H.X. YI
KODAI MATH. J.
13 (1990), 363—372

MEROMORPHIC FUNCTIONS THAT SHARE TWO OR THREE VALUES

BY HONG-XUN YI

1. Introduction and Main Results.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane. If f and g have the same a-points with the same multiplicities, we say f and g share the value a CM. (see [2]). It is assumed that the reader is familiar with the fundamental concepts of Nevanlinna's theory of meromorphic functions and their standard symbols, as found in [3]. It will be convenient to let E denote any set of finite linear measure of $0<r<\infty$ and let I denote any set of infinite linear measure of $0<r<\infty$. The notation $S(r, f)$ denotes any quantity satisfying $S(r, f)=o(T(r, f))$ $(r\to\infty,\ r\notin E)$.

M. Ozawa proved the following result.

THEOREM A (see [5]). *Let f and g be entire functions of finite order such that f and g share 0, 1 CM. If $\delta(0, f)>1/2$, then $f\cdot g=1$ unless $f=g$.*

In [9] H. Ueda showed that in Theorem A the order restriction of f and g can be removed. He proved more generally the following result.

THEOREM B. *Let f and g be meromorphic functions such that f and g share 0, 1, ∞ CM. If*

$$\limsup_{r\to\infty}\frac{N(r, 1/f)+N(r, f)}{T(r, f)}<\frac{1}{2},$$

then $f=g$ or $f\cdot g=1$.

Recently the present author proved the following result.

THEOREM C (see [13]). *Let f and g be meromorphic functions such that f and g share 0, 1, ∞ CM. If*

$$\bar{N}\left(r, \frac{1}{f}\right)+\bar{N}(r, f)<(\lambda+o(1))T(r, f) \qquad (r\in I),$$

where $\lambda<1/2$, then $f=g$ or $f\cdot g=1$.

Received July 3, 1989; Revised March 12, 1990.

In order to state our first theorem, we introduce the following notations.

Let $f(z)$ be a meromorphic function. We denote by $n_1(r, 1/f)$ the number of simple zeros of f in $|z|\leqq r$ and by $n_1(r, f)$ the number of simple poles of f in $|z|\leqq r$. $N_1(r, 1/f)$ and $N_1(r, f)$ are defined in terms of $n_1(r, 1/f)$ and $n_1(r, f)$ respectively in the usual way.

Let $f(z)$ and $g(z)$ be meromorphic functions. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$.

In this paper we prove the following result which is an improvement of the above results.

THEOREM 1. *Let f and g be meromorphic functions such that f and g share 0, 1, ∞ CM. If*

$$N_1\left(r, \frac{1}{f}\right)+N_1(r, f)<(\lambda+o(1))T(r) \qquad (r\in I), \tag{1}$$

where $\lambda<1/2$, then $f=g$ or $f\cdot g=1$.

By Theorem 1 we immediately obtain the following corollary.

COROLLARY 1. *Let f and g be meromorphic function such that f and g share 0, 1, ∞ CM. If*

$$\limsup_{r\to\infty} \frac{N_1(r, 1/f)+N_1(r, f)}{T(r)}<\frac{1}{2},$$

then $f=g$ or $f\cdot g=1$.

In [7] H. Ueda proved the following result.

THEOREM D. *Let f and g be entire functions such that f and g share 0, 1 CM. If all zero-points of f excepting at most finite number have multiplicities $\geqq 2$, then $f=g$ or $f\cdot g=1$.*

From Theorem 1 we immediately deduce the following corollary which is an improvement of Theorem D.

COROLLARY 2. *Let f and g be meromorphic functions such that f and g share 0, 1, ∞ CM. If all zero-points and pole-points of f excepting at most finite number have multiplicities $\geqq 2$, then $f=g$ or $f\cdot g=1$.*

In [5] M. Ozawa proved the following theorem.

THEOREM E. *Let f and g be entire functions such that f and g share 1 CM. If $\delta(0, f)>0$ and 0 is lacunary for g, then $f=g$ or $f\cdot g=1$.*

Recently the present author proved the following result which is an extension of Theorem E.

