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Meromorphic functions of the form $f(z) = \sum_{n=1}^{\infty} a_n/(z - z_n)$

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Abstract

We prove some results on the zeros of functions of the form $f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z-z_n}$, with complex a_n, using quasiconformal surgery, Fourier series methods, and Baernstein's spread theorem. Our results have applications to fixpoints of entire functions.

1. Introduction

A number of recent papers [7, 11, 21] have concerned zeros of meromorphic functions represented as infinite sums

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z - z_n}, \quad z_n, a_n \in \mathbb{C}, \quad z_n \to \infty, \quad \sum_{z_n \neq 0} \left| \frac{a_n}{z_n} \right| < \infty. \tag{1.1}$$

We assume throughout that $a_n \neq 0$ and that $z_n \to \infty$ without repetition. By (1.1),

$$n(r) = \sum_{|z_n| \leq r} |a_n| = o(r), \quad r \to \infty. \tag{1.2}$$

If the z_n are all non-zero and the a_n are all real and positive, then (1.1) gives

$$f = u_x - iu_y, \quad u(z) = \sum_{n=1}^{\infty} a_n \log|1 - z/z_n|, \quad \lim_{r \to \infty} \frac{T(r, u)}{r} = 0, \tag{1.3}$$

in which u is subharmonic in the plane. We will need the following fundamental result [13, p. 327] on functions of the form (1.1) with complex a_n. Here we use standard notation from [16].

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Theorem 1.1 ([13]) *Let f be given by* (1.1)*, and let $0 < p < 1$. Then*

$$m(r,f) = o(1), \quad \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = o(1), \quad r \to \infty, \tag{1.4}$$

so that in particular $\delta(\infty, f) = 0$. If f has finite lower order then $\delta(a, f) = 0$ for all $a \in \mathbb{C} \setminus \{0\}$, and the same conclusion holds for $a = 0$ if in addition

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} a_n \neq 0. \tag{1.5}$$

We state next some of the main results from [7, 11].

Theorem 1.2 ([7, 11]) *Assume that f is given by* (1.1) *with all the a_n real, and let $n(r)$ be defined by* (1.2).

(a) If all the a_n are integers and $a_n \geq -1$ for all but finitely many n then f has infinitely many zeros.

(b) If all the a_n are integers and $n(r) = o(\sqrt{r})$ as $r \to \infty$ then f has infinitely many zeros.

(c) If $\inf\{a_n : n \in \mathbb{N}\} > 0$ then f has infinitely many zeros.

Theorem 1.2 (c) represents a substantial step in the direction of the following conjecture from [7].

Conjecture 1.1 *If all the a_n are real and positive in* (1.1) *then f has infinitely many zeros. Equivalently, subharmonic functions u as in* (1.3)*, with a_n as in* (1.1)*, have infinitely many critical points.*

It was conjectured further in [7] that if the a_n in (1.1) are real and $n(r) = o(\sqrt{r})$ as $r \to \infty$ then f must have zeros, but we give counter-examples to this in Examples 2.2 and 2.3.

The key fact used in [11] to prove Theorem 1.2, (c) is that $\inf\{a_n\} > 0$ implies that $T(r,f) = O(r)$ as $r \to \infty$. The first main result of the present paper refines Theorem 1.2, (c) to allow finitely many complex a_n, and this turns out to require application of the Ahlfors spiral theorem [18, p. 600]. Our theorem also establishes Conjecture 1.1 when f has finite order and all but finitely many of the z_n lie close to the real axis.

Theorem 1.3 *Let f be given by* (1.1)*, with the a_n real and positive. Assume that f has finite order, that*

$$\sum_{n=1}^{\infty} a_n = \infty, \tag{1.6}$$

and that either (i)

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} < \infty, \tag{1.7}$$

or (ii) there exists $\varepsilon > 0$ with

$$|z_n - \overline{z}_n| \leq |z_n|^{1-\varepsilon} \tag{1.8}$$

for all large n. Let $S(z)$ be a rational function. Then $f(z) - S(z)$ has infinitely many zeros.

In Theorem 1.3 and some of our subsequent results we consider the zeros of $f - S$, with S a rational function, rather than of f. The effect of this is to allow in particular finitely many residues which are not real and positive. In Example 2.1 we show that the hypothesis (1.6) is not redundant in Theorem 1.3. On the other hand, if f has finite lower order and $\sum a_n$ is absolutely convergent then the proof of Theorem 1.1 from [13] goes through to give $\delta(0, f - S) = 0$ for some choices of rational S: see Proposition 3.1.

Next, we make two remarks about Theorem 1.3, (ii). The first applies a theorem of Miles [23]. Suppose that the exponent of convergence of the z_n is infinite, but that of the non-real z_n is finite. Then if f is given by (1.1) and S is meromorphic of finite order, $f - S$ has zeros with infinite exponent of convergence. If this is not the case, we can write

$$\frac{1}{f - S} = FG$$

in which F is an entire function of infinite order, with real zeros, and G is meromorphic of finite order. Miles' result [23] gives $N(r, 1/F) = o(T(r, F))$ on a set of logarithmic density 1. This implies the existence of a sequence $r_m \to \infty$ with

$$\frac{\log T(r_m, F)}{\log r_m} \to \infty,$$
$$m(r_m, f - S) \geq m(r_m, 1/F) - m(r_m, G) \geq (1 - o(1))T(r_m, F),$$

which contradicts (1.4) and the fact that S has finite order.

Second, Ostrowski proved in [27] that if f is given by (1.1) with

$$\sum_{z_n \neq 0} \left| \frac{\mathrm{Im}(z_n)}{z_n^2} \right| < \infty \tag{1.9}$$

and $\delta(a,f) > 0$ for some a, then $T(r,f) = O(r)$ as $r \to \infty$. Ostrowski's result may therefore, if the a_n are positive, be combined with the method of Theorem 1.2, (c) to show that f has infinitely many zeros. However, (1.8) only implies (1.9) if $\sum_{z_n \neq 0} |z_n|^{-1-\varepsilon} < \infty$.

We turn our attention next to zeros of f as given by (1.1), with the a_n complex. The following conjecture, which obviously implies Conjecture 1.1, seems likely to be true.

Conjecture 1.2 *If*

$$\sup\{|\arg a_n| : n \in \mathbb{N}\} < \pi/2 \tag{1.10}$$

then f as given by (1.1) *has infinitely many zeros.*

Conjecture 1.2 is certainly true if f has finite lower order and $\sum a_n$ is absolutely convergent, by Theorem 1.1. For the case in which $\sum |a_n| = \infty$ and f has order at most $\frac{1}{2}$, we have the following result in support of Conjecture 1.2.

Theorem 1.4 *Let f be given by* (1.1)*, of order $\sigma \leq \frac{1}{2}$, and write*

$$a_n = x_n^+ - x_n^- + iy_n, \quad x_n^+ \geq 0, \quad x_n^- \geq 0, \quad y_n \in \mathbb{R}, \tag{1.11}$$

and

$$n^+(r) = \sum_{|z_n| \leq r} x_n^+, \quad n^-(r) = \sum_{|z_n| \leq r} x_n^-. \tag{1.12}$$

Assume that

$$\lim_{r \to \infty} n(r) = \sum_{n=1}^{\infty} |a_n| = \infty, \tag{1.13}$$

and that there exist positive constants δ, d_1 such that

$$n^-(r) \leq (1-\delta) n^+(r), \quad n(r) \leq d_1 n^+(r) \tag{1.14}$$

for all large r. Let $S(z)$ be a rational function. Then:

$$\delta(0, f-S) \leq 1 - \cos \pi\sigma, \quad \sigma < \frac{1}{2}; \quad \delta(0, f-S) < 1, \quad \sigma = \frac{1}{2}. \tag{1.15}$$

Conditions (1.11) and (1.14) are obviously satisfied if (1.10) holds. Theorem 1.4 allows infinitely many non-real a_n, but the proof is totally dependent on minimum modulus results for functions of order at most $\frac{1}{2}$, and in particular on results for the extremal case of the $\cos\pi\rho$ theorem [8]. For f of order between $\frac{1}{2}$ and 1 and with $\arg a_n$ sufficiently small, the following rather weaker result is applicable, based on the method of quasiconformal surgery [5, 6, 28]. The statement of the theorem is somewhat technical, but both it and Theorem 1.4 have subsequent applications to fixpoints of entire functions.

