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Research Article

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Meromorphic exact solutions of the (2 + 1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation

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Abstract: In this article, meromorphic exact solutions for the (2 + 1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff (gCBS) equation are obtained by using the complex method. With the applications of our results, traveling wave exact solutions of the breaking soliton equation are achieved. The dynamic behaviors of exact solutions of the (2 + 1)-dimensional gCBS equation are shown by some graphs. In particular, the graphs of elliptic function solutions are comparatively rare in other literature. The idea of this study can be applied to the complex nonlinear systems of some areas of engineering.

Keywords: differential equation, complex method, meromorphic function

MSC 2020: 30D35, 34A05

1 Introduction and main results

A lot of nonlinear integral equations with constant coefficients are used to explain the physical phenomena. Many researchers have studied (1 + 1) dimensional nonlinear integrable systems with constant coefficients. The analysis of higher dimensional systems is an important subject in nonlinear integrable systems. In recent years, many mathematicians and physicists studied the nonlinear integrable systems that occur in various fields such as biology, fluid dynamics, quantum and plasma physics, thermal engineering and optics. Plenty of methods have been developed for getting exact solutions to nonlinear differential equations such as the modified extended tanh method [1,2], the improved F-expansion method [3], the modified simple equation method [4], the complex method [5–8], the generalized (G'/G)-expansion method [9–11], the $\exp(-\psi(z))$-expansion method [12–16], the $(m + 1/G')$-expansion method [17], the sine-Gordon expansion method [18–24], the extended sine-Gordon expansion method [25–27], the extended rational sinh-cosh method [28], the modified Kudryashov method [29] and other methods [30–32].

The Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$u_t + uu_y + \frac{1}{2}u_x \partial_x^{-1} u_y + \frac{1}{4}u_{xxy} = 0, \tag{1.1}$$

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where $\partial_x^{-1} f = \int f \, dx$, was obtained in two different ways by Bogoyavlenskii and Schiff [33–35]. Bogoyavlenskii applied the modified Lax formalism, while Schiff by reducing the self-dual Yang-Mills equation achieved the same equation [36–38].

The (2 + 1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff (gCBS) equation [39] is the potential form of Eq. (1.1) that is given as:

$$au_{xt} + bu_x u_{xy} + cu_y u_{xx} + u_{xxxy} = 0, \tag{1.2}$$

where a, b, c are constants. In the past decade, several methods have been used to obtain the exact solutions of the gCBS equation, for example, Chen and Ma [40] by considering the Hirota bilinear form of the gCBS equation explicitly generated a class of lump solutions; Li and Chen [41] by using the generalized Riccati equation expansion method found some exact analytical solutions; Al-Amr [42] applied the modified simple equation method to construct the exact solutions of the gCBS equation; Wang and Yang [43] implemented the Hirota bilinear method for the construction of the quasi-periodic wave solutions in terms of theta functions; Najafi et al. [44] established the (G'/G)-expansion method to find travelling wave solutions for the gCBS equation; and Zhang et al. [45] studied the integrability of this equation by the Painlevé test and derived its symmetry reductions.

Recently, Yuan et al. [46] have proposed an effective method named complex method to seek the exact solutions of nonlinear differential equations. The complex method is based on complex differential equations and complex analysis. More details of the complex method can be seen in [47–50]. In this article, the complex method is utilized to obtain meromorphic exact solutions of the (2 + 1)-dimensional gCBS equation.

Substituting traveling wave transform

$$u(x, y, t) = U(z), \quad z = kx + ly - rt,$$

into Eq. (1.2), and then integrating it we get

$$k^3 l U''' + \frac{k^2 l(b + c)}{2}(U')^2 - karU' + d = 0, \tag{1.3}$$

where d is the integration constant.

If a meromorphic function ξ is a rational function of z, or an elliptic function, or a rational function of $e^{\mu z}$, $\mu \in \mathbb{C}$, then ξ is said to belong to the class W.

Theorem 1.1. *If $b + c \neq 0$, then the meromorphic solutions of* Eq. (1.3) *belong to the class W. Moreover,* Eq. (1.3) *has the following classes of solutions where c_i $(i = 1, 2, 3, 4)$ are the integration constants.*

(*i*) *The rational function solutions*

$$U_r(z) = \frac{12k}{b + c}\frac{1}{z - z_0} + \frac{ar}{kl(b + c)}(z - z_0) + c_1,$$

where $d = \frac{a^2 r^2}{2(b + c)l}$, $z_0 \in \mathbb{C}$.

