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MEROMORPHIC EXTENSIONS OF A CLASS OF ZETA FUNCTIONS FOR TWO-DIMENSIONAL BILLIARDS WITHOUT ECLIPSE

Dedicated to Professor Yoichiro Takahashi on his sixtieth birthday

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Abstract. The main purpose of the present paper is to show that a class of dynamical zeta functions associated with the so-called two-dimensional open billiard without eclipse have meromorphic extensions to the half-plane consisting of all complex numbers whose real parts are greater than a certain negative number. As an application, we verify that the zeta function for the length spectrum of the corresponding billiard table has the same property.

1. Introduction.

Let $Q_1, Q_2, \dots, Q_J$, $J \geq 3$, be a finite number of bounded domains in Euclidean 2-space $\boldsymbol{R}^2$ with boundaries $\partial Q_1, \partial Q_2, \dots, \partial Q_J$, each of which is called a scatterer. Throughout the paper, we assume that these scatterers are located without having eclipses. Precisely, the following Ikawa conditions (H.1) and (H.2) are satisfied (see Figure 1; see also [7]).

(H.1) (Dispersing) For each j, the boundary ∂Q_j of the domain Q_j is a strictly convex simply closed curve of class C^3.

(H.2) (No eclipse) For any triplet of distinct indices (j_1, j_2, j_3), we have

$$\mathrm{conv}(\overline{Q_{j_1}} \cup \overline{Q_{j_2}}) \cap \overline{Q_{j_3}} = \emptyset\,,$$

where $\mathrm{conv}(A)$ denotes the convex hull of the set A. Consider the exterior of these scatterers $Q = \boldsymbol{R}^2 \setminus \bigcup_{j=1}^J \overline{Q_j}$. Clearly, $\partial Q = \bigcup_{j=1}^J \partial Q_j$. For $q \in \partial Q$, $n(q)$ denotes the inward unit normal of ∂Q at q. Let us consider the billiard flow S^t on $\overline{Q}$, that is, the Euclidean geodesic flow on the manifold $\overline{Q}$ obeying the law of reflections at the boundary (cf. [3], [4] and [18]).

Let $S\boldsymbol{R}^2 = \boldsymbol{R}^2 \times S^1$ denote the unit tangent bundle of $\boldsymbol{R}^2$ and $\pi : S\boldsymbol{R}^2 \to \boldsymbol{R}^2$; $(q, v) \mapsto q$ the natural projection. The state space M of the billiard flow is given by

$$M = \pi^{-1}(Q) \cup (\pi^{-1}(\partial Q)/\sim)\,,$$

where the equivalence relation $\sim$ on $\pi^{-1}(\partial Q)$ means that $(q, v) \sim (p, w)$ if and only if $q = p$ and $w = v - 2\langle v, n(q)\rangle n(q)$. Namely, the state of incidence and the state of reflection are identified. Therefore, by selecting the states of reflection as representatives, we may identify $\pi^{-1}(\partial Q)/\sim$ with

$$M^+ = \{x = (q, v);\, q \in \partial Q,\, \langle v, n(q)\rangle \geq 0\}\,.$$

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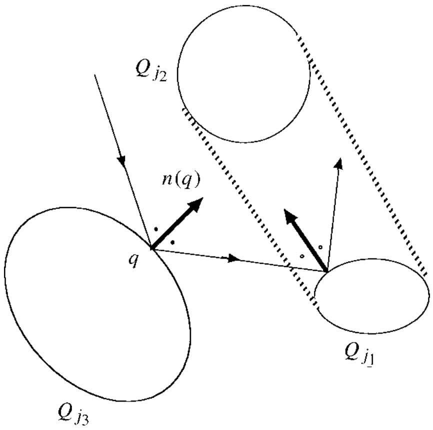

FIGURE 1.

As shown in [9], the non-wandering set $\Omega \subset M$ of the flow S^t coincides with the set of initial states x for which $\pi(S^t x) \in \partial Q$ for infinitely many $t > 0$ and infinitely many $t < 0$ as well. Moreover, it has a sort of hyperbolic structure quite similar to the basic sets for Axiom A flows. Thus, we can investigate the dynamical properties of S^t by constructing a suspension flow over an appropriate discrete dynamical system. To be more precise, consider the set $\Omega^+ = M^+ \cap \Omega$. The first collision time $t^+ : \Omega^+ \to \boldsymbol{R}$ and the first collision map $T : \Omega^+ \to \Omega^+$ are defined by

$$t^+(x) = \inf\{t > 0\,;\, S^t x \in \Omega^+\}\,, \quad Tx = S^{t^+(x)}x\,.$$

Then the billiard flow restricted to Ω can be represented by the suspension flow over the discrete dynamical system (Ω^+, T) with ceiling function t^+. Usually, the first collision map T is called the billiard map.

Given a function $V : \Omega^+ \to \boldsymbol{C}$, we introduce a formal function $\zeta_V(s)$ of complex variable s, which is called the zeta function for T with potential function V, as follows:

$$\zeta_V(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x : T^n x = x} \exp\left(-s \sum_{i=0}^{n-1} V(T^i x)\right)\right),$$

where $\sum_{x:T^n x = x}$ means the sum taken over all points $x \in \Omega^+$ such that $T^n x = x$.

The main result in this paper is the following.

THEOREM 1.1. *Assume that the potential function V on Ω^+ satisfies the following three conditions*:

(A.1) *V is Lipschitz continuous with respect to the Euclidean metric restricted to Ω^+.*

(A.2) *V is eventually positive, i.e., there exists a positive integer n_0 such that*

$$\sum_{i=0}^{n-1} V(T^i x) > 0$$

for any $x \in \Omega^+$ *and* $n \geq n_0$;

(A.3) *for each* $x \in \Omega^+$, V *is constant along the local stable curve of* x.

Then there are positive constants α_V *and* β_V *satisfying the following.*

(a) *The series in the definition of* ζ_V *is absolutely convergent in the half-plane* $\operatorname{Re} s > \alpha_V$ *and* ζ_V *defines an analytic function without zero.*

(b) ζ_V *has a meromorphic extension without zero in a domain containing the closed half-plane* $\operatorname{Re} s \geq \alpha_V$. *In particular,* $s = \alpha_V$ *is a simple pole.*

(c) *Moreover,* ζ_V *has a meromorphic extension without zero in the half-plane* $\operatorname{Re} s > -\beta_V$.

Since we can construct a Hölder continuous conjugacy between the dynamical system (Ω^+, T) and the mixing subshift of finite type (Σ, σ) (see [9] and [12]), assumptions (A.1) and (A.2) enable us to apply the general theory of thermodynamic formalism for mixing subshifts of finite type (see [2], [14] and [17]) to (Ω^+, T), where

$$\Sigma = \{w = (w_n)_{n \in \mathbf{Z}} \in \{1, 2, \dots, J\}^{\mathbf{Z}} \,;\, w_n \neq w_{n+1} \text{ for any } n \in \mathbf{Z}\}\,.$$

Thus, we see that the validity of Assertions (a) and (b) with number α_V is characterized by the equation $P(-\alpha_V V) = 0$, where $P(U)$ denotes the so-called topological pressure of a function U. Therefore, the substantial part of Theorem 1.1 is Assertion (c) for which assumption (A.3) plays a crucial role in the proof. We note that the general theory is not strong enough to derive Assertion (c) even if we additionally assume assumption (A.3) on V. This is because the symbolic dynamics (Σ, σ) do not inherit the information on the regularity of the invariant foliation of (Ω^+, T) constructed in [12]. So we have to introduce a new idea to investigate (Ω^+, T) directly (see Remark 7.2 for the details). It should be remarked that many functions V satisfy the assumptions in Theorem 1.1. For instance, given any positive valued Lipschitz continuous function F on Ω^+, we obtain a desired function V by taking a kind of average along each local stable curve (see Section 6).

Theorem 1.1 plays a significant role in the study of distribution of the length spectrum of the billiard table Q. In this case, the zeta function which is our main concern is ζ_{t^+}, because it plays the same role as the Riemann zeta function for the distribution of the prime numbers. For example, we have the Euler product formula

$$\zeta_{t^+}(s) = \prod_{\tau} (1 - \exp(-s l(\tau)))^{-1}\,,$$

where $\prod_\tau$ means the product taken over all prime closed orbits of the billiard flow, and $l(\tau)$ denotes the Euclidean length of the orbit. It is plausible that the special value of ζ_{t^+} at the origin yields an invariant of the billiard table Q, for example, it must have the information on the number of scatterers. However, unless Theorem 1.1 is established for ζ_{t^+}, one cannot consider $\zeta_{t^+}(0)$ at all. Since t^+ satisfies (A.1) and (A.2) but not (A.3), some modification is necessary to apply Theorem 1.1 to t^+. A well-known technique in thermodynamic formalism allows us to obtain a function g on Ω^+ cohomologous to t^+ satisfying (A.3). Clearly, $\zeta_{t^+} = \zeta_g$ holds. In this procedure, however, the regularity of the resulting function is no better than that of the original function, and is possibly much worse in general. Fortunately, we can verify that the corresponding function g to t^+ can be constructed so that it also satisfies both (A.1) and (A.2). Indeed, we can show the following.

THEOREM 1.2. *There exists a function g satisfying Assumptions* (A.1), (A.2) *and* (A.3) *such that it is cohomologous to the function t^+, i.e., there exists a real-valued continuous function h on Ω^+ such that $g(x) = t^+(x) + h(Tx) - h(x)$ holds for any $x \in \Omega^+$.*

Note that even if t^+ is positive, the function g cohomologous to t^+ is not necessarily positive. We only see that it is eventually positive. Thus, we prove Theorem 1.1 under the Assumption of eventual positivity rather than positivity. Consequently, as a corollary to Theorem 1.2 we obtain the following.

THEOREM 1.3. *Assertions* (a), (b) *and* (c) *in Theorem* 1.1 are valid for ζ_{t^+}.

Finally, we note that the present paper consists of the unpublished results obtained in [11] and the second half of [10]. One finds that the first half of [10] is devoted to the construction of the K-stable foliation for the billiard map T and the second half to Theorem 1.1 above. Theorems 1.2 and 1.3 are proved in [11]. In order to make these results more comprehensive, we reorganize [10] and [11] to obtain [12] and the present paper.

The present paper is organized as follows. In Section 2, we recall some basic definitions and fundamental results for the billiard map. In Section 3 we explain how to reduce our problem of T to that of a one-dimensional expanding map. In Section 4 we introduce a family of transfer operators that plays a principal role in our argument. Section 5 is devoted to the proof of Theorem 1.1. In Section 6, by a sort of averaging, we construct a function satisfying the Assumptions in Theorem 1.1. Finally, we prove Theorems 1.2 and 1.3 in Section 7.

2. Preliminaries. In this section we summarize the basic facts and results in the previous work [12] without proof. As mentioned in Section 1, the original forms of all of the results and their proofs in this section can also be found in the first half of [10].

We consider the billiard flow S^t on the exterior Q of scatterers satisfying the Ikawa conditions (H.1) and (H.2). The state space M of S^t is regarded as

$$M = \pi^{-1}Q \cup M^+ ,$$

where $\pi : \boldsymbol{R}^2 \times S^1 \to \boldsymbol{R}^2$ is the natural projection and M^+ is the totality of the states of reflection

$$M^+ = \{x = (q, v)\,; q \in \partial Q,\ \langle v, n(q)\rangle \geq 0\} .$$

We introduce a convenient local coordinates system to $\pi^{-1}\partial Q$. Choose a base point $q(j)$ for each $j = 1, 2, \dots, J$, and define the following quantities for $x = (q, v) \in \pi^{-1}\partial Q$ (see Figure 2):

- $w_0(x) = j$ if $q \in \partial Q_j$;
- $r(x)$ is the arclength from $q(w_0(x))$ to q measured counterclockwise along the curve ∂Q_j;

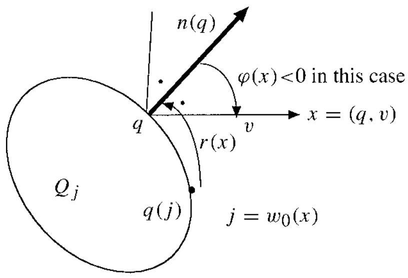

FIGURE 2.

- $\varphi(x)$ is the angle between the vector v and the inward unit normal $n(q)$ measured counterclockwise from $n(q)$ to v.

The coordinates above will be called the (r, φ)-coordinates. For $j = 1, 2, \dots, J$, set

$$M_j^+ = \{x \in M^+ \,;\, w_0(x) = j\},$$

each of which will be called a connected component of M^+. Note that the change of the base point $q(j)$ causes only the translation along the r-coordinate and preserves the φ-coordinate. In other words, M_j^+ can be obtained from $\boldsymbol{R} \times [-\pi/2, \pi/2]$ by identifying the points (r, φ) with $(r + l(\partial Q_j), \varphi)$, where $l(\partial Q_j)$ denotes the perimeter of the simple closed curve ∂Q_j. Therefore, we often regard M_j^+ as the fundamental domain $[0, l(\partial Q_j)) \times [-\pi/2, \pi/2]$. In addition, we often use the (r, φ)-coordinates without specifying base points and often drop the first coordinate j when working on a fixed connected component M_j^+. We also abuse the notation $x = (q, v) = (r, \varphi)$ if there is no possibility of confusion. Under this convention, the totality of reflection states M^+ is expressed as

$$M^+ = \{x \in \pi^{-1}\partial Q \,;\, -\pi/2 \le \varphi(x) \le \pi/2\}.$$

Next, we define the first and the last collision times for the billiard flow at $x \in M^+$ as follows:

$$\begin{cases} t^+(x) = \inf\{t > 0 \,;\, S^t x \in M^+\}, \\ t^-(x) = \sup\{t < 0 \,;\, S^t x \in M^+\}, \end{cases}$$

where $t^+(x)$ (resp. $t^-(x)$) is regarded as $+\infty$ (resp. $-\infty$) if the set in question above is empty. Set

$$\mathcal{D}_1 = \{x \in M^+ \,;\, t^+(x) < \infty\}, \quad \mathcal{D}_{-1} = \{x \in M^+ \,;\, t^-(x) > -\infty\}.$$

We define the first and the last collision maps $T : \mathcal{D}_1 \to M^+$ and $T^{-1} : \mathcal{D}_{-1} \to M^+$ by

$$\begin{cases} Tx = S^{t^+(x)}x & \text{if } \ x \in \mathcal{D}_1, \\ T^{-1}x = S^{t^-(x)}x & \text{if } \ x \in \mathcal{D}_{-1}, \end{cases}$$

respectively. The first collision map is usually called the billiard ball map for S^t. T (resp. T^{-1}) turns out to be a C^2-diffeomorphism from $\operatorname{int}\mathcal{D}_1$ (resp. $\operatorname{int}\mathcal{D}_{-1}$) onto $\operatorname{int}\mathcal{D}_{-1}$

(resp. int $\mathcal{D}_1$). For each positive integer n, we define $\mathcal{D}_n$, $\mathcal{D}_{-n}$, T^n and T^{-n} inductively by

$$\mathcal{D}_{n+1} = \{x \in \mathcal{D}_n ; t^+(T^n x) < \infty\}, \quad T^{n+1}x = T(T^n x) \quad \text{for } x \in \mathcal{D}_{n+1},$$

$$\mathcal{D}_{-(n+1)} = \{x \in \mathcal{D}_{-n} ; t^-(T^{-n} x) > -\infty\}, \quad T^{-(n+1)}x = T^{-1}(T^{-n} x) \quad \text{for } x \in \mathcal{D}_{-(n+1)}.$$

Although T^n and T^{-n} are independently defined in the context above, it is clear that $T^{-n} = (T^n)^{-1}$ holds for any $n \geq 1$. The non-wandering set Ω is given by

$$\Omega = \{x \in M ; \pi(S^t x) \in \partial Q \text{ holds for infinitely many } t > 0 \text{ and } t < 0\}.$$

The set $\Omega^+ = \Omega \cap M^+$ is expressed as $\Omega^+ = \bigcap_{n \in \boldsymbol{Z}} \mathcal{D}_n$ and T is clearly invertible on Ω^+, where $\mathcal{D}_0$ is regarded as M^+ for the sake of convenience. We see that the flow S^t restricted to Ω can be represented as a suspension flow with base transformation (Ω^+, T) and ceiling function t^+. More precisely, set

$$\Omega^{+,t^+} = \{(x, s) ; x \in \Omega^+, 0 \leq s \leq t^+(x)\}/\sim,$$

where '$\sim$' means that $(x, t^+(x))$ is identified with $(Tx, 0)$. We can define a flow $T^t_{t^+}$ so that

$$T^t_{t^+}(x, s) = (x, s + t)$$

holds for (s, x) with $0 \leq s + t \leq t^+(x)$. Then the flows (Ω, S^t) and $(\Omega^{+,t^+}, T^t_{t^+})$ are conjugate each other in such a way that the corresponding periodic orbits have the same periods.

