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Meromorphic Functions Sharing a Nonzero Polynomial IM

Pulak Sahoo
Department of Mathematics, University of Kalyani, West Bengal-741235, India.
e-mail : sahoopulak@yahoo.com; sahoopulak1@gmail.com

ABSTRACT. We study the uniqueness of meromorphic functions concerning nonlinear differential polynomials sharing a nonzero polynomial IM. Though the main concern of the paper is to improve a recent result of the present author [12], as a consequence of the main result we also generalize two recent results of X. M. Li and L. Gao [11].

1. Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [15] and [16]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM (counting multiplicities), provided that $f-a$ and $g-a$ have the same set of zeros with the same multiplicities. Similarly, we say that f and g share a IM (ignoring multiplicities), provided that $f-a$ and $g-a$ have the same set of zeros ignoring multiplicities.

In 1959, W. K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem:

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Corresponding to Theorem A, C. C. Yang and X. H. Hua [14] proved the following result.

Theorem B. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$,*

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$g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

In 2000, M. L. Fang [4] proved the following result:

Theorem C. *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

Corresponding to Theorem C, the following result was proved by M. L. Fang and H. L. Qiu [5].

Theorem D. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1c_2)^{n+1}c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2003, W. Bergweiler and X.C. Pang [3] proved the following result:

Theorem E. *Let f be a transcendental meromorphic function, and let $R \not\equiv 0$ be a rational function. If all zeros and poles of f are multiple, except possibly finitely many, then $f' - R = 0$ has infinitely many solutions.*

The question arises:

Question 1. Is there exist a uniqueness theorem corresponding to Theorem E, similar to Theorems B and D?

Recently X. M. Li and L. Gao [11] proved the following uniqueness theorems that deals with Question 1.

Theorem F. *Let f and g be two transcendental meromorphic functions, let $n \geq 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)d\eta$.*

Theorem G. *Let f and g be two transcendental meromorphic functions, let $n \geq 15$ be a positive integer, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1))' - P$ and $(g^n(g-1))' - P$ share 0 CM and $\Theta(\infty, f) > 2/n$, then $f = g$.*

However questions arise in one's mind which are the motive of the author.

Question 2. Is it possible to obtain the similar result as in Theorems F and G if the sharing value is relaxed from CM to IM ?

Question 3. What happened if one consider kth derivative instead of first in Theorems F and G?

Considering k th derivative, recently the present author [12] proved the following theorem.

Theorem H. *Let f and g be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be three integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share the value 1 IM. Then one of the following holds:*
(i) when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > \frac{2}{n}$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$;
(ii) when $m \geq 2$ and $n > 9k+4m+16$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^n(x-1)^m - y^n(y-1)^m.$$

The possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ does not arise for $k = 1$.

So it is natural to ask the question:

Question 4. What can be said if we replace the sharing value 1 in the above theorem by a nonzero polynomial?

In the paper, we shall try to find out the possible solution of the above three questions. We will prove two theorems of which second one will not only improve and generalize Theorem H and at the same time provide a supplementary and generalize result of Theorem G. Our first theorem will supplement and generalize Theorem F. The following theorems are the main results of the paper.

Theorem 1. *Let f and g be two transcendental meromorphic functions, let n, k be two positive integers such that $n \geq 9k + 15$, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq n - 1$. Let $(f^n)^{(k)} - P$ and $(g^n)^{(k)} - P$ share 0 IM. Then*
(i) if $k = 1$, either $f = tg$ for a complex number t satisfying $t^n = 1$ or $f = c_1e^{cQ}$ and $g = c_2e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^nc^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)d\eta$;
(ii) if $k \geq 2$, either $(f^n)^{(k)}(g^n)^{(k)} = P^2$ or $f = tg$ for a complex number t satisfying $t^n = 1$.

Theorem 2. *Let f and g be two transcendental meromorphic functions, let n, m, k be three positive integers, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1)^m)^{(k)} - P$ and $(g^n(g-1)^m)^{(k)} - P$ share 0 IM, then each of the following holds:*
(i) when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$;
(ii) when $m \geq 2$ and $n > 9k+4m+16$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^n(x-1)^m - y^n(y-1)^m.$$

The possibility $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ does not arise for $k = 1$.

