

## Meromorphic Functions Sharing a Small Function with Their Derivatives

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ABSTRACT. In this paper, we investigate uniqueness problems of meromorphic functions that share a small function with one of their derivatives, and give some results to improve some previous results.

### 1. Introduction and results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions such as  $T(r, f)$ ,  $N(r, f)$ ,  $m(r, f)$  (see e.g., [5], [8]). For any nonconstant meromorphic function  $f$ , we denote by  $S(r, f)$  any quantity satisfying

$$\lim_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)} = 0,$$

possibly outside of a set of finite linear measure in  $R_+$ . A meromorphic function  $a(z)$  is said to be a small function of  $f$ , provided  $T(r, a) = S(r, f)$ .

We say that two meromorphic functions  $f$  and  $g$  share a small function  $a$  IM (ignoring multiplicities) when  $f - a$  and  $g - a$  have the same zeros. If  $f - a$  and  $g - a$  have the same zeros with the same multiplicities, then we say that  $f$  and  $g$  share  $a$  CM (counting multiplicities).

The uniqueness theory of entire and meromorphic functions has grown up to an extensive subfield of the value distribution theory, see e.g. the monograph [8] by Yang and Yi. A widely studied subtopic of the uniqueness theory has been to considering shared value problems relative to a meromorphic function  $f$  and its derivative  $f^{(k)}$ . Some of the basic papers in this direction are due to Rubel and Yang [7], Gundersen [3], Mues and Steinmetz [6] and Yang [9].

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Received 22 January 2007; accepted 17 March 2008.

2000 Mathematics Subject Classification: 30D35, 30D20.

Key words and phrases: uniqueness theorems, shared value, small function.

This work was supported by the NNSF of China(No.10671109). This work is also part of a research project supported by the Academy of Finland (grant 210245).

Recently, L. Z. Yang and the present author [10] considered value sharing relative to a power of a meromorphic function  $F = f^n$  and its derivative  $F'$ , proving the following 2 theorems.

**Theorem A.** *Let  $f$  be a nonconstant entire function,  $n \geq 7$  be an integer. Denote  $F = f^n$ . If  $F$  and  $F'$  share 1 CM, then  $F = F'$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where  $c$  is a nonzero constant.

**Theorem B.** *Let  $f$  be a nonconstant meromorphic function and  $n \geq 12$  be an integer. Denote  $F = f^n$ . If  $F$  and  $F'$  share 1 CM, then  $F = F'$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where  $c$  is a nonzero constant.

In this paper, we improve Theorem A and B by obtaining the following results.

**Theorem 1.1.** *Let  $f$  be a nonconstant entire function,  $n, k$  be positive integers and  $a(z)$  be a small meromorphic function of  $f$  such that  $a(z) \not\equiv 0, \infty$ . If  $f^n - a$  and  $(f^n)^{(k)} - a$  share the value 0 CM and  $n > k + 4$ , then  $f^n = (f^n)^{(k)}$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

**Theorem 1.2.** *Let  $f$  be a nonconstant meromorphic function,  $k, n (\geq k)$  be positive integers and  $a(z)$  be a small meromorphic function of  $f$  such that  $a(z) \not\equiv 0, \infty$ . If  $f^n - a$  and  $(f^n)^{(k)} - a$  share the value 0 CM and*

$$(1.1) \quad (n - k - 1)(n - k - 4) > 3k + 6,$$

then  $f^n = (f^n)^{(k)}$ , and  $f$  assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

**Corollary 1.3.** *Let  $f$  be a nonconstant entire function and  $n \geq 6$  be an integer. Denote  $F = f^n$ . If  $F$  and  $F'$  share 1 CM, then  $F = F'$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where  $c$  is a nonzero constant.

**Corollary 1.4.** *Let  $f$  be a nonconstant meromorphic function and  $n \geq 7$  be an*

integer. Denote  $F = f^n$ . If  $F$  and  $F'$  share 1 CM, then  $F = F'$ , and  $f$  assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant.

**Remark.** Obviously, Corollary 1.3 and Corollary 1.4 improve Theorem A and Theorem B respectively.

For any  $a \in \mathbf{C} \cup \{\infty\}$ , we denote by  $E_l(a, f)$  the set of  $a$ -points of  $f$  with the multiplicity  $m \leq l$ , counting multiplicities.