Theorem 1.5 *Let* $0<\sigma<1$ *and let* f *be given by* (1.1) *with order* $\rho<1$ *and*

$$\text{(1.16)}\quad \mathrm{Re}(a_n) > \frac{1}{2}+\sigma,\,,\ 1<r_n=|a_n|<1/\sigma,\ \ t_n=\arg a_n\in(-\pi/2,\pi/2),$$

and assume that, for all n,

$$\text{(1.17)}\qquad \frac{\left|\tan^{-1}\left(\frac{\sin t_n}{r_n-\cos t_n}\right)\right|}{\sqrt{\left(\log\left(\frac{r_n^2}{1+r_n^2-2r_n\cos t_n}\right)\right)^2+\left(\tan^{-1}\left(\frac{\sin t_n}{r_n-\cos t_n}\right)\right)^2}} < k_0 < \frac{1-\rho}{1+\rho}.$$

Let $S(z)$ *be a rational function. Then* $f(z)-S(z)$ *has infinitely many zeros.*

Note that (1.16) automatically gives (1.13). The hypotheses (1.16) and (1.17) are required in order to facilitate quasiconformal surgery and to control the dilatation arising therefrom, and these conditions seem very unlikely to be sharp. Obviously if a_n is real and positive and satisfies (1.16) then (1.17) holds, and Theorem 1.5 provides a result applying when the a_n are sufficiently close to the positive real axis.

Next, for f as in (1.1) but of possibly larger growth than in Theorems 1.4 and 1.5, we have the following result, based on Baernstein's spread theorem [2].

Theorem 1.6 *Let* $0<\sigma\le 1$. *Let* f *be given by* (1.1), *of finite lower order* μ, *such that* (1.13) *holds and*

$$\text{(1.18)}\quad |\arg z_n|<b<C(\mu,\sigma)=\frac{2}{\mu}\sin^{-1}\sqrt{\frac{\sigma}{2}}\ ,\ \ |\arg a_n|+|\arg z_n|<c<\frac{\pi}{2},$$

for all n. *Let* $S(z)$ *be a rational function. Then* $\delta(0,f-S)<\sigma$.

Corollary 1.1 *Let* f *be transcendental entire, of at most order* 1, *convergence class, and with zero sequence* (z_n). *If* $\lim_{n\to\infty}\arg z_n=0$, *then* $\delta(0,f'/f)=0$.

Corollary 1.1 follows at once from Theorem 1.6, since f'/f has a representation (1.1), and establishes a conjecture of Fuchs [11, 25] in the case where the zeros of f accumulate at a single ray.

We observe next that the hypotheses on $\arg a_n$ in Theorems 1.4, 1.5 and 1.6 are not redundant. In Examples 2.2 and 2.3 we construct functions f of the form (1.1), with real a_n, such that $f(z)$ has no zeros. Thus results for complex a_n require, in general, some condition on the lines of (1.5) or some hypothesis on $\arg a_n$.

Our methods have an application to the fixpoints of entire functions of order less than 1. Whittington [31] proved that if F is a transcendental entire function with $T(r,F) = o(\sqrt{r})$ as $r \to \infty$ then F has infinitely many fixpoints z with either $F'(z) = 1$ or $|F'(z)| > 1$, the proof based on applying the $\cos \pi\rho$ theorem [3, 18] to F and the residue theorem to $1/(z - F(z))$. In the same paper Whittington gave an example of an entire function F of order $\frac{1}{2}$ with only attracting fixpoints i.e. $F(z) = z$ implies $|F'(z)| < 1$ (see also [14]). We prove here the following theorem.

Theorem 1.7 *Let F be transcendental and meromorphic in the plane, with finitely many poles and of order at most $\frac{1}{2}$. Let $0 < c < 1$. Then F has infinitely many fixpoints z with $F(z) = z, |F'(z)| \geq c$.*

Theorem 1.7 is proved by writing $1/(z - F(z))$ in the form (1.1) and applying Theorem 1.4. Using again the method of quasiconformal surgery [5, 6, 28] we establish the following result on the multipliers at fixpoints of functions of order between $\frac{1}{2}$ and 1.

Theorem 1.8 *Let F be transcendental and meromorphic with finitely many poles in the plane, and with order $\rho \in (\frac{1}{2}, 1)$. Let*

$$0 < d < 1, \qquad \frac{\pi}{\sqrt{16(\log 1/d)^2 + \pi^2}} < \frac{1-\rho}{1+\rho}.$$

Then F has infinitely many fixpoints u_n with

$$F(u_n) = u_n, \quad |F'(u_n)| > d.$$

The dependence of d on ρ in Theorem 1.8 seems unlikely to be sharp, in particular as $\rho \to \frac{1}{2}$. However, the function $z + 1 - e^z$ has order 1 and only superattracting fixpoints.

We conclude the paper by proving some results when the denominators in (1.1) are replaced by a larger power.

Theorem 1.9 *Let $k \geq 2$ be an integer, and let*

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n}{(z - z_n)^{k+1}}, \quad z_n \to \infty, \quad \sum_{z_n \neq 0} \left| \frac{a_n}{z_n^{k+1}} \right| < \infty. \tag{1.19}$$

Then, as $r \to \infty$ outside a set of finite measure,

$$(1 - o(1))T(r, F) < \left(\frac{k+1}{k-1} \right) N(r, 1/F). \tag{1.20}$$

Example 2.4 shows that the constant in (1.20) is sharp, and that Theorem 1.9 fails for $k = 1$, even if

$$\sum_{z_n \neq 0} \left| \frac{a_n}{z_n^{1+d}} \right| < \infty \quad \forall d > 0. \tag{1.21}$$

However, if we assume that (z_n) has finite exponent of convergence, and that $\sum_{z_n \neq 0} |a_n/z_n| < \infty$, then we do get a result for $k = 1$.

Theorem 1.10 *Let F be as in* (1.19)*, with $k = 1$, and assume that*

$$\sum_{z_n \neq 0} |z_n|^{-L} < \infty, \quad \sum_{z_n \neq 0} \left| \frac{a_n}{z_n} \right| < \infty \tag{1.22}$$

for some $L > 0$. Then $\delta(0, F) < 1$.

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2. Examples

Example 2.1

The following example shows that (1.6) is not redundant in Theorem 1.3. Let

$$g(z) = \frac{i}{z^2(e^{iz} - 1)},$$

and let S be the principal part of $-g$ at 0, so that

$$S(\infty) = 0, \quad g(z) + S(z) = O(1), \quad z \to 0.$$

Let

$$f(z) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{4\pi^2 n^2 (z - 2\pi n)}, \quad h(z) = f(z) - g(z) - S(z).$$

Then f satisfies (1.1), (1.7) and (1.8), using (1.4), and h is entire. Since $g(z) = o(1)$ as $z \to \infty$ in the union of the circles $|z| = \pi(2m + 1), m \in \mathbb{N}$, we have $h \equiv 0$ by (1.4). Thus $f - S$ has no zeros.

Example 2.2

Let z_n be real and positive with z_1 large and $z_{n+1} > 4z_n$ for each $n \geq 1$, and let

$$g(z) = \prod_{n=1}^{\infty}(1 - z/z_n), \quad r_n = 2|z_n|.$$

We estimate $g'(z_m)$ in the following standard way. For $|z| = r_m$ with m large we have

$$\left|\frac{g'(z)}{g(z)}\right| = \left|\sum_{n=1}^{\infty}\frac{1}{z - z_n}\right| \leq \sum_{n=1}^{\infty}\frac{2}{r_n} < \infty, \quad \log|g(z)| > \log M(r_m, g) - O(1).$$

Let $h(z) = g(z)/(z - z_m)$. Applying the maximum principle to $1/h(z)$ in $r_{m-1} \leq |z| \leq r_m$ shows that

$$\log|g'(z_m)| = \log|h(z_m)| > \frac{1}{2}\log M(r_{m-1}, g)$$

and it follows that

$$\sum_{n=1}^{\infty}|a_n| < \infty, \quad a_n = 1/g'(z_n) \in \mathbb{R}.$$

Let f be defined by (1.1). Then $f(z) - 1/g(z)$ is entire. But on the circle $|z| = r_n$, for large n, the function $g(z)$ is large, while

$$|f(z)| \leq 2r_n^{-1}\sum_{m=1}^{\infty}|a_m| = o(1).$$

It follows that $f(z) - 1/g(z) \equiv 0$, and so f has no zeros. Thus a function $f(z)$ as in (1.1) may have real residues a_n and arbitrarily small growth, but fails to have zeros.