For $x \in M^+$, we put

$$w_i(x) = w_0(T^i x) \quad \text{if} \quad T^i \text{ is defined.}$$

For integers k and l with $-\infty \leq k \leq l \leq \infty$, a sequence $\{w_i\}_{i=k}^l$ is called the itinerary of $x \in M^+$ from time k to time l if $w_i = w_i(x)$ holds for each $i \in \boldsymbol{Z} \cap [k, l]$. The number $l-k+1$, possibly ∞, is called the length of the itinerary. If $k = -\infty$ and $l = \infty$, the sequence $\{w_i(x)\}_{i=-\infty}^{\infty}$ is simply called the itinerary of x and is denoted by $w(x)$. On the other hand, for non-negative integer n, a sequence $\{w_i\}_{i=0}^n \in \{1, 2, \ldots, J\}^{n+1}$ is called admissible or an admissible word if there exists $\xi \in \Sigma$ such that $\xi_i = w_i$ holds for each $i \in \boldsymbol{Z} \cap [0, n]$. The number $n + 1$ is called the length of the word. The totality of admissible words with length n is denoted by $\mathcal{W}_n$.

In [10] and [12] (see also [9]) it is shown that the map

$$w(\cdot) : \Omega^+ \to \Sigma, \quad x \mapsto w(x)$$

is a topological conjugacy between the topological dynamical systems (Ω^+, T) and the shift (Σ, σ).

We reduce our problem to a one-dimensional expanding map in Section 3. To this end we need more investigations. For each $(w_0, w_1, \ldots, w_n)$ and $(w_{-n}, w_{-(n-1)}, \ldots, w_0)$ in $\mathcal{W}_{n+1}$, set

$$\mathcal{D}_n(w_0 w_1 \cdots w_n) = \{x \in M^+ ; w_i(x) = w_i \text{ for } i = 0, 1, \ldots, n\} \quad \text{and}$$

$$\mathcal{D}_{-n}(w_{-n} w_{-(n-1)} \cdots w_0) = \{x \in M^+ ; w_i(x) = w_i \text{ for } i = 0, -1, \ldots, -n\}.$$

The definition domains $\mathcal{D}_n$ of T^n and $\mathcal{D}_{-n}$ of T^{-n} are expressed as

$$\mathcal{D}_n = \bigcup_{w \in \mathcal{W}_{n+1}} \mathcal{D}_n(w) \quad \text{(disjoint union)} \quad \text{and}$$
$$\mathcal{D}_{-n} = \bigcup_{w \in \mathcal{W}_{n+1}} \mathcal{D}_{-n}(w) \quad \text{(disjoint union)}\,.$$

Put

$$k_{\max} = \max\{k(q);\, q \in \partial Q\}, \quad k_{\min} = \min\{k(q);\, q \in \partial Q\}\,,$$
$$t_{\min} = \min\{\operatorname{dist}(\overline{Q_{j_1}}\,, \overline{Q_{j_2}})\,;\, j_1 \neq j_2\}\,,$$
$$K_{\max} = k_{\max} + \frac{1}{t_{\min}}\,, \quad \theta = \frac{1}{1 + t_{\min}k_{\min}}\,,$$

where $k(x) = k(q)$ denotes the curvature of ∂Q at q with $x = (q, v)$.

We introduce the notion of increasing curves and decreasing curves. A curve γ in M^+, expressed as $r = r(\varphi), \alpha \le \varphi \le \beta$, is said to be increasing (resp. decreasing) if $r(\cdot)$ is increasing (resp. decreasing) as a function of φ. When a curve is expressed as $\varphi = \varphi(r), a \le r \le b$, we also say it is increasing or decreasing according to whether $\varphi(\cdot)$ is increasing or decreasing as a function of r. Increasing curves and decreasing curves are occasionally called monotone curves for convenience. For a curve γ in M^+, $\Theta(\gamma)$ denotes the variation of φ-coordinate along γ. Clearly, if γ is a monotone curve which is expressed as $r = r(\varphi),\ \alpha \le \varphi \le \beta$ (resp. $\varphi = \varphi(r),\ a \le r \le b$), then $\Theta(\gamma)$ is given by

$$\Theta(\gamma) = \beta - \alpha \quad (\text{resp. } \Theta(\gamma) = |\varphi(b) - \varphi(a)|)\,. \tag{2.1}$$

An increasing (resp. decreasing) curve is called K-increasing (resp. K-decreasing) if

$$\frac{1}{K_{\max}} \le \frac{r(\psi) - r(\varphi)}{\psi - \varphi} \le \frac{1}{k_{\min}} \quad \left(\text{resp.} -\frac{1}{k_{\min}} \le \frac{r(\psi) - r(\varphi)}{\psi - \varphi} \le -\frac{1}{K_{\max}}\right)$$

holds for any φ and ψ with $\alpha \le \varphi < \psi \le \beta$.

We employ the following notation. For $x = (q, v) = (r, \varphi)$, k_i, r_i, φ_i, c_i, t_i^+ and t_i^-, denote $k(T^i x)$, $r(T^i x)$, $\varphi(T^i x)$, $c(T^i x)$, $t^+(T^i x)$ and $t^-(T^i x)$, respectively, where $c = c(x) = \cos\varphi$.

We summarize the useful formulas in the following.

LEMMA 2.1. *Let γ be a curve of class C^1 which is expressed as $\{(j, r, \varphi); \varphi = \varphi(r),\ a \le r \le b\}$, where $\varphi(\cdot)$ is a C^1 function in r. Assume that T and T^{-1} are defined on γ. If the images $\gamma_1 = T\gamma$ and $\gamma_{-1} = T^{-1}\gamma$ are expressed as $\{(j_1, r_1, \varphi_1); \varphi_1 = \varphi_1(r_1),\ a_1 \le r_1 \le b_1\}$ and $\{(j_{-1}, r_{-1}, \varphi_{-1}); \varphi_{-1} = \varphi_{-1}(r_{-1}),\ a_{-1} \le r_{-1} \le b_{-1}\}$, where $\varphi_1(\cdot)$ and $\varphi_{-1}(\cdot)$*

are C^1 functions in r_1 and r_{-1}, respectively, then we have the formulas:

$$\frac{d\varphi_1}{dr_1} = k_1 + c\frac{c_1}{c}\frac{1}{\dfrac{t^+}{c} + \dfrac{1}{\dfrac{d\varphi}{dr} + k}}, \quad \frac{d\varphi_{-1}}{dr_{-1}} = -k_{-1} + \frac{c_{-1}}{c}\frac{1}{\dfrac{t^-}{c} + \dfrac{1}{\dfrac{d\varphi}{dr} - k}},$$

$$\frac{dr_1}{dr} = -\frac{c}{c_1}\left(1 + \frac{t^+(d\varphi/dr + k)}{c}\right), \quad \frac{dr_{-1}}{dr} = -\frac{c}{c_{-1}}\left(1 + \frac{t^-(d\varphi/dr - k)}{c}\right),$$

$$\frac{d\varphi_1}{d\varphi} = -k_1\frac{c}{c_1}\frac{dr}{d\varphi} - \left(1 + \frac{t^+k}{c_1}\right)\left(1 + k\frac{dr}{d\varphi}\right),$$

$$\frac{d\varphi_{-1}}{d\varphi} = k_{-1}\frac{c}{c_{-1}}\frac{dr}{d\varphi} - \left(1 - \frac{t^-k}{c_{-1}}\right)\left(1 - k\frac{dr}{d\varphi}\right),$$

$$\frac{dt^+}{dr} = \sin\varphi_1\frac{dr_1}{dr} - \sin\varphi\,, \quad \frac{dt^-}{dr} = \sin\varphi_{-1}\frac{dr_{-1}}{dr} - \sin\varphi\,.$$

These formulas have meaning even when $d\varphi/dr = 0$ and we obtain similar formulas if the role of the r-coordinate and that of the φ-coordinate are exchanged in the representations of curves γ, γ_1 and γ_{-1}.

Combining Lemma 2.1 with the fact that the boundary ∂Q is of class C^3, we can easily show the following lemma.

LEMMA 2.2. *Let γ be a C^2 curve in M^+ which is expressed as $\{(r,\varphi); \varphi = \varphi(r),\ a < r < b\}$. Assume that γ is increasing (resp. decreasing) and T (resp. T^{-1}) is defined on γ. Then $T\gamma$ (resp. $T^{-1}\gamma$) turns out to be a C^2 curve which is expressed as $\{(r_1,\varphi_1); \varphi_1 = \varphi_1(r_1),\ a_1 < r_1 < b_1\}$ (resp. $\{(r_{-1},\varphi_{-1}); \varphi_{-1} = \varphi_{-1}(r_{-1}),\ a_{-1} < r_{-1} < b_{-1}\}$) satisfying*

$$k_{\min} \le \frac{d\varphi_1}{dr_1} \le K_{\max} \quad \left(\text{resp.} - K_{\max} \le \frac{d\varphi_{-1}}{dr_{-1}} \le -k_{\min}\right).$$

In addition, we have

$$\Theta(T\gamma) \ge \theta^{-1}\Theta(\gamma) \quad (\text{resp. } \Theta(T^{-1}\gamma) \ge \theta^{-1}\Theta(\gamma))\,,$$

where $\Theta(\gamma)$ denotes the variation of φ-coordinate along γ (see (2.1)).

We make further investigations of the structure of the definition domain $\mathcal{D}_n$ (resp. $\mathcal{D}_{-n}$) of T^n (resp. T^{-n}). For $j = 1, 2, \ldots, J$, define

$$S_j^+ = \{x \in M^+\,; w_0(x) = j,\ \varphi(x) = \pi/2\}, \quad S_j^- = \{x \in M^+\,; w_0(x) = j,\ \varphi(x) = -\pi/2\}$$

and put

$$S^+ = \bigcup_{j=1}^{J} S_j^+\,, \quad S^- = \bigcup_{j=1}^{J} S_j^- \quad \text{and} \quad S = S^- \cup S^+\,.$$

If (i, j) is admissible, we can show that $\mathcal{D}_1(ij)$ is a closed domain in M_i^+ enclosed by four curves $T^{-1}S_j^+$, $\varphi = -\pi/2$, $T^{-1}S_j^-$, and $\varphi = \pi/2$. Similarly, $\mathcal{D}_{-1}(ij)$ is a closed domain in M_j^+ enclosed by four curves TS_i^-, $\varphi = -\pi/2$, TS_i^+, and $\varphi = \pi/2$. Since $\cos\varphi_1 = 0$

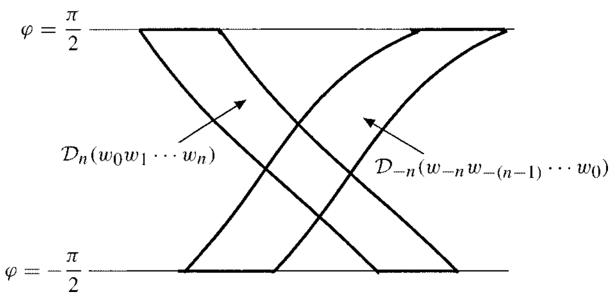

FIGURE 3.

(resp. $\cos\varphi_{-1} = 0$) holds on $T^{-1}S$ (resp. TS), $T^{-1}S_j^+$ and $T^{-1}S_j^-$ (resp. TS_i^- and TS_i^+) are K-decreasing curves (resp. K-increasing curves) expressed by the equation of the form

$$\frac{d\varphi}{dr} = -k - \frac{\cos\varphi}{t^+} \quad \left(\text{resp. } \frac{d\varphi}{dr} = k - \frac{\cos\varphi}{t^-}\right).$$

We call such a closed domain enclosed by a pair of increasing curves and a pair of decreasing curves a quadrilateral. Combining these facts with Lemma 2.2, we can show inductively the following.

LEMMA 2.3 (see Figure 3). *Assume that Q is the exterior domain of the scatterers satisfying Conditions* (H.1) *and* (H.2). *Let $w_0w_1\cdots w_n$ and $w_{-n}w_{-(n-1)}\cdots w_0$ be admissible words of length $n+1$ for a positive integer n. Then we have the following.*

(1) *The set $\mathcal{D}_n(w_0w_1\cdots w_n)$ (resp. $\mathcal{D}_{-n}(w_{-n}w_{-(n-1)}\cdots w_0)$) is a quadrilateral enclosed by a pair of K-decreasing curves (resp. K-increasing curves) and $\varphi = \pm\pi/2$.*

(2) *T^n (resp. T^{-n}) is a homeomorphism from $\mathcal{D}_n(w_0w_1\cdots w_n)$ onto $\mathcal{D}_{-n}(w_0w_1\cdots w_n)$ (resp. $\mathcal{D}_{-n}(w_0w_1\cdots w_n)$ onto $\mathcal{D}_n(w_0w_1\cdots w_n)$) and a diffeomorphism of class C^2 from $\operatorname{int}\mathcal{D}_n(w_0w_1\cdots w_n)$ onto $\operatorname{int}\mathcal{D}_{-n}(w_0w_1\cdots w_n)$ (resp. from $\operatorname{int}\mathcal{D}_{-n}(w_0w_1\cdots w_n)$ onto $\operatorname{int}\mathcal{D}_n(w_0w_1\cdots w_n)$).*

(3) *The Hausdorff distance with respect to (r,φ)-coordinates between two K-decreasing curves (resp. K-increasing curves) lying in the boundary of $\mathcal{D}_n(w_0\cdots w_n)$ (resp. $\mathcal{D}_{-n}(w_{-n}\cdots w_0)$) is not greater than $C_1\theta^n$ for some positive number C_1 depending only on the domain Q.*

(4) *Each K-decreasing curve lying along the boundary of $\mathcal{D}_n(w_0\cdots w_n)$ intersects each K-increasing curve lying along the boundary of $\mathcal{D}_{-n}(w_{-n}\cdots w_0)$. Moreover, the diameter of the set $\mathcal{D}_{-n}(w_{-n}\cdots w_0)\cap\mathcal{D}_n(w_0\cdots w_n)$ is not greater than $C_2\theta^n$ for some positive number C_2 depending only on the domain Q.*

Now we recall the itinerary problem studied in [9]. By the itinerary problem we mean the problem finding a point $x \in \Omega^+$ which satisfies the equation

$$w(x) = w$$

for a sequence w in Σ given beforehand. In virtue of Theorem 2.4 below, the itinerary problem has a unique solution. Thus, we denote by $x(w)$ the point having the given itinerary w. For $w,\ w' \in \Sigma$, put $d_\theta(w, w') = \theta^n$, where $n = \min\{i \geq 0\,;\, w_{-i} \neq w'_{-i}$ or $w_i \neq w'_i\}$. Then d_θ is a metric on Σ which introduces the same topology as that induced by the product topology of $\{1,\ 2, \ldots, J\}^{\mathbf{Z}}$. In virtue of Lemma 2.3, we can show the d_θ-Lipschitz well-posedness of the itinerary problem as follows.