We now explain some definitions and notations which are used in the paper.

Definition 1([9]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r, a; f \mid\leq$

p) the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f\mid\leq p)$ we denote the corresponding reduced counting function.

Analogously we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.

Definition 2([8]). Let p be a positive integer or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$. Then

$$N_p(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+...+\overline{N}(r,a;f\mid\geq p).$$

Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 3. For $a\in\mathbb{C}\cup\{\infty\}$ we define

$$\delta_p(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N_p(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)},$$

where p is an arbitrary nonnegative integer.

Remark 1. From the above definition it is clear that

$$0\leq\delta_p(a,f)\leq\delta_{p-1}(a,f)\leq\delta_1(a,f)\leq\Theta(a,f)\leq 1.$$

Definition 4([1, 2])**.** Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g, where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g, where $p=q=1$, by $\overline{N}_E^{(2}(r,1;f)$ the reduced counting function of those 1-points of f and g, where $p=q\geq 2$. Similarly we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1([13]). *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f)=a_nf^n(z)+a_{n-1}f^{n-1}(z)+...+a_1f(z)+a_0,$$

where $a_n(\neq 0)$, a_{n-1}, ... , a_1, a_0 are complex numbers. Then

$$T(r,P_n(f))=nT(r,f)+O(1).$$

Lemma 2([17]). *Let f and g be two nonconstant meromorphic functions, and let p, k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3([7, 15])**.** *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 4([7]). *Let f be a nonconstant meromorphic function, k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\le \overline{N}(r,\infty;f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\le \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \ne 0$.

Lemma 5. *Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share 0 IM, where k is a positive integer, $P \not\equiv 0$ is a polynomial. If*

$$\begin{aligned} \Delta_1 &= (2k+4)\Theta(\infty,f) + (2k+3)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ &\quad +3\delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4k+13 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \Delta_2 &= (2k+4)\Theta(\infty,g) + (2k+3)\Theta(\infty,f) + \Theta(0,g) + \Theta(0,f) \\ &\quad +3\delta_{k+1}(0,g) + 2\delta_{k+1}(0,f) > 4k+13, \end{aligned} \tag{2.2}$$

then either $f^{(k)}g^{(k)} = P^2$ or $f = g$.

Proof. Since f and g are two transcendental meromorphic functions, $f^{(k)}$ and $g^{(k)}$ are also two transcendental meromorphic functions. Let

$$F = \frac{f^{(k)}}{P}, \quad G = \frac{g^{(k)}}{P},$$

and let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.3}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a common simple zero of $f^{(k)} - P$ and $g^{(k)} - P$. Then z_0 is a common simple zero of $F - 1$ and $G - 1$. Substituting their Taylor series at z_0 into (2.3), we see that z_0 is a zero of H. Thus we have

$$\begin{aligned} N_E^{1)}(r, 1; F) &\leq N(r, 0; H) \leq T(r, H) + O(1) \\ &\leq N(r, \infty; H) + S(r, F) + S(r, G). \end{aligned} \tag{2.4}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a pole of H. Then from (2.3) we can see that H have poles only at the zeros of F' and G', 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G, and poles of f and g. Hence we have

$$\begin{aligned} N(r, \infty; H) \leq\ & \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_L(r, 1; F) \\ & + \overline{N}_L(r, 1; G) + N_0(r, 0; F') + N_0(r, 0; G') + O(\log r), \end{aligned} \tag{2.5}$$

where $N_0(r, 0; F')$ denotes the counting function of those zeros of F' which are not the zeros of $f(F - 1)$, $N_0(r, 0; G')$ is similarly defined. Since f is a transcendental meromorphic functions we have

$$T(r, P) = o\{T(r, f)\}. \tag{2.6}$$

By Lemma 4, we have

$$\begin{aligned} T(r, f) \leq\ & \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}(r, 1; F) \\ & - N_0(r, 0; F') + S(r, f). \end{aligned} \tag{2.7}$$

Similarly

$$\begin{aligned} T(r, g) \leq\ & \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g) + \overline{N}(r, 1; G) \\ & - N_0(r, 0; G') + S(r, g). \end{aligned} \tag{2.8}$$