Obviously, if  $E_l(a, f) = E_l(a, g)$  and  $l = \infty$ , then  $f$  and  $g$  share  $a$  CM. It is natural to ask what happens if  $F - a$  and  $F' - a$  share 0 CM is replaced by  $E_l(0, F - a) = E_l(0, F' - a)$  in Theorem A and B? Corresponding to this question, we obtain the following results.

**Theorem 1.5.** *Let  $f$  be a nonconstant entire function,  $n, k$  be positive integers and  $a(z)$  be a small meromorphic function of  $f$  such that  $a(z) \not\equiv 0, \infty$ . If  $E_3(0, f^n - a) = E_3(0, (f^n)^{(k)} - a)$  and  $n > k + 4$ , then  $f^n = (f^n)^{(k)}$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

From Theorem 1.5, we can easily get Theorem 1.1.

**Theorem 1.6.** *Let  $f$  be a nonconstant meromorphic function,  $n, k$  be positive integers and  $a(z)$  be a small meromorphic function of  $f$  such that  $a(z) \not\equiv 0, \infty$ . If  $E_3(0, f^n - a) = E_3(0, (f^n)^{(k)} - a)$  and*

$$(1.2) \quad n \geq \begin{cases} 8 & \text{if } k = 1, \\ 10 & \text{if } k = 2, \\ \left\lfloor \frac{3k}{2} \right\rfloor + 8 & \text{if } k \geq 3, \end{cases}$$

then  $f^n = (f^n)^{(k)}$ , and  $f$  assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

## 2. Some lemmas

Let  $F$  and  $G$  be two non-constant meromorphic functions. We denote by  $N_E^{(1)}(r, \frac{1}{F-1})$  the counting function of common simple 1-points of  $F$  and  $G$ .

**Lemma 2.1**([11], Lemma 3). *Let*

$$(2.1) \quad H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where  $F$  and  $G$  are two nonconstant meromorphic functions. If  $H \neq 0$ , then

$$(2.2) \quad N_E^1\left(r, \frac{1}{F-1}\right) \leq N(r, H) + S(r, F) + S(r, G).$$

Let  $p$  be a positive integer and  $a \in \mathbf{C} \cup \{\infty\}$ . We denote by  $N_p\left(r, \frac{1}{f-a}\right)$  the counting function of the zeros of  $f-a$  with the multiplicities less than or equal to  $p$ , and by  $N_{(p+1)}\left(r, \frac{1}{f-a}\right)$  the counting function of the zeros of  $f-a$  with the multiplicities larger than  $p$ . And we use  $\bar{N}_p\left(r, \frac{1}{f-a}\right)$  and  $\bar{N}_{(p+1)}\left(r, \frac{1}{f-a}\right)$  to denote the corresponding reduced counting functions (ignoring multiplicities). However,  $N_p\left(r, \frac{1}{f-a}\right)$  denotes the counting function of the zeros of  $f-a$  where  $m$ -fold zeros are counted  $m$  times if  $m \leq p$  and  $p$  times if  $m > p$ .

**Lemma 2.2** ([12], Lemma 3). *Suppose that  $f$  is a nonconstant meromorphic function and  $k, p$  are positive integers. Then*

$$(2.3) \quad N_p\left(r, 1/f^{(k)}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 1/f) + S(r, f),$$

$$(2.4) \quad N_p\left(r, 1/f^{(k)}\right) \leq k\bar{N}(r, f) + N_{p+k}(r, 1/f) + S(r, f).$$

**Lemma 2.3.** *Suppose that  $f$  is a nonconstant meromorphic function and  $a$  is a small meromorphic function of  $f$  such that  $a(z) \neq 0, \infty$ . Let*

$$(2.5) \quad V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right),$$

where  $F = \frac{f^n}{a}$ ,  $G = \frac{(f^n)^{(k)}}{a}$  and  $n, k$  are positive integers. If  $V = 0$  and  $n \geq 2$ , then  $F = G$ .

*Proof.* From  $V = 0$ , we get

$$(2.6) \quad 1 - \frac{1}{F} = B - \frac{B}{G},$$

where  $B$  is a non-zero constant. We discuss the following two cases.