(*ii*) *The simply periodic solutions*

$$U_{1s}(z) = \frac{6k\mu}{b + c}\coth\frac{\mu(z - z_0)}{2} + \frac{3k\mu}{b + c}\ln\left(\frac{\coth\frac{\mu}{2}(z - z_0) - 1}{\coth\frac{\mu}{2}(z - z_0) + 1}\right) + \frac{2k^2\mu^2 l + ar}{k(b + c)l}(z - z_0) + c_2$$

and

$$U_{2s}(z) = \frac{6k\mu}{b + c}\tanh\frac{\mu(z - z_0)}{2} + \frac{3k\mu}{b + c}\ln\left(\frac{\tanh\frac{\mu}{2}(z - z_0) - 1}{\tanh\frac{\mu}{2}(z - z_0) + 1}\right) + \frac{2k^2\mu^2 l + ar}{k(b + c)l}(z - z_0) + c_3,$$

where $d = \frac{-\mu^4 k^4 l^2 + a^2 r^2}{2(b + c)l}$, $z_0 \in \mathbb{C}$.

(*iii*) *The elliptic function solutions*

$$U_d(z) = \frac{12k}{b+c}[\zeta(z) - \zeta(z_0)] + \frac{6k}{b+c}\frac{\wp'(z) + E}{\wp(z) - F} + \frac{ar}{kl(b+c)}(z - z_0) + c_4,$$

where $E^2 = 4F^3 - g_2F - g_3$, $g_2 = \frac{a^2r^2 - 2bdl - 2cdl}{12k^4l^2}$, g_3 *is arbitrary and* $z_0 \in \mathbb{C}$.

Breaking soliton equation is given by

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0. \tag{1.4}$$

Theorem 1.2. *The breaking soliton equation has the following forms traveling wave exact solutions where* c_{1i} $(i = 1, 2, 3, 4)$ *are the integration constants.*

(*i*) *The rational function solutions*

$$U_{11}(x, y, t) = -\frac{2k}{kx + ly - rt - x_0} - \frac{r(kx + ly - rt - x_0)}{6kl} + c_{11},$$

where $d = -\frac{r^2}{16l}$.

(*ii*) *The simply periodic solutions*

$$\begin{aligned} U_{12}(x, y, t) &= -k\mu \coth\frac{\mu(kx + ly - rt - x_0)}{2} - \frac{2k^2\mu^2 l + r}{6kl}(kx + ly - rt - x_0) \\ &\quad - \frac{k\mu}{2}\ln\left(\frac{\coth\frac{\mu}{2}(kx + ly - rt - x_0) - 1}{\coth\frac{\mu}{2}(kx + ly - rt - x_0) + 1}\right) + c_{12} \end{aligned}$$

and

$$\begin{aligned} U_{13}(x, y, t) &= -k\mu \tanh\frac{\mu(kx + ly - rt - x_0)}{2} - \frac{2k^2\mu^2 l + r}{6kl}(kx + ly - rt - x_0) \\ &\quad - \frac{k\mu}{2}\ln\left(\frac{\tanh\frac{\mu}{2}(kx + ly - rt - x_0) - 1}{\tanh\frac{\mu}{2}(kx + ly - rt - x_0) + 1}\right) + c_{13}, \end{aligned}$$

where $d = \frac{\mu^4k^4l^2 - r^2}{16l}$.

(*iii*) *The elliptic function solutions*

$$U_{14}(x, y, t) = -2k[\zeta(kx + ly - rt) - \zeta(x_0)] - \frac{k(\wp'(kx + ly - rt) + E)}{\wp(kx + ly - rt) - F} - \frac{r(kx + ly - rt - x_0)}{6kl} + c_{14},$$

where $E^2 = 4F^3 - g_2F - g_3$, $g_2 = \frac{r^2 + 16dl}{12k^4l^2}$ *and* g_3 *is arbitrary.*

2 Preliminary theory

The constants a, b, c may differ in different places. Let $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $r_i \in \{0, 1, 2, \ldots\}$, $i = 0, 1, \ldots, m$, $r = (r_0, r_1, \ldots, r_m)$ and

$$M_r[V](z) := \prod_{i=0}^{m} [V^{(i)}(z)]^{r_i}. \tag{2.1}$$

The degree of $M_r[V]$ is denoted by $d(r) := \sum_{i=0}^{m} r_i$. The differential polynomial is given by:

$$P(V, V', \ldots, V^{(m)}) := \sum_{r \in \tau} a_r M_r[V],$$

where τ is a finite index set, and then $\deg P(V, V', \ldots, V^{(m)}) := \max_{r\in\tau}\{d(r)\}$ is the degree of $P(V, V', \ldots, V^{(m)})$.