THEOREM 2.4. *For any sequence $w \in \Sigma$, there exists a unique $x \in \Omega^+$ such that $w(x) = w$. Moreover, there exists a positive constant C_3 depending only on the domain Q such that*

$$|r(x(w)) - r(x(w'))| \leq C_3 d_\theta(w, w'), \quad |\varphi(x(w)) - \varphi(x(w'))| \leq C_3 d_\theta(w, w')$$

hold for any $w,\ w' \in \Sigma$.

It is an easy consequence of Theorem 2.4 that $w(\cdot) : \Omega^+ \to \Sigma$ gives a topological conjugacy between (Ω^+, T) and (Σ, σ).

Next we summarize the facts on the structure of the local stable curve and the local unstable curve for $x \in \Omega^+$ as Theorem 2.5.

THEOREM 2.5. *Given $x \in \Omega^+$, let*

$$\gamma^s(x) = \{y \in M^+\,;\, w_n(y) = w_n(x) \text{ for any } n \geq 0\},$$
$$(\text{resp. } \gamma^u(x) = \{y \in M^+\,;\, w_n(y) = w_n(x) \text{ for any } n \leq 0\}).$$

Then $\gamma^s(x)$ (resp. $\gamma^u(x)$) yields a K-decreasing curve (resp. K-increasing curve) of class C^2 except for the endpoints, and satisfies

$$\gamma^s(x) = \bigcap_{n=1}^{\infty} \mathcal{D}_n(w_0(x), \ldots, w_n(x)),$$
$$\left(\text{resp. } \gamma^u(x) = \bigcap_{n=1}^{\infty} \mathcal{D}_{-n}(w_{-n}(x), \ldots, w_0(x))\right).$$

In the rest of this section we give the existence theorem of a K-stable foliation for the billiard map (Ω^+, T). From Lemma 2.3 and Theorem 2.5 we can easily notice the existence of a horseshoe-like structure. In particular, we have seen that the set $\Gamma = \bigcup_{x \in \Omega^+} \gamma^s(x)$ forms an invariant lamination $\mathcal{L}$ with the following properties.

($\mathcal{L}$.1) Each leaf of $\mathcal{L}$ is a K-decreasing curve of class C^2.

($\mathcal{L}$.2) Each leaf is a local stable curve for some point $x \in \Omega^+$.

($\mathcal{L}$.3) For any point $x \in \Gamma$ the leaf $\mathcal{L}(x)$ containing x satisfies $T\mathcal{L}(x) \subset \mathcal{L}(Tx)$.

The main theorem in the first half of [10] asserts that the invariant lamination can be extended to a Lipschitz continuous invariant foliation supported on the set $\mathcal{D}_1$. Precisely we have the following.

THEOREM 2.6 (see [12]). *With the same notation as above, we can construct a foliation $\mathcal{F}$ supported on the set $\mathcal{D}_1$ satisfying the following.*

($\mathcal{F}$.1) *Each leaf of $\mathcal{F}$ is a K-decreasing curve.*

($\mathcal{F}$.2) *For any $x \in \Omega^+$, the leaf $\mathcal{F}(x)$ containing x coincides with the local stable curve $\gamma^s(x)$.*

($\mathcal{F}$.3) *For any point $x \in \mathcal{D}_2$, $T\mathcal{F}(x) \subset \mathcal{F}(Tx)$ holds.*

($\mathcal{F}$.4) *$\mathcal{F}$ is a Lipschitz continuous foliation on $\mathcal{D}_1$ with respect to the Euclidean distance in the (r, φ)-coordinates.*

We call the foliation in Theorem 2.6 a K-stable foliation for (Ω^+, T).

REMARK 2.7. (1) In this paper, a foliation is said to be Lipschitz continuous if it has a bi-Lipschitz continuous foliation chart. Precisely, we have the following. For each $(i, j) \in \mathcal{W}_2$, we identify $\mathcal{D}_1(ij)$ with a quadrilateral in the (r, φ)-plane which is enclosed by two K-decreasing curves and two lines $\varphi = \pi/2$ and $\varphi = -\pi/2$. Then there exist numbers $a = a(ij)$ and $b = b(ij)$ with $a < b$ and a bijection

$$\Phi = \Phi_{ij} : [a, b] \times [-\pi/2, \pi/2] \to \mathcal{D}_1(ij)$$

with the following properties.

(i) Φ is Lipschitz continuous with respect to the usual Euclidean distance on $[a, b] \times [-\pi/2, \pi/2]$ and that on the (r, φ)-plane.

(ii) Φ^{-1} is Lipschitz continuous with respect to the Euclidean distance on the (r, φ)-plane and that on $[a, b] \times [-\pi/2, \pi/2]$.

(iii) For each $r \in [a, b]$, Φ maps $\{r\} \times [-\pi/2, \pi/2]$ homeomorphically to a leaf of $\mathcal{F}$.

(2) It will be convenient if we can extend $\mathcal{F}$ to a Lipschitz continuous foliation on the whole of M^+. We can obtain such an extension in the following way.

If $\mathcal{E}$ is a quadrilateral in the (r, φ)-plane enclosed by two K-decreasing curves γ_0 and γ_1 and two lines $\varphi = -\pi/2$ and $\varphi = \pi/2$, then we can construct a Lipschitz continuous foliation on $\mathcal{E}$ whose leaves are K-decreasing in the following way. We may assume that γ_i is expressed as $r = r_i(\varphi)$ with $\varphi \in [-\pi/2, \pi/2]$ for $i = 0, 1$. If we define the curves γ_t, $t \in [0, 1]$, expressed by $r = r_t(\varphi) = (1 - t)r_0(\varphi) + tr_1(\varphi)$, then they yield the leaves of the desired foliation.

Recall that for each j, M_j^+ can be obtained from $\boldsymbol{R} \times [-\pi/2, \pi/2]$ by identifying the points (r, φ) with $(r + l(\partial Q_j), \varphi)$, where $l(\partial Q_j)$ denotes the perimeter of the simple closed curve ∂Q_j. Therefore, we can regard M_j^+ as the fundamental domain $[0, l(\partial Q_j)) \times [-\pi/2, \pi/2]$. Thus, it is easy to see from the argument above that we can fill up $M_j^+ \setminus (\bigcup_{i \neq j} \mathcal{D}_1(ji))$ by K-decreasing curves so that the resulting foliation can be a Lipschitz continuous extension of $\mathcal{F}$ on M^+.

In what follows, the K-stable foliation $\mathcal{F}$ means the foliation obtained by such an extension procedure, and the K-stable foliation $\mathcal{F}$ is assumed to be the foliation supported on the whole of M^+ whose restriction to $\mathcal{D}_1$ satisfies the Assertions in Theorem 2.6 unless otherwise stated.

3. Reduction to a one-dimensional expanding map. In this section we introduce a one-dimensional expanding map and reduce the analysis of our zeta functions to that of the zeta functions associated with the one-dimensional map.

First, for each $j = 1, 2, \ldots, J$, choose a point $x(j) \in \Omega^+$ with $w_0(x(j)) = j$ and set $\gamma(j) = \gamma^u(x(j))$. $\gamma(j)$ is a K-increasing curve joining S_j^+ and S_j^- passing through each of $\mathcal{D}_1(ji)$ with $i \neq j$. It is easy to see that we can choose a base point $q(j)$ so that $\gamma(j)$ and $(J-1)$ domains $\mathcal{D}_1(ji)$, $i \neq j$, can be identified with a curve and domains in a (r, φ)-plane P_j, in terms of the corresponding (r, φ)-coordinates. From now on we also fix such a choice of base points. Unless otherwise stated, we employ such an identification in what follows. This enables us to carry out our investigation as if $\gamma(j)$ and $\mathcal{D}_1(ji)$ with $i \neq j$ are lying in the (r, φ)-plane P_j. Recall that $\mathcal{D}_1 \cap M_j^+ = \bigcup_{i:i \neq j} \mathcal{D}_1(ji)$. Let $\mathcal{D}(j)$ be the minimal quadrilateral in M_j^+ among all quadrilaterals containing $\mathcal{D}_1 \cap M_j^+$ such that two of their four sides are parallel to the r-axis. Note that one of the other sides of $\mathcal{D}(j)$ is necessarily the right-hand side of $\mathcal{D}_1(jk)$ which is located in the right end of $\mathcal{D}(j)$, and the other is the left-hand side of $\mathcal{D}_1(jl)$ which is located in the left end of $\mathcal{D}(j)$ for some k and l (see Figure 4). As mentioned in Remark 2.7, we consider the K-stable foliation supported on the whole of M^+. For any subset A of M^+, we denote the foliation restricted to the set A by $\mathcal{F} \cap A$. Set $\mathcal{D} = \bigcup_{j=1}^J \mathcal{D}(j)$. Let $\bar{\gamma}$ and $\hat{\gamma}$ be increasing curves in M_j^+ such that any leaf of $\mathcal{F}$ that intersects $\bar{\gamma}$ also intersects $\hat{\gamma}$. Then we can define a map $\Pi_{\bar{\gamma},\hat{\gamma}} : \bar{\gamma} \cap \mathcal{F} \to \hat{\gamma} \cap \mathcal{F}$ so that $\Pi_{\bar{\gamma},\hat{\gamma}} x$ is the unique point in $\hat{\gamma} \cap \mathcal{F}(x)$. $\Pi_{\bar{\gamma},\hat{\gamma}}$ is called the holonomy map (or the canonical projection) from $\bar{\gamma}$ to $\hat{\gamma}$ along the leaf of $\mathcal{F}$. Note that $\Pi_{\bar{\gamma},\hat{\gamma}}$ depends on the choice of $\mathcal{F}$ but $\Pi_{\bar{\gamma},\hat{\gamma}}|_{\bar{\gamma} \cap \Omega^+}$ does not.

In the sequel, we use the following notation. For x, y in the same connected component of M^+, $l(x, y)$ denotes the Euclidean length between the points x and y with respect to the (r, φ)-coordinates. For positive numbers a, b and c with $c > 1$, we write $a \in [c^{-1}, c]b$ if $c^{-1}b \leq a \leq cb$ holds.

The following fact will be used frequently in what follows.

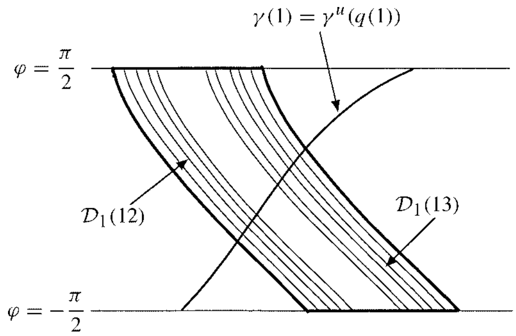

FIGURE 4. This illustrates the case when $J = 3$ and $j = 1$. $D(1)$ is the quadrilateral with a black border.

LEMMA 3.1. *Let $\mathcal{F}$ be the K-stable foliation as above. For $j \in \mathcal{W}_1$, consider two increasing curves $\bar{\gamma}$ and $\hat{\gamma}$ in M_j^+ such that any leaf of $\mathcal{F}$ that intersects $\bar{\gamma}$ also intersects $\hat{\gamma}$. Then there exists a positive number $C_4 > 1$ depending only on Q such that $l(\Pi_{\bar{\gamma},\hat{\gamma}}x, \Pi_{\bar{\gamma},\hat{\gamma}}y) \in [C_4^{-1}, C_4]l(x, y)$ holds for any $x,\ y \in \bar{\gamma}$. In particular, both $\bar{\gamma}$ and $\hat{\gamma}$ are K-increasing, there exists $C_5 > 1$ such that for any segments $\bar{\delta} \subset \bar{\gamma}$, we have $\Theta(\Pi_{\bar{\gamma},\hat{\gamma}}\bar{\delta}) \in [C_5^{-1}, C_5]\Theta(\bar{\delta})$, where $\Theta(\gamma)$ denotes the variation of φ-coordinate along the curve γ (see* (2.1)).

PROOF. The second Assertion is an easy consequence of the first. Thus, we just prove the first Assertion. Without loss of generality, we may assume that $r(x) \le r(y)$.

First we consider the special case when $\hat{\gamma}$ is a curve parallel to the r-axis and passing through x. This means $x = \Pi_{\bar{\gamma},\hat{\gamma}}x$. Let z be the point where the line passing through y which is perpendicular to $\hat{\gamma}$ intersects $\hat{\gamma}$. Note that $x \le z < \Pi_{\bar{\gamma},\hat{\gamma}}y$. Since the leaf $\mathcal{F}(y)$ is K-decreasing, we easily see that

$$\begin{aligned} l(x, y) &\le |r(x) - r(y)| + |\varphi(x) - \varphi(y)| \\ &\le (1 + K_{\max})|r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(x)| = (1 + K_{\max})l(\Pi_{\bar{\gamma},\hat{\gamma}}x, \Pi_{\bar{\gamma},\hat{\gamma}}y)\,. \end{aligned}$$

On the other hand, if $r(y) - r(x) > (1/2)(r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(x))$, we have $l(x, y) > (1/2)(r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(x))$ and if $r(y) - r(x) \le (1/2)(r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(x))$, we have $l(x, y) \ge \varphi(y) - \varphi(x) \ge k_{\min}(r(y) - r(z)) \ge k_{\min}(1/2)(r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(x))$, since $\mathcal{F}(y)$ is K-decreasing. Anyway, we have

$$l(x, y) \ge \frac{\min(1, k_{\min})}{2} l(\Pi_{\bar{\gamma},\hat{\gamma}}x, \Pi_{\bar{\gamma},\hat{\gamma}}y)\,.$$

Therefore, we have proved the Assertion in this case.