Example 2.3

Let

$$H(z) = \frac{1}{z\cos z}, \quad J(z) = \frac{1}{z} + \sum_{k\in\mathbb{Z}}\frac{(-1)^{k+1}}{w_k(z - w_k)}, \quad w_k = \frac{(2k+1)\pi}{2}.$$

Then it is evident that $J(z)$ satisfies (1.1), and there is an entire function G such that $H - J = G$. However, since there exist arbitrarily large r such that $H(z)$ is small on the whole circle $|z| = r$, estimate (1.4) shows that $G \equiv 0$ and so J has no zeros. Again J has real residues a_n, this time with

$$\sum_{|z_n|\leq r}|a_n| = O(\log r), \quad r \to \infty.$$

Example 2.4

We show here that the constant in (1.20) is sharp, and that Theorem 1.9 fails for $k = 1$. Let $g(z) = 1/(e^z - 1)$. A simple induction argument shows that, for each $k \in \mathbb{N}$,

$$g^{(k)}(z) = \frac{e^z P_{k-1}(e^z)}{(e^z-1)^{k+1}},$$

in which P_{k-1} is a polynomial of degree at most $k-1$. Expanding the numerator in powers of $e^z - 1$ we see that there exists $d_k > 0$ such that

$$m(r, g^{(k)}) \leq d_k m\left(r, \frac{1}{e^z-1}\right) = o(r),$$

and so

$$T(r, g^{(k)}) \sim (k+1)T(r, e^z), \quad N(r, 1/g^{(k)}) \leq (k-1)T(r, e^z) + O(1).$$

In particular, g' has no zeros. By periodicity there exists a constant c_k such that

$$g^{(k)}(z) = \frac{c_k}{(z-2\pi i n)^{k+1}} + O(1), \quad z \to 2\pi i n, \quad n \in \mathbb{Z}.$$

Set

$$G_k(z) = c_k \sum_{n \in \mathbb{Z}} \frac{1}{(z-z_n)^{k+1}}, \quad z_n = 2\pi i n.$$

Then each $G_k, k \in \mathbb{N}$, is of the form (1.19). Also $g^{(k)} - G_k$ is entire of order at most 1. Let δ be small and positive and let z be large with $|\arg z| \leq \pi/2 - \delta$ or $|\pi - \arg z| \leq \pi/2 - \delta$. Then $g^{(k)}(z)$ is small and, with the d_j positive constants,

$$|z - z_n| \geq d_1 \max\{|z|, |z_n|\}, \quad |G_k(z)| \leq d_2 |z|^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} |z_n|^{-k-\frac{1}{2}},$$

so that $G_k(z)$ is also small. Applying the Phragmén-Lindelöf principle now gives $g^{(k)} \equiv G_k$. Clearly G_1, which has no zeros, satisfies (1.21).

3. Preliminaries

We need the following lemmas.

Lemma 3.1 ([21]) *Suppose that $d > 1$ and that G is transcendental and meromorphic in the plane of order less than d. Let $R_0 > 0$. Then there exist uncountably many $R > R_0$ such that the length $L(r, R, G)$ of the level curves $|G(z)| = R$ lying in $|z| \leq r$ satisfies*

$$L(r, R, G) \leq r^{(3+d)/2}, \quad r \geq \log R. \tag{3.1}$$

Lemma 3.2 ([9]) *Let $1 < r < R < \infty$ and let g be meromorphic in $|z| \leq R$. Let $I(r)$ be a subset of $[0, 2\pi]$ of Lebesgue measure $\mu(r)$. Then*

$$\frac{1}{2\pi}\int_{I(r)} \log^+ |g(re^{i\theta})| d\theta \leq \frac{11R\mu(r)}{R-r}\left(1 + \log^+ \frac{1}{\mu(r)}\right) T(R, g).$$

Lemma 3.3 ([17]) *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ and lower order μ. Let $A > 1, B > 1$. Then $G = \{r \geq r_0 : S(Ar) \geq BS(r)\}$ satisfies*

$$\overline{\mathrm{logdens}} G \leq \rho \left(\frac{\log A}{\log B}\right), \quad \underline{\mathrm{logdens}} G \leq \mu \left(\frac{\log A}{\log B}\right).$$

The next lemma is a standard application of Tsuji's estimate for harmonic measure [30].

Lemma 3.4 *Let u be subharmonic and non-constant in the plane, and let U be a domain such that $u \equiv 0$ on ∂U and $\sup\{u(z) : z \in U\} > 0$. For $t > 0$ let $\theta_U^*(t)$ be the angular measure of the intersection of U with the circle $|z| = t$, except that $\theta_U^*(t) = \infty$ if the whole circle $|z| = t$ lies in U. Then there exists $R_0 \geq 1$ with*

$$\int_{R_0}^{r} \frac{\pi dt}{t\theta_U^*(t)} \leq \log B(2r, u) + O(1), \quad r \to \infty,$$

in which $B(2r, u) = \sup\{u(z) : |z| = 2r\}$.

Lemma 3.5 *Let f be as in (1.1), of finite lower order, and suppose that $\delta(0, f - S) > 0$, for some rational function S. Then $S(\infty) = 0$.*

Proof. Since $f - S$ has finite lower order Lemmas 3.2 and 3.3 give $c_0 > 0$ and arbitrarily large r such that $f - S$ is small on a subset E_r of the circle $|z|=r$ of angular measure at least c_0. If $S(\infty)\neq 0$, this contradicts (1.4). ∎

Since several of our results (Theorems 1.3, 1.4, 1.5 and 1.6) rely on the assumption that $\sum |a_n|$ diverges, we include for completeness the following immediate extension of the method of Theorem 1.1 from [13].

Proposition 3.1 *Suppose that f is given by (1.1), with $\sum_{n=1}^{\infty} |a_n| < \infty$, and that f has finite lower order. If S is a rational function with*

$$\lim_{z\to\infty} zS(z) \neq \lambda = \sum_{n=1}^{\infty} a_n, \tag{3.2}$$

then $\delta(0, f - S) = 0$.

Examples 2.2, with $S = \lambda = 0$, and 2.1 show that (3.2) is not redundant in Proposition 3.1.

Proof. Assume that f and S are as in the hypotheses, but that $\delta(0,f-S)>0$. Then $S(\infty)=0$, by Lemma 3.5. Following [13, p. 333], write

$$z(f(z)-S(z)) = g(z)+\lambda - zS(z),$$

in which

$$g(z)=\sum_{n=1}^{\infty}\frac{a_n z_n}{z-z_n}$$

satisfies the requirements of (1.1). The assumption (3.2) gives

$$\lim_{z\to\infty}(\lambda - zS(z))\neq 0.$$

Now (1.4), applied to g, shows that

$$g(z)=o(1), \quad \frac{1}{f(z)-S(z)}=O(r)$$

for all z on $|z|=r$, apart from a set $I(r)$ of angular measure $o(1)$ as $r\to\infty$. Lemma 3.2 gives

$$(\delta(0,f-S)-o(1))T(r,f-S)\le o(T(2r,f-S))$$

which contradicts Lemma 3.3, since f has finite lower order. ∎

4. Proof of Theorem 1.3

To prove Theorem 1.3 assume that f, a_n, z_n are as in (1.1) and (1.6) with the a_n real and positive. Assume further that f satisfies at least one of (1.7) and (1.8), but that $f-S$ has finitely many zeros, for some rational function S. We may assume that all z_n are non-zero. Let G and the positive integer N satisfy

$$N>3+\rho(f), \quad f(z)-S(z)=\frac{1}{z^N G(z)}. \tag{4.1}$$

Then G is transcendental and meromorphic in the plane, of finite order and with finitely many poles. By Lemma 3.5, we have $S(\infty)=0$.

Let r_0 be large and positive, so large that neither G nor S has poles in $r_0\le|z|<\infty$. Thus $\log|G(z)|$ is subharmonic in $|z|>r_0$. Using Lemma 3.1 and (4.1), choose a large positive R, in particular with

$$R>M(2r_0,G), \quad L(r,R,G)\le r^{N-1} \quad \forall r\ge \log R. \tag{4.2}$$

We may also assume that G has no multiple points with $|G(z)|=R$.

Lemma 4.1 *The set* $\{z : |G(z)| > R\}$ *has finitely many unbounded components* V_j. *If* U *is one of the* V_j *then* U *lies in* $|z| > 2r_0$. *Further, the finite boundary* ∂U *of* U *is the union of countably many pairwise disjoint level curves of* G, *each either simple and going to infinity in both directions, or simple closed. Finally, we have*

$$I = \int_{\partial U} |f(z) - S(z)| \, |dz| < 1, \tag{4.3}$$

and $B(0, 2r_0)$ *lies in an unbounded component of* $\mathbb{C} \setminus U$.