Considering the following ordinary differential equation:

$$P(V, V', \ldots, V^{(m)}) = aV^n + d, \tag{2.2}$$

where $a \neq 0$, d are constants, and $n \in \mathbb{N}$.

Set $p,\ q \in \mathbb{N}$, and assume that meromorphic solutions v of Eq. (2.2) have at least one pole. Substituting the Laurent series

$$V(z) = \sum_{k=-q}^{\infty} \beta_k z^k,\ \beta_{-q} \neq 0, \quad q > 0, \tag{2.3}$$

into Eq. (2.2), if it can determine p distinct Laurent singular parts:

$$\sum_{k=-q}^{-1} \beta_\tau z^k,$$

then Eq. (2.2) is said to satisfy the weak $\langle p, q \rangle$ condition.

$\wp(z) := \wp(z, g_2, g_3)$ is the Weierstrass elliptic function with double periods satisfying:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Weierstrass zeta function $\zeta(z)$ is a meromorphic function which satisfies

$$\wp(z) = -\zeta'(z).$$

These two Weierstrass functions admit the addition formulas as follows:

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2,$$

$$\zeta(z - z_0) = \zeta(z) - \zeta(z_0) + \frac{1}{2}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right].$$

The following mth order Briot-Bouquet equation (BBEq) had been investigated by Eremenko et al. [51],

$$P(V, V^{(m)}) = \sum_{j=0}^{n} P_j(V)(V^{(m)})^j = 0, \tag{2.4}$$

where $P_j(V)$ are the constant coefficient polynomials, and $m \in \mathbb{N}$.

Lemma 1.1. [52–54] *Let $p,\ m,\ \varsigma,\ n \in \mathbb{N}$, $\deg P(V, V^{(m)}) < n$. If a mth order BBEq,*

$$P(V, V^{(m)}) = aV^n + d,$$

satisfies the weak $\langle p, q \rangle$ condition, then any meromorphic solution v belongs to the class W. Suppose for some values of parameters such solution v exists, then any other meromorphic solution forms one parameter family $V(z - z_0)$, $z_0 \in \mathbb{C}$. Additionally, any elliptic solution $V(z)$ with pole at origin is of the form

$$\begin{aligned} V(z) \ &= \sum_{i=1}^{\varsigma-1}\sum_{j=2}^{q} \frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z) + D_i}{\wp(z) - B_i}\right]^2 - \wp(z)\right) \\ &+ \sum_{i=1}^{\varsigma-1} \frac{\beta_{-i1}}{2}\frac{\wp'(z) + D_i}{\wp(z) - B_i} + \sum_{j=2}^{q} \frac{(-1)^j\beta_{-\varsigma j}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \end{aligned} \tag{2.5}$$

where $D_i^2 = 4B_i^3 - g_2B_i - g_3$, β_{-ij} are determined by (2.3) *and* $\sum_{i=1}^{\varsigma}\beta_{-i1} = 0$.

Every rational function solution has the form

$$R(z) = \sum_{i=1}^{\varsigma}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z-z_i)^j} + \beta_0, \tag{2.6}$$

and it has $\varsigma(\le p)$ distinct poles of multiplicity q.