THEOREM F (see [11]). *Let f and g be meromorphic functions such that f and g share 1 CM. If $\delta(0, f)+\delta(0, g)>1$ and $\delta(\infty, f)=\delta(\infty, g)=1$, then $f=g$ or $f\cdot g=1$.*

In this paper we prove the following result which is an improvement of the above theorems.

THEOREM 2. *Let f and g be meromorphic functions such that f and g share $1, \infty$ CM. If*

$$N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)<(\mu+o(1))T(r) \qquad (r\in I), \tag{2}$$

where $\mu<1$, then $f=g$ or $f\cdot g=1$.

By Theorem 2 we immediately obtain the following corollary.

COROLLARY 3. *Let f and g be meromorphic functions such that f and g share $1, \infty$ CM. If $\delta(0, f)+\delta(0, g)+2\Theta(\infty, f)>3$, then $f=g$ or $f\cdot g=1$.*

Let $f(z)=2e^z(1-2e^z)$, $g(z)=(1/4)e^{-z}(2-e^{-z})$. It is easy to see that this example shows that the theorems and corollaries in this paper are sharp.

2. Some Lemmas.

The following lemmas will be needed in the proof of our theorems.

LEMMA 1. *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1f+c_2g=c_3$, then*

$$T(r, f)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+\bar{N}(r, f)+S(r, f).$$

Proof. By the second fundamental theorem, we have

$$\begin{aligned} T(r, f)&<N\left(r, \frac{1}{f}\right)+N\left(r, \left(f-\frac{c_3}{c_1}\right)^{-1}\right)+\bar{N}(r, f)+S(r, f) \\ &=N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+\bar{N}(r, f)+S(r, f), \end{aligned}$$

which proves Lemma 1.

LEMMA 2 (see [4]). *Let $f_1, f_2, \cdots, f_n$ be linearly independent meromorphic functions satisfying $\sum_{i=1}^n f_i=1$. Then for $j=1, 2, \cdots, n$, we have*

$$T(r, f_j)<\sum_{i=1}^{n}N\left(r, \frac{1}{f_i}\right)+N(r, f_j)+N(r, D)-\sum_{i=1}^{n}N(r, f_i)$$

$$-N\left(r, \frac{1}{D}\right)+O(\log r+\log T_n(r)) \qquad (r\notin E),$$

where D denotes the Wronskian

$$D=\begin{vmatrix} f_1 & , f_2 & , \cdots, f_n \\ f_1' & , f_2' & , \cdots, f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)}, & f_2^{(n-1)}, & \cdots, f_n^{(n-1)} \end{vmatrix}$$

and $T_n(r)$ denotes the maximum of $T(r, f_i)$, $i=1, 2, \cdots, n$.

LEMMA 3. *Let f_1, f_2 and f_3 be three nonconstant meromorphic functions satisfying $\sum_{i=1}^{3}f_i=1$, and let $g_1=-f_3/f_2$, $g_2=1/f_2$, $g_3=-f_1/f_2$. If f_1, f_2 and f_3 are linearly independent, then g_1, g_2 and g_3 are linearly independent.*

Proof. Suppose that g_1, g_2 and g_3 are linearly dependent. Then there exist three constants $(c_1, c_2, c_3)\neq(0, 0, 0)$ such that

$$c_1g_1+c_2g_2+c_3g_3=0,$$

that is

$$c_1f_3+c_3f_1=c_2. \tag{3}$$

If $c_2=0$, then $c_1\neq0$, $c_3\neq0$, and

$$c_1f_3+c_3f_1=0,$$

which contradicts our assumption.

If $c_2\neq0$, from (3) we have

$$\frac{c_1}{c_2}f_3+\frac{c_3}{c_2}f_1=1. \tag{4}$$

Noting $\sum_{i=1}^{3}f_i=1$, from (4) we get

$$\left(1-\frac{c_3}{c_2}\right)f_1+f_2+\left(1-\frac{c_1}{c_2}\right)f_3=0,$$

which is impossible.

This completes the proof of Lemma 3.