Next we consider the case when both $\bar{\gamma}$ and $\hat{\gamma}$ are parallel to the r-axis. In virtue of the Lipschitz continuity, we see that there exists a constant $C \ge 1$ depending only on the domain Q such that

$$r(y) - r(x) \in [C^{-1}, C](r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(\Pi_{\bar{\gamma},\hat{\gamma}}x))\,.$$

Since $l(x, y) = r(y) - r(x)$ and $l(\Pi_{\bar{\gamma},\hat{\gamma}}x, \Pi_{\bar{\gamma},\hat{\gamma}}y) = r(\Pi_{\bar{\gamma},\hat{\gamma}}y) - r(\Pi_{\bar{\gamma},\hat{\gamma}}x)$ in this case, we have

$$l(x, y) \in [C^{-1}, C]l(\Pi_{\bar{\gamma},\hat{\gamma}}x, \Pi_{\bar{\gamma},\hat{\gamma}}y)\,.$$

Combining the results in both cases, we arrive at the desired Assertion. □

For each $j = 1,\ 2, \dots, J$, let $Z(j)$ be $\gamma(j) \cap \mathcal{D}(j)$. In other words, $Z(j)$ is the minimal curve segment of $\gamma(j)$ containing $\gamma(j) \cap (\bigcup_{i \ne j} \mathcal{D}_1(ji))$. Next, for each admissible word $w_0w_1 \cdots w_n$, $n \ge 1$, we define a curve segment $Z(w_0w_1 \cdots w_n)$ as follows. Choose any $y \in \Omega^+$ satisfying $w_{-n}(y)w_{-(n-1)}(y) \cdots w_0(y) = w_0w_1 \cdots w_n$. Then the curve segment $\Pi_{\gamma^u(T^{-n}y),\gamma(w_0)}T^{-n}\Pi_{\gamma(w_n),\gamma^u(y)}Z(w_n)$ in $Z(w_0)$ is independent of the choice of such a y in virtue of Condition ($\mathcal{F}$.3). Now we set

$$Z(w_0w_1 \cdots w_n) = \Pi_{\gamma^u(T^{-n}y),\gamma(w_0)}T^{-n}\Pi_{\gamma(w_n),\gamma^u(y)}Z(w_n)\,.$$

By definition, it is obvious that

$$\Pi_{\gamma^u(T^i x),\gamma(w_i)}T^i Z(w_0w_1 \cdots w_n) = Z(w_i \cdots w_n)$$

holds for any $i = 0, 1, \dots, n$ and any $x \in Z(w_0 w_1 \cdots w_n)$, where $\gamma^u(y)$ denotes the local unstable curve containing y.

Taking this fact into consideration, we define a one-dimensional local map S on $X = \bigcup_{j=1}^{J} Z(j)$ by

$$Sx = \Pi_{\gamma^u(Tx),\gamma(w_1)} Tx \quad \text{if} \quad x \in Z(w_0) \cap \mathcal{D}_1(w_0 w_1)\,.$$

It is clear that if $x \in Z(w_0) \cap \mathcal{D}_n(w_0 w_1 \cdots w_n)$, then $S^n x$ can be defined as well as $T^n x$ can and

$$S^n x = \Pi_{\gamma^u(T^n x),\gamma(w_n)} T^n x$$

holds. Now we notice that the holonomy map $\Pi_{\gamma^u(Tx),\gamma(w_1(x))}$ in the above is determined by the information of x only. Therefore, even if we simply write $Sx = \Pi T x$, one can easily recognize that Π means $\Pi_{\gamma^u(Tx),\gamma(w_1(x))}$ from the context.

For each non-negative integer n, put

$$\mathcal{P}_n = \{Z(w);\, w \in \mathcal{W}_{n+1}\} \quad \text{and} \quad X_n = \bigcup_{Z \in \mathcal{P}_n} Z\,.$$

Obviously, $X_0 = X$. For $n \geq 1$, X_n is the definition domain of S^n, and $\mathcal{P}_n$ can be considered as a partition of the definition domain of S^n. For $Z \in \mathcal{P}_n$, we denote $T^n|_Z$ and $S^n|_Z$ by T_Z^n and S_Z^n, respectively. The inverse of $T_Z^n : Z \to T_Z^n Z$ and that of $S_Z^n : Z \to S_Z^n Z$ are denoted by T_Z^{-n} and S_Z^{-n}, respectively.

We can verify that the maps T_Z^n are quite similar to those studied in [8] and [15].

LEMMA 3.2. *For any $n \geq 1$ and for any $Z \in \mathcal{P}_n$, let $r = r(\varphi)$ and $r_n = r_n(\varphi_n(\varphi))$, $\alpha \leq \varphi \leq \beta$, be the representations of Z and $T_Z^n Z$ as K-increasing curves, respectively. Then we have the following.*

(1) (C^2-*regular*) *φ_n can be extended to a C^2 function in some open interval in the φ-axis containing $[\alpha, \beta]$.*

(2) (*Uniformly expanding*)

$$\inf_{Z \in \mathcal{P}_n} \inf_{x \in Z} \left| \frac{d\varphi_n}{d\varphi}(\varphi(x)) \right| \geq \theta^{-n}$$

holds.

(3) (*Finite distortion* (*Rényi condition*).) *There exists a positive number C_6 depending only on Q such that*

$$\sup_{Z \in \mathcal{P}_n} \sup_{x \in Z} \left| \frac{d^2\varphi_n}{d\varphi^2}(\varphi(x)) \right| \left| \left(\frac{d\varphi_n}{d\varphi}(\varphi(x)) \right)^2 \right|^{-1} < C_6\,.$$

(4) *There is a positive number $C_7 > 1$ depending only on Q such that*

$$\left| \left(\frac{d\varphi_n}{d\varphi}(\varphi(x)) \right)^{-1} \right| \in [C_7^{-1}, C_7]\Theta(Z)$$

holds for any $x \in Z$.

PROOF. Since it is easy to see that $-\pi/2 < \alpha < \beta < \pi/2$, Assertion (1) follows from Theorem 2.5. Assertion (2) is an easy consequence of Lemma 2.2.

Next we denote by $R(n)$ the left-hand side of Assertion (3). In virtue of Lemma 2.2 and Theorem 2.5, we can easily see by the chain rule that $R(n) \leq R(1) + \theta R(n-1)$ holds for each $n \geq 2$. Therefore, Assertion (3) is valid.

Finally, we prove Assertion (4). Since

$$\Pi_{\gamma^u(x),\gamma(w_0(x))}(\gamma^u(x) \cap \mathcal{D}(w_0(x))) = Z(w_0(x))$$

holds for any $x \in \Omega^+$, Lemma 3.1 implies that

$$\Theta(\gamma^u(x) \cap \mathcal{D}(w_0(x))) \in [C_5^{-1}, C_5]\Theta(Z(w_0(x)))\,.$$

Thus, we have

$$\Delta = \inf_{x\in\Omega^+} \Theta(\gamma^u(x) \cap \mathcal{D}(w_0(x))) > 0\,.$$

In addition, we have

$$\Delta \leq \Theta(T_Z^n Z) = \int_\alpha^\beta \left|\frac{d\varphi_n}{d\varphi}\right| d\varphi \leq \sup_{x\in Z}\left|\frac{d\varphi_n}{d\varphi}(\varphi(x))\right| \Theta(Z)\,,$$

since $\Theta(Z) = \beta - \alpha$.

On the other hand, we have

$$\pi \geq \Theta(T_Z^n Z) = \int_\alpha^\beta \left|\frac{d\varphi_n}{d\varphi}\right| d\varphi \geq \inf_{x\in Z}\left|\frac{d\varphi_n}{d\varphi}(\varphi(x))\right| \Theta(Z)\,.$$

Hence, we obtain the result in Assertion (4) by choosing $C_7 = \max\{\pi, \Delta^{-1}\}$. □

The following lemma plays a crucial role in extending our zeta function meromorphically to the domain containing the half-plane $\mathrm{Re}\, s \geq 0$.

LEMMA 3.3. *There exist numbers* $\kappa \in (0, 1)$ *and* $C_8 > 0$ *depending only on* Q *such that*

$$\sum_{Z\in\mathcal{P}_n} \Theta(Z) \leq C_8\kappa^n\,.$$

PROOF. Consider an admissible word $w_0w_1\cdots w_n$ of length $n+1$. The totality of the elements in $\mathcal{P}_{n+1}$ contained in $Z(w_0w_1\cdots w_n) \in \mathcal{P}_n$ is $\{Z(w_0w_1\cdots w_n j)\}_{j\neq w_n}$. We compare $\Theta(Z(w_0w_1\cdots w_n))$ with $\sum_{j\neq w_n}\Theta(Z(w_0w_1\cdots w_n j))$. To this end, first we compare $\Theta(T^n Z(w_0w_1\cdots w_n))$ with $\sum_{j\neq w_n}\Theta(T^n Z(w_0w_1\cdots w_n j))$. From the definition of $Z(w_0w_1\cdots w_n)$, $T^n Z(w_0w_1\cdots w_n)$ is the minimal segment of $\gamma^u(T^n x)$ containing $\gamma^u(T^n x) \cap (\bigcup_{i\neq w_n(x)}\mathcal{D}_1(w_n(x)j))$, where x is a point in $Z(w_0w_1\cdots w_n)$. In particular, we have

$$\Pi_{\gamma^u(T^n x),\gamma(w_n)} T^n Z(w_0w_1\cdots w_n) = Z(w_n) \quad \text{and}$$
$$\Pi_{\gamma^u(T^n x),\gamma(w_n)} T^n Z(w_0w_1\cdots w_n j) = Z(w_0 j)\,.$$

Therefore, in virtue of the Lipschitz continuity of the foliation $\mathcal{F}$ in Theorem 2.6, we find a number $\kappa_0 \in (0, 1)$ depending only on Q such that

$$\sum_{j \neq w_n} \Theta(T^n Z(w_0 w_1 \cdots w_n j)) \leq \kappa_0 \Theta(T^n Z(w_0 w_1 \cdots w_n)) .$$

Now, letting $Z(w_0 w_1 \cdots w_n)$ be expressed as $r = r(\varphi)$ and applying Assertion (4) in Lemma 3.2 to it, we have

$$\frac{\Theta(A)}{\Theta(B)} \in [C_7^{-2}, C_7^2] \frac{\Theta(T^n A)}{\Theta(T^n B)}$$

for any measurable sets $A, B \subset Z(w_0 w_1 \cdots w_n)$. Apply this inequality to the case when $A = Z(w_0 w_1 \cdots w_n) \setminus \bigcup_{j \neq w_0} Z(w_0 w_1 \cdots w_n j)$ and $B = Z(w_0 w_1 \cdots w_n)$, we have

$$\frac{\Theta(A)}{\Theta(B)} \geq C_7^{-2}(1 - \kappa_0) .$$

Hence, we have

$$\frac{\sum_{j \neq j_n} \Theta(Z(w_0 w_1 \cdots w_n j))}{\Theta(Z(w_0 w_1 \cdots w_n))} = 1 - \frac{\Theta(A)}{\Theta(B)} \leq 1 - C_7^{-2}(1 - \kappa_0) .$$

Setting $\kappa = 1 - C_7^{-2}(1 - \kappa_0)$, we have

$$\sum_{Z \in \mathcal{P}_{n+1}} \Theta(Z) = \sum_{Z' \in \mathcal{P}_n} \sum_{Z \in \mathcal{P}_{n+1}, : Z \subset Z'} \Theta(Z) \leq \kappa \sum_{Z' \in \mathcal{P}_n} \Theta(Z') .$$

Hence, we can reach the desired inequality with $C_8 = J\pi$. □

Now we introduce the one-dimensional map σ on the parameter space of φ. Each local unstable curve γ^u has an expression $r = r(\varphi)$, $-\pi/2 \leq \varphi \leq \pi/2$, as a K-increasing curve. Thus, we employ the φ-coordinate as a natural parameter (a local coordinate) for γ^u. In terms of such a parametrization, γ^u is identified with the interval $I(\gamma^u)$ $(= [-\pi/2, \pi/2])$. Therefore, it will be convenient to reduce our investigation of the map S to that of a map of the interval.

For each $j = 1, 2, \ldots, J$, let $I(j)$ be the subinterval of $I(\gamma(j)) = I(\gamma^u(x(j)))$ corresponding to $Z(j)$. Define a map $\Phi : \bigcup_{j=1}^J \gamma(j) \to \bigsqcup_{j=1}^J I(\gamma(j))$ by

$$\Phi(x) = \varphi(x) \quad \text{if} \quad x = (w_0(x), r(x), \varphi(x)) \in \gamma(j) , \tag{3.1}$$

where $\bigsqcup$ means the direct sum of sets. For any $x, y \in \gamma(j)$, $l(x, y)$ denotes the Euclidean distance with respect to the (r, φ)-coordinates as before. Then we have

$$|\Phi(x) - \Phi(y)| \leq l(x, y) \leq C_9 |\Phi(x) - \Phi(y)| \tag{3.2}$$

with $C_9 = \sqrt{1 + k_{\min}^{-2}}$, since $\gamma(j)$ is K-increasing.

Recall that $X = \bigcup_{j=1}^J Z(j)$. Put $Y = \bigsqcup_{j=1}^J I(j) = \Phi(X)$. Then we can define a local map $\sigma : Y \to \Phi(\bigcup_{j=1}^J \gamma(j)) = \bigsqcup_{j=1}^J I(\gamma(j))$ so that $\sigma(\Phi x) = \Phi(Sx)$ holds for any $x \in X$. It is easy to see that σ^n is defined at $\Phi(x)$ if and only if S^n is defined at x. Clearly,

$$\sigma^n(\Phi x) = \Phi(S^n x) \tag{3.3}$$

is valid if $\sigma^n x$ is defined. Note that $I(j) = \Phi Z(j)$ for $j = 1, 2, \dots, J$. For a positive integer n, set

$$I(w) = \Phi Z(w) \quad \text{for} \quad w \in \mathcal{W}_{n+1} \quad \text{and} \quad \mathcal{Q}_n = \{I(w); w \in \mathcal{W}_{n+1}\}.$$

Obviously, σ, $I(j)$, $I(w)$ and $\mathcal{Q}_n$ play the roles of S, $Z(j)$, $Z(w)$ and $\mathcal{P}_n$, respectively. In particular, $\mathcal{Q}_n$ is a partition of the definition domain $Y_n = \Phi X_n$ of σ^n. As in the case of S^n, for each $I \in \mathcal{Q}_n$, σ_I^n denotes $\sigma^n|_I$ and its inverse $(\sigma_I^n)^{-1} : \sigma_I^n I \to I$ is denoted by σ_I^{-n}.

Before studying the function on Y which plays the role of V on X, we introduce the following metrics d_X and d_Y to X and $Y = \Phi X$. For $x \in \gamma(i)$ and $y \in \gamma(j)$, set

$$d_X(x, y) = \begin{cases} l(x, y) & \text{if} \quad i = j \\ 1 & \text{if} \quad i \neq j \end{cases}, \quad d_Y(\Phi x, \Phi y) = \begin{cases} |\Phi x - \Phi y| & \text{if} \quad i = j \\ 1 & \text{if} \quad i \neq j \end{cases}.$$

Let $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$ be the totality of complex-valued Lipschitz continuous functions with respect to the metric d_X and d_Y, respectively. For functions $f \in \mathrm{Lip}(X)$ and $g \in \mathrm{Lip}(Y)$, we set

$$\begin{aligned}
\|f\|_{X,\infty} &= \sup_{x \in X} |f(x)|, \quad \|g\|_{Y,\infty} = \sup_{\varphi \in Y} |g(\varphi)|, \\
[f]_{X,j} &= \sup_{x \neq y} \frac{|f(x) - f(y)|}{l(x, y)}, \quad [g]_{Y,j} = \sup_{\varphi \neq \psi} \frac{|g(\varphi) - g(\psi)|}{|\varphi - \psi|}, \\
[f]_X &= \max_{1 \leq j \leq J} [f]_{X,j}, \quad [g]_Y = \max_{1 \leq j \leq J} [g]_{Y,j}, \\
\|f\|_X &= \|f\|_{X,\infty} + [f]_X, \quad \|g\|_Y = \|g\|_{Y,\infty} + [g]_Y.
\end{aligned}$$

Then $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$ become Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. In virtue of the inequality (3.2), $\mathrm{Lip}(Y) \ni g \mapsto g \circ \Phi \in \mathrm{Lip}(X)$ gives an isomorphism with

$$\|g \circ \Phi\|_X \leq \|g\|_Y \leq C_9 \|g \circ \Phi\|_X, \tag{3.4}$$

where C_9 is the same constant as in (3.2).