Every simply periodic solution has $\varsigma(\le p)$ distinct poles of multiplicity q which is a rational function $R(\eta)$ of $\eta = e^{\alpha z}(\alpha \in \mathbb{C})$,

$$R(\eta) = \sum_{i=1}^{\varsigma}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta-\eta_i)^j} + \beta_0. \tag{2.7}$$

3 Proof of main results

Proof of Theorem 1.1. Let $v = U'$, Eq. (1.3) becomes

$$k^3lv'' + \frac{k^2l(b+c)}{2}v^2 - karv + d = 0. \tag{3.1}$$

Substitute (2.3) into Eq. (1.3) to obtain $p = 1$, $q = 2$, $\beta_{-2} = -\frac{12k}{b+c}$, $\beta_{-1} = 0$, $\beta_0 = \frac{ar}{kl(b+c)}$, $\beta_1 = 0$, $\beta_2 = -\frac{a^2r^2-2bdl-2cdl}{20l^2k^3(b+c)}$, $\beta_3 = 0$ and β_4 is arbitrary. Therefore, Eq. (3.1) is a second-order BBEq and its weak $\langle 1, 2\rangle$ condition holds. Thus, by Lemma 1.1, it is known that meromorphic solutions v of Eq. (3.1) belong to W.

From (2.5) of Lemma 1.1, the form of elliptic solutions of Eq. (3.1) is

$$v_{d0}(z) = \beta_{-2}\wp(z) + \beta_{10},$$

with pole at $z = 0$.

Insert $v_{d0}(z)$ into Eq. (3.1), then

$$v_{d0}(z) = -\frac{12k}{b+c}\wp(z) + \frac{ar}{kl(b+c)},$$

where $g_2 = \frac{a^2r^2-2bdl-2cdl}{12k^4l^2}$ and g_3 is arbitrary.

Thus, the elliptic solutions of Eq. (3.1) with arbitrary pole are

$$v_d(z) = -\frac{12k}{b+c}\wp(z-z_0) + \frac{ar}{kl(b+c)},$$

where $z_0 \in \mathbb{C}$.

Then, the solutions of Eq. (1.3)

$$\begin{aligned} U_d(z) &= \int v_d(z)\,dz = \int\left(-\frac{12k}{b+c}\wp(z-z_0) + \frac{ar}{kl(b+c)}\right)dz \\ &= \frac{12k}{b+c}\zeta(z-z_0) + \frac{ar}{kl(b+c)}(z-z_0) + c_4 \\ &= \frac{12k}{b+c}[\zeta(z)-\zeta(z_0)] + \frac{6k}{b+c}\frac{\wp'(z)+E}{\wp(z)-F} + \frac{ar}{kl(b+c)}(z-z_0) + c_4, \end{aligned}$$

in which $E^2 = 4F^3 - g_2F - g_3$, $g_2 = \frac{a^2r^2-2bdl-2cdl}{12k^4l^2}$, g_3 is arbitrary and c_4 is an integral constant.

By (2.6) of Lemma 1.1, it can be inferred that the indeterminate rational solutions of Eq. (3.1) with pole at $z = 0$ are

$$R_1(z) = \frac{\beta_{11}}{z^2} + \frac{\beta_{12}}{z} + \beta_{20}.$$

Substituting $R_1(z)$ into Eq. (3.1) yields

$$-kar\beta_{10}+1/2k^2lb\beta_{10}{}^2+1/2k^2lc\beta_{10}{}^2+d+\frac{bk^2l\beta_{10}\beta_{11}+ck^2l\beta_{10}\beta_{11}-akr\beta_{11}}{z}+\frac{-kar\beta_{12}+k^2lb\beta_{12}\beta_{10}+1/2k^2lb\beta_{11}{}^2+k^2lc\beta_{12}\beta_{10}+1/2k^2lc\beta_{11}{}^2}{z^2}+\frac{bk^2l\beta_{11}\beta_{12}+ck^2l\beta_{11}\beta_{12}+2k^3l\beta_{11}}{z^3}+\frac{6lk^3\beta_{12}+1/2k^2lb\beta_{12}{}^2+1/2k^2lc\beta_{12}{}^2}{z^4}=0,$$

then $\beta_{12}=-\frac{12k}{b+c}$, $\beta_{11}=0$ and $\beta_{10}=\frac{ar}{kl(b+c)}$.

Therefore, we can determine that

$$R_1(z)=-\frac{12k}{b+c}\frac{1}{z^2}+\frac{ar}{kl(b+c)},$$

where $d=\frac{a^2r^2}{2(b+c)l}$.