Proof. There exist finitely many V_j, since G has finite order, and each lies in $|z| > 2r_0$, by the choice of R. The assertion concerning the components of ∂U holds since the level curves $|G(z)| = R$ do not intersect. To prove (4.3) let $T = \log R$, and partition ∂U into its intersections with the disc $|z| \leq T$ and the annuli $2^m T < |z| \leq 2^{m+1} T, m \in \mathbb{Z}, m \geq 0$. Since (4.1) gives

$$|f(z) - S(z)| \leq R^{-1} |z|^{-N} \leq R^{-1}, \quad z \in U \cup \partial U, \tag{4.4}$$

we get

$$I \leq R^{-1} \left(T^{N-1} + \sum_{m=0}^{\infty} 2^{(N-1)(m+1)} T^{N-1} 2^{-Nm} T^{-N} \right) < 1,$$

using (4.2), since r_0 and R are large.

We prove the last assertion of the lemma by contradiction, and thus assume that $B(0, 2r_0)$ lies in a bounded component of the complement of U. Then there exists a simple closed curve γ_1, a component of ∂U, such that $B(0, 2r_0)$ lies in the interior U_1 of γ_1. Since z_n, for large n, is a pole of $f - S$ with residue $a_n > 0$ we get, using (4.3),

$$1 \geq \int_{\gamma_1} \quad |f(z) - S(z)| \quad |dz| \geq 2\pi \sum_{z_n \in U_1} a_n - O(1) \geq 2\pi \sum_{|z_n| < r_0} a_n - O(1),$$

which contradicts (1.6) provided r_0 was chosen large enough. ■

Lemma 4.2 *Let* U *be one of the* V_j, *and let* C_1 *be the unbounded component of* ∂U *which separates* U *from* $B(0, 2r_0)$. *Let* D *be the component of* $\mathbb{C} \setminus C_1$ *which contains* U. *Fix* $w_0 \in U$ *and define a single valued branch of* $\log z$, *continuous on the closure of* D, *with* $|\arg w_0| \leq \pi$. *Then we have*

$$\log z = O(\log |z|) \qquad \text{as } z \to \infty$$

in the closure of D.

Proof. Define a subharmonic function $v(z)$ on $\mathbb{C}$ as follows:

$$v(z) = \log|G(z)| - \log R, \quad z \in U \quad ; \quad v(z) = 0, \quad z \notin U. \tag{4.5}$$

Then v has finite order, since G has. The boundary of D is the curve C_1, and $v(z) = 0$ there. Thus the Ahlfors spiral theorem [18, pp. 600–608] gives $\arg z = O(\log|z|)$ as z tends to infinity on ∂D. Since $\arg z$ is monotone on arcs of circles centred at the origin, the result follows. ■

Lemma 4.3 *Let u be defined by* (1.3), *and let U be one of the V_j. Then $u(z) = O(\log|z|)$ as $z \to \infty$ in the closure of U.*

Proof. Using $U = V_j, w_0, D$ and the same branch of $\log z$ as in Lemma 4.2, there exist a constant c_1 and a function S_1, analytic and bounded in $|z| > r_0$, such that

$$S(z) = \frac{d}{dz}\left(c_1 \log z + S_1(z)\right) \tag{4.6}$$

for z in the closure of D. By (1.3) and (4.1) we have

$$u_x - iu_y = f(z) = S(z) + \frac{1}{z^N G(z)}. \tag{4.7}$$

For each w in U, join w_0 to w by a path σ_w consisting of part of the ray $\arg z = \arg w_0$, part of the circle $|z| = |w|$, and part of the boundary of U. Using Lemma 4.1 and (4.4) we get a constant c_2 such that

$$\int_{\sigma_w} |f(z) - S(z)|\, |dz| < c_2$$

for all w in U. This gives, using (4.6) and (4.7),

$$\left| \int_{\sigma_w} (u_x - iu_y)(dx + idy) \right| \le |c_1 \log w| + O(1)$$

and the result follows using Lemma 4.2. ■

Lemma 4.4 *There exist positive constants k_1, k_2 with the following property. With u as in* (1.3), *define:*

$$u_1(z) = \max\{u(z) - k_1 \log|z| - k_2, 0\}, \quad |z| > 2r_0 \quad ; \quad u_1(z) = 0, \quad |z| \le 2r_0.$$

Then u_1 is non-constant and subharmonic in the plane, with $u_1(z) = 0$ on the union of the closures of the V_j, and $T(r, u_1) = o(r)$ as $r \to \infty$.

Proof. Choose k_1 and k_2 using Lemma 4.3, so that $u(z) \le k_1 \log|z| + k_2$ on $|z| = 2r_0$ and on the union of the closures of the V_j. Thus u_1 is subharmonic, with $T(r, u_1) = o(r)$ by (1.3). Finally, u_1 is non-constant by (1.6). ■

We may now complete the proof of Theorem 1.3 in case (i), in which (1.7) holds. Let U be one of the V_j, and let W be a component of the set $\{z : u_1(z) > 0\}$. Then (1.7), (4.1) and (4.5) give $\liminf_{r\to\infty} T(r,v)/r < \infty$. Since v vanishes off U, while u_1 vanishes on the closure of U, a standard application of the Cauchy-Schwarz inequality as in [30] gives

$$\frac{1}{\theta^*_U(t)} + \frac{1}{\theta^*_W(t)} \geq \frac{2}{\pi}$$

for large t, and a contradiction arises on applying Lemma 3.4 to v and u_1.

To finish the proof of Theorem 1.3, it remains only to dispose of case (ii), and we assume henceforth that (1.8) holds.

Lemma 4.5 *For $r > 0$ and $0 < t < \pi/2$ let*

$$\begin{aligned} V^+(r,t) &= \{z : |z| = r, t < \arg z < \pi - t\}, \\ V^-(r,t) &= \{z : |z| = r, \pi + t < \arg z < 2\pi - t\}. \end{aligned}$$

Let $\delta > 0$. Then for all r in a set E_δ of lower logarithmic density at least $1 - \delta$, we have $u_1(z) \equiv 0$ on at least one of the sets $V^+(r,\delta), V^-(r,\delta)$.

Proof. We may assume that δ is small. Then we have

$$f(z) - S(z) = O(1), \quad z \to \infty, \quad \delta/4 \leq |\arg z| \leq \pi - \delta/4, \tag{4.8}$$

since $S(\infty) = 0$ and (1.8) gives $|z_n| = O(|z - z_n|)$ for large z as in (4.8).

By Lemma 3.3, there exist $c_3 > 0$ and a set E of lower logarithmic density at least $1 - \delta$ such that

$$T(4r, G) < c_3 T(2r, G), \quad r \in E.$$

For $r \in E$, since δ is assumed small, Lemma 3.2 and (4.1) now give

$$2\log|G(z)| > T(2r, G), \quad 4\log|f(z) - S(z)| < -T(2r, G), \quad z \in I_r, \tag{4.9}$$

in which I_r is a subset of the circle $|z| = 2r$, of angular measure at least $8\delta_1 > 0$.

Let $\delta_2 = \min\{\delta, \delta_1\}$ and let r be large, with $r \in E$. Without loss of generality $I_r \cap V^+(2r, \delta_2/2)$ has angular measure at least δ_2, and we apply the two-constants theorem [26] to $\log|f(z) - S(z)|$ in the interior Ω of the region $r/2 \leq |z| \leq 2r, \delta_2/4 \leq \arg z \leq \pi - \delta_2/4$. This gives positive c_4, c_5 independent of r such that, using (4.8) and (4.9), for $z \in V^+(r, \delta_2)$,

$$4\log|f(z) - S(z)| \leq -T(2r, G)\omega(z, I_r, \Omega) + O(1) \leq -c_4 T(2r, G) + c_5,$$

where $\omega(z, I_r, \Omega)$ denotes harmonic measure, and so $V^+(r, \delta_2) \subseteq V_j$, for some j, using (4.1) again. Lemma 4.5 now follows from Lemma 4.4. ∎

Lemma 4.6 *u_1 has lower order at least* 1.

