency constraints.

Seven sets of examples consisting of a total of 31 problems are given. The problems range from a modest truss example with 5 design variables under displacement constraints to a difficult intermediate complexity wing problem with different element types, 320 constraints, and 57 linked design variables. The load conditions, mass distributions, stress, displacement, and frequency constraints are chosen to ensure that, at optimum, several behavior constraints are active. For example, one three-bar truss problem results in a total of nine active constraints at optimum, consisting of one frequency, two displacement, and six stress constraints. In all cases, the objective is to minimize the weight of the structure.

A. Five-Bar Truss

The five-bar truss reported in reference 4 consists of a set of four problems. The truss is made of aluminum, with a Young's modulus E of 10 000 ksi, a Poisson's ratio ν of 0.3, and a weight density ρ of 0.1 lb/in.^{3} Two related problems are also included in this first example set. Displacement limitations are the only behavior constraints. The structures are depicted in figures 5 and 6, whereas the loads and constraints are given in tables 1 and 2, respectively. Note that the

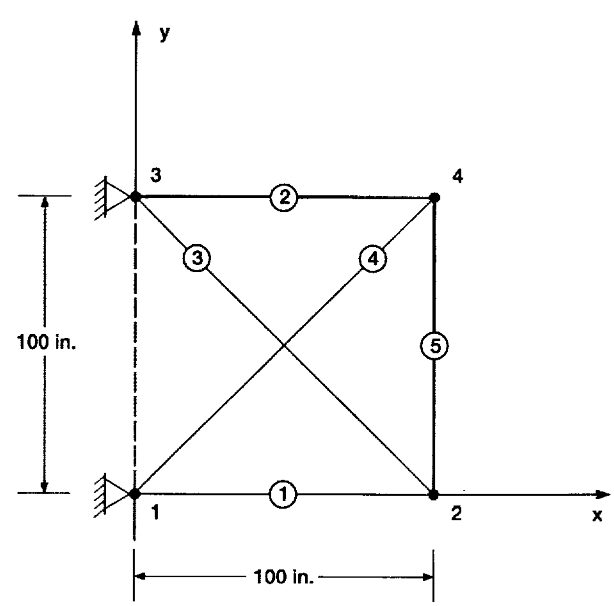


Figure 5.—Five-bar truss. (Elements are circled, nodes are not.)

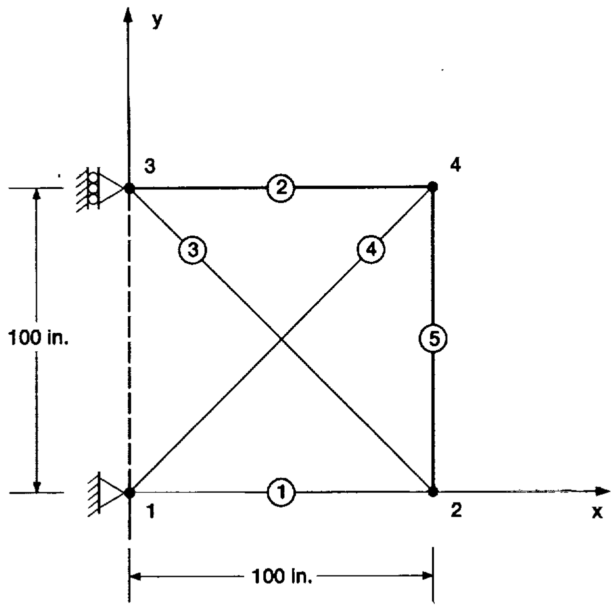


Figure 6.—Modified five-bar truss. (Elements are circled, nodes are not.)

boundary conditions differ between the original four problems and the last two. The examples are designated as Bar5L.1, Bar5L.2, Bar5L.3, Bar5L.4, Bar5L.2a, and Bar5L.2b. The optimal solutions given by SUMT show that all (either one or two) displacement constraints become active for all the problems except problem Bar5L.2, where only one of the two becomes active.

Solutions for the six problems are obtained using all 17 methods (OC-type, hybrid, and FUD), as well as SUMT, and are presented in tables 3 to 8. The results for five of the OC methods (4, 5, 12, 13, and 16) agree well with the SUMT

TABLE 1.—FIVE-BAR TRUSS EXAMPLES: LOAD SPECIFICATIONS

Problem	Number of boundary conditions	Load components, kips		
		Node	P_x	P_y
Bar5L.1	4	4	0.0	100.0
Bar5L.2		4		100.0
Bar5L.3		2		−100.0
		4		100.0
Bar5L.4		2		
		4		
Bar5L.2a	3	2	0.0	100.0
Bar5L.2b	3	2	0.0	100.0

TABLE 2. - FIVE-BAR TRUSS EXAMPLES: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Description of constraints along y direction	
		Node	Magnitude, in.
Bar5L.1	Displacement	4	2.0
Bar5L.2		2	6.0
		4	6.0
Bar5L.3		2	−.1
		4	.1
Bar5L.4		2	2.0
		4	2.0
Bar5L.2a	Displacement	4	2.0
Bar5L.2b		2	6.0
		4	2.0
Bar5L.1 Bar5L.2 Bar5L.3 Bar5L.4 Bar5L.2a Bar5L.2b	Minimum area	χ_i = 0.001 in.^{2} (i = 1,2,...,5)	

TABLE 3.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.1

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	45.016	0.001	1.499	0.001	2.120	0.001
2	1	2	45.016	.001	1.499	.001	2.120	.001
3	1	3	79.970	1.998	.001	2.826	.003	1.998
4	2	1	45.016	.001	1.499	.001	2.120	.001
5	2	2	45.016	.001	1.499	.001	2.120	.001
6	2	3	79.970	1.998	.001	2.826	.003	1.998
7	3	1	45.034	.001	1.500	.001	2.121	.001
8	3	2	45.034	.001	1.500	.001	2.121	.001
9	3	3	45.034	.001	1.500	.001	2.121	.001
10	4	1	45.040	.001	1.524	.001	2.105	.001
11	4	2	45.171	.001	1.624	.001	2.043	.001
12	4	3	45.171	.001	1.624	.001	2.043	.001
13	5	4	45.268	.015	1.488	.022	2.105	.015
14	5	5	45.040	.001	1.524	.001	2.105	.001
15	5	6	45.171	.001	1.624	.001	2.043	.001
16	5	7	45.171	.001	1.624	.001	2.043	.001
17	0	8	62.228	1.068	1.068	1.068	1.080	1.068
SUMT[a]			45.029	.001	1.501	.001	2.119	.001

[a]Sequence of Unconstrained Minimizations Technique.

TABLE 4.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.2

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	20.743	0.356	0.356	0.356	0.356	0.356
2	1	2	20.743	.356	.356	.356	.356	.356
3	1	3	20.743	.356	.356	.356	.356	.356
4	2	1	15.016	.001	.499	.001	.706	.001
5	2	2	15.016	.001	.499	.001	.706	.001
6	2	3	15.016	.001	.499	.001	.706	.001
7	3	1	15.034	.001	.500	.001	.707	.001
8	3	2	15.034	.001	.500	.001	.707	.001
9	3	3	15.034	.001	.500	.001	.707	.001
10	4	1	15.036	.001	.508	.001	.702	.001
11	4	2	15.080	.001	.541	.001	.681	.001
12	4	3	15.080	.001	.541	.001	.681	.001
13	5	4	15.084	.005	.496	.007	.702	.005
14	5	5	15.036	.001	.508	.001	.702	.001
15	5	6	15.080	.001	.541	.001	.681	.001
16	5	7	15.080	.001	.541	.001	.681	.001
17	0	8	20.743	.356	.356	.356	.356	.356
SUMT[a]			15.026	.001	.502	.002	.704	.001

[a]Sequence of Unconstrained Minimizations Technique.

TABLE 5.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.3

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	50.043	0.001	0.001	0.001	0.001	5.000
2	1	2	50.043	.001	.001	.001	.001	5.000
[a]3	1	3	----	---	---	---	---	---
4	2	1	50.047	.001	.001	.001	.001	5.000
5	2	2	50.047	.001	.001	.001	.001	5.000
[a]6	2	3	----	---	---	---	---	---
7	3	1	53.065	.065	.065	.065	.065	4.991
8	3	2	55.971	.122	.122	.131	.131	4.983
9	3	3	60.421	.216	.216	.226	.226	4.971
10	4	1	50.000	.000	.000	.000	.000	5.000
11	4	2	50.000	.000	.000	.000	.000	
12	4	3	50.000	.000	.000	.000	.000	
13	5	4	50.073	.001	.001	.002	.002	
14	5	5	50.000	.000	.000	.000	.000	
15	5	6	50.000	.000	.000	.000	.000	
16	5	7	50.000	.000	.000	.000	.000	↓
17	0	8	257.758	4.422	4.422	4.422	4.422	4.422
SUMT[b]			50.560	.008	.008	.013	.013	5.002

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 6.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.4

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	90.010	1.500	1.500	2.121	2.121	0.001
2	1	2	90.010			2.121	2.121	.001
3	1	3	90.087			2.122	2.122	.009
4	2	1	90.010			2.121	2.121	.001
5	2	2	90.010			2.121	2.121	.001
6	2	3	90.023	↓	↓	2.121	2.121	.002
[a]7	3	1	----	---	---	---	---	---
8	3	2	97.982	1.485	1.769	2.100	2.502	.034
9	3	3	96.051	1.702	1.500	2.407	2.121	.001
10	4	1	90.010	1.500		2.121		
11	4	2	90.010					
12	4	3	90.010					↓
13	5	4	90.060					.006
14	5	5	90.010					.001
15	5	6	90.010					.001
16	5	7	90.010	↓	↓	↓	↓	.001
17	0	8	111.568	1.914	1.914	1.914	1.914	1.914
SUMT[b]			90.063	1.498	1.498	2.124	2.124	.002

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 7.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.2a

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	45.034	0.001	1.500	0.001	2.121	0.001
2	1	2	45.034	.001	1.500	.001		.001
3	1	3	45.106	.003	1.499	.003		.003
4	2	1	45.034	.001	1.500	.001		.001
5	2	2	45.034					
6	2	3	45.042					
7	3	1	45.036					
8	3	2	45.035					
9	3	3	45.035					
10	4	1	45.036					
11	4	2	45.035					
12	4	3	45.035	↓		↓		↓
13	5	4	45.078	.002		.002		.002
14	5	5	45.036	.001		.001		.001
15	5	6	45.035	.001		.001		.001
16	5	7	45.035	.001	↓	.001	↓	.001
17	0	8	111.568	1.914	1.914	1.914	1.914	1.914
SUMT[a]			45.051	.001	1.497	0.001	2.124	.001

[a]Sequence of Unconstrained Minimizations Technique.

TABLE 8.—FIVE-BAR TRUSS EXAMPLES: PROBLEM BAR5L.2b

[Displacement constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	111.568	1.914	1.914	1.914	1.914	1.914
2	1	2	103.459	.197	3.223	.197	4.600	.197
3	1	3	67.518	.225	1.532	.225	3.148	.225
4	2	1	47.540	.001	1.500	.001	2.121	.252
5	2	2	47.539	.001	1.500	.001	2.121	.251
[a]6	2	3	----	---	---	---	---	---
[a]7	3	1	----	---	---	---	---	---
8	3	2	53.208	.001	1.694	.001	2.395	.236
9	3	3	48.446	.001	1.500	.001	2.121	.341
10	4	1	60.217	.009	1.500	.009	2.121	1.500
11	4	2	76.593	.002	2.482	.002	3.510	.209
12	4	3	47.551	.001	1.500	.001	2.121	.253
13	5	4	48.368	.001	1.528	.001	2.162	.248
14	5	5	60.217	.009	1.500	.009	2.121	1.500
15	5	6	76.593	.002	2.482	.002	3.510	.209
16	5	7	47.551	.001	1.500	.001	2.121	.253
17	0	8	111.568	1.914	1.914	1.914	1.914	1.914
SUMT[b]			47.540	.001	1.496	.001	2.125	.250

[a]Method failed for this problem.

[b]Sequence of Unconstrained Minimizations Technique.

answers for all six problems (although method 13 shows a 1.7 percent difference for the last problem). Four additional methods (10, 11, 14, and 15) show results that agree with SUMT for all but the last problem (Bar5L.2b). Among most of the other OC and hybrid methods, agreement is seen for at least half the problems. The fully utilized design approach is not applicable for this problem because no stress constraints are specified, but the results are included for completeness.

B. Tapered Five-Bar Truss Under Stress, Displacement, and Frequency Constraints

A tapered five-bar truss, shown in figure 7, provides three more example problems. It is made of steel, with a Young's modulus E of 30 000 ksi, a Poisson's ratio ν of 0.3, and a weight density ρ of 0.284 lb/in.^{3} The truss is subjected to two load conditions and has five stress and two displacement constraints per load condition, along with a frequency constraint, as shown in tables 9 and 10. These minimum weight design problems are designated by (1) Bar5.a, where only stress constraints are specified (of which eight are active with SUMT's optimal solution) and the results are depicted in table 11; (2) Bar5.b, where stress and displacement constraints are specified (of which three displacement and six stress constraints are active at optimum) and the results are shown in table 12; and (3) Bar5.c, where stress, displacement, and frequency constraints are all specified (of which one frequency, two displacement, and two stress constraints are active at the optimal solution) and the results are given in table 13.

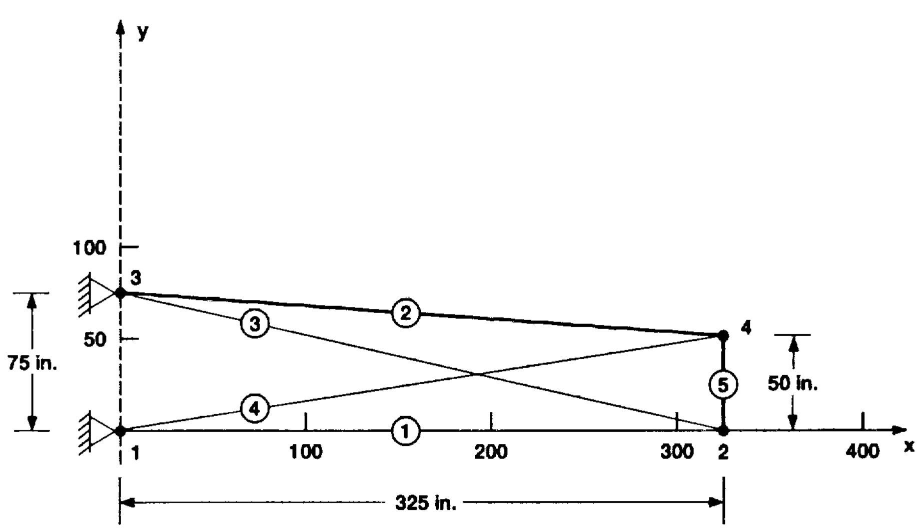


Figure 7.—Tapered five-bar truss. (Elements are circled, nodes are not.)