Now we define the function $U \in \mathrm{Lip}(Y)$ corresponding to the given function V on Ω^+ satisfying Assumptions (A.1), (A.2) and (A.3). First we note that on the set Ω^+, the Lipschitz continuity with respect to the Euclidean distance in the usual coordinates (q, v) is equivalent to that with respect to the Euclidean distance in the (r, φ)-coordinates. This follows from the fact that for some $\varphi_0 \in (0, \pi/2)$ $\cos\varphi \geq \cos\varphi_0$ holds for any $x = (r, \varphi) \in \Omega^+$ in virtue of Condition (H.2). Thus, we carry out our argument using the Euclidean distance induced by the (r, φ)-coordinates in the sequel. Consider the restriction $V|_{\Omega^+ \cap X}$. In virtue of Kirszbraun's theorem (see [6, p. 201]), we obtain a Lipschitz continuous extension with the same Lipschitz constant $\bar{V}$ of V on X with respect to d_X. Define a function $U : Y \to \boldsymbol{R}$ by $U = \bar{V} \circ \Phi^{-1}$. Then we have the following lemma.

LEMMA 3.4. *U is an element in* $\mathrm{Lip}(Y)$ *with* $[U]_Y \leq [V]_X$ *and is eventually positive with respect to* σ *in the following sense. There exist positive constants* $a = a(U)$ *and* $b =$

$b(U)$ such that

$$\sum_{i=0}^{n-1} U(\sigma^i \varphi) \geq an - b \tag{3.5}$$

holds for any $n \geq 1$ and $\varphi \in Y$ whenever $\sigma^{n-1}\varphi$ is defined.

PROOF. The first Assertion is an easy consequence of the definition. We verify the second Assertion. For any x in $Z \in \mathcal{P}_n$ and any y in $Z \cap \Omega^+$, we obtain

$$|\bar{V}(x) - \bar{V}(y)| \leq [V]_X l(x, y) \leq [V]_X C_9 |\Phi(x) - \Phi(y)| \leq [V]_X C_9 \pi \theta^n,$$

in virtue of the inequality (3.2) and Assertion (2) of Lemma 3.2. Thus, we conclude that

$$\begin{aligned} \sum_{i=0}^{n-1} \bar{V}(S^n x) &\geq \sum_{i=0}^{n-1} \bar{V}(S^n y) - \left| \sum_{i=0}^{n-1} \bar{V}(S^i x) - \sum_{i=0}^{n-1} \bar{V}(S^i y) \right| \\ &\geq \sum_{i=0}^{n-1} \bar{V}(S^n y) - \frac{C_9 \pi}{1-\theta}[V]_X = \sum_{i=0}^{n-1} V(T^n y) - \frac{C_9 \pi}{1-\theta}[V]_X \,. \end{aligned}$$

Note that we have used Assumption (A.3) on V in the last equality. Now it is immediate that the second Assertion follows from the eventual positivity (A.2) on V. □

Finally, we verify that the zeta function $\zeta_V(s)$ can be expressed in terms of σ and U. Recall that $\zeta_V(s)$ is formally defined by

$$\zeta_V(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x:T^n x = x} \exp\left(-s \sum_{k=0}^{n-1} V(T^k x) \right) \right).$$

Let $x \in X$ be a point where T^n is defined and let $\varphi = \Phi x$. Then by the definition of S^n, σ^n and the identity (3.3) we have

$$U(\sigma^n \varphi) = \bar{V}(S^n x) = \bar{V}(\Pi T^n x) \,.$$

In addition, if $x \in \Omega^+$, we have

$$\bar{V}(\Pi T^n x) = V(\Pi T^n x) = V(T^n x) \,,$$

in virtue of Assumption (A.3) on V. Moreover, $T^n x = x$, $S^n x = x$ and $\sigma^n \varphi = \varphi$ are mutually equivalent by definition. Thus, we arrive at the identity

$$\sum_{x:T^n x = x} \exp\left(-s \sum_{i=0}^{n-1} V(T^i x) \right) = \sum_{\varphi:\sigma^n x = x} \exp\left(-s \sum_{i=0}^{n-1} U(\sigma^i \varphi) \right)$$

for each n. Hence, we have

$$\zeta_V(s) = \zeta_U(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\varphi:\sigma^n \varphi = \varphi} \exp\left(-s \sum_{i=0}^{n-1} U(\sigma^i \varphi) \right) \right).$$

4. Transfer operators. In this section, we keep the notation $I(j)$, $\mathcal{Q}_n$, Y and σ from the previous section. Since we work only on the space Y and $\mathrm{Lip}(Y)$, we write $[\cdot]$, $\|\cdot\|_\infty$, $\|\cdot\|$ instead of $[\cdot]_Y$, $\|\cdot\|_{Y,\infty}$, $\|\cdot\|_Y$ and so on.

First we introduce a family of transfer operators associated to the one-dimensional dynamical system σ. Let U and f be functions defined on Y. We define $L(s) = L_U(s)$ with $s \in \boldsymbol{C}$ by

$$L(s)f(\varphi) = \sum_{I\in\mathcal{Q}_1:\sigma_I I=I(j)} e^{-sU(\sigma_I^{-1}\varphi)} f(\sigma_I^{-1}\varphi)$$

if $\varphi \in I(j)$. It is clear that if U is continuous, then $L(s)$ becomes a bounded linear operator on the Banach space of all continuous functions $C(Y)$ endowed with supremum norm for each s. We can easily see that $L(s)^n$ is given by

$$L(s)^n f(\varphi) = \sum_{I\in\mathcal{Q}_n:\sigma_I^n I=I(j)} \exp\left(-s\sum_{i=0}^{n-1} U(\sigma^i\sigma_I^{-n}\varphi)\right) f(\sigma_I^{-n}\varphi)$$

if $\varphi \in I(j)$. For later convenience, we set

$$G(s)(\varphi) = \exp(-sU(\varphi)),$$

$$G_n(s)(\varphi) = G(s)(\varphi)G(s)(\sigma\varphi)\cdot\cdots\cdot G(s)(\sigma^{n-1}\varphi) = \exp\left(-s\sum_{i=0}^{n-1} U(\sigma^i\varphi)\right). \tag{4.1}$$

Then we can write

$$L(s)^n f(\varphi) = \sum_{I\in\mathcal{Q}_n:\sigma_I^n I=I(j)} G_n(s)(\sigma_I^{-n}\varphi) f(\sigma_I^{-n}\varphi) \tag{4.2}$$

if $\varphi \in I(j)$.

For each element $I \in \mathcal{Q}_n$, we choose a $\varphi_I \in I$. For a function f on Y, we define a family of operators $K_n(s)$ with $s \in \boldsymbol{C}$ by

$$K_n(s)f = \sum_{I\in\mathcal{Q}_n} f(\varphi_I)L(s)^n\chi_I, \tag{4.3}$$

where χ_A denotes the indicator function of the set A as usual.

In the rest of the section, we consider an eventually positive function $U \in \mathrm{Lip}(Y)$ satisfying the inequality (3.5). For $s \in \boldsymbol{C}$, put

$$\rho(s) = \rho_U(s) = \begin{cases} \exp((-\mathrm{Re}\, s)a) & \text{if } \mathrm{Re}\, s \geq 0, \\ \|G(s)\|_\infty = \|\exp((-\mathrm{Re}\, s)U)\|_\infty & \text{if } \mathrm{Re}\, s < 0, \end{cases} \tag{4.4}$$

where $a = a(U)$ is the constant appearing in (3.5). If we slightly modify arguments in [1], we can show the following.

LEMMA 4.1. *Assume that U is an eventually positive function in* $\mathrm{Lip}(Y)$. *Then $L(s)$ defines an analytic family of bounded operators on the Banach space* $\mathrm{Lip}(Y)$ *satisfying the* *inequality*

$$\|L(s)^n - K_n(s)\| \le C(s)(\rho(s)\kappa)^n \tag{4.5}$$

for some positive number $C(s)$ depending only on s, U and Q. In particular, $C(s)$ can be chosen to be continuous on s.

Lemma 4.1 follows from Lemmas 4.2 and 4.3 below. For the sake of simplicity, we drop the letter s in the sequel. So we write $L = L(s)$, $K_n = K_n(s)$, $G_n = G_n(s)$, $\rho = \rho(s)$ and so on.

LEMMA 4.2. *Assume that U is an eventually positive function in* $\mathrm{Lip}(Y)$ *as above. For any $I \in \mathcal{Q}_n$ with $\sigma_I^n I = I(j)$, we have*

$$\sup_{\varphi \in I(j)} |G_n(\sigma_I^{-n}\varphi)| \le e^{b|s|}\rho^n \quad \text{and} \quad [G_n(\sigma_I^{-n}\cdot)]_j \le \rho^n e^{(3b+[U]\pi)|s|}|s|[U]\frac{C_5}{1-\theta},$$

where C_5 and $b = b(U)$ are the constants appeared in Lemma 3.1 *and the inequality* (3.5), *respectively.*

PROOF. If $\mathrm{Re}\, s \ge 0$, we have

$$|G_n(\varphi)| = \exp\left((-\mathrm{Re}\, s)\sum_{i=0}^{n-1} U(\sigma^i\varphi)\right) \le \exp((-\mathrm{Re}\, s)(an-b)) \le \rho^n e^{b|s|}.$$

On the other hand, if $\mathrm{Re}\, s < 0$, clearly we have

$$|G_n(\varphi)| \le \rho^n \le \rho^n e^{b|s|}.$$

Thus, we obtain the first Assertion.

To see the second Assertion, first we show

$$[G] \le \rho e^{(b+[U]\pi)|s|}|s|[U].$$

This is shown as follows. If $\varphi,\ \psi \in I(j)$ for some j, we have

$$\begin{aligned}|e^{-sU(\varphi)} - e^{-sU(\psi)}| &\le |e^{-sU(\varphi)}|e^{|s||U(\varphi)-U(\psi)|}|s||U(\varphi)-U(\psi)| \\ &\le \rho e^{b|s|+[U]\pi|s|}|s|[U]|\varphi-\psi|.\end{aligned}$$

Here we used an inequality $|e^z - e^w| \le |e^z|e^{|w-z|}|z-w|$.

Now we have

$$\begin{aligned}&|G_n(\sigma_I^{-n}\varphi) - G_n(\sigma_I^{-n}\psi)| \\ &= \sum_{i=0}^{n-1} G_i(\sigma_I^{-n}\varphi)(G(\sigma^i\sigma_I^{-n}\varphi) - G(\sigma^i\sigma_I^{-n}\psi))G_{n-(i+1)}(\sigma^{i+1}\sigma_I^{-n}\psi) \\ &\le \rho^{n-1}e^{2b|s|}[G]\sum_{i=0}^{n-1}|\sigma^i\sigma_I^{-n}\varphi - \sigma^i\sigma_I^{-n}\psi|\end{aligned}$$

$$\leq \rho^n e^{(3b+[U]\pi)|s|}|s|[U]\sum_{i=0}^{n-1}|\sigma^i\sigma_I^{-n}\varphi - \sigma^i\sigma_I^{-n}\psi|$$
$$\leq \rho^n e^{(3b+[U]\pi)|s|}|s|[U]C_5\sum_{i=0}^{n-1}\theta^{n-i}|\varphi - \psi|$$
$$\leq \rho^n e^{(3b+[U]\pi)|s|}|s|[U]\frac{C_5}{1-\theta}|\varphi - \psi|\,.$$

Here the first inequality follows from the first Assertion of the lemma. The second inequality is a consequence of the inequality on $[G]$. Lemma 3.1 and the inequality (2) in Lemma 3.2 are used to obtain the third inequality. Hence, we have the second Assertion of the lemma. □

From Lemma 4.2, it is easy to see that $L(s)$ is an analytic family of bounded linear operators on $\mathrm{Lip}(Y)$. If $I \in \mathcal{Q}_n$, we have $L^n\chi_I = \chi_{I(j)}G_n(\sigma_I^{-n}\cdot)$ for some $j = 1, 2, \ldots, J$. Thus, from Lemma 4.2, we have

$$\|L^n\chi_I\|_\infty \leq \rho^n e^{b|s|} \quad \text{and} \quad [L^n\chi_I] \leq \rho^n e^{(3b+[U]\pi)|s|}|s|[U]\frac{C_5}{1-\theta}\,.$$

Therefore, $L^n\chi_I \in \mathrm{Lip}(Y)$ for each $I \in \mathcal{Q}_n$. Consequently, K_n is an operator on $\mathrm{Lip}(Y)$ of finite rank.

To estimate the norm of $L^n - K_n$, we need the following.

LEMMA 4.3. *Assume that U is an eventually positive function in* $\mathrm{Lip}(Y)$ *as above. Then we have*

$$\sum_{I\in\mathcal{Q}_n}\|L^n((f - f(\varphi_I))\chi_I)\|_\infty \leq C_8 e^{b|s|}\rho^n\kappa^n[f]$$

and

$$\sum_{I\in\mathcal{Q}_n}[L^n((f - f(\varphi_I))\chi_I)] \leq \left(\frac{e^{(3b+[U]\pi)|s|}|s|[U]}{1-\theta} + C_7 e^{b|s|}\right)C_5C_8\rho^n\kappa^n[f]\,,$$

where C_5, C_7 and C_8 are the same as before.

PROOF. By definition we have

$$L^n((f - f(\varphi_I))\chi_I) = \chi_{I(j)}G_n(\sigma_I^{-n}\cdot)(f(\sigma_I^{-n}\cdot) - f(\varphi_I))$$

if $\sigma_I^n I = I(j)$. By Lemma 4.2, we have

$$\|L^n((f - f(\varphi_I))\chi_I)\|_\infty \leq \rho^n e^{b|s|}[f]|I|\,, \tag{4.6}$$

where $|I|$ denote the one-dimensional Lebesgue measure of the set I. On the other hand, for any φ and ψ in $I(j)$

$$\begin{aligned}
&L^n((f - f(\varphi_I))\chi_I)(\varphi) - L^n((f - f(\varphi_I))\chi_I)(\psi)\\
&\quad = (G_n(\sigma_I^{-n}\varphi) - G_n(\sigma_I^{-n}\psi))(f(\sigma_I^{-n}\varphi) - f(\varphi_I)) + G_n(\sigma_I^{-n}\varphi)(f(\sigma_I^{-n}\varphi) - f(\sigma_I^{-n}\psi))\\
&\quad = A + B\,,
\end{aligned}$$

where A and B denote, respectively, the first and the second terms in the second line in the above. By Lemma 4.2 we obtain

$$|A| \le \rho^n e^{(3b+[U]\pi)|s|}|s|[U]\frac{C_5}{1-\theta}||\varphi-\psi|[f]|I|\,. \tag{4.7}$$

Next, using Lemmas 3.2(4) and 4.2, we have

$$\begin{aligned}|B| &\le \rho^n e^{b|s|}[f]|\Phi T_Z^{-n}\Pi^{-1}\Phi^{-1}\varphi-\Phi T_Z^{-n}\Pi^{-1}\Phi^{-1}\psi| \\ &\le \rho^n e^{b|s|}[f]C_5\sup_{x\in Z}\left|\left(\frac{d\varphi_n}{d\varphi}(\varphi(x))\right)^{-1}\right||\varphi-\psi| \le \rho^n e^{b|s|}[f]C_5C_7|I||\varphi-\psi|\,,\end{aligned} \tag{4.8}$$

where $Z=\Phi^{-1}I$. Combining (4.6) with Lemma 3.3, we obtain the first inequality in the lemma. Combining (4.7), (4.8) and Lemma 4.2, we see the second inequality. □

It is clear that the inequality (4.5) follows from Lemma 4.3. Now the proof of Lemma 4.1 is completed.