Proof. This is a standard application of Lemma 3.4. Let Ω be a component of the set $\{z : u_1(z) > 0\}$. Let $\delta > 0$. Then by Lemma 4.5 we have $\theta^*_\Omega(t) \le \pi + 2\delta$ for all t in a set E_δ of lower logarithmic density at least $1 - \delta$. Lemma 3.4 gives

$$\log B(2r, u_1) - O(1) \ge$$
$$\ge \int_{R_0}^{r} \frac{\pi dt}{t\theta^*_\Omega(t)} \ge \frac{\pi}{\pi + 2\delta} \int_{E_\delta \cap [R_0, r]} \frac{dt}{t} \ge \frac{(1 - \delta - o(1))\pi \log r}{\pi + 2\delta}$$

as $r \to \infty$, and since δ is arbitrary the result follows. ∎

Lemma 4.7 *Let $\delta > 0$. Then there exists $c(\delta) > 0$ such that for all large r we have*

$$|u_1(w)| \le c(\delta)|w|^{1-\varepsilon}, \quad |w| = r, \quad \delta \le |\arg w| \le \pi - \delta. \tag{4.10}$$

Proof. By (1.8) there exist $c_6 = c_6(\delta) > 0$ and $r_1 > 0$ such that

$$|z_n - w| \ge c_6 \max\{|z_n|, |w|\}, \quad \delta \le |\arg w| \le \pi - \delta, \quad |w| = r \ge r_1. \tag{4.11}$$

For w as in (4.11) we get, without loss of generality,

$$0 \le \log|1 - \overline{w}/z_n| - \log|1 - w/z_n| = \log\left|\frac{1 - w/\overline{z}_n}{1 - w/z_n}\right| = \log\left|1 + \frac{w(\overline{z}_n - z_n)}{\overline{z}_n(z_n - w)}\right|,$$

and so (1.8) gives for large n, since $\log|1 + t| \le \log(1 + |t|) \le |t|$,

$$\Big|\log|1 - \overline{w}/z_n| - \log|1 - w/z_n|\Big| \le \frac{|w|}{|z_n|^\varepsilon c_6 \max\{|z_n|, |w|\}}.$$

This gives, for w as in (4.11), using (1.2),

$$|u(\overline{w}) - u(w)| \le$$
$$\le O(\log r) + c_6^{-1} \int_1^r t^{-\varepsilon} dn(t) + c_6^{-1} r \int_r^\infty t^{-1-\varepsilon} dn(t) = O(r^{1-\varepsilon}). \tag{4.12}$$

But, by Lemma 4.5, $u_1(w)$ vanishes and so $u(w) \le O(\log r)$ on at least one of the arcs $V^+(r, \delta), V^-(r, \delta)$, and (4.10) now follows from (4.12). ∎

Let η be small and positive. Since u_1 has finite order Lemma 3.3 gives a positive constant c_7 such that

$$T(2r, u_1) \le c_7 T(r, u_1), \quad r \in F_1, \tag{4.13}$$

in which F_1 has lower logarithmic density at least $1 - \eta/2$.

Let δ be small and positive, in particular so small that

$$\delta < \eta/2, \quad 24\delta c_7 < 1. \tag{4.14}$$

Using Lemma 4.7 we find that, for all r in a set F_2 of lower logarithmic density at least $1-\delta$, we have

$$T(r, u_1) \leq 4\delta B(r, u_1) + O(r^{1-\varepsilon}) \leq 12\delta T(2r, u_1) + O(r^{1-\varepsilon}).$$

This gives, by (4.13) and (4.14),

$$T(r, u_1) = O(r^{1-\varepsilon}), \quad r \in F_3 = F_1 \cap F_2,$$

and F_3 has lower logarithmic density at least $1-\eta$. This contradicts Lemma 4.6, and the proof of Theorem 1.3 is complete.

5. Proof of Theorem 1.4

Suppose that f and S are as in the statement of Theorem 1.4, but that (1.15) fails. Then we have $S(\infty) = 0$, by Lemma 3.5. We may assume that all the z_n are non-zero. Throughout the proof we use c to denote a positive constant, not necessarily the same at each occurrence. We also write

$$z = re^{i\theta}, \quad r = |z|, \quad \theta = \arg z \in [0, 2\pi]. \tag{5.1}$$

Lemma 5.1 *If r is large and positive and none of the z_n lie on $|z| = r$ then*

$$cn(r) \leq \int_0^{2\pi} |zf(z)| d\theta. \tag{5.2}$$

Further,

$$\lim_{s\to\infty} sM(s, H) = \infty, \quad H(w) = f(w) - S(w). \tag{5.3}$$

Proof. We have, by the residue theorem,

$$I = \int_0^{2\pi} zf(z) d\theta = 2\pi \sum_{|z_n|<r} a_n$$

and so, using (1.14),

$$|I| \geq 2\pi \left| \sum_{|z_n|<r} \mathrm{Re}(a_n) \right| = 2\pi(n^+(r) - n^-(r)) \geq cn^+(r) \geq cn(r),$$

which proves (5.2). Now (5.3) follows from (1.13), (5.2) and Lemma 3.5. ■

Since (1.15) fails by assumption, it follows immediately from (5.3) and an application to $1/H$ of the $\cos \pi\rho$ theorem for functions with deficient poles [13, p. 262] (see also [15]) that $\sigma = \frac{1}{2}$. We now write $H = f_1/f_2$, with f_1, f_2 entire functions of order at most $\frac{1}{2}$ with no common zeros. By (1.4), (5.3) and Lemma 3.5 we have

$$T(r, H) \leq N(r, H) + O(1) \leq N(r, 1/f_2) + O(1).$$

Also, since (1.15) fails by assumption, we get

$$\delta(0, H) = 1, \quad N(r, 1/f_1) = N(r, 1/H) = o(T(r, H)) = o(N(r, 1/f_2)).$$

Now (5.3) gives

$$m_0(r, f_2) = \min\{|f_2(z)| : |z| = r\} = o(rM(r, f_1)),$$

and exactly as in [11, pp. 282–284] we obtain

$$\log m_0(r, f_2) \leq o(\log M(r, f_2)), \quad \log M(r, f_1) \leq o(\log M(r, f_2)). \tag{5.4}$$

Lemma 5.2 *There exist a set E_1 of logarithmic density* 1 *and, for each $r \in E_1$, a subset U_r of $[0, 2\pi]$ such that*

$$m(U_r) = o(1), \quad zf(z) = O(1), \; r = |z|, \; \theta = \arg z \in V_r = [0, 2\pi] \setminus U_r, \tag{5.5}$$

in which $m(U_r)$ denotes Lebesgue measure.

Proof. The relations (5.4) imply that f_2 is extremal for the $\cos \pi\rho$ theorem. Let the positive function $\psi(r)$ tend to 0 slowly as $r \to \infty$. Results of Drasin and Shea [8] (see also [19, section II]) give

$$\log |f_2(z)| \geq \psi(r) \log M(r, f_2), \quad |z| = r \in E_1, \quad \arg z \in V_r.$$

Using (5.4) we get $zH(z) = o(1)$ for $z \in V_r$ and (5.5) follows, using (5.3) and the fact that $S(\infty) = 0$. ∎

We now apply to $zf(z)$ the method of [24, Theorem 1]. Since $n(r)$ has order at most 1, we may apply [24, p. 198] with $M = 6$, $\rho = 1$ and $R_0 = \min\{|z_n|\}$ to obtain (2.1) and (2.2) of [24] for all r in a set $E = E_M$ of positive lower logarithmic density. Note that the lemma of [24, p. 198] is stated only for integer-valued functions, but the proof goes through for $n(r)$ as in (1.2). We may assume that $E \subseteq E_1$, with E_1 as in Lemma 5.2, and that $E \cap \{|z_n|\} = \emptyset$.

Assume henceforth that r is large and in E, and that z satisfies (5.1). Write

$$\sum_{|z_n|<r} \frac{za_n}{z - z_n} = \sum_{|z_n|<r} a_n \sum_{m=0}^{\infty} (z_n/z)^m = \sum_{m\in\mathbb{Z},\, m\leq 0} z^m \sum_{|z_n|<r} a_n z_n^{-m}$$

and

$$\sum_{|z_n|>r} \frac{za_n}{z - z_n} = -\sum_{|z_n|>r} \frac{za_n}{z_n} \sum_{k=0}^{\infty} (z/z_n)^k = -\sum_{m=1}^{\infty} z^m \sum_{|z_n|>r} a_n z_n^{-m}, \tag{5.6}$$

the change of order of summation justified since the first double series in (5.6) is plainly absolutely convergent. Thus

$$zf(z) = \sum_{m\in\mathbb{Z}} b_m(r) e^{im\theta}, \tag{5.7}$$

in which

$$b_m(r) = -\sum_{|z_n|>r} a_n (r/z_n)^m, \quad m > 0, \tag{5.8}$$

and

$$b_m(r) = \sum_{|z_n|<r} a_n (r/z_n)^m, \quad m \leq 0. \tag{5.9}$$

Fix a large positive integer q, in particular with $q > 2M(\rho + 1) = 24$. Then (2.1) and (2.2) of [24] may be applied to (5.8) and (5.9) exactly as in [24, p. 202], to give (2.11) and (2.12) of [24], with $b_m(r, F_j)$ and $n_j(r)$ replaced by $b_m(r)$ and $n(r)$ respectively. This leads to

$$\sum_{m\in\mathbb{Z},\, m\notin\{0,1,\ldots,q\}} |b_m(r)|^2 \leq cn(r)^2. \tag{5.10}$$