TABLE 9.—TAPERED FIVE-BAR TRUSS PROBLEM: LOAD SPECIFICATIONS

Problem	Load conditions	Load components, kips		
		Node	P_x	P_y
Bar5.a Bar5.b Bar5.c	I	2 4	120.0 35.0	120.0 25.0
	II	2 4	0.0 0.0	−65.0 −75.0
Bar5.c	Lumped masses	2 4	$m_x = m_y = 114$ lb $m_x = m_y = 165$ lb	

TABLE 10.—TAPERED FIVE-BAR TRUSS: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
Bar5.a	Stress Minimum area	$\sigma_i \le \sigma_0 \quad (i = 1,2,\ldots,5)$ $\sigma_0 = 20$ ksi $\chi_i \ge 0.25$ in.$^2 \quad (i = 1,2,\ldots,5)$
Bar5.b	Stress Displacement Minimum area	Same as Bar5.a Constraints along y direction 1.9 in. at node 2 1.9 in. at node 4 Same as Bar5.a
Bar5.IIc	Stress Displacement Frequency Minimum area	Same as Bar5.a Same as Bar 5.b $f \ge f_0$; $f_0 = 15$ Hz $\chi_i \ge 4.0$ in.$^2 \quad (i = 1,2,\ldots,5)$

TABLE 11.—TAPERED FIVE-BAR TRUSS: PROBLEM BAR5.a
[Stress constraints only.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	7 743.920	20.000	20.000	20.000	20.000	20.000
2	1	2	7 000.653	20.686	17.607	16.565	17.737	16.606
3	1	3	7 001.601	20.597	17.620	16.632	17.744	16.672
4	2	1	12 545.037	44.792	20.564	45.969	7.068	105.110
5	2	2	6 584.949	21.949	14.803	16.873	16.633	2.437
6	2	3	6 836.458	21.263	16.533	15.612	18.490	9.698
[a]7	3	1	------	----	----	----	----	----
8	3	2	11 400.548	55.692	18.233	29.274	19.061	1.366
9	3	3	12 896.357	34.797	3.256	94.752	3.753	4.028
10	4	1	11 319.068	55.268	9.582	51.968	3.634	4.833
11	4	2	8 211.508	39.035	16.123	20.099	12.706	1.794
[a]12	4	3	------	----	----	----	----	----
13	5	4	6 638.117	27.456	10.414	21.437	11.643	1.546
14	5	5	6 465.411	27.024	9.300	21.628	11.121	1.612
15	5	6	6 465.481	27.023	9.301	21.626	11.123	1.611
16	5	7	6 465.405	27.024	9.300	21.628	11.121	6.612
17	0	8	6 465.411	27.024	9.300	21.627	11.121	1.612
SUMT[b]			6 476.737	25.743	10.710	20.392	12.390	1.441

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

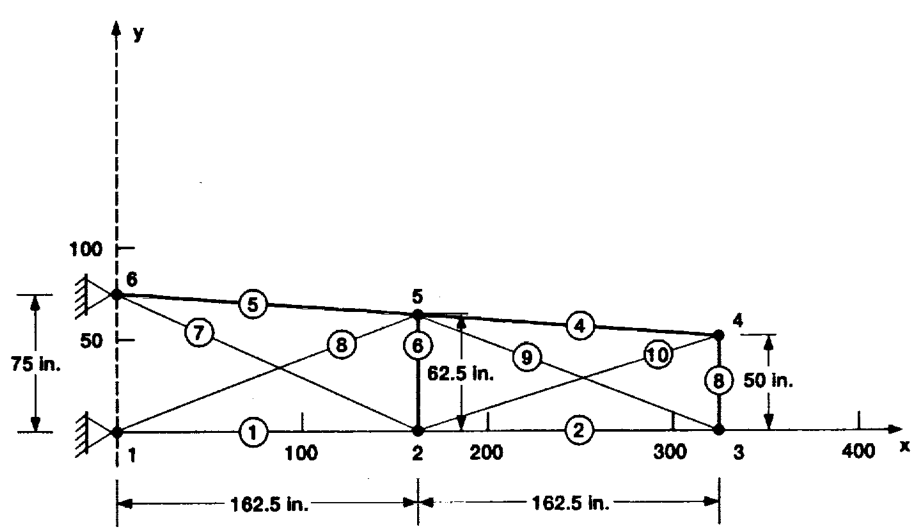


Figure 8.—Ten-bar truss. (Elements are circled, nodes are not.)

From tables 11 to 13, it is observed that method 17, the fully utilized design, yields solutions close to the SUMT results for all the three problems. Methods 5, 6, and 13 show satisfactory performance with less than a 3 percent difference with SUMT (except for method 6, which differs by 5.6 percent in the first problem, Bar5.a). Performance of the three hybrid methods differs depending on the type of constraints considered. For stress constraints only (Bar5.2a), results obtained by all three hybrid methods are good, differences in the optimal weight being less than 1 percent; but results are poor when displacement and/or frequency constraints are included. The performance of the other 10 OC methods for this set of problems is generally poor.

C. 10-Bar Truss Design

The 10-bar aluminum truss, shown in figure 8, is taken as the third example set. In the first two cases, the truss is subjected to two load conditions, with 10 stress and 2 displacement constraints per load condition, along with a frequency constraint, (for a total of 25 constraints), as given in tables 14 and 15. Two design situations are considered. Problem Bar10.a has all 10 bar areas considered as independent design variables. The second case, Bar10.b, is obtained by linking the variables into a set of five independent linked design

TABLE 13.—TAPERED FIVE-BAR TRUSS: PROBLEM BAR5.c

[Stress, displacement, and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	8 083.500	20.878	20.878	20.878	20.878	20.878
2	1	2	7 391.503	20.572	18.983	18.753	19.419	10.251
3	1	3	7 377.293	21.088	18.678	18.555	19.044	14.341
4	2	1	8 083.500	20.878	20.878	20.878	20.878	20.878
5	2	2	6 890.761	23.498	14.643	17.974	16.578	8.143
6	2	3	6 892.336	21.708	16.493	16.206	18.311	6.111
[a]7	3	1	------	----	----	----	----	----
[a]8	3	2	------	----	----	----	----	----
[a]9	3	3	------	----	----	----	----	----
[a]10	4	1	------	----	----	----	----	----
[a]11	4	2	------	----	----	----	----	----
12	4	3	8 521.432	14.827	32.046	9.998	33.685	6.590
13	5	4	6 798.956	28.886	8.747	24.184	10.426	4.126
14	5	5	13 512.923	57.520	16.029	52.576	18.021	4.000
15	5	6	18 037.263	81.186	17.023	74.704	19.638	4.000
[a]16	5	7	------	----	----	----	----	----
17	0	8	6 742.205	26.898	10.664	21.511	12.591	4.141
SUMT[b]			6 958.113	22.492	15.848	19.740	15.937	4.000

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 12.—TAPERED FIVE-BAR TRUSS: PROBLEM BAR5.b

[Stress and displacement constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	7 743.920	20.001	20.001	20.001	20.001	20.001
2	1	2	6 997.886	20.365	17.657	17.073	18.400	10.433
3	1	3	6 985.157	20.420	17.580	16.817	17.999	14.018
4	2	1	9 266.655	43.652	7.979	41.588	5.200	5.200
5	2	2	6 632.620	23.391	13.068	18.703	15.133	5.557
6	2	3	6 763.739	21.412	16.332	15.782	18.232	5.489
7	3	1	12 870.526	5.999	63.995	.080	66.317	13.510
8	3	2	8 561.541	22.726	28.900	9.588	30.009	5.487
[a]9	3	3	------	----	----	----	----	----
10	4	1	7 456.582	31.803	8.796	29.071	9.894	2.053
11	4	2	7 145.541	29.424	9.507	26.777	10.568	1.852
[a]12	4	3	------	----	----	----	----	----
13	5	4	6 549.017	26.469	10.330	21.229	11.973	1.456
14	5	5	14 873.955	67.303	13.761	61.949	15.913	2.377
15	5	6	17 976.304	82.470	15.521	76.026	18.108	2.453
[a]16	5	7	------	----	----	----	----	----
17	0	8	6 549.674	27.376	9.421	21.909	11.266	1.633
SUMT[b]			6 541.559	26.208	10.397	21.327	11.931	1.809

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 14.—BAR TRUSS: LOAD SPECIFICATIONS

Problem	Load conditions	Load components, kips		
		Node	P_x	P_y
Bar10.a Bar10.b	I	2	60.0	120.0
		3	60.0	60.0
		4	17.5	12.5
		5	17.5	25.0
			17.5	25.0
	II	2	0.0	−50.0
		3	5	−25.0
		4	5	−37.5
		5	5	−75.0
Bar10.c Bar10.d	I	2	0.6	6.0
		3	6.0	6.0
		4	1.75	1.25
		5	.17	2.5
Bar10.a	Lumped masses	2	$m_x = m_y =$ 75 lb	
Bar10.b		3	$m_x = m_y =$ 135 lb	
Bar10.c		4	$m_x = m_y =$ 75 lb	
Bar10.d		5	$m_x = m_y =$ 135 lb	

TABLE 15.—10-BAR TRUSS: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
Bar10.a Bar 10.b	Stress	$\sigma_i \le \sigma_0$ $(i = 1,2,\ldots,10)$ $\sigma_0 = 10$ ksi
	Displacement	Constraints along y direction of magnitude: 2.2 in. at node 3 2.2 in. at node 4
	Frequency	$f \ge f_0$; $f_0 = 26$ Hz
	Minimum area	$\chi_i = 4.0$ in.^{2} $(i = 1,2,\ldots,10)$ for Bar10.a $\chi_i = 4.0$ in.^{2} $(i = 1,2,\ldots,5)$ for Bar10.b
Bar10.c Bar10.d	Displacement	Constraints along y direction of magnitude: 1.5 in. at node 3 1.5 in. at node 4
	Frequency	$f \ge f_0$; $f_0 = 15$ Hz
	Minimum area	$\chi_i = 0.10$ in.^{2} $(i = 1,2,\ldots,10)$ for Bar10.c $\chi_i = 0.10$ in.^{2} $(i = 1,2,\ldots,5)$ for Bar10.d

TABLE 16.—10-BAR TRUSS: DESIGN VARIABLE LINKAGE

Problem	Serial number	Design variable	Members linked
Bar10.b	1	1	1,5
	2	2	2,4
	3	3	3,6
	4	4	7,8
	5	5	9,10
Bar10.d	1	1	1
	2	2	2,4
	3	3	5
	4	4	3,9,10
	5	5	6,7,8

TABLE 17.—OPTIMUM DESIGN OF A 10-BAR TRUSS: PROBLEM BAR10.a
[Stress, displacement, and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Normalized weight	Normalized design variables									
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
1	1	1	1.78	1.27	1.69	7.50	4.43	1.69	7.50	1.21	1.47	2.32	3.82
2	1	2	1.94	1.96	.79	3.35	2.03	2.43	3.37	1.97	2.12	1.06	1.77
3	1	3	1.25	1.08	.89	3.90	2.32	1.27	3.93	1.12	1.15	1.22	2.01
4	2	1	1.23	.89	.79	3.51	2.07	1.64	3.51	.57	1.71	1.09	1.79
5	2	2	1.11	1.04	.69	1.96	1.96	1.23	2.32	1.04	1.14	.61	1.91
6	2	3	1.07	1.05	.81	1.49	1.55	1.11	1.49	1.09	1.03	.77	1.61
7	3	1	3.66	3.38	4.81	1.00	3.50	3.48	1.00	3.63	2.89	5.89	3.80
8	3	2	3.69	3.41	4.83	1.00	3.53	3.48	1.00	3.77	2.87	5.92	3.83
[a]9	3	3	Failed	(a)	(a)	(a)	(a)	(a)	(a)	(a)	---	---	---
10	4	1	14.06	13.51	19.32	4.87	13.11	13.60	5.80	11.87	10.73	23.65	14.20
11	4	2	2.28	2.09	3.19	1.00	1.84	2.26	1.00	1.92	1.91	3.92	2.04
[a]12	4	3	Failed	---	---	---	---	---	---	---	---	---	---
13	5	4	1.18	.89	1.65	1.08	.64	1.35	1.82	.73	1.51	2.06	.55
14	5	5	1.70	1.56	2.29	1.00	1.19	1.85	1.15	1.14	1.62	2.86	1.32
15	5	6	1.69	1.55	2.29	1.00	1.18	1.84	1.15	1.14	1.62	2.85	1.32
[a]16	5	7	Failed	---	---	---	---	---	---	---	---	---	---
17	0	8	1.05	1.15	1.18	1.11	.79	.80	1.21	1.35	.87	1.18	.95
SUMT[b]			1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Weight, lb			3 326.93	56.53	17.70	4.00	6.77	46.10	4.00	24.72	23.76	12.91	7.85

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

groups as shown in table 16. At optimum, for both these problems, the SUMT results show that one frequency, two displacement, and seven stress (six, in the Bar10.b case) constraints are active.

Both cases are solved by all 17 methods and SUMT. The results obtained are normalized with respect to the SUMT answers, and are given in tables 17 and 18. Overall, all the methods perform at about the same level for both problems (Bar10.a and Bar10.b). The performance of these methods for the 10-bar truss is similar to that of the last two 5-bar truss problems (Bar5.b and Bar5.c). Methods 5 and 17 yield results that differ by around 10 percent or less, compared with SUMT. Method 6 shows a 7 percent difference with problem Bar10.a, but 17 percent with Bar10.b, whereas method 13 shows an 18 percent difference with Bar10.a, but less than 1 percent difference with Bar10.b.

One effect of linking of design variables (see section III.D) is illustrated by also subjecting this 10-bar truss to a single load condition (see table 14), with displacement and frequency constraints (see table 15). As before, two design situations are considered. Problem Bar10.c has all 10 bar areas considered as independent design variables. The last case, Bar10.d, is obtained by linking the variables into a set of five independent linked design groups as shown in table 16. At optimum, for these last two problems, the SUMT results show that one frequency and two displacement constraints are active.

Once again, in addition to SUMT, all 17 methods are used to solve these problems. The results are given in tables 19 and 20. Methods 1 to 6 perform well, showing less than a 1 percent difference with SUMT for both problems. Method 13 shows less than a 2 percent difference. Methods 7, 10, 11, 14, and 15 also perform well for Problem Bar10.c but do poorly for the linked case (Bar10.d). Additional discussion regarding the effect of linking of design variables is provided in section V.

TABLE 18.—10-BAR TRUSS: PROBLEM BAR10.b LINKED DESIGN CONFIGURATION
[Stress, displacement, and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Normalized weight	Normalized design variables				
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	1.69	1.42	2.12	5.34	1.29	2.69
2	1	2	1.31	1.21	1.39	3.29	1.14	1.75
3	1	3	1.60	1.29	2.02	5.02	1.31	2.55
4	2	1	1.83	1.32	2.56	6.44	1.47	3.25
5	2	2	1.08	1.09	.99	1.40	1.08	1.09
6	2	3	1.17	1.16	.97	1.58	1.21	1.31
[a]7	3	1	---	---	---	---	---	---
[a]8	3	2	---	---	---	---	---	---
[a]9	3	3	---	---	---	---	---	---
[a]10	4	1	---	---	---	---	---	---
[a]11	4	2	---	---	---	---	---	---
[a]12	4	3	---	---	---	---	---	---
13	5	4	1.00	.98	1.01	1.02	1.02	1.05
14	5	5	1.55	1.54	1.80	.93	1.24	2.17
15	5	6	2.18	2.16	2.52	1.20	1.74	3.04
[a]16	5	7	---	---	---	---	---	---
17	0	8	1.00	.98	1.01	1.02	1.02	1.01
SUMT[b]			1.00	1.00	1.00	1.00	1.00	1.00
Weight, lb			3 572.53	53.92	14.37	5.72	25.37	11.33

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

D. 60-Bar Trussed Ring

A 60-bar trussed ring made of aluminum (fig. 4) is subjected to three load conditions (table 21). The constraints considered are specified in table 22. The 60 element areas of the structure are linked into 25 groups each considered as a design variable (table 23). The problem is solved for the following six situations: (a) Bar60.a, stress constraints only; (b) Bar60.b, displacement constraints only; (c) Bar60.c, frequency constraints only; (d) Bar60.d, both stress and displacement constraints; (e) both displacement and frequency constraints; and (f) all three types of constraints. The chart below shows the number of active constraints of each type in SUMT's solutions for all six cases.