Thus, the rational solutions of Eq. (3.1) with arbitrary pole are

$$v_r(z)=-\frac{12k}{b+c}\frac{1}{(z-z_0)^2}+\frac{ar}{kl(b+c)}.$$

Then, the solutions of Eq. (1.3)

$$\begin{aligned}U_r(z)&=\int v_r(z)\mathrm{d}z\\&=\int\left(-\frac{12k}{b+c}\frac{1}{(z-z_0)^2}+\frac{ar}{kl(b+c)}\right)\mathrm{d}z\\&=\frac{12k}{b+c}\frac{1}{z-z_0}+\frac{ar}{kl(b+c)}(z-z_0)+c_1,\end{aligned}$$

where $d=\frac{a^2r^2}{2(b+c)l}$, $z_0\in\mathbb{C}$, and c_1 is an integral constant.

Let $\eta=e^{\mu z}$. Substitute $w=R(\eta)$ into Eq. (3.1) to yield

$$k^3l\mu^2(\eta R'+\eta^2R'')-karR+\frac{k^2l(b+c)}{2}R^2+d=0. \tag{3.2}$$

Substituting $R_2(z)$ into Eq. (3.2) gives

$$R_{21}(z)=-\frac{12k}{b+c}\frac{\mu^2}{(\eta-1)^2}-\frac{12k}{b+c}\frac{\mu^2}{(\eta-1)}+\frac{-k^2\mu^2l+ar}{k(b+c)l} \tag{3.3}$$

and

$$R_{22}(z)=-\frac{12k}{b+c}\frac{\mu^2}{(\eta+1)^2}+\frac{12k}{b+c}\frac{\mu^2}{(\eta+1)}+\frac{-k^2\mu^2l+ar}{k(b+c)l}, \tag{3.4}$$

where $d=\frac{-\mu^4k^4l^2+a^2r^2}{2(b+c)l}$.

Substituting $\eta=e^{\mu z}$ into Eq. (3.3) and (3.4) yields simply periodic solutions to Eq. (3.1) with pole at $z=0$

$$\begin{aligned}v_{1s0}(z)&=-\frac{12k}{b+c}\frac{\mu^2}{(e^{\mu z}-1)^2}-\frac{12k}{b+c}\frac{\mu^2}{(e^{\mu z}-1)}+\frac{-k^2\mu^2l+ar}{k(b+c)l}\\&=-\frac{12k}{b+c}\mu^2\frac{e^{\mu z}}{(e^{\mu z}-1)^2}+\frac{-k^2\mu^2l+ar}{k(b+c)l}=-\frac{3k\mu^2}{b+c}\coth^2\frac{\mu z}{2}+\frac{2k^2\mu^2l+ar}{k(b+c)l}\end{aligned}$$

and

$$\begin{aligned} v_{2s0}(z) &= -\frac{12k}{b+c}\frac{\mu^2}{(e^{\mu z}+1)^2} + \frac{12k}{b+c}\frac{\mu^2}{(e^{\mu z}+1)} + \frac{-k^2\mu^2 l + ar}{k(b+c)l} \\ &= \frac{12k}{b+c}\mu^2 \frac{e^{\mu z}}{(e^{\mu z}+1)^2} + \frac{-k^2\mu^2 l + ar}{k(b+c)l} = -\frac{3k\mu^2}{b+c}\tanh^2\frac{\mu z}{2} + \frac{2k^2\mu^2 l + ar}{k(b+c)l}, \end{aligned}$$

where $d = \frac{-\mu^4k^4l^2 + a^2r^2}{2(b+c)l}$.

Thus, simply periodic solutions of Eq. (1.3) with arbitrary pole are

$$\begin{aligned} U_{1s}(z) &= \int v_{1s}(z)\,\mathrm{d}z \\ &= \int\left(-\frac{3k\mu^2}{b+c}\coth^2\frac{\mu(z-z_0)}{2} + \frac{2k^2\mu^2 l + ar}{k(b+c)l}\right)\mathrm{d}z \\ &= \frac{6k\mu}{b+c}\coth\frac{\mu(z-z_0)}{2} + \frac{3k\mu}{b+c}\ln\left(\frac{\coth\frac{\mu}{2}(z-z_0) - 1}{\coth\frac{\mu}{2}(z-z_0) + 1}\right) + \frac{2k^2\mu^2 l + ar}{k(b+c)l}(z-z_0) + c_2 \end{aligned}$$