Since $f^{(k)} - P$ and $g^{(k)} - P$ share 0 IM, therefore using (2.4) and (2.5) we obtain

$$\begin{aligned} \overline{N}(r, 1; F) + \overline{N}(r, 1; G) =\ & 2N_E^{1)}(r, 1; F) + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) \\ & + 2\overline{N}_E^{(2}(r, 1; F) \\ \leq\ & N_E^{1)}(r, 1; F) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \\ & + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + 3\overline{N}_L(r, 1; F) \\ & + 3\overline{N}_L(r, 1; G) + N_0(r, 0; F') + N_0(r, 0; G') \\ & + 2\overline{N}_E^{(2}(r, 1; F) + S(r, f) + S(r, g). \end{aligned} \tag{2.9}$$

Obviously

$$\begin{aligned} & N_E^{1)}(r, 1; F) + 2\overline{N}_E^{(2}(r, 1; F) + \overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) \\ & \leq N(r, 1; G) + S(r, f) + S(r, g) \\ & \leq T(r, G) + S(r, f) + S(r, g) \\ & \leq T(r, g) + k\overline{N}(r, \infty; g) + S(r, f) + S(r, g). \end{aligned} \tag{2.10}$$

Also by Lemma 3 we have

$$\begin{aligned}(2.11)\qquad \overline{N}_L(r,1;F) &\le N(r,1;F)-\overline{N}(r,1;F)\\ &\le N\left(r,\infty;\frac{F}{F'}\right)\\ &\le N\left(r,\infty;\frac{F'}{F}\right)+S(r,f)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+S(r,f)\\ &\le N_{k+1}(r,0;f)+(k+1)\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Similarly,

$$(2.12)\qquad \overline{N}_L(r,1;G)\le N_{k+1}(r,0;g)+(k+1)\overline{N}(r,\infty;g)+S(r,g).$$

From (2.7) - (2.12), we obtain

$$\begin{aligned}(2.13)\quad T(r,f) \le{}& (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\ &+3N_{k+1}(r,0;f)+2N_{k+1}(r,0;g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly,

$$\begin{aligned}(2.14)\quad T(r,g) \le{}& (2k+4)\overline{N}(r,\infty;g)+(2k+3)\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,0;f)\\ &+3N_{k+1}(r,0;g)+2N_{k+1}(r,0;f)+S(r,f)+S(r,g).\end{aligned}$$

Suppose that there exists a subset $I\subseteq R^+$ satisfying $mesI=\infty$ such that $T(r,g)\le T(r,f)$, $r\in I$. Hence from (2.13) we have

$$\begin{aligned}\Delta_1 ={}& (2k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)\\ &+3\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)\le 4k+13,\end{aligned}$$

contradicting (2.1). Similarly if there exists a subset $I\subseteq R^+$ satisfying $mesI=\infty$ such that $T(r,f)\le T(r,g)$, $r\in I$, from (2.14) we can obtain

$$\begin{aligned}\Delta_2 ={}& (2k+4)\Theta(\infty,g)+(2k+3)\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)\\ &+3\delta_{k+1}(0,g)+2\delta_{k+1}(0,f)\le 4k+13,\end{aligned}$$

contradicting (2.2). We now assume that $H=0$. That is

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$(2.15)\qquad \frac{1}{F-1}=\frac{A}{G-1}+B,$$

where $A(\ne 0)$ and B are finite complex constants. We now discuss the following three cases.

Case 1. Let $B \neq 0$ and $A = B$. If $B = -1$, we obtain from (2.15) $FG = 1$, i.e., $f^{(k)}g^{(k)} = P^2$. If $B \neq -1$, from (2.15) we get

$$\frac{1}{F} = \frac{BG}{(1+B)G-1} \textit{ and } G = \frac{-1}{b(F - \frac{1+B}{B})}.$$

So by Lemma 2 we obtain

$$\text{(2.16)} \qquad \overline{N}\left(r, \frac{1}{1+B}; G\right) \leq \overline{N}(r,0;F) \leq N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) + O(\log r) + S(r,f)$$

and

$$\text{(2.17)} \qquad \overline{N}\left(r, \frac{1+B}{B}; F\right) \leq \overline{N}(r,\infty;g) + O(\log r).$$