Number of active constraints by type			
Problem	Stress	Displacement	Frequency
a	30	--	--
b	---	1	--
c	---	--	1
d	24	1	--
e	---	1	1
f	17	1	1

Results obtained for the six problems by all 17 methods, normalized by the SUMT results are given in table 24. For problem Bar60.b (displacement constraints only), almost all of the OC-type methods (1 to 16) perform well, with differences from SUMT not exceeding 5 percent. For problem Bar60.c (frequency constraints only), 12 of the 16 OC-type methods perform well with differences of about 5 percent. Methods 4, 5, 6, and 13 exhibit larger differences. For problem Bar60.e (displacement and frequency constraints), 11 methods show less than 10 percent differences with

TABLE 19. – OPTIMUM DESIGN OF A 10-BAR TRUSS: PROBLEM BAR10.c
[Displacement and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}									
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
1	1	1	414.064	5.609	3.265	0.144	0.514	7.114	0.144	1.227	3.282	3.127	0.611
2	1	2	413.962	5.616	3.260	.137	.517	7.109	.137	1.232	3.278	3.127	.607
3	1	3	413.702	5.630	3.210	.131	.542	7.116	.131	1.249	3.272	3.086	.629
4	2	1	412.263	5.943	2.388	.103	1.014	7.178	.103	1.620	3.097	2.382	1.048
5	2	2	412.242	5.884	2.498	.106	.943	7.188	.106	1.553	3.140	2.480	.981
6	2	3	412.186	5.878	2.500	.106	.939	7.193	.106	1.548	3.145	2.483	.978
7	3	1	415.584	5.511	3.018	.353	.285	7.817	.134	1.020	3.832	2.913	.390
8	3	2	506.694	7.210	2.952	.100	.469	9.254	.100	1.747	4.300	3.458	.920
9	3	3	586.999	8.933	3.163	.100	1.061	10.668	.100	2.579	4.805	2.875	1.076
10	4	1	413.242	5.948	2.454	.101	.983	7.172	.101	1.610	3.105	2.436	1.022
11	4	2	413.238	5.950	2.426	.101	.984	7.170	.101	1.612	3.103	2.434	1.024
[a]12	4	3	Failed	---	---	---	---	---	---	---	---	---	---
13	5	4	412.942	6.023	2.388	.101	1.045	7.088	.101	1.696	3.019	2.366	1.086
14	5	5	413.242	5.948	2.454	.101	.983	7.172	.101	1.610	3.105	2.436	1.022
15	5	6	413.238	5.950	2.453	.101	.984	7.170	.101	1.612	3.103	2.434	1.024
[a]16	5	7	Failed	---	---	---	---	---	---	---	---	---	---
17	0	8	771.019	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279
SUMT[b]			411.796	6.228	2.009	.103	1.287	7.005	.104	1.910	2.872	2.035	1.295

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 20. – OPTIMUM DESIGN OF A 10-BAR TRUSS: PROBLEM BAR10.d
LINKED DESIGN CONFIGURATION
[Displacement and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}									
	λ	χ		χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
1	1	1	444.351	6.851	1.945	1.770	1.945	6.595	2.224	2.224	2.224	1.770	1.770
2	1	2	444.260	6.853	1.939	1.766	1.939	6.603	2.227	2.227	2.227	1.766	1.766
3	1	3	444.112	6.857	1.930	1.758	1.930	6.618	2.231	2.231	2.231	1.758	1.758
4	2	1	452.937	6.758	2.245	2.019	2.245	6.292	2.116	2.116	2.116	2.019	2.019
5	2	2	454.974	7.177	1.752	1.620	1.752	7.206	2.407	2.407	2.407	1.620	1.620
6	2	3	443.173	6.917	1.782	1.638	1.782	6.903	2.302	2.302	2.302	1.638	1.638
7	3	1	1 263.502	19.029	6.261	5.633	6.261	17.377	5.902	5.902	5.902	5.633	5.633
8	3	2	1 282.120	19.309	6.354	5.716	6.354	17.633	5.989	5.989	5.989	5.716	5.716
9	3	3	1 641.241	24.717	8.133	7.317	8.133	22.572	7.666	7.666	7.666	7.317	7.317
10	4	1	2 035.628	30.657	10.088	9.076	10.088	27.996	9.508	9.508	9.508	9.076	9.076
11	4	2	5 853.529	88.155	29.008	26.097	29.008	80.504	27.342	27.342	27.342	26.097	26.097
12	4	3	756.686	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180
13	5	4	460.207	6.950	2.282	2.029	2.282	6.346	2.156	2.156	2.156	2.029	2.029
14	5	5	2 035.628	30.657	10.088	9.076	10.088	27.996	9.508	9.508	9.508	9.076	9.076
15	5	6	5 853.529	88.155	29.008	26.097	29.008	80.504	27.342	27.342	27.342	26.097	26.097
16	5	7	756.686	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180	5.180
17	0	8	771.019	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279	5.279
SUMT[a]			450.454	7.090	1.725	1.592	1.725	7.188	2.388	2.388	2.388	1.592	1.592

[a]Sequence of Unconstrained Minimizations Technique.

TABLE 21. – 60-BAR TRUSSED RING:
LOAD SPECIFICATIONS

Problem	Load conditions	Load components, kips		
		Node	P_x	P_y
Bar60.a Bar60.b Bar60.c Bar60.d	I	1 7	−10.0 9.0	0.0 0.0
	II	15 18	−8.0 −8.0	3.0 3.0
	III	22	−20.0	10.0
Bar60.c Bar60.e Bar60.f	Lumped masses	4 12	$m_x = m_y = 200$ lb $m_x = m_y = 100$ lb	

TABLE 22. – 60-BAR TRUSSED RING: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
Bar60.a	Stress	$\sigma_i \le \sigma_0$ $(i = 1,2,\ldots,60)$ $\sigma_0 = 10$ ksi
	Minimum area	$\chi_i = 0.5$ in.^{2} $(i = 1,2,\ldots,25)$
Bar60.b	Displacement	Constraints along y direction 1.75 in. at node 4
	Minimum area	Same as Bar60.a
Bar60.c	Frequency	$f \ge f_0; f_0 = 13$ Hz
	Minimum area	Same as Bar60.a
Bar60.d	Stress	Same as Bar60.a
	Displacement	Constraints along both x and y directions 1.25 in. at node 10 1.75 in. at node 4 2.75 in. at node 19 2.25 in. at node 13
	Minimum area	Same as Bar60.a
Bar60.e	Displacement	Constraints along both x and y directions 1.25 in. at node 10 1.75 in. at node 4 2.75 in. at node 19 2.25 in. at node 13
	Frequency	Same as Bar60.c
	Minimum area	Same as Bar60.a
Bar60.f	Stress	Same as Bar60.a
	Displacement	Constraints along y direction 150 in. at node 4
	Frequency	Same as Bar60.c
	Minimum area	Same as Bar60.a

SUMT results. Methods 4, 5, 7, 8, and 9 have large differences. For problem Bar60.a (stress constraints only), the fully utilized design method, as well as all hybrid methods, performs well. Performance of the other methods for Bar60.a is, in general, inadequate. For problem Bar60.d (stress and displacement constraints), performance of method 17 is acceptable with a 5 percent difference in comparison to the SUMT results, method 5 shows less than 10 percent difference, but the performance is poor for other methods. For problem Bar60.f (all three constraint types), method 5 gives a design that is within 5 percent of the SUMT design. The next best method for this last problem is method 13, with a 20 percent heavier design than SUMT.

TABLE 23. – 60-BAR TRUSSED RING: PROBLEMS BAR60.a, BAR60.b, BAR60.c, BAR60.d, BAR60.e, AND BAR60.f DESIGN VARIABLE LINKAGE

Serial number	Design variable	Members linked
1	1	49,50,51,52,53,54, 55,56,57,58,59,60
2	2	1,13
3	3	2,14
4	4	3,15
5	5	4,16
6	6	5,17
7	7	6,18
8	8	7,19
9	9	8,20
10	10	9,21
11	11	10,22
12	12	11,23
13	13	12,24
14	14	25,37
15	15	26,38
16	16	27,39
17	17	28,40
18	18	29,41
19	19	30,42
20	20	31,43
21	21	32,44
22	22	33,45
23	23	34,46
24	24	35,47
25	25	36,48

E. Intermediate Complexity Wing

The Intermediate Complexity Wing (IC Wing) shown in figure 9 is considered next. The finite element model of the IC Wing has 88 grid points and a total of 158 elements consisting of 39 bars, 2 triangular membranes, 62 quadrilateral membranes, and 55 shear panels. The wing is made of aluminum and subjected to two load conditions as given in table 25. The constraints of the problem are given in table 26. The 158 elements are linked to obtain a reduced set of 57 design variables for optimization as given in table 27. Three design cases are considered: (1) IC–Wing.a, for stress constraints; (2) IC–Wing.b, for displacement constraints; and (3) IC–Wing.c, for stress and displacement constraints. For problem IC–Wing.a, the SUMT solution shows 108 active stress constraints. For problem IC–Wing.b, it produces three active displacement constraints. For IC–Wing.c, no displacement constraint becomes active, but there are 108 active stress constraints.

The results obtained for the three design cases by 14 of the OC-type methods, normalized by SUMT, are presented in

TABLE 24. – 60-BAR TRUSSED RING; LINKED DESIGN CONFIGURATION

Methods	Optimality criteria (OC) formulas		Normalized weight					
	λ	χ	Bar60.a (stress)	Bar60.b (displacement)	Bar60.c (frequency)	Bar60.d (stress and displacement)	Bar60.e	Bar60.f
1	1	1	1.59	1.05	1.04	1.48	1.03	1.26
2	1	2	2.24	1.00	1.02	1.42	1.01	2.73
3	1	3	1.80	1.00	1.00	1.76	1.00	3.08
4	2	1	2.16	1.05	(a)	1.69	Failed	1.48
5	2	2	1.30	1.03	1.19	1.09	1.14	1.05
6	2	3	(a)	1.01	(a)	(a)	1.08	1.45
7	3	1	4.63	1.00	1.06	2.44	1.76	3.03
8	3	2	1.81		1.02	2.36	1.32	1.39
9	3	3	(a)		1.00	(a)	1.43	Failed
10	4	1	(a)		1.06	2.29	1.08	4.04
11	4	2	2.60		1.02	3.96	1.07	4.09
12	4	3	(a)		1.00	1.63	1.05	1.32
13	5	4	2.11		2.14	1.39	1.02	1.20
14	5	5	1.00		1.06	1.30	1.08	Failed
15	5	6			1.02	1.29	1.07	Failed
16	5	7		↓	1.00	1.29	1.05	1.32
17	0	8	↓	1.33	1.61	1.05	1.61	1.54
SUMT[b] Weight, lb			1.00 286.862	1.00 294.066	1.00 391.902	1.00 308.730	1.00 391.820	1.00 420.243

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

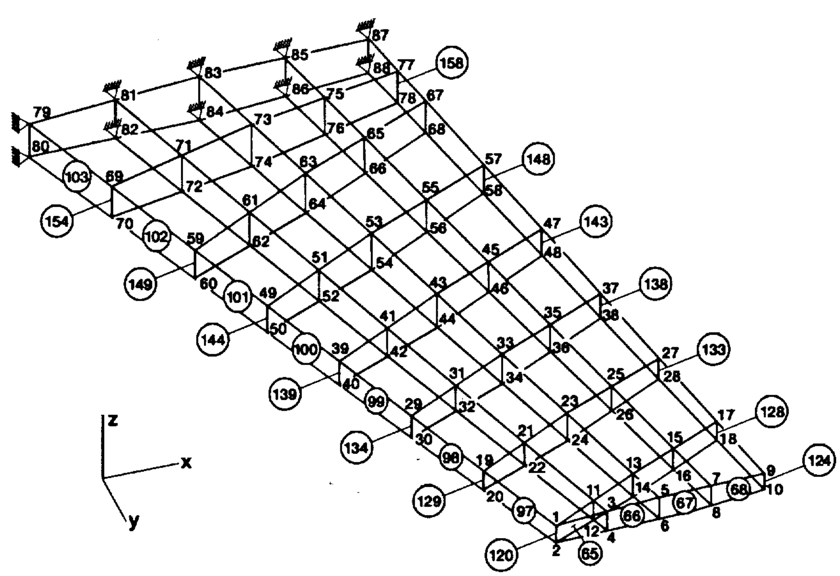


Figure 9.—Intermediate complexity wing. (Representative elements are circled, nodes are not.)

TABLE 25. – INTERMEDIATE COMPLEXITY WING: LOAD SPECIFICATIONS

(a) Load condition I

	Load components, lb				Load components, lb				Load components, lb		
Node	P_x	P_y	P_z	Node	P_x	P_y	P_z	Node	P_x	P_y	P_z
1	205.0	7380.0	926.0	27	−4070.0	1660.0	902.0	53	0	0	404.0
2	−205.0	7380.0	926.0	28	4070.0	−1660.0	902.0	54			404.0
3	0	0	29.0	29	1740.0	−713.0	646.0	55			420.0
4	0	0	29.0	30	−1740.0	713.0	646.0	56	↓	↓	420.0
5	−2800.0	−6960.0	1130.0	31	0	0	340.0	57	−4640.0	1900.0	1120.0
6	2800.0	6960.0	1130.0	32			340.0	58	4640.0	−1900.0	1120.0
7	0	0	90.9	33			352.0	59	2290.0	−937.0	883.0
8	0	0	90.9	34			352.0	60	−2290.0	937.0	883.0
9	−9870.0	−9780.0	1130.0	35			365.0	61	0	0	413.0
10	9870.0	9780.0	1130.0	36	↓	↓	365.0	62			413.0
11	0	0	178.0	37	−4250.0	1740.0	974.0	63			391.0
12			178.0	38	4250.0	−1740.0	974.0	64			391.0
13			214.0	39	1820.0	−743.0	694.0	65			368.0
14			214.0	40	−1820.0	743.0	694.0	66	↓	↓	368.0
15			253.0	41	0	0	365.0	67	−3030.0	1240.0	804.0
16	↓	↓	253.0	42			365.0	68	3030.0	−1240.0	804.0
17	−5680.0	2320.0	1020.0	43			378.0	69	3070.0	−520.0	1040.0
18	5680.0	2320.0	1020.0	44			378.0	70	−3070.0	520.0	1040.0
19	2310.0	−946.0	723.0	45			392.0	71	0	0	433.0
20	−2310.0	946.0	723.0	46	↓	↓	392.0	72			433.0
21	0	0	314.0	47	−4440.0	1820.0	1050.0	73			370.0
22			314.0	48	4440.0	−1820.0	1050.0	74			370.0
23			326.0	49	1890.0	−773.0	742.0	75			304.0
24			326.0	50	−1890.0	773.0	742.0	76	↓	↓	304.0
25			338.0	51	0	0	390.0	77	−1370.0	262.0	446.0
26	↓	↓	338.0	52	0	0	390.0	78	1370.0	−262.0	446.0

