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MEROMORPHIC FUNCTIONS THAT SHARE ONE OR TWO VALUES II

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Abstract

In this paper, we deal with the problem of uniqueness of meromorphic functions that share one or two values IM, and obtain some results that are improvements of that of R. Nevanlinna, G. Brosch, H. X. Yi and other authors.

1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in $\boldsymbol{C}$. We adopt the usual notations in the Nevanlinna theory of meromorphic functions as explained in [1]. We use E to denote any set of finite linear measure of $0 < r < \infty$ and use I to denote any set of infinite linear measure of $0 < r < \infty$. Let f and g be two nonconstant meromorphic functions. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ $(r \to \infty, r \notin E)$. We say that f and g share the value $a \in \hat{\boldsymbol{C}}$ provided that $f(z) = a$ if and only if $g(z) = a$. We will state whether a shared value is by CM (counting multiplicities), or by IM (ignoring multiplicities) (see [2]).

Let h be a nonconstant meromorphic function. We denote by $N_{1)}(r, h)$ the counting function of simple poles of h, and by $\overline{N}_{(2}(r, h)$ the counting function of poles of h with multiplicities ≥ 2, each point in these counting functions is counted only once. Set (see [3])

$$N_2(r, h) = \overline{N}(r, h) + \overline{N}_{(2}(r, h).$$

We define

$$N^*(r, h) = 2N_2(r, h) + 3\overline{N}(r, h). \tag{1.1}$$

Recently, the present author dealt with the problem of uniqueness of meromorphic functions that share one or two values CM and proved the following results.

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THEOREM A (see [4]). *Let f and g be two nonconstant meromorphic functions such that f and g share the value* 1 *CM. If*

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{N_2(r,1/f)+N_2(r,f)+N_2(r,1/g)+N_2(r,g)}{T(r)}<1,$$

then $f\equiv g$ *or* $f\cdot g\equiv 1$.

THEOREM B (see [4]). *Let f and g be two nonconstant meromorphic functions such that f and g share the values* 1 *and* ∞ *CM. If*

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{N_2(r,1/f)+N_2(r,1/g)+2\overline{N}(r,f)}{T(r)}<1,$$

then $f\equiv g$ *or* $f\cdot g\equiv 1$.

In this paper, we shall deal with the problem of uniqueness of meromorphic functions that share one or two values IM.

R. Nevanlinna proved the following well-known theorem.

THEOREM C (see [5]). *Let f and g be two nonconstant meromorphic functions such that f and g share the value* 1 *IM. If* 0 *and* ∞ *are Picard values of f and g, then* $f\equiv g$ *or* $f\cdot g\equiv 1$.

G. Brosch proved the following theorem which is an improvement of Theorem C.

THEOREM D (see [6]). *Let f and g be two nonconstant meromorphic functions such that f and g share the value* 1 *IM. If*

$$\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)=S(r,f) \quad \textit{and} \quad \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)=S(r,g),$$

then $f\equiv g$ *or* $f\cdot g\equiv 1$.

In this paper we improve the above results and obtain the following theorems.

THEOREM 1. *Let f and g be two nonconstant meromorphic functions such that f and g share the value* 1 *IM. If*

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{N^*(r,1/f)+N^*(r,f)+N^*(r,1/g)+N^*(r,g)}{T(r,f)+T(r,g)}<1, \tag{1.2}$$

then $f\equiv g$ *or* $f\cdot g\equiv 1$.

Noting (1.1), by Theorem 1 we immediately obtain the following corollary.

Corollary 1. *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. If*

$$\Theta(0,f)+\Theta(\infty,f)>\frac{13}{7} \quad \textit{and} \quad \Theta(0,g)+\Theta(\infty,g)>\frac{13}{7},$$

then $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem 2. *Let f and g be two nonconstant meromorphic functions such that f and g share the values 1 and ∞ IM. If*

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{N^*(r,1/f)+N^*(r,1/g)+12\overline{N}(r,f)}{T(r,f)+T(r,g)}<1, \tag{1.3}$$

then $f\equiv g$ or $f\cdot g\equiv 1$.

Noting (1.1), from Theorem 2 we immediately deduce the following corollary.

Corollary 2. *Let f and g be two nonconstant meromorphic functions such that f and g share the values 1 and ∞ IM. If*

$$7\Theta(0,f)+6\Theta(\infty,f)>12 \quad \textit{and} \quad 7\Theta(0,g)+6\Theta(\infty,g)>12,$$

then $f\equiv g$ or *$f\cdot g\equiv 1$.*

2. Some lemmas

Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f of order p, a 1-point of g of order q. We denote by $\overline{N}_L(r,1/(f-1))$ the counting function of those 1-points of f where $p>q$, by $N_E^{1)}(r,1/(f-1))$ the counting function of those 1-points of f where $p=q=1$, by $N_E^{(2}(r,1/(f-1))$ the counting function of those 1-points of f where $p=q\geq 2$, each point in these counting functions is counted only once. We denote by $N_0(r,1/f')$ the counting function corresponding to the zeros of f' that are not zeros of f and $f-1$. In the same way, we can define $\overline{N}_L(r,1/(g-1))$, $N_E^{1)}(r,1/(g-1))$, $N_E^{(2}(r,1/(g-1))$ and $N_0(r,1/g')$ (see [7]).