Case 1. Suppose that the counting function of poles of  $f$  is not  $S(r, f)$ . Then there exists a  $z_0$  which is not a zero or pole of  $a$  such that  $\frac{1}{f(z_0)} = 0$ , thus  $\frac{1}{F(z_0)} = \frac{1}{G(z_0)} = 0$ . We get  $B = 1$  from (2.6).

Case 2. Suppose that the counting function of poles of  $f$  is  $S(r, f)$ . If  $B \neq 1$ , then

$N\left(r, \frac{1}{F - \frac{1}{1-B}}\right) = S(r, f)$ . From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F - \frac{1}{1-B}}\right) + S(r, F) \\ &\leq \bar{N}\left(r, \frac{1}{f}\right) + S(r, f), \end{aligned}$$

which is a contradiction since  $n \geq 2$ . Therefore  $B = 1$ . Thus  $F = G$ , completing the proof of Lemma 2.3.  $\square$

**Lemma 2.4.** *Let  $V$  be given by (2.5) and suppose that  $V \neq 0$ . Then the poles of  $f$  are the zeros of  $V$ , and*

$$(n-1)\bar{N}(r, f) \leq N(r, V) + S(r, f).$$

*Proof.* We get from (2.5) that

$$V = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Suppose that  $z_0$  is a pole of  $f$  with the multiplicity  $p$  such that  $a(z_0) \neq 0$  and  $a(z_0) \neq \infty$ . Then  $z_0$  is a zero of  $\frac{F'}{F(F-1)}$  with the multiplicity  $np-1$  and a zero of  $\frac{G'}{G(G-1)}$  with the multiplicity  $np+k-1$ . So  $z_0$  is zero of  $V$  with the multiplicity at least  $n-1$ . Noting that  $m(r, V) = S(r, f)$ , we have

$$(n-1)\bar{N}(r, f) \leq N\left(r, \frac{1}{V}\right) + S(r, f) \leq T(r, V) + S(r, f) \leq N(r, V) + S(r, f).$$

$\square$

**Lemma 2.5.** *Let  $H$  be given by (2.1), where  $F$  and  $G$  are given by Lemma 2.3. If  $H = 0$  and  $n > k + 2$ , then  $F = G$ , and  $f$  assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

*Proof.* By integration, we get from (2.1) that

$$(2.7) \quad \frac{1}{F-1} = \frac{A}{G-1} + B,$$

where  $A(\neq 0)$  and  $B$  are constants. From (2.7) we have

$$(2.8) \quad N(r, F) = N(r, G) = N(r, f) = S(r, f),$$

and

$$(2.9) \quad F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}, \quad G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}.$$

We discuss the following three cases.

Case 1. Suppose that  $B \neq 0, -1$ . From (2.9) we have  $\bar{N}(r, 1/(F - \frac{B+1}{B})) = \bar{N}(r, G)$ . From (2.8) and the second fundamental theorem, we have

$$\begin{aligned} nT(r, f) &\leq T(r, F) + S(r, f) \\ &\leq \bar{N}(r, F) + \bar{N}(r, 1/F) + \bar{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r, f) \\ &\leq \bar{N}(r, 1/f) + \bar{N}(r, F) + \bar{N}(r, G) + S(r, f) \\ &\leq T(r, f) + S(r, f), \end{aligned}$$

which contradicts the assumption  $n \geq 2$ .

Case 2. Suppose that  $B = 0$ . From (2.9) we have

$$(2.10) \quad F = \frac{G + (A-1)}{A}, \quad G = AF - (A-1).$$

If  $A \neq 1$ , from (2.10) we obtain  $\bar{N}(r, 1/(F - \frac{A-1}{A})) = \bar{N}(r, 1/G)$ . By (2.4), (2.8) and the second fundamental theorem, we have

$$\begin{aligned} nT(r, f) &\leq T(r, F) + S(r, f) \\ &\leq \bar{N}(r, F) + \bar{N}(r, 1/F) + \bar{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r, f) \\ &\leq \bar{N}(r, 1/f) + \bar{N}(r, F) + \bar{N}(r, 1/G) + S(r, f) \\ &\leq \bar{N}(r, 1/f) + N_1(r, 1/G) + S(r, f) \\ &\leq (k+2)\bar{N}(r, 1/f) + S(r, f) \\ &\leq (k+2)T(r, f) + S(r, f), \end{aligned}$$