(b) Load condition II

	Load components, lb				Load components, lb				Load components, lb		
Node	P_x	P_y	P_z	Node	P_x	P_y	P_z	Node	P_x	P_y	P_z
1	351.0	−12 600.0	1530.0	27	1210.0	496.0	311.0	53	0	0	334.0
2	−351.0	12 600.0	1530.0	28	1210.0	−496.0	311.0	54			334.0
3	0	0	29.5	29	3990.0	−1630.0	1310.0	55			264.0
4	0	0	29.5	30	−3990.0	1630.0	1310.0	56	↓	↓	264.0
5	−2420.0	−6020.0	979.0	31	0	0	375.0	57	−1380.0	565.0	386.0
6	2420.0	6020.0	979.0	32			375.0	58	1380.0	−565.0	386.0
7	0	0	55.9	33			291.0	59	5300.0	−2170.0	1820.0
8	0	0	55.9	34			291.0	60	−5300.0	2170.0	1820.0
9	−4020.0	−3980.0	474.0	35			230.0	61	0	0	458.0
10	4020.0	3980.0	474.0	36	↓	↓	230.0	62			458.0
11	0	0	194.0	37	−1270.0	518.0	336.0	63			326.0
12			194.0	38	1270.0	−518.0	336.0	64			326.0
13			175.0	39	4160.0	−1700.0	1410.0	65			233.0
14			175.0	40	−4160.0	1700.0	1410.0	66	↓	↓	233.0
15			157.0	41	0	0	402.0	67	−922.0	377.0	287.0
16	↓	↓	157.0	42			402.0	68	922.0	−377.0	287.0
17	−1600.0	653.0	325.0	43			313.0	69	7160.0	−1210.0	2180.0
18	1600.0	−653.0	325.0	44			313.0	70	−7160.0	−1210.0	2180.0
19	5510.0	−2250.0	1550.0	45			247.0	71	0	0	484.0
20	−5510.0	2250.0	1550.0	46	↓	↓	247.0	72			484.0
21	0	0	347.0	47	−1320.0	541.0	361.0	73			310.0
22			347.0	48	1320.0	−541.0	361.0	74			310.0
23			270.0	49	4330.0	−1770.0	1500.0	75			194.0
24			270.0	50	−4330.0	1770.0	1500.0	76	↓	↓	194.0
25			213.0	51	0	0	430.0	77	−451.0	86.0	175.0
26	↓	↓	213.0	52	0	0	430.0	78	451.0	−86.0	175.0

TABLE 26. – INTERMEDIATE COMPLEXITY (IC) WING: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
IC–Wing.a	Stress	$\sigma_i \le \sigma_0 \quad (i = 1,2,\ldots,158)$ $\sigma_0 = 10.5$ ksi
	Minimum area	$\chi_i = 0.001$ in.$^2 \quad (i = 1,2,\ldots,57)$
IC–Wing.b	Displacement	Constraints along traverse direction 10 in. at node 1 10 in. at node 10
IC–Wing.c	Minimum area	Same as IC–Wing.a
	Stress	Same as IC–Wing.a
	Displacement	Same as IC–Wing.b
	Minimum area	Same as IC–Wing.a

TABLE 27. – INTERMEDIATE COMPLEXITY (IC) WING: PROBLEMS IC–WING.a, IC–WING.b, AND IC–WING.c DESIGN VARIABLE LINKAGE

Serial number	Design variable	Members linked	Serial number	Design variable	Members linked
1	1	1,2	32	32	63,64
2	2	3,4	33	33	65,66,67,68,69,70,71,72,73,74,75,
3	3	5,6			76,77,78,79,80,81,82,83,84,85,86,
4	4	7,8			87,88,89,90,91,92,93,94,95,96
5	5	9,10	34	34	97
6	6	11,12	35	35	98
7	7	13,14	36	36	99
8	8	15,16	37	37	100
9	9	17,18	38	38	101
10	10	19,20	39	39	102
11	11	21,22	40	40	103
12	12	23,24	41	41	104
13	13	25,26	42	42	105
14	14	27,28	43	43	106
15	15	29,30	44	44	107
16	16	31,32	45	45	108
17	17	33,34	46	46	109
18	18	35,36	47	47	110
19	19	37,38	48	48	111
20	20	39,40	49	49	112
21	21	41,42	50	50	113
22	22	43,44	51	51	114
23	23	45,46	52	52	115
24	24	47,48	53	53	116
25	25	49,50	54	54	117
26	26	51,52	55	55	118
27	27	53,54	56	56	119
28	28	55,56	57	57	120,121,122,123,124,125,126,127,128,129,
29	29	57,58			130,131,132,133,134,135,136,137,138,139,
30	30	59,60			140,141,142,143,144,145,146,147,148,149,
31	31	61,62			150,151,152,153,154,155,156,157,158

TABLE 28.—INTERMEDIATE COMPLEXITY (IC) WING: LINKED DESIGN CONFIGURATION

Methods	Optimality criteria (OC) formulas		Normalized weight		
	λ	χ	IC–Wing.a (stress)	IC–Wing.b (displacement)	IC–Wing.c (stress and displacement)
1	1	1	(a)	(a)	(a)
2	1	2	(a)	(a)	(a)
3	1	3	(a)	(a)	(a)
4	2	1	(b)	.99	(b)
5	2	2	3.22	1.01	
6	2	3	(b)	1.11	
7	3	1	(b)	1.05	
8	3	2	1.53	1.05	
9	3	3	3.93	1.16	
10	4	1	(b)	1.06	
11	4	2	(b)	1.06	
12	4	3	(b)	1.69	
13	5	4	2.93	1.06	
14	5	5	1.00	1.06	
15	5	6		1.06	
16	5	7		1.69	
17	0	8		1.63	1.00
SUMT[c] Weight, lb			1.00 388.390	1.00 84.114	1.00 391.902

[a]Methods 1 to 3 are not applicable.
[b]Method failed for this problem.
[c]Sequence of Unconstrained Minimizations Technique.

table 28. Note that the linear form of the Lagrange multiplier update formula is not applicable for element types with multiaxial stress fields, such as membranes. For stress constraints only (IC–Wing.a), only the fully utilized design method and the hybrid methods perform satisfactorily. For problem IC–Wing.c, where displacement constraints were added but did not become active in the SUMT solution, even the hybrid methods fail. For all methods in which matrix inversion of a constraint sensitivity matrix is required, matrix singularities are detected during inversion. Floating point divide exceptions are encountered in most of the other methods for this problem. Only the fully utilized design method is successful in this stress and displacement problem. For displacement constraints only (IC–Wing.b), nine of the methods (4, 5, 7, 8, 10, 11, 13, 14, and 15) perform well, two (6 and 9) are marginal, and two (12 and 16) are poor.

F. Forward-Swept Wing

The finite element model of the forward-swept wing (FSW), made of aluminum and shown in figure 10, has 30 grid points and a total of 135 truss elements. The loads are given in table 29. The constraints of the problem are given in table 30. Two design cases are considered. The first one treats all 135 element areas as independent design variables, whereas in the second case the bar areas are linked to obtain a reduced set of 61 linked design variables, as shown in table 31. For each case, three situations are considered: sets FSW.a, FSW.b, and FSW.c include stress, displacement, and frequency constraints, respectively, for the unlinked problem; whereas sets FSW.d, FSW.e, and FSW.f include stress, displacement, and frequency constraints, respectively, for the linked problem. The number of active constraints found by

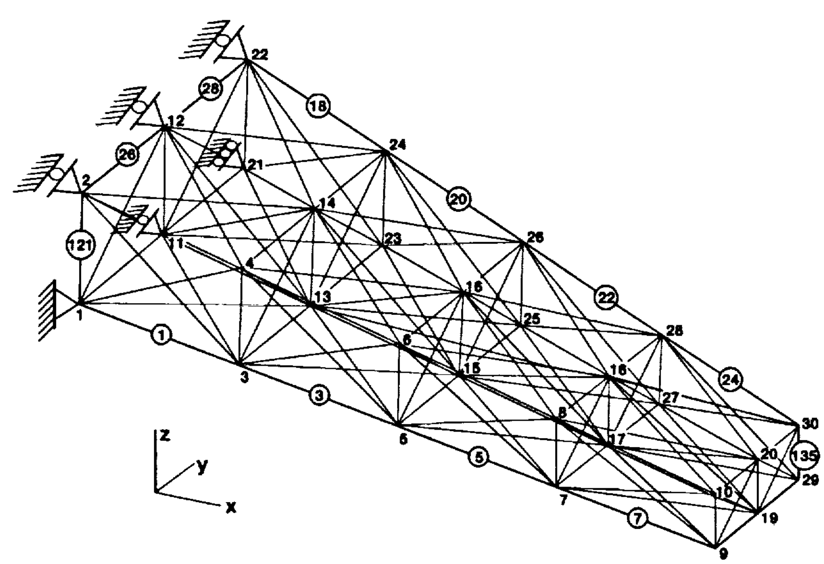


Figure 10.—Forward-swept wing. (Representative elements are circled, nodes are not.)

TABLE 29. – OPTIMUM DESIGN OF FORWARD SWEPT WING: LOAD SPECIFICATIONS

Problem	Load conditions	Load components, kips			
		Node	P_x	P_y	P_y
FSW.a FSW.b FSW.c FSW.d FSW.e FSW.f	I	9 10 29 30	0.0	0.0	40.0 40.0 20.0 20.0
FSW.c FSW.f	Lumped masses	(1,2,...,8,10) 9 (11,12,...,20) (21,22,...,25,27,28,...,30) 26	$m_x = m_y = m_z =$ 15 lb $m_x = m_y = m_z =$ 100 lb $m_x = m_y = m_z =$ 30 lb $m_x = m_y = m_z =$ 18 lb $m_x = m_y = m_z =$ 300 lb		

TABLE 30. – FORWARD-SWEPT WING (FSW): CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
FSW.a FSW.d	Stress	$\sigma_i \le \sigma_0$ $(i = 1,2,\ldots,135)$ $\sigma_0 = 10$ ksi
	Minimum area	$\chi_i = 0.25$ in.^{2} $(i = 1,2,\ldots,135)$ for FSW.a $\chi_i = 0.25$ in.^{2} $(i = 1,2,\ldots,61)$ for FSW.d
FSW.b FSW.e	Displacement	Constraints along traverse direction 10.0 in. at node 10 10.0 in. at node 30
FSW.c FSW.f	Minimum area	Same as FSW.a or FSW.d
	Frequency	$f \ge f_0$; $f_0 = 10$ Hz
	Minimum area	Same as FSW.a or FSW.d

TABLE 31. – FORWARD-SWEPT WING (FSW): PROBLEMS FSW.d, FSW.e, AND FSW.f DESIGN VARIABLE LINKAGE

Serial number	Design variable	Members linked	Serial number	Design variable	Members linked
1	1	121,122,123,124,125, 126,127,128,129,130, 131,132,133,134,135	31 32 33	31 32 33	59,60 61,62 63,64
2	2	1,2	34	34	65,66
3	3	3,4	35	35	67,68
4	4	5,6	36	36	69,70
5	5	7,8	37	37	71,72
6	6	9,10	38	38	73,74
7	7	11,12	39	39	75,76
8	8	13,14	40	40	77,78
9	9	15,16	41	41	79,80
10	10	17,18	42	42	81,82
11	11	19,20	43	43	83,84
12	12	21,22	44	44	85,86,
13	13	23,24	45	45	87,88
14	14	25,26	46	46	89,90
15	15	27,28	47	47	91,92
16	16	29,30	48	48	93,94
17	17	31,32	49	49	95,96
18	18	33,34	50	50	97,98
19	19	35,36	51	51	99,100
20	20	37,38	52	52	101,102
21	21	39,40	53	53	103,104
22	22	41,42	54	54	105,106
23	23	43,44	55	55	107,108
24	24	45,46	56	56	109,110
25	25	47,48	57	57	111,112
26	26	49,50	58	58	113,114
27	27	51,52	59	59	115,116
28	28	53,54	60	60	117,118
29	29	55,56	61	61	119,120
30	30	57,58			

SUMT for the six problems (FSW.a to FSW.f) are 75 stress, 2 displacement, 1 frequency, 68 stress, 2 displacement, and 1 frequency constraint, respectively.

Results for the six problems solved by the 17 methods are given in tables 32 and 33. Many of the first 16 methods perform adequately for problems FSW.b, FSW.c, FSW.e, and FSW.f (displacement or frequency constraints). Most of the methods that include a reciprocal form for the design variable update formula (3, 9, 12, and 16) perform very marginally, at best, for problems with displacement constraints (FSW.b and FSW.e). The only other method that includes a reciprocal form (method 6) has some difficulty with frequency constraints (FSW.c and FSW.f). Method 13 does not perform well on the frequency-constrained problems either. For problems FSW.a and FSW.d (stress constraints), only the hybrid methods and the fully utilized design methods perform well, with differences within 1 percent of the SUMT results.

G. Three-Bar Truss

Several additional features are illustrated with a three-bar truss, as shown in figure 1. The first two problems are con-

TABLE 32. – FORWARD SWEPT WING (FSW)

Methods	Optimality criteria (OC) formulas		Normalized weight		
	λ	χ	FSW.a (stress)	FSW.b (displacement)	FSW.c (frequency)
1	1	1	5.64	1.19	1.09
2	1	2	5.39	1.15	1.08
3	1	3	(a)	2.37	1.07
4	2	1	9.39	1.09	1.00
5	2	2	1.31	1.05	1.00
6	2	3	1.67	1.03	1.18
7	3	1	19.47	1.09	1.01
8	3	2	1.53	1.09	↓
9	3	3	1.98	1.40	↓
10	4	1	8.36	1.07	↓
11	4	2	1.57	1.07	↓
12	4	3	(a)	1.33	↓
13	5	4	1.16	1.08	3.01
14	5	5	1.01	1.07	1.01
15	5	6	↓	1.07	1.01
16	5	7	↓	1.33	1.01
17	0	8	↓	2.80	2.76
SUMT[b] Weight, lb			1.00 2 793.179	1.00 672.574	1.00 230.045

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 33. – FORWARD SWEPT WING (FSW)

Methods	Optimality criteria (OC) formulas		Normalized weight		
	λ	χ	FSW.d (stress)	FSW.e (displacement)	FSW.f (frequency)
1	1	1	5.23	1.17	1.10
2	1	2	3.72	1.13	1.09
3	1	3	(a)	2.34	1.07
4	2	1	5.64	1.08	1.00
5	2	2	1.24	1.05	1.00
6	2	3	1.32	1.03	1.19
7	3	1	24.15	1.10	1.02
8	3	2	1.58	1.10	↓
9	3	3	2.26	1.39	↓
10	4	1	19.76	1.07	↓
11	4	2	1.49	1.07	↓
12	4	3	(a)	1.40	↓
13	5	4	1.60	1.07	2.99
14	5	5	1.00	1.07	1.02
15	5	6	↓	1.07	1.02
16	5	7	↓	1.40	1.02
17	0	8	↓	2.73	2.74
SUMT[b] Weight, lb			1.00 3 009.874	1.00 690.338	1.00 231.803

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.

TABLE 34. – THREE-BAR TRUSS: LOAD SPECIFICATIONS

Problem	Load conditions	Load components, kips		
		Node	P_x	P_y
Bar3.a	I	1	50.0	100.0
	II	1	−50.0	−100.0
	III	1	50.0	0.0
Bar3.b	I	1	50.0	100.0
Bar3.c	I	1	70.0	0.0
	II	1	−35.0	−95.0
Bar3.a Bar3.b	Lumped masses	1	$m_x = m_y = 262.5$ lb	

TABLE 35.—THREE-BAR TRUSS: CONSTRAINT SPECIFICATIONS

Problem	Constraint type	Constraint description
Bar3.a Bar3.b	Stress	$\sigma_i \leq \sigma_0 \quad (i = 1,2,3)$ $\sigma_0 = 20$ ksi
	Displacement	Constraints at node 1 of magnitude: 0.20 in. along the x direction 0.05 in. along the y direction
	Frequency	$f \geq f_0; f_0 = 114.3$ Hz
	Minimum area	$\chi_i = 0.001$ in.^{2} $\quad (i = 1,2,3)$
Bar3.c	Stress	$\sigma_i \leq \sigma_0 \quad (i = 1,2,3)$ $\sigma_0 = 15$ ksi
	Minimum area	Same as Bar3.a

sidered with three load conditions (for Bar3.a) and a single load condition (for Bar3.b), as given in table 34. Note that the load condition in problem Bar3.b is identical to the first load condition in problem Bar3.a. The stress, displacement, and frequency constraints are specified in table 35. SUMT provides an optimal solution with one active frequency, two active displacement, and three active stress constraints for problem Bar3.a, as well as one active frequency, one active displacement, and one active stress constraint for problem Bar3.b. The results of these two problems, shown in tables 36 and 37, are discussed in section V.