Using Lemma 4, (2.16) and (2.17) we obtain

$$\text{(2.18)} \qquad \begin{aligned} T(r,g) &\leq N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;g) \\ &\quad -N_0(r,0;G') + S(r,g) \\ &\leq N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) \\ &\quad +\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \end{aligned}$$

and

$$\text{(2.19)} \qquad \begin{aligned} T(r,f) &\leq N_{k+1}(r,0;f) + \overline{N}\left(r, \frac{1+B}{B}; F\right) + \overline{N}(r,\infty;f) \\ &\quad -N_0(r,0;F') + S(r,f) \\ &\leq N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f). \end{aligned}$$

Suppose that there exists a subset $I \subseteq R^+$ satisfying $mesI = \infty$ such that $T(r,f) \leq T(r,g)$, $r \in I$. So from (2.18) we obtain

$$k\Theta(\infty,f) + \Theta(\infty,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) \leq k+2,$$

which by (2.1) gives

$$\begin{aligned}(k+4)\Theta(\infty,f) + (2k+2)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ +2\delta_{k+1}(0,f) + \delta_{k+1}(0,g) > 3k+11,\end{aligned}$$

a contradiction together with Remark 1. If there exists a subset $I \subseteq R^+$ satisfying $mesI = \infty$ such that $T(r,g) \leq T(r,f)$, $r \in I$, by the same argument we obtain a contradiction from (2.1) and (2.19).

Case 2. Let $B \neq 0$ and $A \neq B$. If $B = -1$, from (2.15) we obtain $F = \frac{A}{-(G-(a+1))}$. If $B \neq -1$, from (2.15) we obtain $F - \frac{1+B}{B} = \frac{-A}{B^2(G+\frac{A-B}{B})}$. Using the same argument as in case 1 we obtain a contradiction in both the cases.

Case 3. Let $B = 0$. Then from (2.15) we get

$$g = Af + (1-A)P_1, \tag{2.20}$$

where P_1 is a polynomial of degree $\gamma_{P_1} \geq k$. If $A \neq 1$, by Lemma 3 and (2.20) we get

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,(1-A)P_1;g) + S(r,g) \\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + S(r,g). \end{aligned} \tag{2.21}$$

Since f and g are transcendental meromorphic function, from (2.20) we have

$$T(r,f) = T(r,g) + O(\log r).$$

So from (2.21), we obtain

$$\Theta(0,f) + \Theta(0,g) + \Theta(\infty,g) \leq 2,$$

which gives by (2.1)

$$(2k+4)\Theta(\infty,f) + (2k+2)\Theta(\infty,g) + 3\delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4k+11,$$

a contradiction together with Remark 1. Thus $A = 1$ and so $f = g$. This proves the lemma. □

Lemma 6. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b) \equiv g^n(ag+b)$$

implies $f \equiv g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [10]. □

Lemma 7([11]). *Let f and g be two transcendental meromorphic functions, let $n \geq 2$ be a positive integer, and let P be a nonconstant polynomial with its degree $\gamma_P \leq n$. If $f^n f' g^n g' = P^2$, then f and g are expressed as $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$ respectively, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)d\eta$.*

Lemma 8. *Let f and g be two transcendental meromorphic functions, let n, m be two positive integers and let P be a nonconstant polynomial. If $m = 1$, $n \geq 6$ or if $m \geq 2$, $n \geq m + 3$, then*

$$(f^n(f-1)^m)'(g^n(g-1)^m)' \neq P^2.$$

Proof. If possible, let

$$(f^n(f-1)^m)'(g^n(g-1)^m)' = P^2. \tag{2.22}$$

We discuss the following two cases.
Case 1. Let $m \geq 2$. Then from (2.22) we obtain

$$f^{n-1}(f-1)^{m-1}(cf-d)f'g^{n-1}(g-1)^{m-1}(cg-d)g' = P^2, \tag{2.23}$$

where $c = n + m$ and $d = n$.
Let $z_0 \notin \{z : P(z) = 0\}$ be a 1-point of f with multiplicity $p_0(\geq 1)$. Then from (2.23) it follows that z_0 is a pole of g. Suppose that z_0 is a pole of g of order $q_0(\geq 1)$. Then we have $mp_0 - 1 = (n+m)q_0 + 1$, i.e., $mp_0 = (n+m)q_0 + 2 \geq n + m + 2$, and so