which contradicts the assumption  $n > k+2$ . Thus  $A = 1$ . From (2.10) we have  $F = G$ , then

$$(2.11) \quad f^n = (f^n)^{(k)}.$$

We claim that 0 is a Picard exceptional value of  $f$ . In fact, if  $z_0$  is a zero of  $f$  with the multiplicity  $p$ , then  $z_0$  is a zero of  $f^n$  with the multiplicity  $np$  and a zero of  $(f^n)^{(k)}$  with the multiplicity  $np - k$ , which is impossible from (2.11). Then from (2.11), we have

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where  $c$  is a nonzero constant and  $\lambda^k = 1$ .

Case 3. Suppose that  $B = -1$ . From (2.9) we have

$$(2.12) \quad F = \frac{A}{-G + (A + 1)}, \quad G = \frac{(A + 1)F - A}{F}.$$

If  $A \neq -1$ , we obtain from (2.12) that  $\overline{N}\left(r, 1/\left(F - \frac{A}{A+1}\right)\right) = \overline{N}(r, 1/G)$ . By the same reasoning discussed in Case 2, we obtain a contradiction. Hence  $A = -1$ . From (2.12), we get  $F \cdot G = 1$ , that is

$$f^n \cdot (f^n)^{(k)} = a^2.$$

From above equation, we have

$$N\left(r, \frac{1}{f}\right) + N(r, f) = S(r, f),$$

and so  $T(r, \frac{f^{(k)}}{f}) = S(r, f)$ . From above two equations, we obtain

$$2T\left(r, \frac{f^n}{a}\right) = T\left(r, \frac{f^{2n}}{a^2}\right) = T\left(r, \frac{a^2}{f^{2n}}\right) + O(1) = T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + O(1) = S(r, f).$$

So  $T(r, f) = S(r, f)$ , which is impossible. This completes the proof of Lemma 2.5.  $\square$

### 3. Proofs of results

*Proof of Theorem 1.6.* Let

$$(3.1) \quad F = \frac{f^n}{a}, \quad G = \frac{(f^n)^{(k)}}{a}.$$

From the conditions of Theorem 1.6, we know that  $E_3(1, F) = E_3(1, G)$  possibly except at the zeros and poles of  $a(z)$ . From (3.1), we have

$$(3.2) \quad T(r, F) = n(T(r, f)) + S(r, f),$$

$$(3.3) \quad \overline{N}(r, F) = \overline{N}(r, G) + S(r, f) = \overline{N}(r, f) + S(r, f).$$

Let  $H$  be defined by (2.1). Suppose that  $H \neq 0$ . By Lemma 2.1 we know that (2.2) holds. From (2.1) and (3.3), we have

$$(3.4) \quad \begin{aligned} N(r, H) &\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) \\ &+ \overline{N}_{(4)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4)}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right), \end{aligned}$$

where  $N_0(r, \frac{1}{F'})$  denotes the counting function corresponding to the zeros of  $F'$  which are not the zeros of  $F$  and  $F - 1$ , and correspondingly for  $G'$ . From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) + T(r, G) &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) \\ (3.5) \quad &+ \bar{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

Noting that  $E_3(1, F) = E_3(1, G)$ , we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) \\ = 2N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(2)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(2)}\left(r, \frac{1}{G-1}\right). \end{aligned}$$

Combining with (2.2) and (3.4), we obtain

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) \\ \leq \bar{N}_{(2)}\left(r, \frac{1}{F}\right) + \bar{N}_{(2)}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) + \bar{N}_{(4)}\left(r, \frac{1}{F-1}\right) \\ + \bar{N}_{(4)}\left(r, \frac{1}{G-1}\right) + N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(2)}\left(r, \frac{1}{F-1}\right) \\ (3.6) \quad + \bar{N}_{(2)}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