The third problem in this set (Bar3.c) considers six cases with stress constraints under two load conditions (see tables 34 and 35). In the first case, the densities of the three bars are taken to be the same (0.1 lb/in.^{3}). In each of the subsequent cases, the densities are allowed to differ. The six cases within this problem are each solved with several optimizers, including OC (refs. 3 and 4), SUMT (ref. 21), the International Mathematical and Statistical Library's

TABLE 36.—OPTIMUM DESIGN OF THREE-BAR TRUSS: PROBLEM BAR3.a WITH THREE LOAD CONDITIONS

[Stress, displacement, and frequency constraints. Case 1: six active constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}			Active Constraints		
	λ	χ		χ_1	χ_2	χ_3	Stress	Displacement	Frequency
1	1	1	118.475	1.993	3.863	3.652	(b)	g_{11},g_{13}	(b)
2	1	2	118.287	1.985	3.866	3.646	(b)	g_{11},g_{13}	
3	1	3	116.807	1.917	3.882	3.598	g_3,g_6	g_{11},g_{13}	
4	2	1	166.293	4.247	2.852	5.495	(b)	g_{11},g_{13}	
5	2	2	109.423	1.008	4.726	3.388	g_3,g_6,g_7	(b)	g_{16}
6	2	3	128.181	2.423	3.700	4.025	(b)	g_{11},g_{13}	(b)
7	3	1	15 460.000	183.667	535.264	531.058		(b)	(b)
8	3	2	15 948.900	191.988	566.978	534.860		(b)	(b)
9	3	3	147 900.000	(a)	271.077	271.077		(b)	g_{16}
10	4	1	123.696	1.080	3.166	5.428		g_{11},g_{13}	(b)
11	4	2	105.878	1.224	3.718	3.634	g_3,g_6,g_7	g_{11},g_{13}	g_{16}
12	4	3	107.537	1.251	3.671	3.757	(b)	g_{11},g_{13}	(b)
13	5	4	117.430	1.445	3.432	4.432		g_{11},g_{13}	
14	5	5	126.122	1.141	3.128	5.565		g_{11},g_{13}	
15	5	6	205.515	6.429	1.939	6.732		g_{11},g_{13}	
16	5	7	14 390.000	.151	360.764	762.766		(b)	
17	0	8	122.221	1.571	3.333	4.714		g_{11},g_{13}	
SUMT[c]			100.078	1.091	3.848	3.264	g_3,g_6,g_7	g_{11},g_{13}	g_{16}

[a]Area greater than 1000 in^2.

[b]No active constraints of this type.

[c]Sequence of Unconstrained Minimizations Technique.

TABLE 37.—OPTIMUM DESIGN OF THREE-BAR TRUSS: PROBLEM BAR3.b WITH ONE LOAD CONDITION

[Stress, displacement, and frequency constraints. Case 2: three active constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Optimum design variables, in.^{2}			Active constraints		
	λ	χ		χ_1	χ_2	χ_3	Stress	Displacement	Frequency
1	1	1	115.125	1.840	3.899	3.543	g_3	g_5	(b)
2	1	2	117.322	1.940	3.876	3.615	(b)	g_5	(b)
3	1	3	117.598	1.953	3.874	3.624	(b)	g_5	(b)
4	2	1	100.085	1.091	3.851	3.262	g_3	g_5	g_6
5	2	2	112.170	1.470	3.661	3.873	(b)	g_5	(b)
6	2	3	150.214	.001	6.070	6.373		(b)	g_6
7	3	1	13 000.000	242.722	274.612	480.441			(b)
8	3	2	14 300.000	142.087	583.845	455.524			
9	3	3	12 400.000	386.746	0.000	494.501			
10	4	1	53 800.000	.001	2 232.740	2 232.470			
11	4	2	46 200.000	1 206.536	1 206.536	1 206.536			
12	4	3	147 000.000	270.881	270.881	(a)			g_6
13	5	4	135.991	.257	4.613	6.097			g_6
14	5	5	180.738	2.931	3.037	7.701			(b)
15	5	6	4 444.091	72.560	155.956	131.407			(b)
16	5	7	611.774	36.218	3.678	4.440		g_5	(b)
17	0	8	150.728	.002	5.023	7.104		(b)	g_6
SUMT[c]			100.074	1.089	3.848	3.266	g_3	g_5	g_6

[a]Area greater than 1000 in^2.
[b]No active constraints of this type.
[c]Sequence of Unconstrained Minimizations Technique.

nonlinear optimization routine based on sequential quadratic programming (IMSL) (ref. 9), another sequential quadratic programming code (SQP) (ref. 7), a sequential linear programming code (SLP) (ref. 8), and feasible directions (FD) (ref. 8). These results are summarized in table 38 and examined in section V.

V. Discussion

In this section the overall performance of the methods are examined on the basis of the numerical examples solved, and some related topics are discussed.

A. Performance of Optimality Criteria Methods

The derivable OC method was developed first for displacement constraints and was later extended for stiffness constraints that can be stated in terms of work quantities. The external work of a virtual load system along the constrained displacement is an example. Convergence, as in the case of the heuristic fully stressed design, is a function only of the sensitivity of the internal forces to changes in the number of truss members. The method exhibits excellent performance if the behavior approximates that of statically determinate structures. Convergence is not a function of the number of design variables. Buckling and vibration frequency constraints also can be stated in energy terms and brought under the OC formulation, but because of the nature of the eigenvalue problem, OC convergence can be dependent on the specific nature of the structure.

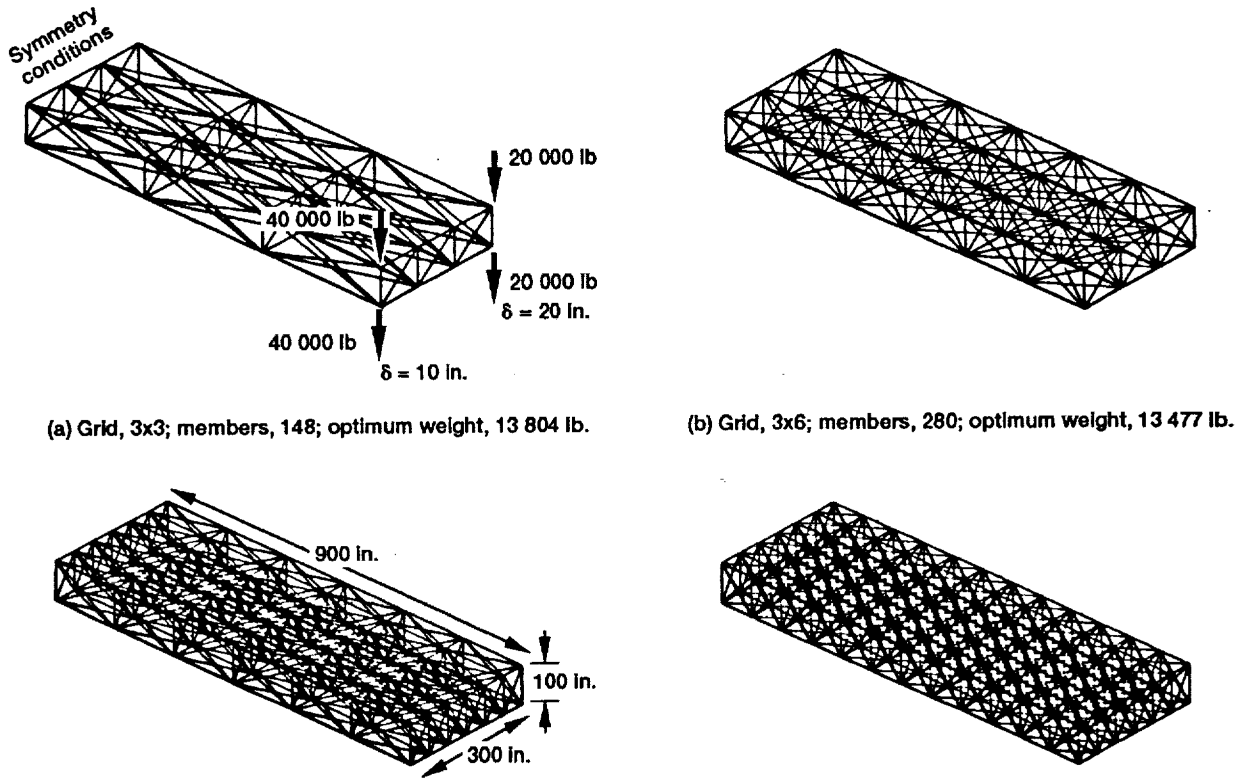
The elegancy of the criteria for displacement constraints is illustrated by considering a set of four large, three-dimensional trusses with 148 to 1027 design variables (figs. 11(a) to (d)). The trusses shown in figure 11(a) have four concentrated loads of different magnitudes (20 and 40 kip) which simultaneously induces twisting and flexing of the cantilevered trussed slabs. Two displacement constraints are imposed: the displacement must not exceed (1) 10 in. at the higher load point and (2) 20 in. at the lower load point (fig. 11(a)), and at optimum both displacement constraints are active. The problem is solved using the OC method. Design weight versus iteration history

TABLE 38. – OPTIMUM DESIGNS OF THREE-BAR TRUSS: PROBLEM BAR3.c WITH TWO LOAD CONDITIONS

[Number of active constraints exceed number of design variables.]

Method[a]	Cost coefficients	Member areas			Active constraints	Optimum weight, lb	Remarks
SUMT	0.1 0.1 0.1	3.291	3.986	3.323	g_1,g_3,g_5,g_6	1.33×10^2	
	3 6 8	3.299	3.998	3.299	↓	7.53×10^3	
	6 12 18	3.299	3.997	3.298	↓	1.59×10^4	
	16 13 25	3.298	3.998	3.299	↓	2.43×10^4	
	1 200 300	67.068	9.111	0.001	g_5,g_6	1.92×10^5	(c)
	1 5 900	56.629	−1.339	−1.937	(b)	-2.39×10^5	(d)
OC 14	.1 .1 .1	3.299	3.999	3.300	g_1,g_3,g_5,g_6	1.33×10^2	
	3 6 8	↓	↓	↓	↓	7.53×10^3	
	6 12 18	↓	↓	↓	↓	1.60×10^4	
	16 13 25	↓	↓	↓	↓	2.43×10^4	
	1 200 300	↓	↓	↓	↓	2.20×10^5	
	1 5 900	↓	↓	↓	↓	4.22×10^5	
OC17	.1 .1 .1	3.299	3.999	3.300	g_1,g_3,g_5,g_6	1.33×10^2	
	3 6 8	↓	↓	↓	↓	7.53×10^3	
	6 12 18	↓	↓	↓	↓	1.60×10^4	
	16 13 25	↓	↓	↓	↓	2.43×10^4	
	1 200 300	↓	↓	↓	↓	2.20×10^5	
	1 5 900	↓	↓	↓	↓	4.22×10^5	
OC 2	.1 .1 .1	5.469	.007	12.655	g_5	2.56×10^2	(c)
	3 6 8	3.234	3.919	3.446	g_1,g_3,g_5,g_6	7.62×10^3	
	6 12 18	3.242	3.936	3.426	g_1,g_3,g_5,g_6	1.62×10^4	
	16 13 25	3.251	3.958	3.407	g_1,g_3,g_5,g_6	2.45×10^4	
	1 200 300	4.839	3.776	3.574	g_5,g_6	2.28×10^5	
	1 5 900	3.340	4.031	3.288	g_1,g_3,g_5,g_6	4.21×10^5	
IMSL	.1 .1 .1	1.000	1.000	1.000	(b)	3.83×10^1	(e)
	3 6 8	3.299	4.000	3.299	g_1,g_3,g_5,g_6	7.53×10^3	
	6 12 18	↓	↓	↓	↓	1.60×10^4	
	16 13 25	↓	↓	↓	↓	2.43×10^4	
	1 200 300	↓	↓	↓	↓	2.20×10^5	
	1 5 900	13.352	11.509	.001	g_6	7.77×10^3	(c)
SQP	.1 .1 .1	2.335	2.503	2.505	(b)	9.35×10^1	
	3 6 8	2.334	↓	↓	↓	5.32×10^3	
	6 12 18	2.334	↓	↓	↓	1.14×10^4	
	16 13 25	2.335	↓	↓	↓	1.74×10^4	
	1 200 300	2.335	↓	↓	↓	1.57×10^5	(e)
	1 5 900	8.144	8.729	.001	g_5	5.64×10^3	(d)
SLP	.1 .1 .1	3.313	3.971	3.323	g_1,g_3,g_5,g_6	1.33×10^2	↓
	3 6 8	3.309	3.963	3.334	↓	7.55×10^3	↓
	6 12 18	3.309	3.962	3.335	↓	1.60×10^4	↓
	16 13 25	3.308	3.961	3.336	↓	2.44×10^4	↓
	1 200 300	3.300	3.967	3.328	↓	2.21×10^5	
	1 5 900	12.630	11.726	.001	g_3,g_6	7.78×10^3	(d)
FD	.1 .1 .1	3.199	2.556	5.102	g_1,g_5	1.42×10^2	(c)
	3 6 8	3.501	2.402	4.682	(b)	8.22×10^3	(d)
	6 12 18	3.635	2.353	4.563	g_6	1.75×10^4	(e)
	16 13 25	3.657	2.316	4.467	g_6	2.71×10^4	(e)
	1 200 300	3.684	2.336	4.512	g_6	2.39×10^5	(c)
	1 5 900	3.167	2.105	3.822	g_1,g_3	4.88×10^5	(c)

[a]Methods defined in appendix B.
[b]No active constraints.
[c]Incomplete.
[d]Method failed for this problem.
[e]Final design is infeasible.



(a) Grid, 3x3; members, 148; optimum weight, 13 804 lb.

(b) Grid, 3x6; members, 280; optimum weight, 13 477 lb.

(c) Grid, 6x6; members, 529; optimum weight, 14 310 lb.

(d) Grid, 6x12; members, 1027; optimum weight, 17 455 lb.

Figure 11.—Three-dimensional truss problems.

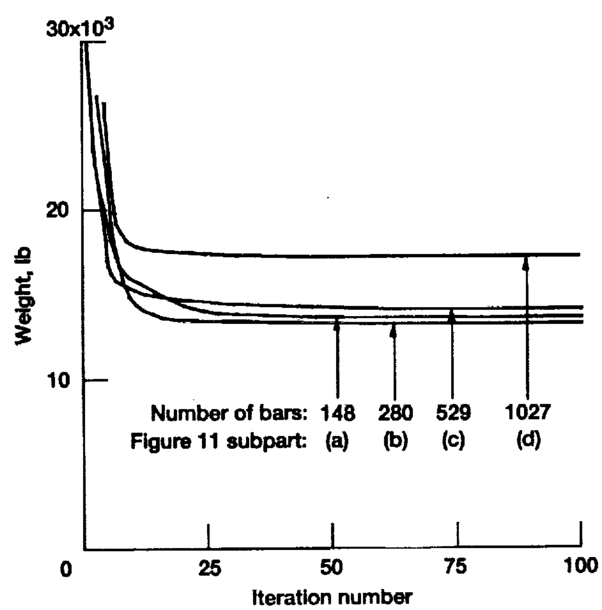


Figure 12.—Convergence curves for three-dimensional truss problems.

(fig. 12) indicates smooth monotonic convergence, and the number of iterations required for convergence also is independent of the number of design variables for all four examples. The OC method performed very well for all four design problems.