and

$$\begin{aligned} U_{2s}(z) &= \int v_{2s}(z)\,\mathrm{d}z \\ &= \int\left(-\frac{3k\mu^2}{b+c}\tanh^2\frac{\mu(z-z_0)}{2} + \frac{2k^2\mu^2 l + ar}{k(b+c)l}\right)\mathrm{d}z \\ &= \frac{6k\mu}{b+c}\tanh\frac{\mu(z-z_0)}{2} + \frac{3k\mu}{b+c}\ln\left(\frac{\tanh\frac{\mu}{2}(z-z_0) - 1}{\tanh\frac{\mu}{2}(z-z_0) + 1}\right) + \frac{2k^2\mu^2 l + ar}{k(b+c)l}(z-z_0) + c_3, \end{aligned}$$

where c_2 and c_3 are the integral constants, $d = \frac{-\mu^4k^4l^2 + a^2r^2}{2(b+c)l}$, $z_0 \in \mathbb{C}$. □

Remark. Let $a = 1$, $b = -4$, $c = -2$, then Eq. (1.2) can be converted to Eq. (1.4). Applying Theorem 1.1, and letting $z = kx + ly - rt$, $z_0 = x_0$, traveling wave exact solutions for the breaking soliton equation are obtained. Therefore, Theorem 1.2 holds.

4 Dynamic behaviors

In this section, by using computer simulations, the dynamical structures of the begotten solutions are demonstrated. Figure 1 shows the rational solutions $U_r(z)$ of the (2 + 1)-dimensional gCBS equation, when $t = -3$, $t = 0$ and $t = 3$ and by choosing parameters as $a = -1$, $b = -1$, $c = 0.1$, $k = 2$, $l = -1$, $r = 1$, $z_0 = 0$ and $c_1 = 0$, the graph of solution $U_r(z)$ contains two components in the opposite directions. Figures 2 and 3 present simply periodic solutions $U_{1s}(z)$, $U_{2s}(z)$ of Eq. (1.2) when $t = -3$, $t = 0$ and $t = 3$ and by choosing parameters for $U_{1s}(z)$, $U_{2s}(z)$ as $a = 1$, $b = 1$, $c = 2$, $k = 1$, $l = 2$, $r = -1$, $z_0 = 0$, $c_2 = 0$ and $a = -1$, $b = 1$, $c = 1$, $k = 1$, $l = 1$, $r = 2$, $z_0 = 0$ and $c_3 = 0$, respectively, the graphs contain oscillation in waves which are moving forward. Figure 4 displays Weierstrass elliptic function solutions $U_d(z)$ of Eq. (1.2) by choosing parameters as $a = 1$, $b = -2$, $c = 1$, $k = 0.4$, $l = -1$, $r = 0.26$, $z_0 = 0$ and $c_4 = 0$ and considering $t = -3$, $t = 0$ and $t = 3$, the graph illustrates that during the time, direction of oscillations of wave is changed.

5 Conclusions

In this article, the complex method is utilized to construct meromorphic solutions to the complex (2 + 1)-dimensional gCBS equation, then exact solutions to the (2 + 1)-dimensional gCBS equation are obtained. By

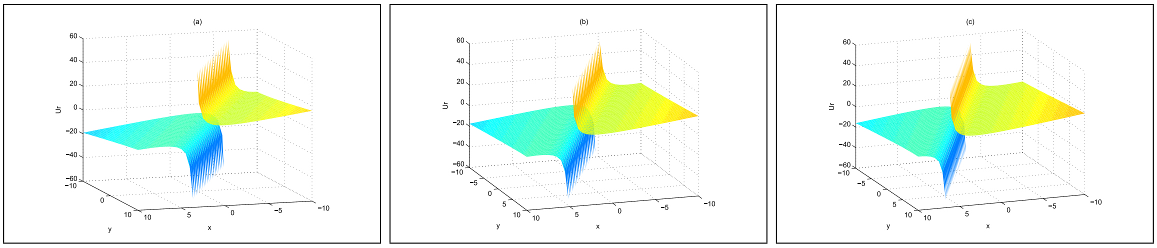

Figure 1: The solution of the (2 + 1)-dimensional gCBS equation corresponding to $U_r(z)$, (a) $t = -3$, (b) $t = 0$ and (c) $t = 3$.