It is easy to see that

$$\begin{aligned} \frac{1}{2}N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(2)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(4)}\left(r, \frac{1}{F-1}\right) \\ (3.7) \quad \leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2}T(r, F) + O(1), \end{aligned}$$

$$\begin{aligned} \frac{1}{2}N_E^{(1)}\left(r, \frac{1}{G-1}\right) + \bar{N}_{(2)}\left(r, \frac{1}{G-1}\right) + \bar{N}_{(4)}\left(r, \frac{1}{G-1}\right) \\ (3.8) \quad \leq \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \leq \frac{1}{2}T(r, G) + O(1). \end{aligned}$$

From (3.5) to (3.8) and (3.3), we have

$$\frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\bar{N}(r, f) + S(r, f).$$



Then

$$(3.9) \quad T(r, F) + T(r, G) \leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 6\bar{N}(r, f) + S(r, f).$$

From (3.1), (3.9) and by using Lemma 2.2, we have

$$\begin{aligned} 2T(r, F) &\leq 2N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_{2+k}\left(r, \frac{1}{f^n}\right) + 6\bar{N}(r, f) + S(r, f) \\ &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_{2+k}\left(r, \frac{1}{f^n}\right) + (6+k)\bar{N}(r, f) + S(r, f). \end{aligned}$$

Then

$$\begin{aligned} T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_{2+k}\left(r, \frac{1}{f^n}\right) + \left(3 + \frac{k}{2}\right)\bar{N}(r, f) + S(r, f) \\ &\leq (k+4)\bar{N}\left(r, \frac{1}{f}\right) + \left(3 + \frac{k}{2}\right)\bar{N}(r, f) + S(r, f). \end{aligned}$$

From (3.2) and above inequality, we get

$$(3.10) \quad nT(r, f) \leq (k+4)\bar{N}\left(r, \frac{1}{f}\right) + \left(3 + \frac{k}{2}\right)\bar{N}(r, f) + S(r, f).$$

We now divide the discussion in two cases:

Case 1. Suppose first that  $k \geq 3$ . We can get a contradiction from (1.2) and (3.10).

Case 2. Suppose next that  $k \leq 2$ . Let  $V$  be given by (2.5). If  $V = 0$ , we get  $F = G$  from Lemma 2.3. From the proof of Lemma 2.5, we obtain the conclusions of Theorem 1.6. Next, we suppose that  $V \neq 0$ . Since  $E_3(1, F) = E_3(1, G)$ , by Lemma 2.4 and (2.5), we obtain

$$\begin{aligned} (n-1)\bar{N}(r, f) &\leq N(r, V) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}_{(4)}\left(r, \frac{1}{F-1}\right) \\ (3.11) \quad &\quad + \bar{N}_{(4)}\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned}$$

Observe that

$$\begin{aligned} \bar{N}_{(4)}\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{3}N\left(r, \frac{F}{F'}\right) \leq \frac{1}{3}N\left(r, \frac{F'}{F}\right) + S(r, f) \\ &\leq \frac{1}{3}\bar{N}(r, 1/F) + \frac{1}{3}\bar{N}(r, F) + S(r, f), \end{aligned}$$

$$\begin{aligned} \bar{N}_{(4)}\left(r, \frac{1}{G-1}\right) &\leq \frac{1}{3}N\left(r, \frac{G}{G'}\right) \leq \frac{1}{3}N\left(r, \frac{G'}{G}\right) + S(r, f) \\ &\leq \frac{1}{3}\bar{N}(r, 1/G) + \frac{1}{3}\bar{N}(r, G) + S(r, f). \end{aligned}$$

From (3.11) and (2.4), we have

$$\begin{aligned}
 (n-1)\overline{N}(r, f) &\leq \frac{4}{3}\overline{N}(r, 1/F) + \frac{4}{3}\overline{N}(r, 1/G) + \frac{2}{3}\overline{N}(r, F) + S(r, f) \\
 &\leq \frac{4}{3}\overline{N}(r, 1/f) + \frac{4}{3}N_1(r, 1/G) + \frac{2}{3}\overline{N}(r, f) + S(r, f) \\
 &\leq \frac{4}{3}\overline{N}(r, 1/f) + \frac{4}{3}((k+1)\overline{N}(r, 1/f) + k\overline{N}(r, f)) + \frac{2}{3}\overline{N}(r, f) + S(r, f) \\
 &= \frac{4(k+2)}{3}\overline{N}(r, 1/f) + \frac{2(2k+1)}{3}\overline{N}(r, f) + S(r, f),
 \end{aligned}$$

and so

$$\left(n-1 - \frac{2(2k+1)}{3}\right)\overline{N}(r, f) \leq \frac{4(k+2)}{3}\overline{N}(r, 1/f) + S(r, f).$$