In this study, the optimality criteria method achieves satisfactory solutions, closely matching the optimal design obtained with the use of the mathematical programming technique SUMT, for problems in which only displacement and/or frequency constraints arise. For each of these problems, at least one of the OC methods produces a result close to the SUMT answer (usually within 1 percent, always within 3 percent). The most consistent method for these problems is OC method 5 (with the use of the exponential form of the Lagrange multiplier update formula and the linearized form of the design variable update formula). This OC method shows a 19 percent heavier design in the worst case (Bar60.c; table 24).

For problems that contain only a few design variables (up to 10) for which stresses are constrained, for which displacements and/or frequencies may also be constrained, the optimality criteria method also appears to be satisfactory. In the worst case (Bar10.a; table 17), the best solution gives a 7 percent heavier design than SUMT.

For larger problems that include stress constraints, the optimality criteria method may not be as effective. The best solution for the best case (Bar60.f; table 24) is 5 percent over-designed. For the worst case (IC–Wing.c), the solution diverged for all of the OC methods (see table 28). The cause of the divergence may, however, be due to linear functional dependence among active constraints (see ref. 16). For a problem with a large number of active constraints that include stress limitations, the optimality criteria method appears to follow a subset (or subsets) of these active constraints, and possibly for this reason yields a heavier design. For example, the optimum solution obtained by SUMT for problem Bar60.f shows a total of 19 active constraints, consisting of 1 frequency, 1 displacement, and 17 stress constraints. The designs of the ring given by the OC methods yield over-designs with fewer active constraints (see table 39). For example, method 1, which over-designs by 26 percent, has only six active stress constraints and neither displacement nor frequency constraints become active. The best solution (method 5) yields a 5 percent over-design with nine active stress constraints, as well as active displacement and frequency constraints.

This observation is seen in all the larger problems in which stress constraints play a role (see table 40). No OC

TABLE 39.—ACTIVE CONSTRAINTS BY TYPE AND NORMALIZED WEIGHTS, 60-BAR TRUSSED RING: PROBLEM BAR60.f
[Stress, displacement, and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Normalized weight	Index of active constraints by type		
	λ	χ		Stress	Displacement	Frequency
1	1	1	1.26	1,15,31,36,43,48	(c)	(c)
2	1	2	2.73	36,48		(c)
3	1	3	3.08	36,48		(c)
4	2	1	1.48	1,13		184
5	2	2	1.05	1,13,26,30,31,36,42,43,48	181	
6	2	3	1.45	(c)	(c)	
7	3	1	3.03	6,18		
8	3	2	1.39	80		
9	3	3	(a)	(c)		
10	4	1	4.04	(c)		
11	4	2	4.09	(c)		(c)
12	4	3	1.32	25,30,36,37,42,48,80		(c)
13	5	4	1.20	25,30,36,37,42,48,80		184
SUMT[b]			1.00	1,7,12,13,25,26,30,31,36, 37,42,43,48,49,55,80,114	181	184

[a]Method failed for this problem.
[b]Sequence of Unconstrained Minimizations Technique.
[c]No active constraints of this type.

TABLE 40.—NUMBER OF ACTIVE CONSTRAINTS AND NORMALIZED WEIGHT

[Active constraint types are stress σ, displacement X, and frequency f.]

Methods	Optimality criteria (OC) formulas		Bar60.a		Bar60.d			Bar60.f				IC-Wing.a		FSW.a		FSW.d	
	λ	χ	Weight	σ	Weight	σ	X	Weight	σ	X	f	Weight	σ	Weight	σ	Weight	σ
1	1	1	1.59	[a]8	1.48	8	0	1.26	6	0	0	(b)	(b)	5.64	2	5.23	2
2	1	2	2.24	2	1.42	9	0	2.73	2	0	0	(b)	(b)	5.39	1	3.72	4
3	1	3	1.80	6	1.76	4	0	3.08	2	0	0	(b)	(b)	(b)	(b)	(b)	(b)
4	2	1	2.16	2	1.69	2	0	1.48	2	0	1	(b)	(b)	9.39	2	5.64	5
5	2	2	1.30	15	1.09	17	1	1.05	9	1	1	3.22	1	1.31	14	1.24	21
6	2	3	(b)	(b)	(b)	(b)	(b)	1.45	0	0	1	(b)	(b)	1.67	26	1.32	19
7	3	1	4.63	5	2.44	2	0	3.03	2	0	1	(b)	(b)	19.47	1	24.15	1
8	3	2	1.81	6	2.36	1	0	1.39	1	0	1	1.53	9	1.53	5	1.58	5
9	3	3	(b)	(b)	(b)	(b)	(b)	(b)	(b)	(b)	(b)	3.93	7	1.98	3	2.26	7
10	4	1	(b)	(b)	2.29	2	0	4.04	0	0	1	(b)	(b)	8.36	2	19.76	1
11	4	2	2.60	4	3.96	0	0	4.09	0	0	0	(b)	(b)	1.57	2	1.49	7
12	4	3	(b)	(b)	1.63	1	0	1.32	7	0	0	(b)	(b)	(b)	(b)	(b)	(b)
13	5	4	2.11	3	1.39	11	0	1.20	7	0	1	2.93	8	1.16	51	1.60	18
SUMT[c]			1.00	30	1.00	24	1	1.00	17	1	1	1.00	108	1.00	75	1.00	68

[a]Number of active constraints.
[b]Method failed for this problem.
[c]Sequence of Unconstrained Minimizations Technique.

method gives the full set of active constraints found by SUMT for any of the larger problems. In each case, the best OC method gives the largest number of active constraints (method 5 for the three 60-bar truss problems and problem FSW.d, method 8 for IC–Wing.a, and method 13 for problem FSW.a). Improving the ability of these methods to drive the solution to contain a more appropriate number of constraints in the active set would probably improve the performance of the OC methods.

B. Performance of Fully Utilized Design Method

For stress constraints alone, the fully utilized design (method 17) produces good results for all the problems that have been solved. Furthermore, for simultaneous stress and displacement constraints, method 17 gave a satisfactory design for problem Bar5.b (table 12), the ring (Bar60.d, table 24), and for problem IC–Wing.c (table 28). For problems Bar5.c (table 13) and the 10-bar truss problems (tables 17 to 20), the addition of a frequency constraint does not cause any further difficulty. However, with problem Bar60.f (table 24), method 17 gives a 54 percent heavier design than obtained with SUMT.

To examine this last observation in more detail, consider, first, situations in which no stress constraints occur (or none become active). In this case, the design is obtained by a single scaling of the design variables, which normally produces only one active constraint. It has been seen that, in the absence of stress constraints, the design can be significantly heavier. In a similar way, when more than one nonstress constraint could become active (such as one displacement and one frequency constraint), the method 17 design is obtained from the fully stressed design by a single uniform rescaling for violated displacement or frequency constraints. This rescaling typically produces significantly fewer active constraints than a mathematical programming technique, and often results in a heavier design.

To further illustrate this aspect of FUD, recall problem Bar3.a (table 36), in which SUMT yields an optimal weight of 100 lb with six active constraints, consisting of one frequency, two displacement, and three stress constraints. The fully utilized design (method 17) yields an over-design (122 lb) with only two active constraints. Neither the frequency constraint nor any of the stress constraints becomes active with FUD.

C. Performance of Hybrid Methods

In the hybrid methods, stress constraints are treated separately from displacement and frequency constraints. The stress ratio technique is adopted for stress constraints, and optimality criteria techniques are followed for displacement and frequency constraints. Thus, when problems contain stress constraints alone, the hybrid methods produce results identical to those of the fully utilized design. Similarly, when problems contain only displacement or frequency constraints, the hybrid methods usually produce results identical to those of OC methods 10, 11, and 12. We expected the hybrid methods to work better than FUD or OC on problems in which both stress and nonstress constraint types appear. With these problems, however, the hybrid methods give unsatisfactory results. This may be partially explained by the generally poor results of methods 10, 11 and 12 in those situations. The hybrid methods sometimes perform somewhat better than the corresponding modified OC method, but sometimes worse (see, for example, Bar60.d and Bar60.f in table 24). Use of the exponential or linear form of the Lagrange multiplier update formulas (designated 2 and 1, respectively) may improve results.

D. Convergence Characteristics

Examination of intermediate results each time a structural analysis is completed during optimization of the 60-bar trussed ring under displacement and frequency constraints (Bar60.e) provides some insight into the convergence characteristics of the optimality criteria methods. Figure 13 depicts the weight of the design at each call to the analyzer for three of the OC methods, as well as for SUMT. Table 41 gives the total number of calls to the analyzer, as well as the total CPU time required to arrive at the final design when the analysis is run on one processor of a Cray-YMP8/8128 running version 6.0 of the Unicos operating system and using version 5.0 of the cft77 Fortran compiler.

All four methods show oscillations initially. The SUMT oscillations show ever-decreasing amplitude and a slowly increasing mean until convergence to 392 lb is reached after 282 analyzer calls. The OC method 16 proceeds to a point near 150 calls, then shows very small oscillations around 413 lb for another 250 calls. Method 2 shows a smooth convergence to a low-weight infeasible design after 264 calls, then jumps to 398 lb to regain feasibility. Method 5 continues to oscillate throughout the optimization procedure around a somewhat heavier design, ending with a final weight of 449 lb after 429 analyzer calls. The larger CPU times used by the OC methods were partly due to the larger number of calls to the analyzer, and partly due to an increased fraction of those calls that required constraint gradient information. Overall, the convergence characteristics for some of the optimality criteria techniques appear to be competitive with other nonlinear optimization methods.

E. Number of Design Variables and Number of Constraints

For an optimal design problem with n independent design variables, the consideration of a maximum of n active constraints is sufficient to establish the optimal point. Active

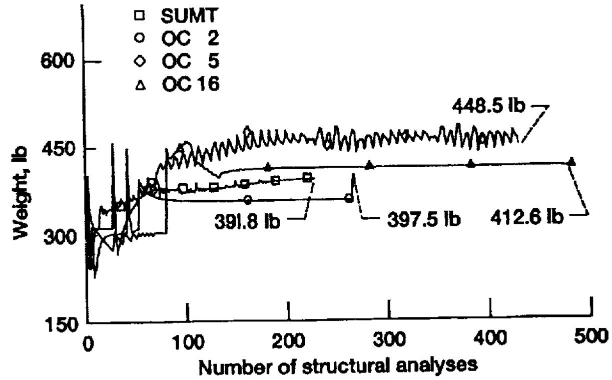


Figure 13.—Sixty-bar trussed ring: problem Bar 60.e. Displacement analyzer, displacement, and frequency constraints. (SUMT, Sequence of Unconstrained Minimizations Technique; OC, optimality criteria.)

TABLE 41.—60-BAR TRUSSED RING: PROBLEM BAR60.e
[Displacement and frequency constraints.]

Methods	Optimality criteria (OC) formulas		Optimum weight, lb	Number of structural analyses	Cray–YMP CPU time, sec
	λ	χ			
2	1	2	397.505	265	136.774
5	2	2	448.536	429	223.209
16	5	7	412.579	483	251.177
SUMT[a]			391.815	228	85.000

[a]Sequence of Unconstrained Minimizations Technique.

constraints in excess of these n, which have been termed as follower constraints, can be ignored without affecting the solution of the optimization problem (see section II.B.1). This feature is numerically illustrated by the three-bar truss problem Bar3.a (see tables 34 to 37). As mentioned earlier, of the 16 specified constraints, the optimal design given by SUMT has 6 active constraints, consisting of 1 frequency, 2 displacement, and 3 stress constraints. The optimal weight is 100.042 lb, and the design variables are 1.089, 3.847, and 3.265 in.^{2} For this problem, three active constraints are sufficient to establish the optimal point. When solved by limiting the active constraints to three, the problem yields, via SUMT, a virtually identical optimal design, with a weight of 100.074 lb and a design of 1.089, 3.847, and 3.266 in.^{2} This indicates that for this problem of three design variables, three active constraints are sufficient to fix the optimal design point.

The observation that the objective function does not participate in the optimization process when the number of active constraints are equal to or greater than the number of design variables (see section II.B.1) is numerically illustrated by the three-bar truss problem Bar3.c (see tables 34, 35, and 38). For the problem with equal weight densities of 0.1 lb/in.^{3} for each of the three elements, the optimal areas obtained are 3.299, 3.999, and 3.300 in.^{2}. The optimal design has four active constraints. Observing the results found as the cost coefficients are varied over a wide range of weight densities (from 0.1 to 900 lb/in.^{3}; see table 38) shows that the optimal areas do not change with respect to changes in the cost coefficients as long as the number of active constraints are greater than or equal to the number of design variables (three). Because of the failure of SUMT in some cases for this problem, several other mathematical programming techniques were used in addition to SUMT and OC. It is interesting to note the performance of the optimization methods. The hybrid OC (method 14) and the FUD (method 17) perform well for all variations of the cost coefficients. As indicated, SUMT fails for two cases, one of which produces a negative weight. The IMSL routine fails for two cases, whereas the other SQP code fails for all six cases. The SLP code fails once, and the FD code fails for three cases and yields heavy designs for the other three cases.

F. Efficiency of Analysis Methods

The 60-bar trussed ring with stress and displacement constraints (Bar60.d) is considered in further detail to examine the effects of using different analysis methods. The problem is solved with three OC methods and SUMT, each using the three analyzers discussed in section III.C: (1) the displacement method, (2) IFM, and (3) a simplified IFM. The final weights, number of structural analyses required, and the CPU times to obtain the solution on a Cray-YMP8/8128 are given in table 42. Differences in final weights range from less than 1 percent for SUMT to 23 percent for method 16. The number of structural analyses do not show significant differences, except for method 16. The CPU times reflect the slightly higher cost for using the implementation of IFM found in CometBoards and show a significantly lower cost for using the simplified IFM at each call to the analyzer.

The significantly fewer number of analysis cycles with the use of the simplified integrated force method for method 16 and the lower final weight can be explained by plotting the weight of each design versus the number of calls to the analyzer (see fig. 14). Initially, the results are indistinguishable. Near call 40, however, both the displacement method and IFM cause OC to move from an infeasible design to near the optimal design found by SUMT. These paths then oscillated near the optimum until near call 200, where the solution moved to a heavier design. This movement may have been

TABLE 42.—60-BAR TRUSSED RING: PROBLEM BAR60.d
[Stress and displacement constraints.]

Methods	Optimality criteria (OC) formulas λ	χ	Displacement method	Integrated force method	Simplified integrated force method
			Optimum weight		
2	1	2	435.522	431.838	441.228
5	2	2	345.041	360.779	400.335
16	5	7	398.102	398.102	324.300
SUMT[a]			308.730	308.729	308.896
			Number of structural analyses required		
2	1	2	69	69	71
5	2	2	805	805	805
16	5	7	433	433	90
SUMT[a]			81	81	78
			Cray-YMP CPU time, sec		
2	1	2	27.342	35.881	15.702
5	2	2	322.279	421.115	174.789
16	5	7	173.184	224.857	19.563
SUMT[a]			23.511	41.953	17.267

[a]Sequence of Unconstrained Minimizations Technique.

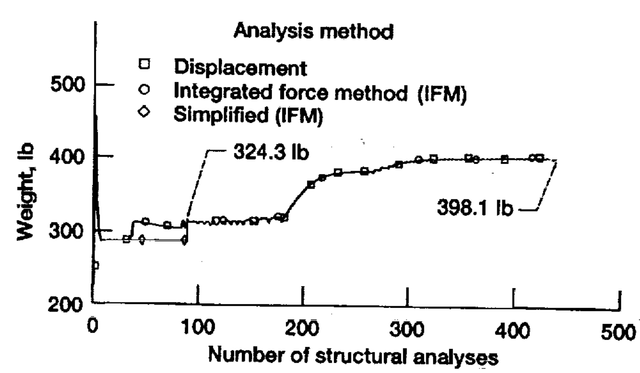


Figure 14.—Sixty-bar trussed ring: problem Bar 60.d. Hybrid optimality criteria method 16 (Lagrange multiplier update 5 with design variable update 7). Stress and displacement constraints.

caused by linear functional dependence among active constraints (see ref. 16).