In the rest of this section, we prove another inequality that plays an important role in the meromorphic continuation of our zeta functions.

For each $I\in\mathcal{Q}_n$, we select $\psi_I\in I$. Define Y_I by

$$Y_I=\begin{cases}L^n\chi_I-G(\psi_I)L^{n-1}\chi_{\sigma I}, & \text{if} \quad n\ge 2\,,\\ L\chi_I\,, & \text{if} \quad n=1\,.\end{cases} \tag{4.9}$$

It is easily verified that

$$L^n\chi_I=\sum_{i=0}^{n-1}G_i(\psi_I)Y_{\sigma^i I} \tag{4.10}$$

holds.

LEMMA 4.4. *Assume that U is a non-negative valued function in* Lip(Y). *Then* $\{Y_I;\, I\in\mathcal{Q}_n\}$ *satisfies the following inequalities*

$$\sum_{I\in\mathcal{Q}_n}\|Y_I\|_\infty \le |s|[U]e^{(2b+[U]\pi)|s|}C_5C_8\rho^n\kappa^n$$

and

$$\sum_{I\in\mathcal{Q}_n}[Y_I]\le e^{(3b+[U]\pi)|s|}|s|[U]\left(C_7+\frac{e^{(b+[U]\pi)|s|}|s|[U]}{1-\theta}\right)C_5C_8\rho^n\kappa^n\,.$$

PROOF. Note that

$$Y_I(\varphi)=\begin{cases}(G(\sigma_I^{-n}\varphi)-G(\psi_I))G_{n-1}(\sigma_{\sigma I}^{-(n-1)}\varphi), & \text{if} \quad n\ge 2\,,\\ G(\sigma_I^{-1}\varphi), & \text{if} \quad n=1\,,\end{cases}$$

if $\varphi\in\sigma_I^n I$ by definition. Therefore, we have $\|Y_I\|_\infty\le\rho e^{b|s|}$ if $n=1$, and

$$\|Y_I\|_\infty\le[G]C_5|I|\rho^{n-1}e^{b|s|}\le\rho^n|s|[U]e^{(2b+[U]\pi)|s|}C_5|I|$$

if $n\ge 2$. Here we used the inequality $[G]\le\rho e^{(b+[U]\pi)|s|}|s|[U]$ and the first inequality in Lemma 4.2. This yields the first inequality.

Next, if $n \geq 2$, we have

$$\begin{aligned} Y_I(\varphi) - Y_I(\psi) &\leq (G(\sigma_I^{-n}\varphi) - G(\sigma_I^{-n}\psi))G_{n-1}(\sigma_{\sigma I}^{-(n-1)}\varphi) \\ &\quad + (G(\sigma_I^{-n}\psi) - G(\psi_I))(G_{n-1}(\sigma_{\sigma I}^{-(n-1)}\varphi) - G_{n-1}(\sigma_{\sigma I}^{-(n-1)}\varphi)) \\ &= A + B \end{aligned}$$

for any φ and ψ in $\sigma_I^n I$, where A and B denote the first and the second terms in the above inequality, respectively. Thus, we have

$$\begin{aligned} |A| &\leq [G]|\sigma_I^{-n}\varphi - \sigma_I^{-n}\psi|\rho^{n-1}e^{b|s|} \\ &\leq \rho e^{(b+[U]\pi)|s|}|s|[U]\rho^{n-1}e^{b|s|}C_5C_7|I||\varphi - \psi| \\ &= e^{(2b+[U]\pi)|s|}|s|[U]\rho^n C_5C_7|I||\varphi - \psi| \end{aligned}$$

by Lemmas 3.1 and 3.2(4) in the same way as in (4.8). In addition,

$$\begin{aligned} |B| &\leq [G]|I|\rho^{n-1}e^{(3b+[U]\pi)|s|}|s|[U]\frac{C_5}{1-\theta}|\varphi - \psi| \\ &\leq \rho^n e^{(4b+2[U]\pi)|s|}(|s|[U])^2\frac{C_5}{1-\theta}|I||\varphi - \psi| \end{aligned}$$

in virtue of Lemma 4.2. Therefore, we reach the inequality

$$[Y_I] \leq e^{(3b+[U]\pi)|s|}|s|[U]\left(C_7 + \frac{e^{(b+[U]\pi)|s|}|s|[U]}{1-\theta}\right)C_5\rho^n|I|$$

if $n \geq 2$. The estimate for $n = 1$ is just same as that for I above. Consequently, we obtain the second inequality. □

5. Meromorphic extensions of zeta functions. The purpose of this section is to prove our main result Theorem 1.1. In virtue of the reduction made in Section 3, it suffices to prove the following.

THEOREM 5.1. *Let $\sigma : Y \to \bigsqcup_{j=1}^{J} I(\gamma(j))$ be the same as in Section* 3 *and let U be an function in* $\mathrm{Lip}(Y)$ *which is eventually positive in the sense of* (3.5)*. Consider the formally defined zeta function*

$$\zeta_U(s) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\varphi:\sigma^n\varphi=\varphi}\exp\left(-s\sum_{i=0}^{n-1}U(\sigma^i\varphi)\right)\right).$$

Then there are positive constants α_U and β_U satisfying the following.

(a) *The series in the definition of ζ_U is absolutely convergent in the half-plane* $\mathrm{Re}\, s > \alpha_U$ *and defines an analytic function without zero.*

(b) *ζ_U has a meromorphic extension without zero in a domain containing the closed half-plane* $\mathrm{Re}\, s \geq \alpha_U$*. In particular, $s = \alpha_U$ is a simple pole.*

(c) *Moreover, ζ_U has a meromorphic extension without zero in the half-plane* $\mathrm{Re}\, s > -\beta_U$*.*

As noticed just after the statement of Theorem 1.1, Assertions (a) and (b) are consequences of the general theory of thermodynamic formalism. In particular, the positivity of α_U follows from the condition that U is eventually positive. Therefore, we may assume their validity and we devote ourselves to the proof of Assertion (c).

For each $I \in \mathcal{Q}_n$, we take φ_I so that φ_I is a unique fixed point of σ_I^n on I if $\sigma_I^n I \supset I$. Then we can easily see that

$$\sum_{\varphi:\sigma^n\varphi=\varphi} \exp\left(-s\sum_{i=0}^{n-1} U(\sigma^i\varphi)\right) = \sum_{I\in\mathcal{Q}_n} (L(s)^n\chi_I)(\varphi_I)\,, \tag{5.1}$$

where $L(s) = L_U(s)$ is the transfer operator defined in Section 4. Assume for a while that $s_0 \in \boldsymbol{C}$ satisfies $\rho(s_0)\kappa < 1$. By Lemma 4.1 (see also [13]), for such an $s_0 \in \boldsymbol{C}$ with $\rho(s_0)\kappa < 1$, the transfer operator $L(s_0)$ is quasicompact and the spectrum in $|z| > \rho(s_0)\kappa$ consists of eigenvalues with finite multiplicity. Select $r > \rho(s_0)\kappa$ such that there is no eigenvalue of modulus r. From the general spectral theory for linear operators (see [5, Chapter VII]), there exists an open disc $D(s_0) \subset \boldsymbol{C}$ centered at s_0 such that $L(s)$ does not have eigenvalues of modulus r for any $s \in D(s_0)$. Thus, we can define the following projections by using the Dunford integral,

$$\begin{aligned} R(s,r) &= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=r} (zI - L(s))^{-1}\,dz\,,\\ P(s,r) = I - R(s,r) &= \frac{1}{2\pi\sqrt{-1}}\left(\int_{|z|=\bar r} - \int_{|z|=r}\right)(zI - L(s))^{-1}\,dz\,, \end{aligned} \tag{5.2}$$

where $\bar r$ is any number greater than $\sup_{s\in D(s_0)} \|L(s)\|$. In particular, $P(s,r)$ and $R(s,r)$ depend analytically on s in $D(s_0)$. Since $P(s,r)L(s) = L(s)P(s,r) : \mathrm{Lip}(Y) \to \mathrm{Lip}(Y)$ is an operator of finite rank, the trace of $L(s)^n P(s,r)$ is given by the spectral trace

$$\mathrm{tr}\, L(s)^n P(s,r) = \sum_{\lambda:|\lambda|>r} \lambda^n\,.$$

On the other hand, the determinant of $I - L(s)P(s,r)$ is given by

$$\det(I - L(s)P(s,r)) = \prod_{\lambda:|\lambda|>r} (1-\lambda)\,.$$

In the above, the sum $\sum_{\lambda:|\lambda|>r}$ and the product $\prod_{\lambda:|\lambda|>r}$ are taken over all eigenvalues λ of $L(s)$ with $|\lambda| > r$. If $\mathrm{Re}\, s$ is large enough (precisely, $\mathrm{Re}\, s > \alpha_U$), the spectral radius of $L(s)$ is less than 1. Therefore, we have the formula

$$\det(I - L(s)P(s,r)) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{tr}(L(s)^n P(s,r))\right) \quad \text{for} \quad \mathrm{Re}\, s > \alpha_U\,. \tag{5.3}$$

Note that the left-hand side of (5.3) is defined without assuming $\mathrm{Re}\, s > \alpha_U$ and depends analytically on s in a neighborhood $N(s_0, r) \subset \boldsymbol{C}$ of s_0.

Combining (5.1) with (5.3), we can write at least formally

$$\zeta_U(s)\det(I - L(s)P(s.r)) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{I\in\mathcal{Q}_n}(L(s)^n\chi_I)(\varphi_I) - \mathrm{tr}(L(s)^n P(s,r))\right)\right). \tag{5.4}$$

Inspired by the above observation, we can prove the following lemma.

LEMMA 5.2. *s_0 and r are as above. Then there exist a neighborhood $W(s_0, r) \subset \boldsymbol{C}$ of s_0 and a positive number $C(s_0, r)$ depending only on the domain Q, the function U and s_0 such that*

$$\left|\sum_{I\in\mathcal{Q}_n}(L(s)^n\chi_I)(\varphi_I) - \mathrm{tr}(L(s)^n P(s,r))\right| \le C(s_0, r)r^n$$

holds for any $s \in W(s_0, r)$.

PROOF. We prove that there exist a neighborhood $W(s_0, r)$ and positive numbers $C_1(s_0, r)$ and $C_2(s_0, r)$ such that

$$\left|\sum_{I\in\mathcal{Q}_n}(P(s,r)L(s)^n\chi_I)(\varphi_I) - \mathrm{tr}(L(s)^n P(s,r))\right| \le C_1(s_0, r)r^n \tag{5.5}$$

and

$$\left|\sum_{I\in\mathcal{Q}_n}(R(s,r)L(s)^n\chi_I)(\varphi_I)\right| \le C_2(s_0, r)r^n \tag{5.6}$$

hold.

We choose a neighborhood $W(s_0, r)$ so that the following hold for any $s \in W(s_0, r)$:

(i) $\|L(s)\| \le \bar{r}$;

(ii) $|P(s,r) - P(s_0, r)\| < \min(\|P(s,r)\|^{-1}, \|P(s_0, r)\|^{-1})$;

(iii) the spectral radius of $R(s,r)L(s)$ is less than r_1 for some positive number $r_1 < r$ independent of $s \in W(s_0, r)$.

From the general perturbation theory (see [5, Section VII-6]), $\dim P(s,r)\,\mathrm{Lip}(Y) = \dim P(s_0, r)\,\mathrm{Lip}(Y)$, say $d = \dim P(s_0, r)\,\mathrm{Lip}(Y)$. We proceed with our argument by fixing $s \in W(s_0, r)$. So we drop the letter s and r and write L, P, R, etc., instead of $L(s)$, $P(s)$, $R(s)$, etc., for the sake of simplicity.

PROOF OF (5.5). First we notice that we can choose a basis $e_1, \dots, e_d$ of $P\,\mathrm{Lip}(Y)$ and the elements $\hat{e}_1, \dots, \hat{e}_d$ in $\mathrm{Lip}(Y)^*$ such that

$$\|e_i\| = 1\,, \quad \|\hat{e}_j\| \le 2^d \quad \text{and} \quad \hat{e}_i(e_j) = \delta_{ij} \tag{5.7}$$

for $1 \le i,\ j \le d$, where δ_{ij} denotes the Kronecker delta.

Indeed, we can choose a basis $e_1, \dots, e_d$ of $P\,\mathrm{Lip}(Y)$ satisfying

$$\|e_i\| = 1 \quad \text{for} \quad i \ge 1 \quad \text{and} \quad \mathrm{dist}(e_{i+1}, [e_1, \dots, e_i]) \ge 1 \quad \text{for} \quad 1 \le i \le d-1$$

in virtue of the finite-dimensional Riesz lemma, where $[e_1, \ldots, e_i]$ denotes the linear subspace spanned by $e_1, \ldots, e_i$. If we take elements $e_1', \ldots, e_d'$ in $P\,\mathrm{Lip}(Y)^*$ satisfying $e_i'(e_j) = \delta_{ij}$, then it is not hard to see that $\|e_i'\| \le 2^d$. Thus, in virtue of the Hahn-Banach theorem, we can extend them to bounded linear functionals on $\mathrm{Lip}(Y)$ without changing their norms.

We can write

$$\mathrm{tr}\, PL^n = \sum_{i=1}^{d} \hat{e}_i(PL^n e_i)\,. \tag{5.8}$$

On the other hand, since $PL^n\chi_I \in P\,\mathrm{Lip}(Y)$ for any $I \in \mathcal{Q}_n$, we have

$$PL^n\chi_I = \sum_{i=1}^{d} \hat{e}_i(PL^n\chi_I)e_i\,.$$

Therefore, we have

$$PL^n\chi_I(\varphi_I) = \sum_{i=1}^{d} \hat{e}_i(PL^n(e_i(\varphi_I)\chi_I))\,. \tag{5.9}$$

Then (5.8) and (5.9) imply that

$$\sum_{I\in\mathcal{Q}_n}(PL^n\chi_I)(\varphi_I) - \mathrm{tr}(L^nP) = \sum_{i=1}^{d}\sum_{I\in\mathcal{Q}_n} \hat{e}_i(PL^n((e_i(\varphi_I) - e_i)\chi_I))\,.$$

In virtue of (5.7) and Lemma 4.3, we have

$$\begin{aligned}\sum_{i=1}^{d}\sum_{I\in\mathcal{Q}_n} \|\hat{e}_i(PL^n((e_i(\varphi_I) - e_i)\chi_I))\| &\le d2^d\|P\| \sum_{I\in\mathcal{Q}_n} \|L^n((e_i(\varphi_I) - e_i)\chi_I)\| \\ &\le C_3(s_0)\rho^n\kappa^n\,,\end{aligned}$$

where $C_3(s_0)$ is a positive number depending only on the domain Q, the function U and the neighborhood $W(s_0, r)$. □

PROOF OF (5.6). For each $j = 1, 2, \ldots, J$, choose $\omega(j) \in I(j)$ and define ψ_I by $\sigma_I^{-n}\omega(j)$ if $\sigma_I^n I = I(j)$. By definition we have $\sigma^i\psi_I = \psi_{\sigma^i I}$ for any $I \in \mathcal{Q}_n$ and any $i = 1, 2, \ldots, n-1$.