Write

$$zf(z) = P(z) + s(r, \theta), \quad P(z) = \sum_{m=0}^{q} b_m(r) e^{im\theta}. \tag{5.11}$$

Then (5.10) gives

$$\|s\| = \|s\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |s(r, \theta)|^2 d\theta \right)^{\frac{1}{2}} \leq cn(r). \tag{5.12}$$

Set

$$|P(z)|^2 = \sum_{k=-q}^{q} h_k(r)e^{ik\theta}, \quad h_k(r) = \frac{1}{2\pi}\int_0^{2\pi} |P(z)|^2 e^{-ik\theta} d\theta, \tag{5.13}$$

so that

$$|h_k(r)| \le h_0(r) = \|P\|^2 = \sum_{m=0}^{q} |b_m(r)|^2. \tag{5.14}$$

By (5.13), since $[0, 2\pi] = U_r \cup V_r$,

$$\int_{V_r} |P(z)|^2 d\theta = h_0(r)m(V_r) - \int_{U_r} \sum_{k\neq 0} h_k(r)e^{ik\theta} d\theta$$

and so (5.5) and (5.14) give

$$\int_{V_r} |P(z)|^2 d\theta \ge h_0(r)(m(V_r) - 2qm(U_r)) \ge (2\pi - o(1))h_0(r). \tag{5.15}$$

But on V_r we have, using (5.5) and (5.11),

$$zf(z) = O(1), \quad P(z) = -s(r,\theta) + O(1),$$

and so (5.15) gives

$$h_0(r) \le c\int_{V_r} (|s| + O(1))^2 d\theta \le c\int_0^{2\pi} (|s| + O(1))^2 d\theta.$$

Thus, using (5.12),

$$h_0(r) \le c\,(\|s\| + O(1))^2 \le cn(r)^2. \tag{5.16}$$

Now (1.13), (5.2) and (5.5) give

$$n(r)^2 \le c\left(\int_{U_r} |zf(z)|d\theta\right)^2 \le cm(U_r)\int_{U_r} |zf(z)|^2 d\theta$$

and so, using (5.5) again and (5.7), (5.10), (5.14) and (5.16), we get

$$n(r)^2 \le cm(U_r)\|zf(z)\|^2 \le cm(U_r)\sum_{m\in\mathbb{Z}} |b_m(r)|^2 \le o(n(r)^2),$$

which is obviously a contradiction. Theorem 1.4 is proved.

6. A theorem on zeros of entire functions

Theorem 1.3 leads to a result on the zeros of entire functions of small growth, which will be required for the proof of Theorem 1.5. We begin with:

Lemma 6.1 *Let h be transcendental and meromorphic of finite order in the plane, with finitely many poles and with*

$$\liminf_{r\to\infty} \frac{T(r,h)}{r} = 0. \tag{6.1}$$

Assume that (z_n) is a non-zero sequence tending to infinity without repetition such that all but finitely many zeros of h lie in the set $\{z_n\}$. Assume further that $1/a_n = h'(z_n) \neq 0$ for each n, and that

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{z_n} \right| < \infty. \tag{6.2}$$

Then we may write

$$\frac{1}{h(z)} = f(z) - S(z), \tag{6.3}$$

where f is given by (1.1) and S is a rational function with $S(\infty) = 0$.

Proof. Assume that h is as in the statement. Using (6.2), define $f(z)$ by (1.1). Then there exists a rational function S, with $S(\infty) = 0$, such that

$$g(z) = \frac{1}{h(z)} - f(z) + S(z) \tag{6.4}$$

is entire. By (1.4) we have

$$T(r,f) \leq N(r,1/h) + o(1), \quad T(r,g) \leq 2T(r,h) + O(\log r). \tag{6.5}$$

By (6.4) we need only show that $g \equiv 0$.

Choose $\rho_1 > \rho(h)$, and suppose that r is large and positive and the circle $|z| = r$ does not meet the union U_0 of the discs $B(z_n, |z_n|^{-\rho_1})$. Then for $|z| = r$ we have

$$|f(z)| \leq \sum_{|z_n| \leq r/2} \left| \frac{a_n}{z_n} \right| + \sum_{|z_n| \geq 2r} 2 \left| \frac{a_n}{z_n} \right| + (2r)^{\rho_1+1} \sum_{r/2 < |z_n| < 2r} \left| \frac{a_n}{z_n} \right|.$$

Hence using (6.2) there exist a set E_0 of finite measure and a positive integer ρ_2 such that

$$|f(z)| \leq r^{\rho_2}, \quad |z| = r \notin E_0. \tag{6.6}$$

Suppose first that g is transcendental, and set $g_1(z) = g(z)z^{-\rho_2}$. Let $r_0 > 1$ be so large that h has no poles in $|z| \geq r_0$ and

$$M(r_0, g_1) > 4, \quad M(r_0, h) > 4 \quad \text{and} \quad |S(z)| < 1 \quad \text{for} \quad |z| \geq r_0. \tag{6.7}$$

Choose $K > M(r_0, g_1) + M(r_0, h)$ and let U_1 be an unbounded component of the set $\{z : |g_1(z)| > K\}$, and U_2 an unbounded component of the set $\{z : |h(z)| > K\}$. Then both U_j lie in $|z| > r_0$. For $t > 0$ and $j = 1, 2$, let $\theta_j(t)$ be the angular measure of the intersection of U_j with the circle $|z| = t$, and let $\theta_j^*(t) = \theta_{U_j}^*(t)$, as defined in Lemma 3.4.

Then by (6.4), (6.6) and (6.7) we have $\theta_1(t) + \theta_2(t) \leq 2\pi$ for $r \notin E_0$ and so

$$\frac{\pi}{\theta_1^*(t)} + \frac{\pi}{\theta_2^*(t)} \geq 2, \quad r \notin E_0. \tag{6.8}$$

Using (6.1) and (6.5) and the fact that E_0 has finite measure, a standard application of Lemma 3.4 and (6.8) now gives a contradiction.

Suppose finally that g is a polynomial. Lemmas 3.2 and 3.3 give arbitrarily large r such that h is large on a subset E_r of the circle $|z| = r$ of angular measure at least $c_3 > 0$. Using (1.4), (6.4) and the fact that $S(\infty) = 0$ we get $g(z) = o(1)$ for at least one $z \in E_r$. Thus $g \equiv 0$. This proves Lemma 6.1. ∎

Theorem 6.1 *Let h be as in Lemma 6.1, and assume in addition that $h'(z_n)$ is real and positive for each n. Then $\sum_{n=1}^{\infty} 1/h'(z_n) < \infty$.*

To prove Theorem 6.1, assume that h is as in the statement, but that $\sum_{n=1}^{\infty} a_n = \infty$, in which $a_n = 1/h'(z_n)$. Then we have (6.3) and, by (6.1) and (6.5), f satisfies the hypotheses of Theorem 1.3. Thus $f - S$ has infinitely many zeros and this contradicts (6.3).

The hypothesis (6.1) is not redundant in Theorem 6.1. In [20] an entire function E is constructed with zero sequence (z_n) such that $|z_{n+1}/z_n| > 2$ and $E'(z_n) = 1$ for all n, while $T(r, E) = O(r)$ as $r \to \infty$.

Corollary 6.1 *Let $0 < c_1 < c_2 < \infty$ and let h be transcendental and meromorphic with finitely many poles in the plane. Assume that all but finitely many zeros z of h have $h'(z) \in \mathbb{R}, c_1 < h'(z) < c_2$. Then h has order $\rho(h) \geq 1$.*

To prove Corollary 6.1, assume that h has order less than 1. Then we may choose a sequence (z_n) such that all but finitely many zeros of h lie in $\{z_n\}$, and $h'(z_n) \in (c_1, c_2)$, and such that $\sum |z_n|^{-1} < \infty$. Since $\sum 1/h'(z_n)$ obviously diverges, this contradicts Theorem 6.1.

The obvious example $h(z) = e^z - 1$ shows that Corollary 6.1 is sharp.