LEMMA 4. *Let $h(z)$ be a nonconstant entire function. Then*

$$T(r, h')=o(T(r, e^h)) \qquad (r\notin E).$$

Proof. We have

$$T(r, h') \leqq T(r, h) + S(r, h).$$

On other hand, by Clunie's result (see, [3, pp 54]), we have

$$T(r, h) = o(T(r, e^h)).$$

Thus

$$T(r, h') = o(T(r, e^h)) \qquad (r \notin E),$$

which proves Lemma 4.

3. Proof of Theorem 2.

By assumption, we have

$$f - 1 = e^h(g-1), \tag{5}$$

where h is an entire function. Let $f_1 = f$, $f_2 = e^h$, $f_3 = -e^h g$ and $T_3(r)$ denote the maximum of $T(r, f_i)$, $i = 1, 2, 3$. From (5) we have

$$\textstyle\sum_{i=1}^3 f_i = 1, \tag{6}$$

$$\textstyle\sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) = N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right), \tag{7}$$

and

$$T_3(r) = O(T(r)). \tag{8}$$

We discuss the following two cases.

a) Suppose that f_1, f_2 and f_3 are linearly independent. By Lemma 2 and (8), we have

$$T(r, f) < \textstyle\sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) + N(r, D) - N(r, f_2) - N(r, f_3) + o(T(r)) \quad (r \notin E), \tag{9}$$

where

$$D = \begin{vmatrix} f_1, & f_2, & f_3 \\ f_1', & f_2', & f_3' \\ f_1'', & f_2'', & f_3'' \end{vmatrix}. \tag{10}$$

From (6) and (10) we get

$$D = \begin{vmatrix} f_2', & f_3' \\ f_2'', & f_3'' \end{vmatrix}$$

and hence

$$N(r, D) - N(r, f_2) - N(r, f_3) \leqq N(r, g'') - N(r, g) = 2\overline{N}(r, g) = 2\overline{N}(r, f). \tag{11}$$

From (7), (9) and (11) we obtain

$$T(r, f)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)+o(T(r)) \qquad (r \notin E). \tag{12}$$

Let $g_1=-f_3/f_2=g$, $g_2=1/f_2=e^{-h}$, $g_3=-f_1/f_2=-e^{-h}f$. From (6) we obtain

$$\sum_{i=1}^{3} g_i=1.$$

By Lemma 3 we know that g_1, g_2 and g_3 are linearly independent. In a similar manner we get

$$T(r, g)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)+o(T(r)) \qquad (r \notin E). \tag{13}$$

From (12) and (13) we deduce

$$T(r)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)+o(T(r)) \qquad (r \notin E). \tag{14}$$

Combining (2) and (14) we get

$$(1-\mu)T(r)<o(T(r)) \qquad (r \in I), \tag{15}$$

which is impossible.

b) Suppose that f_1, f_2 and f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$c_1 f_1+c_2 f_2+c_3 f_3=0. \tag{16}$$

If $c_1=0$, from (16) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3=-\frac{c_2}{c_3} f_2$$

and hence

$$g=\frac{c_2}{c_3},$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1=-\frac{c_2}{c_1} f_2-\frac{c_3}{c_1} f_3. \tag{17}$$

Now combining (6) and (17) we get

$$\left(1-\frac{c_2}{c_1}\right) f_2+\left(1-\frac{c_3}{c_1}\right) f_3=1. \tag{18}$$

We discuss the following three subcases.

b_1) Assume $c_1=c_2$. From (18) we have $c_1 \neq c_3$ and

$$f_3=\frac{c_1}{c_1-c_3}, \tag{19}$$

that is

$$g=-\frac{c_1}{c_1-c_3}e^{-h}. \tag{20}$$

From (6) and (19) we get

$$f_1+f_2=-\frac{c_3}{c_1-c_3},$$

that is

$$f+e^h=-\frac{c_3}{c_1-c_3}. \tag{21}$$

If $c_3\neq 0$, from (20) and (21) we have

$$T(r)=T(r,e^h)+O(1)$$

and

$$N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)+2\bar{N}(r,f)=T(r,e^h)+S(r,f)=(1+o(1))T(r) \quad (r\notin E),$$

which contradicts our assumption. Thus $c_3=0$. From (20) and (21) we deduce $g=-e^{-h}$ and $f=-e^h$ and hance $f\cdot g=1$.