$$p_0 \geq \frac{n+m+2}{m}.$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $cf - d$ with multiplicity $p_1(\geq 1)$. Then from (2.23) it follows that z_1 is a pole of g. Suppose that z_1 is a pole of g of order $q_1(\geq 1)$. Then we have $2p_1 - 1 = (n+m)q_1 + 1$, and so

$$p_1 \geq \frac{n+m+2}{2}.$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of f with multiplicity $p_2(\geq 1)$. Then it follows from (2.23) that z_2 is a pole of g. Suppose that z_2 is a pole of g of order $q_2(\geq 1)$. Then we have

$$np_2 - 1 = (n+m)q_2 + 1. \tag{2.24}$$

From (2.24) we get $mq_2 + 2 = n(p_2 - q_2) \geq n$, i.e., $q_2 \geq \frac{n-2}{m}$. Thus from (2.24) we obtain $np_2 = (n+m)q_2 + 2 \geq \frac{(n+m)(n-2)}{m} + 2$, and so

$$p_2 \geq \frac{n+m-2}{m}.$$

Let $z_3 \notin \{z : P(z) = 0\}$ be a pole of f. Then it follows from (2.23) that z_3 is a zero of $g(g-1)(cg-d)$ or a zero of g'. So we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\le \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}\left(r,\frac{d}{c};g\right)+\overline{N}_0(r,0;g')\\ &\quad +S(r,f)+S(r,g)\\ &\le \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)T(r,g)+\overline{N}_0(r,0;g')\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)(cg-d)$.
By the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}(2.25)\quad 2T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}\left(r,\frac{d}{c};f\right)+\overline{N}(r,\infty;f)\\ &\quad -\overline{N}_0(r,0;f')+S(r,f)\\ &\le \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0(r,0;f')\\ &\quad +\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}$$

Similarly,

$$\begin{aligned}(2.26)\quad 2T(r,g) &\le \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')\\ &\quad -\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}$$

Adding (2.25) and (2.26) we obtain

$$\left(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

contradicting the fact that $n\ge m+3$.
Case 2. Let $m=1$. Then from (2.22) we obtain

$$f^{n-1}(af-b)f'g^{n-1}(ag-b)g'=P^2, \tag{2.27}$$

where $a=n+1$ and $b=n$.
Let $z_4\notin\{z:P(z)=0\}$ be a pole of f. Then it follows from (2.27) that z_4 is a zero of $g(ag-b)$ or a zero of g'. Then proceeding in a manner similar to Case 1 we obtain

$$\left(1-\frac{2}{n-1}-\frac{4}{n+3}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

which contradicts the fact that $n\ge 6$. This proves the lemma. □

3. Proof of the Theorems

Proof of Theorem 1. We consider $F_1 = f^n$ and $G_1 = g^n$. Then we see that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share the value 0 IM. Using Lemma 1, we have

$$\begin{aligned}\Theta(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F_1)}{T(r, F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f)}{nT(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\ &= \frac{n-1}{n}.\end{aligned} \tag{3.1}$$

Similarly,

$$\Theta(0, G_1) \geq \frac{n-1}{n}. \tag{3.2}$$

$$\begin{aligned}\Theta(\infty, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; F_1)}{T(r, F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f)}{nT(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\ &= \frac{n-1}{n}.\end{aligned} \tag{3.3}$$

Similarly,

$$\Theta(\infty, G_1) \geq \frac{n-1}{n}. \tag{3.4}$$

$$\begin{aligned}\delta_{k+1}(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; F_1)}{T(r, F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; f^n)}{nT(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(k+1)T(r, f)}{nT(r, f)} \\ &= \frac{n-k-1}{n}.\end{aligned} \tag{3.5}$$

Similarly,

$$\delta_{k+1}(0, G_1) \geq \frac{n-k-1}{n}. \tag{3.6}$$

Using (2.1), (2.2) and (3.1)-(3.6) we obtain

$$\Delta_1 \geq (4k+14) - \frac{9k+14}{n} \quad \textit{and} \quad \Delta_2 \geq (4k+14) - \frac{9k+14}{n}.$$