7. Application of quasiconformal surgery

Lemma 7.1 *Let $0 \leq \rho < \infty, 0 < \kappa < 1$, and let g be quasimeromorphic with finitely many poles in the plane and with the following properties:*

(i) there exist Jordan curves σ_n, τ_n in $\mathbb{C}$ such that σ_n lies in the interior domain V_n of τ_n, for each $n \in \mathbb{N}$, and $V_{n'} \cup \tau_{n'}$ lies in the exterior domain of τ_n, for $n' \neq n$;

(ii) g has no poles in $V_n \cup \tau_n$ and maps V_n into the interior domain U_n of σ_n;

(iii) g is conformal on U_n, for each n, and meromorphic on

$$Y = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} (V_n \cup \tau_n);$$

(iv) we have

$$\log M(r, g) < r^{\rho + o(1)}, \quad r \to \infty; \tag{7.1}$$

(v) we have

$$|g_{\overline{z}}| \leq \kappa |g_z| \quad a.e. \tag{7.2}$$

Set $K = \frac{1+\kappa}{1-\kappa}$. Then there exist a K-quasiconformal homeomorphism ϕ of the extended plane, fixing $0, 1, \infty$, and a function h meromorphic in the plane of order at most ρK, with finitely many poles, such that $g \equiv \phi^{-1} \circ h \circ \phi$. Further, ϕ is conformal on $W = \bigcup_{n=1}^{\infty} U_n$ and on the interior of $\mathbb{C} \setminus \bigcup_{m=1}^{\infty} g^{-m}(W)$.

Proof. This is basically Shishikura's lemma on quasiconformal surgery [5, 28]. Let

$$W_0 = W, \quad W_{m+1} = g^{-m-1}(W) \setminus g^{-m}(W), \quad m \geq 0, \quad H = \mathbb{C} \setminus \bigcup_{m=0}^{\infty} g^{-m}(W).$$

Define a Beltrami coefficient $\mu(z)$ on $\mathbb{C}$ as follows. For $z \in W_0 \cup H$ set $\mu = 0$. Assuming that μ has been defined on W_m, define μ for $w \in W_{m+1}$ by

$$\mu(w) = \frac{\mu_g(w) + \mu(g(w))A(w)}{1 + \mu(g(w))\overline{\mu_g(w)}A(w)}, \quad A = \frac{\overline{g_w}}{g_w}. \tag{7.3}$$

Thus μ is defined inductively a.e. in $\mathbb{C}$. We assert next that (7.2) and (7.3) give

$$|\mu(w)| \leq \kappa \quad a.e. \tag{7.4}$$

This is obviously true for $w \in W_0 \cup H$, and we have (7.4) for $w \in V_n$ using (7.2), since $\mu(g(w)) = 0$ for such w, by (ii). Now suppose that

$$w \in W_m, \quad w \notin E = \bigcup_{n=1}^{\infty} (V_n \cup \tau_n).$$

Then $\mu_g(w) = 0$, by (iii), and so (7.3) gives $|\mu(w)| \leq |\mu(g(w))|$. Thus induction gives (7.4) on the W_m.

We then define ϕ to be a quasiconformal homeomorphism of the extended plane fixing $0, 1, \infty$, and with complex dilation μ, and (7.3) gives $\mu_{\phi \circ g} = \mu_\phi$ a.e., so that $\phi \circ g = h \circ \phi$, with h meromorphic, with finitely many poles. It remains only to prove that h has order at most ρK. But this follows from (7.1) and the fact that $h = \phi \circ g \circ \phi^{-1}$, using the standard estimate [1, 22]

$$|\phi(z)| \leq c_1 |z|^K, \quad |\phi^{-1}(z)| \leq c_2 |z|^K, \quad z \to \infty. \qquad \blacksquare$$

Next, for $0 < R < S < \infty$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in [-\pi/2, \pi/2]$, define $\psi(z) = \psi(\alpha, \beta, R, S, z)$ by

$$\psi(z) = \psi(\alpha, \beta, R, S, z) = \alpha z \quad (|z| \leq R), \quad \psi(z) = \alpha z e^{i\beta} \quad (|z| \geq S), \tag{7.5}$$

and

$$\psi(z) = \psi(\alpha, \beta, R, S, z) = \alpha z \exp\left(\frac{i\beta \log |z|/R}{\log S/R}\right) \quad (R \leq |z| \leq S). \tag{7.6}$$

In the domain $\{z \in \mathbb{C} : R < |z| < S, 0 < \arg z < 2\pi\}$ write

$$\begin{aligned} \zeta &= \log z = \sigma + i\theta, \\ \phi(z) &= \log \psi(z)/\alpha = \sigma + i(\theta + \beta(\sigma - \log R)(\log S/R)^{-1}) \end{aligned}$$

with σ, θ real. Then

$$2\phi_{\overline{\zeta}} = \phi_\sigma + i\phi_\theta = i\beta(\log S/R)^{-1}, \quad 2\phi_\zeta = \phi_\sigma - i\phi_\theta = 2 + i\beta(\log S/R)^{-1}.$$

Thus ψ is quasiconformal in the plane, with

$$|\mu_\psi|^2 \leq \frac{\beta^2}{4(\log S/R)^2 + \beta^2} \quad a.e. \tag{7.7}$$

Lemma 7.2 *Let $0 < c < 1$ and let F be meromorphic with finitely many poles in the plane, and with order $\rho < 1$. Suppose that (u_n) tends to infinity without repetition, and that all but finitely many fixpoints of F lie in the set $\{u_n : n \in \mathbb{N}\}$. Suppose further that, for each $n \in \mathbb{N}$,*

$$F(u_n) = u_n, \quad F'(u_n) = \lambda_n e^{i\theta_n}, \quad 0 \leq \lambda_n \leq c < 1, \quad \theta_n \in \mathbb{R}, \tag{7.8}$$

and that θ_n satisfies

$$\min\{|\theta_n|, |\theta_n - \pi|\} \leq \delta_n \leq \pi/2. \tag{7.9}$$

Then

$$\limsup_{n\to\infty} q_n \geq \frac{1-\rho}{1+\rho} \quad , \quad q_n = \frac{\delta_n}{\sqrt{4(\log 1/\lambda_n)^2 + \delta_n^2}}. \tag{7.10}$$

In (7.10) we set $q_n = 0$ whenever $\lambda_n = 0$.

Proof. Assume that F, u_n are as in the statement, but that (7.10) fails. Then we may assume that there exists $\kappa_0 \in (0,1)$ such that, for all n,

$$q_n = \frac{\delta_n}{\sqrt{4(\log 1/\lambda_n)^2 + \delta_n^2}} \leq \kappa_0, \quad \rho K_0 < 1, \quad K_0 = \frac{1+\kappa_0}{1-\kappa_0}. \tag{7.11}$$

We define a quasimeromorphic function g by modifying F as follows. Let u_n be such that $\lambda_n \neq 0$. Then (7.8) and Schröder's functional equation [29, p. 66] give a neighbourhood $B(u_n, \rho_n)$, with ρ_n small and positive, and a function ϕ_n defined and conformal on $B(0, \rho_n)$ such that

$$\phi_n(0) = 0, \quad \phi_n'(0) = 1, \quad F(z) - u_n = \phi_n^{-1}(\lambda_n e^{i\theta_n}\phi_n(z - u_n)), \tag{7.12}$$

for $z \in B(u_n, \rho_n)$. Take a small positive r_n, with $B(0, r_n) \subseteq \phi_n(B(0, \frac{1}{2}\rho_n))$, and define ψ_n by

$$\psi_n(w) = \psi(\lambda_n e^{ip_n}, \theta_n - p_n, \lambda_n r_n, r_n, w),$$

with ψ as in (7.5) and (7.6). Here p_n is 0 or π and is chosen according to (7.9) so that $|\theta_n - p_n| \leq \delta_n$. Then (7.7) gives $|\mu_{\psi_n}| \leq \kappa_0$, with κ_0 as in (7.11). Set

$$U_n = \{u_n + v : \phi_n(v) \in B(0, \lambda_n r_n)\}, \quad V_n = \{u_n + v : \phi_n(v) \in B(0, r_n)\},$$

so that $U_n \subseteq V_n \subseteq B(u_n, \frac{1}{2}\rho_n)$, and set

$$g(z) - u_n = \phi_n^{-1}(\psi_n(\phi_n(z - u_n))), \quad z \in V_n.$$

Then g is conformal on U_n with

$$g(z) - u_n = \phi_n^{-1}(\pm\lambda_n(\phi_n(z - u_n))), \quad g'(u_n) = \pm\lambda_n \in [-c, c].$$

Also g maps V_n into U_n. For z not in the union of the V_n, we set $g(z) = F(z)$. Thus g is quasimeromorphic and has the same poles and fixpoints as F.