Overall, the three analysis methods appear to give similar solutions in most cases, but the simplified integrated force method is more efficient. The optimality criteria methods also may be more sensitive to small differences in constraint gradients than SUMT is.

G. Linking of Design Variables

The last two example problems (Bar10.c and Bar10.d) of the 10-bar truss shown in figure 8 (see tables 14 to 16) are considered to examine the effect of design variable linkage. Recall that each of the 10 elements is taken as a design variable in Problem Bar10.c, but these elements are linked to form only five design variables in Problem Bar10.d. The optimal designs and the convergence histories are given in tables 19 and 20, as well as in figures 15 and 16.

First observe that the overall optimal weight for the linked case (Bar10.d) is heavier than for the unlinked case by 9.4 percent. This is expected, because the area of any given element within a linked design variable group is determined on the basis of the most critical bar element of the group. It is not allowed to be reduced even if that particular element's contribution to the carrying of the loads is much less important.

Next, consider the optimal design itself. In particular, the unlinked case has two element areas near their minimum bounds (0.103 in.^{2} for element area 3, and 0.104 in.^{2} for element area 6). The structure obtained when these two elements, with negligible areas, are removed becomes a determinant structure (see fig. 17). Even though the design is a

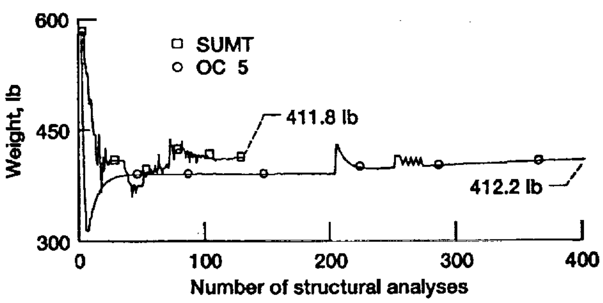


Figure 15.—Ten-bar trussed ring: problem Bar 10.c. Displacement analyzer, displacement, and frequency constraints. (SUMT, Sequence of Unconstrained Minimizations Technique; OC, optimality criteria.)

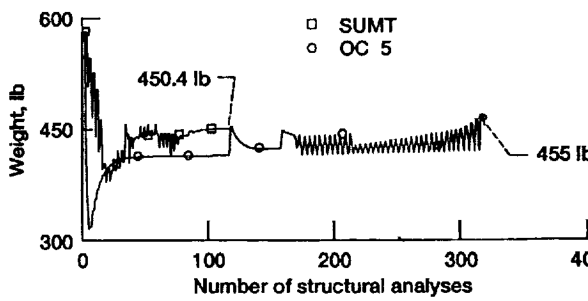


Figure 16.—Ten-bar truss: problem Bar 10.d. Displacement analyzer linked design configuration displacement, and frequency constraints. (SUMT, Sequence of Unconstrained Minimizations Technique; OC, optimality criteria.)

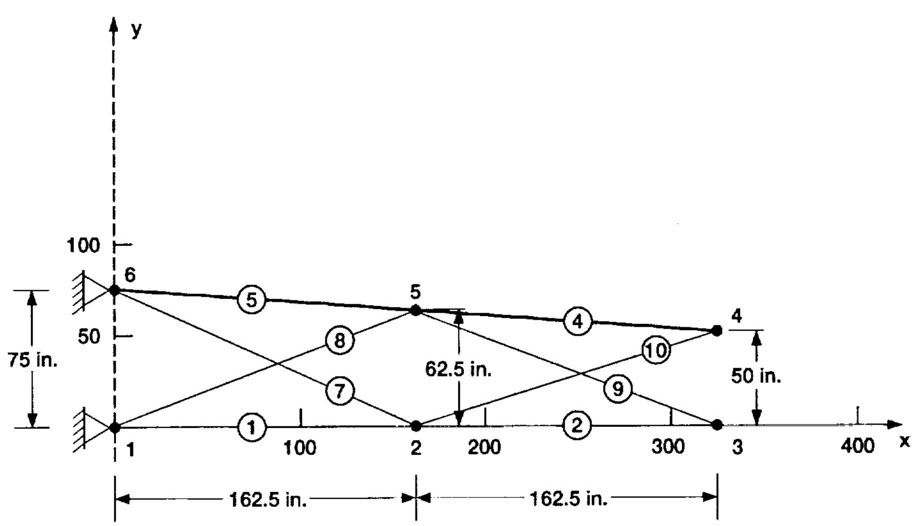


Figure 17.—Eight-bar determinant structure formed by removing two elements from the ten-bar truss shown in figure 8. (Elements are circled, nodes are not.)

few pounds lighter, the potential for instability arises because the structure can become a mechanism when one of its elements fails and, therefore, may not be considered safe. For the linked case, the smallest element area (1.592 $in.^2$) is well separated from the minimum bound of 0.1 $in.^2$ The optimum structure is indeterminant and does not become a mechanism when any one element fails. Even though the design with linked design variables is a few pounds heavier, it represents a more practical and a safer design.

In terms of computational requirements, the linked case (Bar10.d) requires fewer analyses than the unlinked case (Bar10.c)—perhaps because fewer decision (design) variables need to be determined. In both cases, the same number of constraints need to be satisfied.

The convergence characteristics of these two cases do not show major differences, although the linked case seems to produce more oscillations (see figs. 15 and 16). More numerous oscillations could be caused by an increased difficulty in adjusting the structure evenly because of changes to the areas of one element of a linked design variable group that are necessary to satisfy some constraint producing residual effects caused by associated changes to the areas of other elements of the same linked design group.

VI. Conclusions

When only displacement constraints are used, the OC method is satisfactory even for large structural systems with many design variables. However, the convergence behavior can become unpredictable if a problem has internal forces that are highly sensitive to design variable changes or if a fully stressed design (which can be considered as a heuristic OC) is mixed with the derivable exact stiffness OC method.

When extended for general design application (i.e., stress and frequency constraints are included in addition to displacement constraints), the OC method satisfactorily provided optimal designs for small and large structures under displacement and/or frequency constraints. It also is adequate for structures with a small number of design variables, even in the presence of stress constraints. However, the presence of large numbers of design variables and behavior constraints, the optimality criteria method follows a subset of constraints, resulting in a heavier design.

The fully utilized design methodology was an adequate design tool when stress constraints dominated the design. Hybrid methods, as formulated, were unsatisfactory, but further research could be fruitful.

The computational efficiency of the OC methods was similar to some mathematical programming techniques for displacement and frequency constraints, and the simplified integrated force method was more computationally efficient than the displacement method or IFM. All three analysis methods gave generally similar solutions, except, possibly, when the optimizer was especially sensitive to small changes in analysis results.

It is both acceptable, and in some cases necessary, to limit the active set of constraints to no more than the number of design variables. Such behavior constraints need to constitute a linearly functionally independent set (ref. 16).

The linking of design variables produces designs that are somewhat heavier, but it gives the designer added flexibility.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, May 25, 1993

Appendix A—Symbols

$\vec{A}$	areas
A_i	areas of individual elements
$[1/A]$	diagonal matrix whose elements are $1/A_i$
a,b,c,d	constants used in the melange form of the design variable update formula
B	force equilibrium matrix
C	compatibility matrix
$\vec{C}_i$	ith column of $[C]$
C_{ja}	actual value of displacement at a particular node or stress in a particular element
C^*_{ja}	maximum specified permissible value of displacement at a particular node or stress in a particular element
$\vec{D}$	rescaling vector
D_i	ith component of the rescaling vector
$[D]$	matrix used in IFM for calculation of sensitivities
E	Young's modulus
E_i	moduli of elasticity of individual elements
e_{ij}	variables used in the illustration of a design variable update formula derivation
$\vec{F}$	internal forces
F_i	ith component of the internal force vector
$[\tilde{F}]$	diagonal matrix used in illustrating the sensitivity matrix simplification
$[\bar{F}]$	diagonal matrix whose elements are F_i
$[F/A^2]$	diagonal matrix whose elements are F_i/A_i^2

$\vec{f}_\ell$	left eigenvectors of eigenvalue problem
$\vec{f}_m$	IFM "force mode shape" of eigenvalue problem
f_n	natural frequencies of structure
f_{n0}	limitations on natural frequencies of structure
$\vec{f}_r$	right eigenvectors of eigenvalue problem
$f(\overline{X})$	objective function (typically weight)
G	concatenated flexibility matrix
$\overline{G}_i$	partial derivative of the concatenated flexibility matrix with respect to the ith element area
$[\bar{\bar{G}}]$	diagonal matrix whose elements are $\ell_i/(A_i^2 E_i)$
g_f	constraints on frequencies
g_{ii}	value of ith diagonal element of flexibility matrix
$\bar{g}_{ii}$	$\ell_i/(A_i^2 E_i)$ (used in IFM)
g_{is}	value of stress constraint corresponding to ith element
g_{jad}	constraint value for specific "active" displacement constraint
g_{jaf}	constraint value for specific "active" frequency constraint
g_{js+j}	jth displacement constraint
g_j^*	jth active constraint
$g_j(\overline{X})$	jth constraint function
$g_{\max}$	value of largest violated displacement or frequency constraint
$\overline{J}_i$	partial derivative of the deformation coefficient matrix with respect to the ith element area
$[J]$	deformation coefficient matrix defined as ($[J] = m$ rows of $[S]^{-T}$)
K	number of iterations required to reach a solution
$\mathcal{L}$	Lagrangian
ℓ_i	length of individual elements
$[M]$	mass matrix
$[M^*]$	IFM mass matrix
m	number of inequality constraints; number of displacement degrees of freedom
n	number of design variables; number of internal forces
$njad$	number of displacement constraints in "active" set
$njaf$	number of frequency constraints in "active" set
P_x	specified load in x direction
P_y	specified load in y direction
$\vec{P}$	m component load vector
$\vec{P}^*$	n component IFM load vector
p_0	initial value of an update parameter, typically 0.5
q_0	acceleration parameters that form exponent of D_i
$R_{\sigma i}$	scaling factor for ith design variable based on the maximum stress component associated with that variable
r	number of compatibility conditions
S_i	partial derivative of the IFM governing matrix with respect to the ith element area
$[S]$	IFM governing matrix
$[S_{dg}]$	diagonal matrix whose elements are $(-g_{ii}F_i)/A_i$
t_k	constraint thickness at kth iteration
t_0	initial prescribed constraint thickness
u_j	jth displacement component
u_{j0}	displacement limitation for jth displacement component
u_1	displacement at node 1 in x direction
u_2	displacement at node 1 in y direction
w_i	product of the ith bar element's length with its density
$\overline{X}$	m component displacement vector
α	step length; acceleration parameter used to modify update parameter, often taken as 1.0
α_k	step length at the kth iteration
β	acceleration parameter that is used to form the exponent of D_i
$\vec{\beta}$	deformation vector
$\vec{\beta}_0$	initial deformation vector
γ	$\rho_1\rho_2 + 2\rho_1\rho_3 + \rho_2\rho_3$ (used in the illustration of a design variable update formula derivation
$\delta\vec{R}$	effective initial deformation vector
$\delta(\cdot)/(\delta A_i)$	partial derivative with respect to the ith element area
$\lambda, \bar{\lambda}$	Lagrange multipliers
λ_{ja}^k	value of Lagrange multiplier associated with jath "active" constraint at the kth iteration
$\vec{\lambda}^*$	vector of Lagrange multipliers associated with active constraints

ν	number of follower constraints; Poisson's ratio
ρ	weight density
ρ_i	variables used in the illustration of a design variable update formulation
σ_j	design stress for jth element
σ_{j0}	permissible stress for jth element
$\sigma_{1i},\sigma_{2i},\ldots,\sigma_{Li}$	stress values for each element associated with χ_i under each load condition
τ	factor by which constraint thickness is reduced at each iteration
$\vec{\varphi}$	direction
$\vec{\varphi}^k$	direction vector at kth iteration
χ_i	ith component of design variable
χ_i^k	ith component of design variable at kth iteration
$\chi_i^{k+1/2}$	ith component of an intermediate design variable
χ_i^{LB}	ith component of the design variable lower bound
χ_i^{UB}	ith component of the design variable upper bound
$\chi_i^{\sigma,k}$	ith component of the fully stressed design at the kth iteration
χ_j^L	jth representative linked design variable
$\vec{\chi}$	design variable
$\vec{\chi}^{\,\mathrm{opt}}$	optimal design
$\vec{\chi}^{\,\sigma,\mathrm{opt}}$	fully stressed design
$\vec{\chi}^{\,0}$	initial design
ω	circular frequency
∇f	gradient of the objective function
∇f_i	ith component of the gradient of the objective function
$\nabla \vec{G}_i$	ith column of the stress gradient (or sensitivity) matrix
∇g_j^*	gradient of jth active constraint
$\left[\nabla \vec{g}^{\,*}\right]$	constraint gradient (or sensitivity) matrix
$[\nabla g_{ja}]$	gradient of jath active constraint
$[\nabla g_{ja}]_i$	ith component of the gradient of the jath active constraint
$[\nabla X]$	gradient matrix for displacement constraints
$[\nabla X]^a$	approximate gradient matrix for displacement constraints
$[\nabla \sigma]$	sensitivity matrix for stress constraints
$[\nabla \sigma]^a$	approximate sensitivity matrix for stress constraints
$\vec{0}$	zero vector
$[\ ]^{-1}$	inverse of a matrix
$[\ ]^T$	transpose of a matrix
$[\ ]^{-T}$	inverse transpose of a matrix

Subscripts:

i,j	ith and jth variables
js	total number of stress constraints
k	value at kth iteration
max	maximum
0	initial; permissible

Superscripts:

a	approximation to sensitivity matrix
k	value at kth iteration (for vectors and vector components, e.g., $\vec{\varphi}^k$; λ_{ja}^k, χ_i^k); raised to the power of k (for scalars, e.g., α^k, β^k, τ^{k-1})
L	linked
LB	lower bound
UB	upper bound
v	virtual
0	initial
*	constraints within active set

Appendix B—Acronyms and Initialisms

ANALYZ/DANLYZ	Analyze/Dynamic Modes Analyze
CometBoards	Comparative Evaluation Test Bed of Optimization and Analysis Routines for the Design of Structures
FD	feasible directions
FSW	forward-swept wing
FUD	fully utilized design
IFM	integrated force method
IMSL	International Mathematical and Statistical Library code

NAC	number of active constraints
NDV	number of design variables
OC	optimality criteria method
OC 2	OC method using Lagrange multiplier update method 1 and design variable update method 2
OC 14	OC method using Lagrange multiplier update method 5 and design variable update method 5
OC 17	fully utilized design
SLP	sequential linear programming code
SQP	sequential quadratic programming code
SUMT	Sequence of Unconstrained Minimizations Technique
VM/CMS	Virtual Machine/Conversational Monitor System

Appendix C—Derivation of Optimality Criteria Formulas for Displacement Constraints

In this appendix, the three-bar truss shown in figure 1 is used as an example to derive the exponential form of the optimality criteria design variable update formula for displacement constraints (eq.(18)). The simplification suggested for the calculation of the sensitivity matrix via the integrated force method (see eqs. (31) and (33)) is also illustrated.