Using the identity (4.10), we can write

$$\begin{aligned}&\sum_{I\in\mathcal{Q}_n} RL^n\chi_I(\varphi_I) \\ &\quad= \sum_{I\in\mathcal{Q}_n}\sum_{i=0}^{n-1} G_i(\psi_I)RY_{\sigma^i I}(\varphi_I) \\ &\quad= \sum_{i=0}^{n-1}\sum_{I\in\mathcal{Q}_n} G_i(\psi_I)(RY_{\sigma^i I}(\varphi_I) - RY_{\sigma^i I}(\psi_I)) + \sum_{i=0}^{n-1}\sum_{I\in\mathcal{Q}_n} G_i(\psi_I)RY_{\sigma^i I}(\psi_I)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \sum_{I' \in \mathcal{Q}_{n-i}} \sum_{I \in \mathcal{Q}_n : \sigma^i I = I'} G_i(\psi_I)(RY_{\sigma^i I}(\varphi_I) - RY_{\sigma^i I}(\psi_I)) \\
&\quad + \sum_{i=0}^{n-1} \sum_{I' \in \mathcal{Q}_{n-i}} \sum_{I \in \mathcal{Q}_n : \sigma^i I = I'} G_i(\psi_I) RY_{\sigma^i I}(\psi_I) \\
&= A + B ,
\end{aligned}
$$

where A and B denote the first and the second quantities in the last equality, respectively. First, from Lemma 4.4, we have

$$
\begin{aligned}
|A| &\leq \sum_{i=0}^{n-1} \sum_{I' \in \mathcal{Q}_{n-i}} \sum_{I \in \mathcal{Q}_n : \sigma^i I = I'} \rho^i \|R\| [Y_{I'}] |I| \\
&\leq C_4(s_0) \sum_{i=0}^{n-1} \rho^i \sum_{I' \in \mathcal{Q}_{n-i}} [Y_{I'}] \kappa^n \\
&\leq C_5(s_0) \rho^n \kappa^n ,
\end{aligned}
\tag{5.10}
$$

where $C_4(s_0)$ and $C_5(s_0)$ are positive numbers depending only on Q, U and $W(s_0, r)$.

Next, we see from the choice of ψ_I that

$$
B = \sum_{i=0}^{n-1} \sum_{I' \in \mathcal{Q}_{n-i}} (L^n RY_{I'})(\psi_{I'}) .
$$

Therefore, by Lemma 4.4 and the choice of $W(s_0, r)$, we obtain

$$
\begin{aligned}
|B| &\leq C_6(s_0) r_1^n \sum_{i=0}^{n-1} \sum_{I' \in \mathcal{Q}_{n-i}} \|RY_{I'}\| \\
&\leq C_7(s_0) r_1^n \sum_{i=0}^{n-1} \rho^{n-1} \kappa^{n-i} \\
&\leq C_8(s_0) r_1^n ,
\end{aligned}
\tag{5.11}
$$

where $C_6(s_0)$, $C_7(s_0)$ and $C_8(s_0)$ are positive numbers depending only on Q, U and $W(s_0, r)$. The desired inequality (5.6) follows from (5.10) and (5.11). □

Now we are in a position to prove Assertion (c) in Theorem 5.1.

PROOF OF THEOREM 5.1(c). Let U be the function satisfying the Assumptions in Theorem 5.1.

Set

$$
\beta_U = -\frac{\log \kappa}{\|U\|_\infty} .
$$

Recall the definition (4.4) of $\rho(s)$. If $\mathrm{Re}\, s \geq 0$, obviously we have $\rho(s) \leq 1$ and if $0 > \mathrm{Re}\, s > -\beta_U$, we have

$$\rho(s)\kappa \leq \exp(-(\mathrm{Re}\, s)\|U\|_\infty + \log \kappa) < \exp(\beta_U \|U\|_\infty + \log \kappa) = 1\,.$$

Therefore, we have $\rho(s)\kappa < 1$ whenever Re s $> -\beta_U$, consequently, $L(s) : \mathrm{Lip}(Y) \to \mathrm{Lip}(Y)$ is quasicompact.

Let D_0 denote the half-plane $\mathrm{Re}\, s > -\beta_U$. In virtue of Lemma 5.2, we see the following. For any $s_0 \in D_0$, and for any $r > \rho(s_0)\kappa$ such that there is no eigenvalue of modulus r, we can find a neighborhood $W(s_0, r)$ of s_0 such that

$$\det(I - L(s)P(s,r))$$

and

$$\eta_r(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{I \in \mathcal{Q}_n} (L(s)^n \chi_I)(\varphi_I) - \mathrm{tr}(L(s)^n P(s,r))\right)\right)$$

are analytic functions in $s \in W(s_0, r)$. In particular, η_r does not have zero.

Let D be the maximal subdomain of D_0 with the following properties: ζ_U has a meromorphic extension such that for each $s_0 \in D$, it is given by

$$\eta_r(s) \det(I - L(s)P(s,r))^{-1} \tag{5.12}$$

in the neighborhood $W(s_0, r)$, where r is any number greater than $\rho(s_0)\kappa$ such that there is no eigenvalue of $L(s_0)$ of modulus r.

Now we prove that D coincides with D_0. It is clear that this fact completes the proof. By Assertion (a), the half-plane $\mathrm{Re}\, s > \alpha_U$ is contained in D. Thus, D is not empty. From the connectedness of D_0, it suffices to show that D is open and closed in D_0. Clearly D is open. So it remains to show the closedness.

Let s_n be a sequence in D and $s_n \to s_0 \in D_0$ as $n \to \infty$. Then we can find $r > \rho(s_0)\kappa$ and a neighborhood of s_0 such that $\eta_r(s)$ is analytic and has no zero in $W(s_0, r)$, and $\det(I - L(s)P(s,r))$ is analytic. Thus, the function defined by (5.12) is meromorphic in $W(s_0, r)$ without zero. On the other hand, if n is sufficiently large, s_n is an interior point in $W(s_0, r)$. For the same r, there is a neighborhood $W(s_n, r)$ of s_n in which $\zeta_U(s)$ is given by (5.12). Therefore, D must contain $W(s_0, r)$. Thus, D is closed in D_0. □

6. Construction of functions satisfying the Assumptions in Theorem 1.1. In this section, we show that one can construct a function V satisfying Assumptions (A.1) and (A.3), starting with any Lipschitz continuous function F on Ω^+. We give two constructions. Both of them are the same in principle, because we define the value $V(x)$ by taking a sort of average of values of F along the local stable curve $\gamma^s(x)$. These constructions guarantee that if F is positive valued, then so is V. Therefore, starting with any positive-valued Lipschitz continuous function on Ω^+, we can obtain a function satisfying Assumptions (A.1), (A.2) and (A.3) by such averaging methods.

Since the Euclidean metric in the (q, v)-coordinates and that in the (r, φ)-coordinates are equivalent on the set Ω^+, we may assume that F is Lipschitz continuous with respect to the latter metric.

Let F be any positive-valued Lipschitz continuous function on Ω^+. Note that the Lipschitz continuity here means the Lipschitz continuity in the Euclidean distance with respect to the (r, φ)-coordinates.

CONSTRUCTION 1. Assume that for each local stable curve γ^s, a probability measure μ_{γ^s} supported on $\gamma^s \cap \Omega^+$ is assigned. Further we assume that the family $\mu = \{\mu_{\gamma^s}\}$ is a transverse measure for the unstable lamination for Ω^+ in the following sense. Let γ and γ' be local stable curves contained in the same connected component of M^+. Then we have

$$\mu_{\gamma'}(\Pi^{(u)}_{\gamma,\gamma'}B) = \mu_\gamma(B)$$

for any Borel subset of γ, where the unstable lamination for Ω^+ means that consisting of all local unstable curves, and $\Pi^{(u)}_{\gamma,\gamma'}$ denotes the holonomy map from $\gamma \cap \Omega^+$ to $\gamma' \cap \Omega^+$ along the local unstable lamination.

Define a function V_μ on Ω^+ by

$$V_\mu(x) = \int_{\gamma^s(x)} F(y)\, \mu_{\gamma^s(x)}(dy)\,. \tag{6.1}$$

Then V_μ satisfies Assumptions (A.1), (A.2) and (A.3).

CONSTRUCTION 2. We can extend F to a Lipschitz continuous function $\bar F$ on M^+ in virtue of Kirszbraun's theorem. Define $\hat F$ by $\hat F = \max(\bar F, \min_{x\in\Omega^+} F(x))$.

Note that any local stable curve is a K-decreasing curve expressed as $r = r(\varphi),\ -\pi/2 \le \varphi \le \pi/2$ and $\varphi = \varphi(r),\ a \le r \le b$, where $a = r(\pi/2)$ and $b = r(-\pi/2)$. For $x \in \Omega^+$, define $V_1(x)$ and $V_2(x)$ by

$$\begin{aligned} V_1(x) &= \int_{\gamma^s(x)} \hat F d\varphi = \int_{-\pi/2}^{\pi/2} \hat F(r(\varphi)\,, \varphi) d\varphi = -\int_a^b \hat F(r, \varphi(r)) \frac{d\varphi}{dr} dr\,, \\ V_2(x) &= -\int_{\gamma^s(x)} \hat F dr = \int_a^b \hat F(r, \varphi(r)) dr = -\int_{-\pi/2}^{\pi/2} \hat F(r(\varphi)\,, \varphi) \frac{dr}{d\varphi} d\varphi\,. \end{aligned} \tag{6.2}$$

Clearly, V_μ, V_1 and V_2 satisfy Assumptions (A.3) in Theorem 1.1. In addition, if F is positive, then they are also positive valued and, hence, satisfy Assumption (A.2). So, in the rest of this section we show the validity of (A.1).

In virtue of Kirszbraun's theorem, we may assume that F itself is Lipschitz continuous on M^+. We prove the Lipschitz continuity of the functions V_μ, V_1 and V_2 on Ω^+ by substituting F for $\bar F$ in (6.1) and for $\hat F$ in (6.2).

Let x and y be points in Ω^+ contained in the same connected component of M^+. Let $\gamma^s(x)$ and $\gamma^s(y)$ denote local stable curves of x and y, respectively. We denote by $\Pi^{(u)} : \gamma^s(x) \cap \Omega^+ \to \gamma^s(y) \cap \Omega^+$ the holonomy map from $\gamma^s(x)$ to $\gamma^s(y)$ along the local unstable lamination. Assume that $\gamma^s(x)$ is expressed as $r = u(\varphi)$, $-\pi/2 \le \varphi \le \pi/2$, and $\varphi = \tau(r)$, $a \le r \le b$ and $\gamma^s(y)$ is expressed as $r = v(\varphi)$, $-\pi/2 \le \varphi \le \pi/2$, and $\varphi = \psi(r)$, $c \le r \le d$.

Note that the inequalities

$$|u(\varphi) - v(\varphi)| \le C_4 l(x, y) \quad \text{and} \quad l(z, \Pi^{(u)} z) \le C_4 l(x, y) \tag{6.3}$$

are valid for any $\varphi \in [-\pi/2, \pi/2]$ and for any $z \in \gamma^s(x) \cap \Omega^+$, where $C_4 > 1$ is the same constant as in Lemma 3.1. Indeed, if the line segment γ joining x and y is increasing, the first inequality can be obtained by applying Lemma 3.1 to γ and any curve parallel to the r-axis and the second inequality can be obtained by applying Lemma 3.1 to γ and the local unstable curve $\gamma^u(z)$. If the line segment γ is decreasing, we consider the line segment γ' joining $x = (u(\varphi(x)), \varphi(x))$ and $y' = (v(\varphi(x)), \varphi(x))$. If we use γ' instead of γ, we can obtain (6.3) with y replaced by y'. On the other hand, it is clear that $l(x, y') = |u(\varphi(x)) - v(\varphi(x))| \le l(x, y)$, since $\gamma^s(y)$ is decreasing. Therefore, we obtain (6.3) even in the case when γ is decreasing.

Now we have

$$\begin{aligned}
|V_\mu(x) - V_\mu(y)| &= \left| \int_{\gamma^s(x)} F(z)\, \mu_{\gamma^s(x)}(dz) - \int_{\gamma^s(y)} F(z)\, \mu_{\gamma^s(y)}(dz) \right| \\
&= \left| \int_{\gamma^s(x)} F(z)\, \mu_{\gamma^s(x)}(dz) - \int_{\gamma^s(x)} F(\Pi^{(u)} z)\, \mu_{\gamma^s(x)}(dz) \right| \\
&\le \int_{\gamma^s(x)} |F(z) - F(\Pi^{(u)} z)|\, \mu_{\gamma^s(x)}(dz) \le C_4\, [F] l(x, y)\,.
\end{aligned}$$

Here we used the fact that μ is a transverse measure to obtain the second equality. The inequality in the above is clearly due to (6.3).

Next, from (6.3), we have

$$\begin{aligned}
|V_1(x) - V_1(y)| &= \left| \int_{-\pi/2}^{\pi/2} F(u(\varphi), \varphi) - F(v(\varphi), \varphi) d\varphi \right| \\
&\le \int_{-\pi/2}^{\pi/2} |F(u(\varphi), \varphi) - F(v(\varphi), \varphi)| d\varphi \le C_4 [F] \pi l(x, y)\,.
\end{aligned}$$

Finally, we show the Lipschitz continuity of V_2. Without loss of generality we may assume that $r(x) < r(y)$. Consider the case when $r(y) - r(x) < \pi/K_{\max}$ holds. In this case we can easily see that $a < c < b < d$ holds since $\gamma^s(x)$ and $\gamma^s(y)$ are K-decreasing. Then we have

$$\begin{aligned}
&|V_2(x) - V_2(y)| \\
&\quad = \left| \int_a^b F(r, \tau(r)) dr - \int_c^d F(r, \psi(r)) dr \right| \\
&\quad \le \int_a^c |F(r, \tau(r))| dr + \int_c^b |F(r, \tau(r)) - F(r, \psi(r))| dr + \int_b^d |F(r, \psi(r))| dr \\
&\quad = I + II + III\,.
\end{aligned}$$

In virtue of (6.3), we obtain

$$I \le \|F\|_\infty (c-a) \le \|F\|_\infty (b-a) \le \|F\|_\infty C_4 l(x,y)\,,$$
$$III \le \|F\|_\infty (d-b) \le \|F\|_\infty (d-c) \le \|F\|_\infty C_4 l(x,y)\,,$$

and

$$II \le [F](b-c)\max_{c\le r\le b}|\varphi(r)-\psi(r)| \le [F]\pi C_4 l(x,y)\,.$$

Next, consider the case when $r(y)-r(x)\ge \pi/K_{\max}$ holds. Then we have

$$\begin{aligned}|V_2(x)-V_2(y)| &\le \|F\|_\infty (b-a+d-c) \le 2\|F\|_\infty (\pi/k_{\min})\\ &\le 2\|F\|_\infty \frac{K_{\max}}{k_{\min}}(r(y)-r(x)) \le 2\|F\|_\infty \frac{K_{\max}}{k_{\min}} l(x,y)\,.\end{aligned}$$

Hence, we have verified (A.1) for V_μ, V_1 and V_2.