By Lemma 7.1 there exist a quasiconformal homeomorphism ϕ of the extended plane, fixing $0, 1, \infty$, and a function H meromorphic with finitely many poles in the plane, and with order at most $\rho K_0 < 1$, with K_0 as in (7.11), such that $g \equiv \phi^{-1} \circ H \circ \phi$. Also ϕ is conformal on the union W of the U_n, and on the interior of $\mathbb{C} \setminus \bigcup_{m=1}^{\infty} g^{-m}(W)$.

Obviously z is a fixpoint of g if and only if $\phi(z)$ is a fixpoint of H. If u_n has $\lambda_n \neq 0$ then g and ϕ are conformal on U_n and we get

$$H(w_n) = w_n, \quad H'(w_n) = g'(u_n) = \pm\lambda_n \in [-c, c], \quad w_n = \phi(u_n).$$

Suppose next that u_n has $\lambda_n = 0$. Then u_n lies in an open disc Y_n disjoint from the closures of the V_m, with $F(Y_n) \subseteq Y_n$, and $g = F$ on Y_n. Hence g and ϕ are analytic on Y_n and in this case we get $H'(w_n) = 0$.

The function $h(z) = z - H(z)$ thus satisfies the hypotheses of Corollary 6.1, since

$$h(w_n) = 0, \quad h'(w_n) \in [1 - c, 1 + c].$$

But h has order less than 1, and this contradiction proves Lemma 7.2. ■

8. Proof of Theorem 1.5

Assume that f and S are as in the statement, but that $f - S$ has finitely many zeros. Define F by

$$\frac{1}{z - F(z)} = f(z) - S(z).$$

Then F is transcendental and meromorphic in the plane, with finitely many poles, and with order $\rho < 1$. Further, each pole z_n of f is, for large n, a fixpoint of F with

$$F'(z_n) = b_n, \quad b_n = 1 - \frac{1}{a_n} = 1 - r_n^{-1} e^{-it_n}.$$

By (1.16) we have

$$\lambda_n^2 = |b_n|^2 = \frac{1 + r_n^2 - 2r_n \cos t_n}{r_n^2} < M_0 < 1, \quad \mathrm{Re}(b_n) = 1 - r_n^{-1} \cos t_n > 0$$

and

$$\arg b_n = \theta_n \in (-\pi/2, \pi/2), \quad \tan(\theta_n) = \frac{\sin t_n}{r_n - \cos t_n}.$$

Applying Lemma 7.2 gives (7.10), which contradicts (1.17).

9. Proof of Theorem 1.6

Assume that f and σ are as in the statement, but that $\delta(0, f - S) \geq \sigma$, for some rational function S. By Lemma 3.5 we have $S(\infty) = 0$. The second condition of (1.18) allows us to assume that $b < \pi/2$. Choose b_0, b_1, b_2 with

$$b < b_0 < b_1 < b_2 < \min\{\pi/2, C(\mu, \sigma)\}.$$

Lemma 9.1 *There exists $M_1 > 0$ such that for all large z lying outside the region $|\arg z| < b_0$ we have $|f(z)| \leq M_1$.*

Proof. This follows from (1.1), since there exists $M_2 > 0$ such that for such z we have $|z - z_n| \geq M_2 |z_n|$ for all n. ∎

Since $\delta(0, f - S) \geq \sigma$, Baernstein's spread theorem [2] gives a sequence $r_m \to \infty$ and, for each m, a subset I_m of the circle $|z| = r_m$, of angular measure at least

$$\min\{2\pi, 2C(\mu, \sigma)\} - o(1) \geq 2b_2,$$

such that

$$\log|f(z) - S(z)| < -T(r_m, f)^{\frac{1}{2}}, \quad z \in I_m.$$

For large m, we consider the subharmonic function $v(z) = \log|f(z) - S(z)|$ on the domain

$$\Omega = \{z : r_m/4 < |z| < r_m, b_0 < \arg z < 2\pi - b_0\},$$

and v is bounded above on Ω, by Lemmas 3.5 and 9.1. Since the intersection J_m of I_m with the arc $\{z : |z| = r_m, b_1 < \arg z < 2\pi - b_1\}$ has angular measure at least $2(b_2 - b_1)$, the two-constants theorem [26] and a standard estimate via conformal mapping for the harmonic measure of J_m at $-r_m/2$ now give

$$r_m(f(-r_m/2) - S(-r_m/2)) \to 0, \quad m \to \infty.$$

Since $S(\infty) = 0$, applying the next lemma gives a contradiction.

Lemma 9.2 *We have $\lim_{r \in \mathbb{R}, r \to +\infty} r|f(-r)| = \infty$.*

Proof. If $r > 0$ then (1.18) gives $|\arg(r + z_n)| \leq |\arg z_n|$ and so there exists $M_3 > 0$ such that

$$\mathrm{Re}\left(\frac{a_n}{r + z_n}\right) > M_3 \left|\frac{a_n}{r + z_n}\right|.$$

This gives, as $r \to \infty$, using (1.13),

$$r|f(-r)| \geq M_3 r \sum_{|z_n| \leq r} \left|\frac{a_n}{r + z_n}\right| \geq \frac{M_3}{2} \sum_{|z_n| \leq r} |a_n| \to \infty.$$

∎

10. Proof of Theorem 1.7

Assume that F is transcendental and meromorphic with finitely many poles in the plane, of order at most $\frac{1}{2}$, and that all but finitely many fixpoints w of F have $|F'(w)| < c < 1$. Set $h(z) = z - F(z)$. Then h has infinitely many zeros w, of which all but finitely many have

$$|1 - h'(w)| < c < 1, \quad 1 - c < |h'(w)| < 1 + c.$$

By Lemma 6.1 we have (6.3), in which S is a rational function and

$$a_n = 1/h'(z_n), \quad |a_n| \geq 1/(1 + c), \quad \sup\{|\arg a_n| : n \in \mathbb{N}\} < \pi/2,$$

and each z_n is a zero of h and so a fixpoint of F. The function f thus satisfies the hypotheses of Theorem 1.4. But this implies that $f - S$ has infinitely many zeros, contradicting (6.3). This proves Theorem 1.7.

11. Proof of Theorem 1.8

Assume that F, ρ and d are as in the hypotheses. Since $\rho < 1$ and F is transcendental with finitely many poles, F has infinitely many fixpoints u. If we assume that all but finitely many of these have $|F'(u)| \leq d$, then applying Lemma 7.2 with $\delta_n = \pi/2$ gives a contradiction.

12. Proof of Theorem 1.9

We need the following special case of a result of Frank and Weissenborn.

Theorem 12.1 ([12]) *Let f be transcendental and meromorphic in the plane with only simple poles, and let $k \geq 2$ be an integer. Then*

$$(k - 1)N(r, f) < N(r, 1/f^{(k)}) + o(T(r, f))$$

as $r \to \infty$ outside a set of finite measure.

Assume now that F is as in (1.19). Then we have

$$F = f^{(k)}, \quad f(z) = \sum_{n=1}^{\infty} \frac{a_n(-1)^k z^k}{(z - z_n)k!z_n^k}.$$

Applying (1.4) to $f(z)z^{-k}$, with a_n replaced by $a_n(-1)^k(k!)^{-1}z_n^{-k}$, and using (1.19), we get

$$m(r, f) \leq o(1) + k \log r, \quad T(r, f) \leq N(r, f) + O(\log r).$$

Thus (1.4), (1.19) and Theorem 12.1 give

$$\begin{aligned} T(r,F) &\leq N(r,F) + S(r,f) \leq (k+1+o(1))N(r,f) \\ &\leq \left(\frac{k+1}{k-1} + o(1)\right) N(r,1/F) \end{aligned}$$

outside a set of finite measure. This proves Theorem 1.9.

13. Proof of Theorem 1.10

Assume that F is as in the statement of Theorem 1.10, but with $\delta(0,F)=1$. The assumptions (1.22) give

$$F(z) = f'(z), \quad f(z) = -\sum_{n=1}^{\infty} \frac{a_n}{z-z_n},$$

in which f is meromorphic of finite order, using (1.4). Then

$$N(r,1/f') = N(r,1/F) = o(T(r,F)) = o(T(r,f))$$

and so the counting function of the multiple points of f is $o(T(r,f))$.

By a theorem of Eremenko [10], the function f has positive order ρ and sum of deficiencies 2, and at least one deficient value, a say, of f is non-zero. But then the results of [10] show that there exist arbitrarily large r such that $f(z)$ is close to a on a subset of the circle $|z|=r$ of angular measure close to π/ρ, and this contradicts (1.4).

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