A. Derivation of Optimality Criteria Design Update Formula

First, consider the derivation of the design variable update formula. The original derivation of the optimality criteria formulas utilized the classical force method (refs. 3 and 4). Here the integrated force method of analysis is used instead to establish the relationship between the bar-element forces $\vec{F}$ and the bar-element areas $\vec{A}$ (which are taken to be the design variables) for any load condition ("real" or "virtual"). The matrices and vectors associated with the governing equations of the integrated force method (eq.(26)) are given next for the three-bar truss shown in figure 1. The force equilibrium matrix is

$$B = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{C1}$$

The compatibility matrix and concatenated flexibility matrix are

$$C = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{C2}$$

$$G = \begin{bmatrix} \frac{1}{\rho_1} & 0 & 0 \\ 0 & \frac{1}{\rho_2} & 0 \\ 0 & 0 & \frac{1}{\rho_3} \end{bmatrix} \tag{C3}$$

where $\rho_i = \frac{A_i E_i}{\ell_i}$, and A_i, E_i, and ℓ_i are the areas, Young's moduli, and lengths of each of the three bar elements.

Combining these to form the matrix S and inverting, gives

$$S^{-1} = \frac{1}{\gamma} \begin{bmatrix} \sqrt{2}\rho_1(\rho_2+\rho_3) & -\sqrt{2}\rho_1\rho_3 & \sqrt{2}\rho_1\rho_2\rho_3 \\ \rho_2(\rho_3-\rho_1) & -\rho_2(\rho_1+\rho_3) & -2\rho_1\rho_2\rho_3 \\ -\sqrt{2}\rho_3(\rho_1+\rho_2) & -\sqrt{2}\rho_1\rho_3 & \sqrt{2}\rho_1\rho_2\rho_3 \end{bmatrix} \tag{C4}$$

where $\gamma = \rho_1\rho_2 + 2\rho_1\rho_3 + \rho_2\rho_3$.

For specified loads P_x and P_y in the x and y directions, respectively, and with no initial deformations, the right side of equation (26b) becomes

$$\vec{P}^* = \begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix} \tag{C5}$$

Finally, dropping the zero terms causes the forces to become

$$\vec{F} = S^{-1}\vec{P}^* = \frac{1}{\gamma} \begin{bmatrix} \sqrt{2}\rho_1(\rho_2+\rho_3) & -\sqrt{2}\rho_1\rho_3 \\ \rho_2(\rho_3-\rho_1) & -\rho_2(\rho_1+\rho_3) \\ -\sqrt{2}\rho_3(\rho_1+\rho_2) & -\sqrt{2}\rho_1\rho_3 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix} \tag{C6}$$

Note, here, that the matrix premultiplying the load vector is the transpose of the matrix J seen in equations (31) and (33).

Consider, now, the Principle of Virtual Work for the three-bar truss, which can be written as

$$\begin{bmatrix} P_x^v P_y^v \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1^v F_2^v F_3^v \end{bmatrix} \vec{\beta} = \begin{bmatrix} F_1^v F_2^v F_3^v \end{bmatrix} G\vec{F} \tag{C7}$$

Here, u_1 and u_2 are the displacements at node 1 in the x and y directions, respectively, the superscript v represents "virtual," and the last equality comes from the discussion immediately following equation (26).

For u_1, taking $P_x = 1$ and $P_y = 0$ and using equation (C6), we have

$$\begin{bmatrix} F_1^v \\ F_2^v \\ F_3^v \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} \sqrt{2}\rho_1(\rho_2+\rho_3) \\ \rho_2(\rho_3-\rho_1) \\ -\sqrt{2}\rho_3(\rho_1+\rho_2) \end{bmatrix} \tag{C8}$$

So, substituting into equation (C7) yields

$$u_1 = \frac{1}{\gamma}\left[\sqrt{2}\rho_1(\rho_2+\rho_3)\rho_2(\rho_3-\rho_1) - \sqrt{2}\rho_3(\rho_1+\rho_2)\right]G\vec{F} \tag{C9}$$

or

$$\begin{aligned} u_1 = &\frac{1}{A_1}\frac{\ell_1}{E_1}\frac{\sqrt{2}}{\gamma}\rho_1(\rho_2+\rho_3)F_1 \\ &+\frac{1}{A_2}\frac{\ell_2}{E_2}\frac{1}{\gamma}\rho_2(\rho_3-\rho_2)F_2 \\ &-\frac{1}{A_3}\frac{\ell_3}{E_3}\frac{\sqrt{2}}{\gamma}\rho_3(\rho_1+\rho_2)F_3 \end{aligned} \tag{C10}$$

Now, let

$$\begin{aligned} e_{11} &= \frac{\ell_1}{E_1}\frac{\sqrt{2}}{\gamma}\rho_1(\rho_2+\rho_3)F_1 \\ e_{21} &= \frac{\ell_2}{E_2}\frac{1}{\gamma}\rho_2(\rho_3-\rho_1)F_2 \\ e_{31} &= -\frac{\ell_3}{E_3}\frac{\sqrt{2}}{\gamma}\rho_3(\rho_1+\rho_2)F_3 \end{aligned} \tag{C11}$$

Then, clearly

$$u_1 = \frac{e_{11}}{A_1}+\frac{e_{21}}{A_2}+\frac{e_{31}}{A_3} \tag{C12}$$

It is now straightforward to show that

$$\frac{\partial u_1}{\partial A_i} = -\frac{e_{i1}}{A_i^2} \qquad (i = 1,2,3) \tag{C13}$$

Similar remarks can be made with respect to the displacements in the y direction.

$$u_2 = \frac{e_{12}}{A_1}+\frac{e_{22}}{A_2}+\frac{e_{32}}{A_3} \tag{C14}$$

where

$$\begin{aligned} e_{12} &= -\frac{\ell_1}{E_1}\frac{\sqrt{2}}{\gamma}\rho_1\rho_3F_1 \\ e_{22} &= -\frac{\ell_2}{E_2}\frac{1}{\gamma}\rho_2(\rho_1+\rho_3)F_2 \\ e_{32} &= -\frac{\ell_3}{E_3}\frac{\sqrt{2}}{\gamma}\rho_1\rho_3F_3 \end{aligned} \tag{C15}$$

and

$$\frac{\partial u_2}{\partial A_i} = -\frac{e_{i2}}{A_i^2} \qquad (i = 1,2,3) \tag{C16}$$

For weight minimization, the objective function can be written as

$$f(\vec{A}) = \sum_{i=1}^{3} w_iA_i \tag{C17}$$

where w_i are the product of the bar elements' lengths with their densities.

Four cases can now be examined: when both u_1 and u_2 are positive, when they are both negative, and two cases when they have opposite signs. Consider the case when both displacements u_1 and u_2 are positive. (The other cases become trivial extensions of this case.) When equations (C13), (C16), (C17), and (3) are used, the stationary condition of the Lagrangian, as given by equation (9), becomes

$$w_i - \sum_{j=1}^{2}\frac{\lambda_j}{u_{j0}}\frac{e_{ij}}{A_i^2} = 0 \qquad (i = 1,2,3) \tag{C18}$$

Solving for the areas gives

$$A_i = \sqrt{\sum_{j=1}^{2}\frac{\lambda_j}{u_{j0}}\frac{e_{ij}}{w_i}} \qquad (i = 1,2,3) \tag{C19}$$

The variables e_{ij} are eliminated between equations (C13), (C16), and (C19) to yield the element areas, for u_j bounded away from zero, as

$$\mathrm{A_i} = \sqrt{-\frac{\sum_{j+1}^{2} \frac{\lambda_j \mathrm{A}_i^2 \nabla u_j}{u_{j0}}}{w_i}} = A_i\sqrt{-\frac{\sum_{j=1}^{2} \lambda_j \frac{\nabla u_j}{u_{j0}}}{\nabla f_i}} \qquad \text{(C20)}$$

Equation (C20) in essence represents the exponential form of the design variable update formula (eq. (18)), with $\chi_i = A_i$, $\beta = 1.0$, and $q_0 = 2.0$. Note the definition of D_i in equation (16), and of g_j in equation (3).

Having used the stationary condition of the Lagrangian with respect to the design variables (eq. (9)) to obtain three equations with which to update the three design variables, it is natural to look to the stationary condition of the Lagrangian with respect to the Lagrange multipliers (eq. (10)) to obtain two equations with which to update the two Lagrange multipliers. Although the theoretical basis for Lagrange multipliers updates remains somewhat unresolved, note that the constraint equations give

$$\left|\frac{u_j}{u_{j0}}\right| = 1 \qquad (j = 1,2) \qquad \text{(C21)}$$

These can then be used to provide what is essentially the exponential form of the Lagrange multiplier update formula (eq. (13)):

$$\lambda_j^{k+1} = \lambda_j^k \sqrt{\left|\frac{u_j}{u_{j0}}\right|} \qquad \text{(C22)}$$

where $C_{ja} = u_j$, $C_{ja}^* = u_{j0}$, $\alpha = 1.0$, and $p_0 = 0.5$. See references 3 and 4 for more details.

B. Illustration of Simplied Sensitivity Matrix Calculation

The simplification for the calculation of the sensitivity matrix suggested by equation (33) is considered next. In particular, the product of the matrices $[J]$ $[G]$ $[D]$ in equation (31) is set to the null matrix. The justification for this substitution can be illustrated by obtaining the matrix D, which is defined immediately following equation (30), for the three-bar truss. The matrix S^{-1} is given by equation (C4), the matrix C is given by equation (C2), and the elements of $[\bar{\bar{G}}]$ are $\frac{\ell_i}{A_i^2 E_i}$. For these conditions, D becomes

$$D = \frac{1}{\gamma}\begin{bmatrix} \frac{E_1}{\ell_1}\frac{\rho_1\rho_2\rho_3}{\rho_1^2} & -\sqrt{2}\frac{E_2}{\ell_2}\frac{\rho_1\rho_2\rho_3}{\rho_2^2} & \frac{E_3}{\ell_3}\frac{\rho_1\rho_2\rho_3}{\rho_3^2} \\ -\sqrt{2}\frac{E_1}{\ell_1}\frac{\rho_1\rho_2\rho_3}{\rho_1^2} & 2\frac{E_2}{\ell_2}\frac{\rho_1\rho_2\rho_3}{\rho_2^2} & -\sqrt{2}\frac{E_3}{\ell_3}\frac{\rho_1\rho_2\rho_3}{\rho_3^2} \\ \frac{E_1}{\ell_1}\frac{\rho_1\rho_2\rho_3}{\rho_1^2} & -\sqrt{2}\frac{E_2}{\ell_2}\frac{\rho_1\rho_2\rho_3}{\rho_2^2} & \frac{E_3}{\ell_3}\frac{\rho_1\rho_2\rho_3}{\rho_3^2} \end{bmatrix}$$

$$\times\left[\bar{F}\right] = \frac{1}{\gamma}\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}\left[\tilde{F}\right] \qquad \text{(C23)}$$

where $[\tilde{F}]$ is a diagonal matrix, whose diagonal elements are given by $\frac{E_i}{\ell_i}\frac{\rho_1\rho_2\rho_3}{\rho_i^2}F_i$.

Using J formed by transposing the matrix in equation (C6) and using G from equation (C3) gives

$$[J][G][D] =$$

$$\frac{1}{\gamma^2}\begin{bmatrix} \sqrt{2}\rho_1(\rho_2+\rho_3) & \rho_2(\rho_3-\rho_1) & -\sqrt{2}\rho_3(\rho_1+\rho_3) \\ -\sqrt{2}\rho_1\rho_3 & -\rho_2(\rho_1+\rho_3) & -\sqrt{2}\rho_1\rho_3 \end{bmatrix}$$

$$\times\begin{bmatrix} \frac{1}{\rho_1} & 0 & 0 \\ 0 & \frac{1}{\rho_2} & 0 \\ 0 & 0 & \frac{1}{\rho_3} \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}\left[\tilde{F}\right]$$

$$= \frac{1}{\gamma^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left[\tilde{F}\right] = [0]$$

In other words, in the calculation of the sensitivities of the displacement constraints, the term associated with the increment in the forces can be set to zero.

Appendix D—Equations for Various Update Methods

A. Lagrange Multiplier

The equations for the Lagrange update method follow:

$$\lambda_{ja}^{k+1} = \lambda_{ja}^{k}\left[1.0 + \alpha^{k} p_0\left(C_{ja} - C_{ja}^{*}\right)\right] \tag{12}$$

$$\lambda_{ja}^{k+1} = \lambda_{ja}^{k}\left(\frac{C_{ja}}{C_{ja}^{*}}\right)^{\alpha^{k} p_0} \tag{13}$$

$$\bar{\lambda}^{*} = \left[\left[\nabla\bar{g}^{*}\right]^{T}\nabla\bar{g}^{*}\right]^{-1}\left[\nabla\bar{g}^{*}\right]^{T}\nabla f \tag{14}$$

Here, λ_{ja}^{k} represents the value of the Lagrange multiplier at the kth iteration and ja represents only those Lagrange multipliers associated with "active" constraints. Note that in equation (14) the Lagrange multiplier does not depend directly on its value at the previous iteration. Lagrange multiplier update method 1 is given by equation (12), method 2 is given by equation (13), and methods 3, 4, and 5 are given by variants of equation (14). See section II.D.1 for more details.

B. Rescaling Vector and Design Variable

Equations for the rescaling vector and design variable update method follow:

$$D_i = -\frac{\sum_{ja} \lambda_{ja}\left(\nabla g_{ja}\right)_i}{\nabla f_i} \tag{16}$$

$$\chi_i^{k+1} = \chi_i^{k} D_i^{\left(1/\beta^{k} q_0\right)} \tag{18}$$

$$\chi_i^{k+1} = \chi_i^{k}\left(1.0 + \frac{1}{\beta^{k} q_0}\left(D_i - 1.0\right)\right) \tag{19}$$

$$\chi_i^{k+1} = \frac{\chi_i^{k}}{1.0 - \frac{1}{\beta^{k} q_0}\left(D_i - 1.0\right)} \tag{20}$$

$$\chi_i^{k+1/2} = a\chi_i^{k} + \frac{b}{njad}\sum_{jad=1}^{njad}\chi_i^{k}\left(1.0 + g_{jad}\right) + \frac{c}{njaf}\sum_{jaf=1}^{njaf}\chi_i^{k}\left(1.0 + g_{jaf}\right) + d\chi_i^{k} D_i \tag{21a}$$

$$\chi_i^{k+1} = \frac{1}{2}\left[\chi_i^{k+1/2} + \chi_i^{k}\left(1.0 + g_{is}\right)\right] \quad \text{or} \quad \chi_i^{k+1/2} \tag{21b}$$

$$\chi_i^{k+1/2} = \chi_i^{k}\left(1.0 + g_{is}\right) \tag{22}$$

Here, D_i is the rescaling vector k and χ_i^k is the ith component of the design variable at the kth iteration. Design variable update methods 1, 2, and 3 are given by equations (18), (19), and (20), respectively. Method 4 uses variants of equations (21a) and (21b). The three hybrid methods are given by combining equations (18), (19), and (20), respectively, with equation (22). See section II.D.2 for more details.

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The merits and limitations of the optimality criteria (OC) method for the minimum weight design of structures subjected to multiple load conditions under stress, displacement, and frequency constraints were investigated by examining several numerical examples. The examples were solved utilizing the Optimality Criteria Design Code that was developed for this purpose at NASA Lewis Research Center. This OC code incorporates OC methods available in the literature with generalizations for stress constraints, fully utilized design concepts, and hybrid methods that combine both techniques. Salient features of the code include multiple choices for Lagrange multiplier and design variable update methods, design strategies for several constraint types, variable linking, displacement and integrated force method analyzers, and analytical and numerical sensitivities. The performance of the OC method, on the basis of the examples solved, was found to be satisfactory for problems with few active constraints or with small numbers of design variables. For problems with large numbers of behavior constraints and design variables, the OC method appears to follow a subset of active constraints that can result in a heavier design. The computational efficiency of OC methods appears to be similar to some mathematical programming techniques.

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