Lemma 1. *Let f and g be two nonconstant meromorphic functions such that f and g share the vlaue 1 IM. Then*

$$\begin{aligned} T(r,f)\leq{}& \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+N_E^{1)}\left(r,\frac{1}{f-1}\right)\\ &+\overline{N}_L\left(r,\frac{1}{f-1}\right)-N_0\left(r,\frac{1}{f'}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r). \end{aligned} \tag{2.1}$$

Proof. By the second fundamental theorem, we have

$$(2.2)\quad T(r,f)+T(r,g)\le \bar N\left(r,\frac{1}{f}\right)+\bar N(r,f)+\bar N\left(r,\frac{1}{f-1}\right)-N_0\left(r,\frac{1}{f'}\right)$$
$$+\bar N\left(r,\frac{1}{g}\right)+\bar N(r,g)+\bar N\left(r,\frac{1}{g-1}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r).$$

Noting that f and g share the value 1 IM, we have

$$\bar N\left(r,\frac{1}{f-1}\right)+\bar N\left(r,\frac{1}{g-1}\right)=N_E^{1)}\left(r,\frac{1}{f-1}\right)+\bar N_L\left(r,\frac{1}{f-1}\right)+\bar N_L\left(r,\frac{1}{g-1}\right)$$
$$+\bar N_E^{(2}\left(r,\frac{1}{g-1}\right)+\bar N\left(r,\frac{1}{g-1}\right)$$
$$\le N_E^{1)}\left(r,\frac{1}{f-1}\right)+\bar N_L\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{g-1}\right)$$
$$\le N_E^{1)}\left(r,\frac{1}{f-1}\right)+\bar N_L\left(r,\frac{1}{f-1}\right)+T(r,g)+O(1).$$

Combining this and (2.2), we obtain the conclusion of Lemma 1.

LEMMA 2. *Let*

$$(2.3)\qquad H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right),$$

where f and g are two nonconstant meromorphic functions. If f and g share the value 1 *IM and $H\not\equiv 0$, then*

$$(2.4)\qquad N_E^{1)}\left(r,\frac{1}{f-1}\right)\le N(r,H)+S(r).$$

Proof. Suppose that z_0 is a simple 1-point of both f and g. Then an elementary calculation (see [4, Lemma 2]) gives that $H(z)=O(z-z_0)$, which proves that z_0 is a zero of H. Thus,

$$(2.5)\qquad N_E^{1)}\left(r,\frac{1}{f-1}\right)\le N\left(r,\frac{1}{H}\right)\le T(r,H)+O(1).$$

From (2.3) we obtain $m(r,H)=S(r)$. Combining this and (2.5), we obtain (2.4).

LEMMA 3 (see [4]). *Let*

$$h=\frac{f''}{f'}-\frac{2f'}{f-1},$$

where f is a nonconstant meromorphic function. If z_0 is a simple pole of f, then h is regular at z_0.

LEMMA 4. *Let H be given by* (2.3) *and $H \not\equiv 0$. If f and g share the value* 1 *IM, then*

$$T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + 2\overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right) + S(r). \tag{2.6}$$

Proof. Since f and g share 1 IM, by Lemma 1 and Lemma 2 we can obtain (2.1) and (2.4). By Lemma 3, we have from (2.3)

$$N(r,H) \le \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) + \overline{N}_{(2}(r,g) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right) + N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r). \tag{2.7}$$

Combining (2.1), (2.4) and (2.7), we get (2.6).

LEMMA 5. *If, in addition to the assumptions of Lemma* 4, *f and g share the value ∞ IM, then*

$$T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + 3\overline{N}(r,f) + 2\overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right) + S(r). \tag{2.8}$$

Proof. Since f and g share the value 1 IM, by Lemma 1 and Lemma 2 we can obtain (2.1) and (2.4). Noting that f and g share the value ∞ IM, we have from (2.3)

$$N(r,H) \le \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right) + N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r). \tag{2.9}$$

From (2.1), (2.4) and (2.9), we get (2.8).

LEMMA 6. *Suppose that H be given by* (2.3) *and $H \equiv 0$, then f and g share the value* 1 *CM. If further suppose that f and g share the value ∞ IM, then f and g share the value ∞ CM.*

Proof. Since $H \equiv 0$, by integration we have from (2.3)

$$\frac{f'}{(f-1)^2} \equiv A\frac{g'}{(g-1)^2},$$

where A is a nonzero constant. From this we obtain the conclusion of Lemma 6.