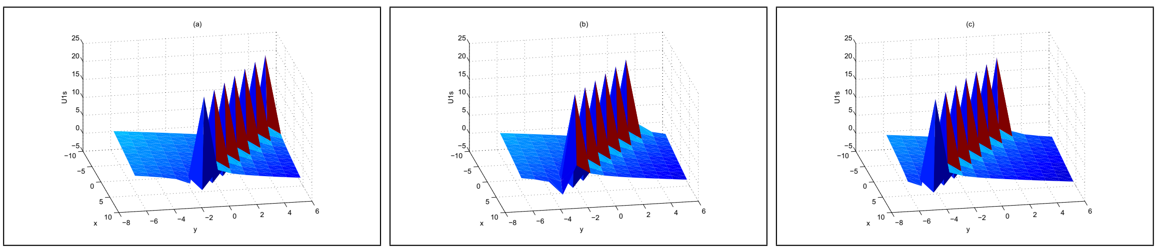

Figure 2: The solution of the (2 + 1)-dimensional gCBS equation corresponding to $U_{1s}(z)$, (a) $t = -3$, (b) $t = 0$ and (c) $t = 3$.

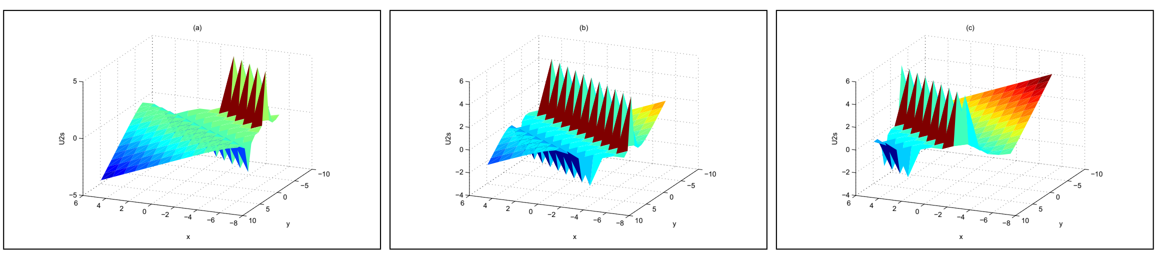

Figure 3: The solution of the (2 + 1)-dimensional gCBS equation corresponding to $U_{2s}(z)$, (a) $t = -3$, (b) $t = 0$ and (c) $t = 3$.

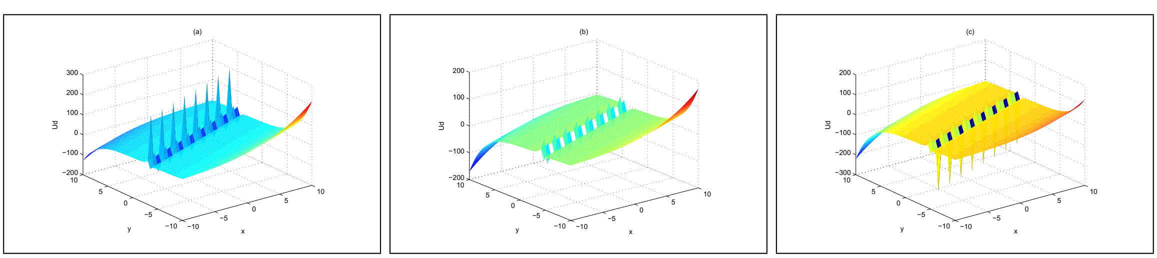

Figure 4: The solution of the (2 + 1)-dimensional gCBS equation corresponding to $U_d(z)$, (a) $t = -3$, (b) $t = 0$ and (c) $t = 3$.

the applications of our results, traveling wave exact solutions to the breaking soliton equation are achieved. To our knowledge, the solutions of this study have not been reported in former literature. The dynamic behaviors of these solutions are shown by some graphs. Figures 1–4 obviously present soliton phenomena and show that soliton interaction will contain oscillations. Figures 1–3 also display that by changing the time, the solutions of the (2 + 1)-dimensional gCBS equation continue to move forward.

The complex method is an efficient method to get the solutions of a BBEq through its undetermined forms. This method is applied to obtain the exact solutions of the (2 + 1)-dimensional gCBS equation and breaking soliton equation which enrich the studies of the mentioned equations. However, for the differential equation which is not a BBEq, how to solve it by the complex method? It will be considered in future studies.

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