7. Theorems 1.2 and 1.3. In this section, we prove Theorems 1.2 and 1.3. Once we establish Theorem 1.2, it is easy to see the formal equation $\zeta_{t^+} = \zeta_g$, and hence we obtain Theorem 1.3 in virtue of Theorem 1.1. Following the usual way in thermodynamic formalism (see [2], [12] and [14]), a candidate of such an function h in Theorem 1.2 is given as follows. For each $x\in\mathcal{D}$ (see Section 3), the leaf $\mathcal{F}(x)$ of $\mathcal{F}$ containing x intersects $\gamma(w_0(x))$ at exactly one point, say $\hat{x}$. We also call the map $\mathcal{D}\ni x\mapsto \hat{x}\in\bigcup_{j=1}^{J}\gamma(j)$ the holonomy map along the leaf of $\mathcal{F}$ and denote it by Π. We define h by

$$h(x)=\sum_{i=0}^{\infty}(t^+(T^k x)-t^+(T^k\Pi x))$$

for any $x\in\Omega^+$. By definition, we see that $g=t^+ + h\circ T - h$ satisfies Assumptions (A.1) and (A.3). However, this construction of h loses the Lipschitz continuity in general. Fortunately, we are in a special situation. As shown later, h can be represented by

$$-\int_{r(\Pi x)}^{r(x)} \sin\varphi\, dr\,.$$

Combining this fact and the Lipschitz continuity of the K-stable foliation, we can verify the Lipschitz continuity of h. Thus, we can arrive at Theorem 1.2. Precisely, we prove the following theorem whose statement is slightly stronger than Theorem 1.2.

THEOREM 7.1. *Let $\mathcal{F}$ be the K-stable foliation. Then there exists a real-valued function g on M^+ satisfying the following.*

(g.1) *g is Lipschitz continuous on M^+ in the Euclidean metric with respect to the (r,φ) coordinates.*

(g.2) *There exists a positive number C_{10} depending only on the domain Q such that if x is in $\mathcal{D}_n$, then*

$$\left|\sum_{k=0}^{n-1} t^+(T^k x) - \sum_{k=0}^{n-1} g(T^k x)\right| \le C_{10}\,.$$

(g.3) *g is constant along each leaf of $\mathcal{F}$.*

(g.4) *$g|_{\Omega^+}$ is cohomologous to $t^+|_{\Omega^+}$, that is, there exists a real-valued continuous function h on Ω^+ such that $g(x) = t^+(x) + h(Tx) - h(x)$ holds for any $x \in \Omega^+$.*

PROOF. Let $\Pi : \mathcal{D} \to \bigcup_{j=1}^{J} \gamma(j)$ be the holonomy map along the leaf of $\mathcal{F}$ as defined above.

We first work on the set $\mathcal{D}_2$. For $x \in \mathcal{D}_2$, we set

$$n(x) = \sup\{k \geq 1\,;\, T^{k+1}x \text{ is defined}\}\,.$$

Note that if $x \in \mathcal{D}_n$ with $n \geq 2$, $T^k\mathcal{F}(x) \subset \mathcal{F}(T^k x)$ holds for $k = 1, 2, \ldots, n-1$ in virtue of the property ($\mathcal{F}$.3). Let $\varphi = \varphi(r)$ and $\varphi_k = \varphi_k(r_k)$ be the expressions of $\mathcal{F}(x)$ and $\mathcal{F}(T^k x)$, respectively, where $r(x)$ denotes the r-coordinate of x as usual. Note that Π is constant along the leaf of $\mathcal{F}$. Thus, by using Lemma 2.1, we can differentiate $t^+(T^k x) - t^+(T^k \Pi x)$ along $\mathcal{F}(x)$ to obtain

$$\begin{aligned} \frac{d}{dr}(t^+(T^k\cdot) - t^+(T^k\Pi\cdot))(x) &= \frac{dt^+}{dr}(x) \\ &= \sin\varphi_{k+1}\frac{dr_{k+1}}{dr}(r(x)) - \sin\varphi_k\frac{dr_k}{dr}(r(x))\,. \end{aligned} \tag{7.1}$$

Since $\mathcal{F}(x)$ is in $\mathcal{D}_n$ and $T^k\mathcal{F}(x)$, $k = 1, 2, \ldots, n-1$, are all K-decreasing, there exists $C_{11} > 0$ depending only on Q such that

$$\left|\frac{dr_k}{dr}(r(x))\right| \leq C_{11}\theta^k \tag{7.2}$$

for $k = 1, \ldots, n-1$. Define $u : \mathcal{D}_2 \to \boldsymbol{R}$ by

$$u(x) = \sum_{k=0}^{n(x)-1}(t^+(T^k x) - t^+(T^k\Pi x))\,.$$

From (7.1) and (7.2), $u(x)$ is convergent and of class C^1 along $\mathcal{F}(x)$ even if $n(x) = \infty$. Moreover, we have

$$u(x) = \begin{cases} \displaystyle\int_{r(\Pi x)}^{r(x)} \sin\varphi_{n(x)}\frac{dr_{n(x)}}{dr}\,dr - \int_{r(\Pi x)}^{r(x)}\sin\varphi\,dr & \text{if} \quad n(x) < \infty\,, \\ \displaystyle -\int_{r(\Pi x)}^{r(x)}\sin\varphi\,dr & \text{if} \quad n(x) = \infty\,. \end{cases}$$

Put

$$v(x) = -\int_{r(\Pi x)}^{r(x)}\sin\varphi\,dr\,.$$

Then, from (7.2) we can find positive numbers C_{12} and C_{13} depending only on Q such that

$$|u(x)| \leq C_{12} \quad \text{and} \quad |u(x) - v(x)| \leq C_{13}\theta^{n(x)}\,. \tag{7.3}$$

Following the well-known method (see [2], [14] and [17]), it is natural to consider the function $g_1 : \mathcal{D}_2 \to \boldsymbol{R}$ defined by

$$g_1(x) = t^+(x) + u(Tx) - u(x)\,.$$

It is easy to see that g_1 satisfies $(g.2)$, $(g.3)$ and $(g.4)$ on $\mathcal{D}_2$. However, we cannot expect the Lipschitz continuity. So we need the following modification. Define $g_2 : \mathcal{D}_2 \to \boldsymbol{R}$ by

$$g_2(x) = t^+(x) + v(Tx) - v(x)\,.$$

We claim that the following hold.

$(g_2.1)$ g_2 is Lipschitz continuous on $\mathcal{D}_2$.

$(g_2.2)$ There exists a positive number C_{14} depending only on Q such that for any $x \in \mathcal{D}_2$ and for any $y,\ z \in \mathcal{F}(x)$,

$$|g_2(y) - g_1(z)| \le C_{14}\theta^{n(x)}$$

is satisfied.

$(g_2.3)$ $g_2(x) = g_1(x)$ for any $x \in \bigcap_{n=2}^{\infty} \mathcal{D}_n$.

Clearly, the second inequality in (7.3) implies both $(g_2.2)$ and $(g_2.3)$. Therefore, we prove $(g_2.1)$. To prove $(g_2.1)$ it suffices to show the Lipschitz continuity of v. Let $x,\ y$ be points in a connected component $\mathcal{D}_2(j_0 j_1 j_2)$ of $\mathcal{D}_2$. Assume that the leaves $\mathcal{F}(x)$ and $\mathcal{F}(y)$ are expressed as $\varphi = \varphi(r)$ and $\psi = \psi(r)$, respectively. Without loss of generality, we just treat the case when $r(\Pi x) \le r(\Pi y)$ and $r(x) \le r(y)$. The other cases are treated in the same way. Further, we consider the following cases separately.

(I) The case when $r(\Pi x) \le r(x) \le r(\Pi y) \le r(y)$, $r(x) \le r(\Pi x) \le r(y) \le r(\Pi y)$, $r(\Pi x) \le r(x) \le r(y) \le r(\Pi y)$ or $r(x) \le r(\Pi x) \le r(\Pi y) \le r(y)$ occur.

Since the absolute value of the integrand is not greater than 1, we have in this case

$$\begin{aligned} |v(x) - v(y)| &= \left| \int_{r(\Pi y)}^{r(y)} \sin\psi\, dr - \int_{r(\Pi x)}^{r(x)} \sin\varphi\, dr \right| \\ &\le \int_{r(\Pi x)}^{r(\Pi y)} 1\, dr + \int_{r(x)}^{r(y)} 1\, dr = |r(x) - r(y)| + |r(\Pi x) - r(\Pi y)|\,. \end{aligned}$$

In virtue of the Lipschitz continuity of the foliation $\mathcal{F}$, we know that $|r(\Pi x) - r(\Pi y)| \le C_4 l(x, y)$ holds (see the inequality of (6.3)). Clearly, $|r(x) - r(y)| \le l(x, y)$. Thus, there exists a positive number C_{15} depending only on Q such that $|v(x) - v(y)| \le C_{15} l(x, y)$ holds.

(II) The case when $r(x) \le r(y) \le r(\Pi x) \le r(\Pi y)$ or $r(\Pi x) \le r(\Pi y) \le r(x) \le r(y)$ occur.

We just consider the former case. Then we have

$$
\begin{aligned}
|v(x)-v(y)| &= \left| \int_{r(\Pi y)}^{r(y)} \sin\psi\, dr - \int_{r(\Pi x)}^{r(x)} \sin\varphi\, dr \right| \\
&\le \int_{r(x)}^{r(y)} 1\, dr + \left| \int_{r(y)}^{r(\Pi x)} (\sin\psi - \sin\varphi) dr \right| + \int_{r(\Pi x)}^{r(\Pi y)} 1\, dr \\
&\le |r(x)-r(y)| + \int_{r(y)}^{r(\Pi x)} |\sin\psi - \sin\varphi| dr + |r(\Pi x) - r(\Pi y)| \,.
\end{aligned}
$$

From the Lipschitz continuity of $\mathcal{F}$, we know that $|\varphi(r) - \psi(r)| \le C_4 l(x, y)$ holds for $r \in [r(y), r(\Pi x)]$ (see the inequality (6.3)). Consequently, we can find a positive number C_{16} depending only on Q such that $|v(x) - v(y)| \le C_{16} l(x, y)$ holds. Hence, (g_2.1) follows.

Next, we extend g_2 to a Lipschitz function g_3 on M^+ by using Kirszbraun's theorem. We define g by means of the averaging method appeared in Section 6. For any $x \in M^+$, let $r = r(\varphi)$, $-\pi/2 \le \varphi \le \pi/2$, be the representation of the leaf $\mathcal{F}(x)$ as a K-decreasing curve. We define g by

$$
g(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} g_3(r(\varphi), \varphi) d\varphi \,.
$$

Then the Lipschitz continuity (g.1) is shown in the same way as in Construction 2 in Section 6.

Assertion (g.2) can be proved as follows. It suffices to consider the case when $n \ge 2$, i.e., $x \in \mathcal{D}_2$. Note that $n(T^k x) = n(x) - k$ for $k = 0, \dots, n(x) - 1$. In virtue of (g_2.2), we obtain

$$
\begin{aligned}
|g(T^k x) - g_1(T^k x)| &\le \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |g_3(r_k(\varphi_k), \varphi_k) - g_1(T^k x)| d\varphi_k \\
&= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |g_2(r_k(\varphi_k), \varphi_k) - g_1(T^k x)| d\varphi_k \\
&\le C_{14} \theta^{n(x)-k}
\end{aligned}
$$

whenever $T^k x \in \mathcal{D}_2$, i.e., $k = 0, \dots, n(x) - 1$, where $r_k = r_k(\varphi_k)$ is the representation of the leaf $\mathcal{F}(T^k x)$ as a K-decreasing curve. Combining this inequality and the fact that g_1 has the desired property on $\mathcal{D}_2$, one can easily obtain (g.2).

(g.3) is trivial. Since g_1 is constant along the leaf $\mathcal{F}(x)$ and $g_1(x) = g_2(x) = g_3(x)$ for any $x \in \bigcap_{n=2}^{\infty} \mathcal{D}_n$, ($g$.4) is true. □

Finally, we give the following remark.

REMARK 7.2. In order to see the advantage of our method, we consider the case when we apply the well-known theory of thermodynamic formalism directly to the zeta function ζ_{t^+}. Recall that the map $w(\cdot) : \Omega^+ \to \Sigma$ assigning the itinerary to each $x \in \Omega^+$ gives a topological conjugacy between the dynamical system (Ω^+, T) and the shift (Σ, σ) (see Theorem 2.4). If we define $f : \Sigma \to \boldsymbol{R}$ so that $f(w(x)) = t^+(x)$ for $x \in \Omega^+$, then f is d_θ-Lipschitz continuous and $\zeta_{t^+} = \zeta_f$. The general theory applied to the ζ_f (see [14] and [16]) tells us that $\zeta_{t^+}(s)$ can be extended meromorphically to the domain

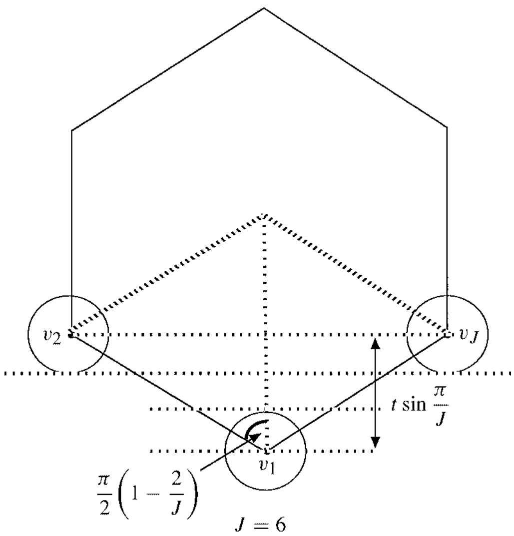

FIGURE 5.

$\{s \in \boldsymbol{C}; P((-\operatorname{Re} s)t^{+})\sqrt{\theta} < 1\}$, where $P((-\operatorname{Re} s)t^{+}) = P((-\operatorname{Re} s)f)$ is the topological pressure. Therefore, we need $P(0)\sqrt{\theta} < 1$ in order that the domain contains the half-plane Re s ≥ 0. It is easy to see that $P(0) = \log(J-1)$, since it coincides with the topological entropy of (Σ, σ). Thus, $\theta(J-1)^2 < 1$ is necessary to obtain the same result as Theorem 1.3. On the other hand, we choose θ so that $\theta = (1 + k_{\min} t_{\min})^{-1}$. Since we do not know whether we can choose it smaller than it is, we keep on our discussion with this choice of θ.

For example, consider a regular J-gon P, each side of which has length t. Label the vertices of P, $v_1, \ldots, v_J$ counterclockwise for our convenience. We assume that scatterers $Q_1, \ldots, Q_J$ are the discs with radius 1 centered at $v_1, \ldots, v_J$, respectively. It is easy to see that no eclipse condition (H.2) is satisfied if and only if $t \sin(\pi/J) > 2$ (see Figure 5).

On the other hand, $\theta(J-1)^2 < 1$ yields $t > J^2 - 2J$ in this case. However, we have

$$J^2 - 2J = (J-2)J > \left(\frac{2/J}{\sin(2/J)}\right)J = \frac{2}{\sin(2/J)}$$

if we apply the well-known inequality $\sin x > (2/\pi)x$ for $(0 < x < \pi/2)$ to $x = \pi/J$. This implies that we have a difficulty in obtaining the meromorphic extension of ζ_{t^+} to the domain containing the half-plane $\operatorname{Re} s \geq 0$ if $2/\sin(\pi/J) < t < J^2 - 2J$.

In the above, we avoid such a difficulty by using the Lipschitz continuity of the K-stable foliation as follows. We first show the length of the intersection of any local unstable curve and $\mathcal{D}_n$ decays with a rate not slower than κ^n for some $0 < \kappa < 1$ (equivalently, the area of $\mathcal{D}_n$ in the (r, φ)-coordinates decays with a rate not slower than κ^n for some $0 < \kappa < 1$). Then we construct a Lipschitz continuous function g cohomologous to t^+ and prove that the zeta function $\zeta_g = \zeta_{t^+}$ can be extended meromorphically to the domain $\{s \in \boldsymbol{C}; \kappa\rho_g(s) < 1\}$.

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