From (1.2), we can easily get  $n-1 - \frac{2(2k+1)}{3} > 0$ . From (3.10) and above inequality, we have

$$\begin{aligned}
 nT(r, f) &\leq \left(k+4 + \frac{(2k+12)(k+2)}{3n-4k-5}\right)\overline{N}(r, 1/f) + S(r, f) \\
 &\leq \left(k+4 + \frac{(2k+12)(k+2)}{3n-4k-5}\right)T(r, f) + S(r, f),
 \end{aligned}$$

which contradicts the assumption (1.2) of Theorem 1.6. Thus,  $H = 0$ . From (1.2), we have  $n > k + 2$ . By Lemma 2.5, we get the conclusions of Theorem 1.6. This completes the proof of Theorem 1.6.  $\square$

*Proof of Theorem 1.5.* The proof of Theorem 1.6 applies, since  $f$  is an entire function, we get from (3.10)

$$nT(r, f) \leq (k+4)\overline{N}(r, 1/f) + S(r, f),$$

which contradicts the assumption  $n > k + 4$ . Hence  $H = 0$ . By the same reasoning as in the proof of Theorem 1.6, we obtain the results of Theorem 1.5, and we complete the proof of Theorem 1.5.  $\square$

*Proof of Theorem 1.2.* The proof of Theorem 1.6 applies. Since  $f^n - a$  and  $(f^n)^{(k)} - a$  share the value 0 CM, then  $F$  and  $G$  share 1 CM except possibly at the zeros and poles of  $a(z)$ . We obtain

$$N\left(r, \frac{1}{F-1}\right) = N\left(r, \frac{1}{G-1}\right) + S(r, f),$$

and

(3.12)

$$N(r, H) \leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f).$$

So

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{F-1}\right) \\ (3.13) \qquad \qquad \qquad &\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + T(r, F) + O(1). \end{aligned}$$

From (2.2), (3.5), (3.12) and (3.13), we have

$$(3.14) \qquad T(r, G) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, f) + S(r, f).$$

By Lemma 2.2 and (3.14), we get

$$(3.15) \qquad T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_{2+k}\left(r, \frac{1}{f^n}\right) + 3\overline{N}(r, f) + S(r, f).$$

Let  $V$  be given by (2.5). If  $V = 0$ , we get  $F = G$  by Lemma 2.3. From Case 2 in the proof of Lemma 2.5, we obtain the conclusions of Theorem 1.2. Next, we suppose that  $V \neq 0$ . Since  $F$  and  $G$  share 1 CM except at the zeros and poles of  $a(z)$ , by Lemma 2.4 and Lemma 2.2, we obtain

$$\begin{aligned} (n-1)\overline{N}(r, f) &\leq N(r, V) + S(r, f) \\ &\leq \overline{N}(r, 1/F) + \overline{N}(r, 1/G) + S(r, f) \\ &\leq \overline{N}(r, 1/f) + (k+1)\overline{N}(r, 1/f) + k\overline{N}(r, f) + S(r, f), \end{aligned}$$

that is

$$(3.16) \qquad (n-k-1)\overline{N}(r, f) \leq (k+2)\overline{N}(r, 1/f) + S(r, f).$$

Since  $n \geq k$ , we get from (1.1)

$$(3.17) \qquad n > \frac{2k+5+\sqrt{12k+33}}{2} > k+4.$$

Combining with (3.15) and (3.16), we obtain

$$nT(r, f) \leq \left(k+4 + \frac{3k+6}{n-k-1}\right)\overline{N}(r, 1/f) + S(r, f),$$

which contradicts the assumption (1.1) of Theorem 1.2. Thus,  $H = 0$ . By Lemma 2.5 and (3.17), we obtain the conclusions of Theorem 1.2.  $\square$

**Acknowledgment.** The author would like to thank the referee for his/her valuable suggestions.

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