b$_2$) Assume $c_1=c_3$. From (18) we have $c_1\neq c_2$ and

$$f_2=\frac{c_1}{c_1-c_2}$$

that is

$$e^h=\frac{c_1}{c_1-c_2}. \tag{22}$$

From (6) and (22) we get

$$f-\frac{c_1}{c_1-c_2}g=-\frac{c_2}{c_1-c_2}. \tag{23}$$

If $c_2\neq 0$, by Lemma 1 we have

$$T(r)<N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)+\bar{N}(r,f)+S(r,f). \tag{24}$$

By (2) and (24) we get

$$(1-\mu)T(r)<o(T(r)) \qquad (r\in I), \tag{25}$$

which is impossible. Thus $c_2=0$. From (23) we deduce $f=g$.

b$_3$) Assume $c_1\neq c_2$ and $c_1\neq c_3$. From (18) we have

$$g=\frac{c_1-c_2}{c_1-c_3}-\frac{c_1}{c_1-c_3}e^{-h}. \tag{26}$$

Now combining (17) and (26), we get

$$f=-\frac{c_2-c_3}{c_1-c_3}e^h-\frac{c_3}{c_1-c_3}. \tag{27}$$

From (26) and (27) we have

$$T(r)=T(r,\ e^h)+O(1)$$

and

$$N\left(r,\frac{1}{g}\right)=T(r,\ e^h)+S(r,\ g)=(1+o(1))T(r) \qquad (r\notin E),$$

which contradicts our assumption.

This completes the proof of Theorem 2.

4. Proof of Theorem 1.

Suppose that $f\neq g$. By assumption we have with two entire functions α and β,

$$f=e^{\alpha}\cdot g\,, \qquad f-1=e^{\beta}\cdot(g-1). \tag{28}$$

Since $f\neq g$, then $e^{\beta}\neq 1$ and $e^{\beta-\alpha}\neq 1$. Thus from (28) we get

$$f=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}} \tag{29}$$

and

$$T(r,\ e^{\alpha})+T(r,\ e^{\beta})=O(T(r)). \tag{30}$$

If $e^{\beta}=c$, where c ($\neq 0$, 1) is a constant, then from (29) we have

$$N\left(r,\frac{1}{f}\right)=0. \tag{31}$$

If e^{β} is not a constant, let $\{z_n\}$ be all the roots of $f=0$ with multiplicity$\geqq 2$, then from (29) $\{z_n\}$ are the roots of $(1-e^{\beta})'=-\beta' e^{\beta}=0$. Thus

$$N\left(r,\frac{1}{f}\right)-N_1\left(r,\frac{1}{f}\right)\leqq 2N\left(r,\frac{1}{\beta'}\right)\leqq 2T(r,\ \beta')+O(1).$$

By Lemma 4 and (30) we have

$$N\left(r,\frac{1}{f}\right)\leqq N_1\left(r,\frac{1}{f}\right)+o(T(r)) \qquad (r\notin E). \tag{32}$$

If $e^{\beta-\alpha}=c$ ($\neq 0$, 1), then from (29) we have

$$N(r,\ f)=0\,. \tag{33}$$

If $e^{\beta-\alpha}$ is not a constant, let $\{t_n\}$ be all the roots of $1/f=0$ with multiplicity $\geqq 2$, then from (29) $\{t_n\}$ are the roots of $(1-e^{\beta-\alpha})'=-(\beta'-\alpha')e^{\beta-\alpha}=0$. Thus

$$N(r, f)-N_1(r, f)\leqq 2N\left(r, \frac{1}{\beta'-\alpha'}\right)\leqq 2T(r, \alpha')+2T(r, \beta')+O(1).$$

By Lemma 4 and (30) we have

$$N(r, f)\leqq N_1(r, f)+o(T(r)) \qquad (r\notin E). \tag{34}$$

Noting $N(r, 1/g)=N(r, 1/f)$ and $N(r, g)=N(r, f)$, from (31), (32), (33) and (34) we deduce

$$N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)<2N_1\left(r, \frac{1}{f}\right)+2N_1(r, f)+o(T(r)) \quad (r\notin E). \tag{35}$$

Now combining (1) and (35) we obtain

$$N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)<(2\lambda+o(1))T(r) \qquad (r\in I).$$

By Theorem 2 we deduce the conclusion of Theorem 1.