LEMMA 7 (see [8]). *Let h be a nonconstant meromorphic function. Then*

$$N\left(r,\frac{1}{h'}\right) \le N\left(r,\frac{1}{h}\right) + \bar{N}(r,h) + S(r,h).$$

LEMMA 8. *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Then*

$$\bar{N}_L\left(r,\frac{1}{f-1}\right) \le \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}(r,f) + S(r), \tag{2.10}$$

and

$$\bar{N}_L\left(r,\frac{1}{g-1}\right) \le \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}(r,g) + S(r). \tag{2.11}$$

Proof. Obviously,

$$\left(N\left(r;\frac{1}{f-1}\right) - \bar{N}\left(r,\frac{1}{f-1}\right)\right) + \left(N\left(r,\frac{1}{f}\right) - \bar{N}\left(r,\frac{1}{f}\right)\right) \le N\left(r,\frac{1}{f'}\right).$$

From this and Lemma 7, we obtain

$$N\left(r,\frac{1}{f-1}\right) - \bar{N}\left(r,\frac{1}{f-1}\right) \le \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}(r,f) + S(r). \tag{2.12}$$

Noting

$$\bar{N}_L\left(r,\frac{1}{f-1}\right) \le N\left(r,\frac{1}{f-1}\right) - \bar{N}\left(r,\frac{1}{f-1}\right),$$

from (2.12) we get (2.10). Similarly, we can obtain (2.11).

3. Proofs of Theorem 1 and Theorem 2

3.1. *Proof of Theorem* 1.

Let H be given by (2.3). If $H \not\equiv 0$, by Lemma 4 and Lemma 8, we can obtain (2.6), (2.10) and (2.11). From (2.6), (2.10) and (2.11) we have

$$(3.1)\qquad T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + S(r).$$

Similarly, we have

$$(3.2)\qquad T(r,g) \le N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{g}\right) + 2\overline{N}(r,g) + S(r).$$

Combining (3.1) and (3.2) we get

$$T(r,f) + T(r,g) \le N^*\left(r,\frac{1}{f}\right) + N^*(r,f) + N^*\left(r,\frac{1}{g}\right) + N^*(r,g) + S(r),$$

which contradicts (1.2). Thus, $H \equiv 0$. By Lemma 6 we know that f and g share the value 1 CM. Again by Theorem A we obtain the conclusion of Theorem 1.

3.2. *Proof of Theorem* 2.

Let H be given by (2.3). If $H \not\equiv 0$, by Lemma 5 and Lemma 8, we can obtain (2.8), (2.10) and (2.11). From (2.8), (2.10) and (2.11) we have

$$(3.3)\quad T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + 6\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r).$$

Similarly, we have

$$(3.4)\quad T(r,g) \le N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + 6\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + S(r).$$

Combing (3.3) and (3.4) we get

$$T(r,f) + T(r,g) \le N^*\left(r,\frac{1}{f}\right) + N^*\left(r,\frac{1}{g}\right) + 12\overline{N}(r,f) + S(r),$$

which contradicts (1.3). Thus, $H \equiv 0$. By Lemma 6 we know that f and g share the values 1 and ∞ CM. Again by Theorem B we obtain the conclusion of Theorem 2.

4. Applications of Theorem 1 and Theorem 2

Let h be a nonconstant meromorphic function and S be a subset of distinct elements in $\hat{C}$, and let

$$E_h(S) = \bigcup_{a \in S} \{z \mid h(z) - a = 0\},$$

where each zero of $h(z) - a$ with multiplicity m is repeated m times in $E_h(S)$. The notation $\bar{E}_h(S)$ expresses the set which contains the same points as $E_h(S)$ but without counting multiplicities (see [7]). We shall use ω to denote the constant $\exp((2\pi i)/n)$, where n is a positive integer. Recently, G. D. Song and N. Li (see [9]) proved the following results.

THEOREM E. *Let $S = \{a + b, a + b\omega, \ldots, a + b\omega^{n-1}\}$, where $n > 14$, a and b $(\neq 0)$ are constants. If f and g are nonconstant meromorphic functions such that $\bar{E}_f(S) = \bar{E}_g(S)$, then $f - a \equiv t(g - a)$, where $t^n = 1$, or $(f - a) \cdot (g - a) \equiv s$, where $s^n = b^{2n}$.*

THEOREM F. *Let $S_1 = \{a + b, a + b\omega, \ldots, a + b\omega^{n-1}\}$, where $n > 13$, a and b $(\neq 0)$ are constants, and let $S_2 = \{\infty\}$. If f and g are nonconstant meromorphic functions such that $\bar{E}_f(S_j) = \bar{E}_g(S_j)$ $(j = 1, 2)$, then $f - a \equiv t(g - a)$, where $t^n = 1$, or $(f - a) \cdot (g - a) \equiv s$, where $s^n = b^{2n}$.*

Using Theorem 1, it is easy to give the proof of Theorem E. In fact, let

$$F = \left(\frac{f - a}{b}\right)^n \quad \text{and} \quad G = \left(\frac{g - a}{b}\right)^n,$$

then F and G share 1 IM. It is obvious that

$$N^*\left(r, \frac{1}{F}\right) + N^*(r, F) + N^*\left(r, \frac{1}{G}\right) + N^*(r, G) \leq \frac{14}{n}(T(r, F) + T(r, G)) + O(1).$$

By Theorem 1, we get $F \equiv G$ or $F \cdot G \equiv 1$. From this we can obtain the conclusion of Theorem E.

Using Theorem 2 and proceeding as in the proof of Theorem E, we can prove Theorem F.

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