Since $n \geq 9k+15$, we get $\Delta_1 > 4k+13$ and $\Delta_2 > 4k+13$. So by Lemma 5 we obtain either $F_1^{(k)} G_1^{(k)} = P^2$ or $F_1 = G_1$. Suppose that $F_1^{(k)} G_1^{(k)} = P^2$, i.e.,

$$(f^n)^{(k)} (g^n)^{(k)} = P^2. \tag{3.7}$$

If $k = 1$, then from (3.7) we have $f^{n-1} f' g^{n-1} g' = P^2/n^2$. Applying Lemma 7 we obtain $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^n c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.

If $F_1 = G_1$, then $f = tg$ for a complex number t such that $t^n = 1$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $F_2 = f^n (f-1)^m$ and $G_2 = g^n (g-1)^m$. Then $F_2^{(k)} - P$ and $G_2^{(k)} - P$ share the value 0 IM. Using Lemma 1, we obtain

$$\begin{aligned} \Theta(0, F_2) &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 0; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 0; f^n (f-1)^m)}{(n+m) T(r, f)} \\ &\geq 1 - \limsup_{r \to \infty} \frac{2T(r, f)}{(n+m) T(r, f)} \\ &\geq \frac{n+m-2}{n+m}. \end{aligned} \tag{3.8}$$

Similarly,

$$\Theta(0, G_2) \geq \frac{n+m-2}{n+m}. \tag{3.9}$$

$$\begin{aligned} \Theta(\infty, F_2) &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; f)}{(n+m) T(r, f)} \\ &\geq 1 - \limsup_{r \to \infty} \frac{T(r, f)}{(n+m) T(r, f)} \\ &\geq \frac{n+m-1}{n+m}. \end{aligned} \tag{3.10}$$

Similarly,

$$\Theta(\infty; G_2) \geq \frac{n+m-1}{n+m}. \tag{3.11}$$

$$\begin{aligned} \delta_{k+1}(0, F_2) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; f^n(f-1)^m)}{(n+m)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(k+m+1)T(r, f)}{(n+m)T(r, f)} \\ &\geq \frac{n-k-1}{n+m}. \end{aligned} \tag{3.12}$$

Similarly,

$$\delta_{k+1}(0, G_2) \geq \frac{n-k-1}{n+m}. \tag{3.13}$$

Using (2.1), (2.2) and (3.8)-(3.13) we obtain

$$\Delta_1 \geq (4k+9) + \frac{5n-9k-16}{n+m} \quad \textit{and} \quad \Delta_2 \geq (4k+9) + \frac{5n-9k-16}{n+m}.$$

Since $n \geq 9k + 4m + 17$, we get $\Delta_1 > 4k + 13$ and $\Delta_2 > 4k + 13$. So by Lemma 5, either $F_2^{(k)} G_2^{(k)} = P^2$ or $F_2 = G_2$ holds. Suppose that $F_2^{(k)} G_2^{(k)} = P^2$. Then

$$(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2. \tag{3.14}$$

Also by Lemma 8, (3.14) does not occur when $k = 1$.

Next we suppose that $F_2 = G_2$, i.e.,

$$f^n(f-1)^m = g^n(g-1)^m. \tag{3.15}$$

Let $m = 1$. Then in view of Lemma 6 and (3.15) we obtain $f = g$.
Let $m \geq 2$. Then from (3.15) we obtain

$$\begin{aligned} f^n[f^m + ... + (-1)^i \ {}^mC_i f^{m-i} + ... + (-1)^m] = g^n[g^m \\ + ... + (-1)^i \ {}^mC_i g^{m-i} + ... + (-1)^m]. \end{aligned} \tag{3.16}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.16) we obtain

$$\begin{aligned} g^{n+m}(h^{n+m} - 1) + ... + (-1)^i \ {}^mC_i g^{n+m-i}(h^{n+m-i} - 1) \\ + ... + (-1)^m g^n(h^n - 1) = 0, \end{aligned}$$

which implies $h = 1$. Hence $f = g$.

If h is not a constant, then from (3.15) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(x,y)=x^n(x-1)^m-y^n(y-1)^m.$$

This completes the proof of Theorem 2. □

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