5. An Application of Theorem 1.

Let f be a nonconstant meromorphic function and S be a set in the complex plane, and let

$$E_f(S)=\bigcup_{a\in S}\{z\,|\,f(z)-a=0\},$$

where any z which is a zero of multiplicity m is included in $E_f(S)$, m times.

In [1] F. Gross and C. F. Osgood proved the following theorem.

THEOREM G. *Let* $S_1=\{-1, 1\}$, $S_2=\{0\}$. *If* f *and* g *are entire functions of finite order such that* $E_f(S_i)=E_g(S_i)$ $(i=1, 2)$, *then* $f=\pm g$ *or* $f\cdot g=\pm 1$.

In [10] the present author proved that in the preceding theorem the order restriction of f and g can be removed. The present author [12] and independently K. Tohge [6] proved the following result which is an extension of the above results.

THEOREM H. *Let* $S_1=\{1, \omega, \cdots, \omega^{n-1}\}$, $S_2=\{0\}$ *and* $S_3=\{\infty\}$, *where* n *is an integer* $(\geqq 2)$ *and* $\omega=\cos(2\pi/n)+i\sin(2\pi/n)$. *If* f *and* g *are meromorphic functions such that* $E_f(S_i)=E_g(S_i)$ $(i=1, 2, 3)$, *then* $f^n=g^n$ *or* $f^n\cdot g^n=1$.

Using Theorem 1, it is easy to give the proof of Theorem H. In fact, let $F=f^n$ and $G=g^n$, then F and G share $0, 1, \infty$ CM and $N_1(r, 1/F)+N_1(r, F)=0$. By Theorem 1, we get $F=G$ or $F\cdot G=1$, that is $f^n=g^n$ or $f^n\cdot g^n=1$. This proves Theorem H.

Acknowledement. I am grateful to the referee for valuable comments.

REFERENCES

[1] F. GROSS AND C.F. OSGOOD, Entire functions with common preimages, Factorization Theory of Meromorphic Functions, 19-24, Marcel Dekker, Inc., 1982.
[2] G.G. GUNDERSEN, Meromorphic functions that share three or four values, J. London Math. Soc., (2), **20** (1979), 457-466.
[3] W.K. HAYMAN, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[4] R. NEVANLINNA, Le Théorèms de Picard-Borel et la Théorie des Fonctions Méromorphes, Gauthier-Villars, Paris, 1929.
[5] M. OZAWA, Unicity theorems for entire functions, J. d'Anal. Math., **30** (1976), 411-420.
[6] K. TOHGE, Meromorphic functions covering certain finite sets at the same points, Kodai Math. J., **11** (1988), 249-279.
[7] H. UEDA, Unicity theorems for entire functions, Kodai Math. J., **3** (1980), 212-223.
[8] H. UEDA, Unicity theorems for meromorphic or entire functions, Kodai Math. J., **3** (1980), 457-471.
[9] H. UEDA, Unicity theorems for meromorphic or entire functions II, Kodai Math. J., **6** (1983), 26-36.
[10] HONG-XUN YI, Meromorphic functions with common premages, J. of Math. (PRC), **7** (1987), 219-224.
[11] HONG-XUN YI, Meromorphic functions with two deficient values, Acta Math. Sin., **30** (1987), 588-597.
[12] HONG-XUN YI, On the uniqueness of meromorphic functions, Acta Math. Sin., **31** (1988), 570-576.
[13] HONG-XUN YI, Meromorphic functions that share three values, Chin. Ann. Math., **9A** (1988), 434-440.

DEPARTMENT OF MATHEMATICS
SHANDONG UNIVERSITY
JINAN, SHANDONG, 250100
P.R. CHINA