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MEROMORPHIC FUNCTIONS WITH LARGE SETS OF JULIA POINTS

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1. Introduction. Let $D = \{z: |z| < 1\}$ and $C = \{z: |z| = 1\}$. If W denotes the Riemann sphere equipped the chordal metric X, let $f: D \to W$ be meromorphic. A chord T lying in D except for an endpoint $\gamma \in C$ is called a *Julia segment* for f if for each Stolz angle Δ in D at γ which contains T, f assumes infinitely often in Δ all values of W with at most two exceptions. We call $\gamma \in C$ a *Julia point* for f if every chord in D ending at γ is a Julia segment for f, and we denote by $J(f)$ the set of Julia points of f.

In this paper we show that for a certain class of functions meromorphic in D the sets of Julia points are residual in C. This class of functions lies in the intersection of two previously-studied classes of functions. In [1] K. Barth defined the class A_m: $f \in A_m$ if f is meromorphic in D, and if for each point γ of a set dense in C there exists a curve K in D ending at γ such that $\lim_{z\to\gamma(z\in K)} f(z)$ exists. More recently, in [7] K-F. Tse divided all functions meromorphic in D into two classes in the following way. For each pair of points $z, w \in D$, the hyperbolic distance between z and w is defined by

$$\rho(z, w) = (1/2)\log\{[1 + \sigma(z, w)]/[1 - \sigma(z, w)]\},$$

where $\sigma(z, w) = |z - w|/|1 - \bar{w}z|$. A meromorphic function f is *of the second kind* if there exist a sequence $\{z_n\}_{n=1}^{\infty} \subset D$, $|z_n| \to 1$, a constant $r > 0$, and a point $\alpha \in W$ such that for $\mathscr{D}(r) = \bigcup_{n=1}^{\infty}\{z: \rho(z, z_n) < r\}$, f tends uniformly to α as $|z| \to 1$ in $\mathscr{D}(r)$. And f is *of the first kind* if it is not of the second kind. Tse's results in [7] characterize the functions of the first kind and show how wild their boundary behavior must be.

THEOREM 1. *If $f \in A_m$ is of the first kind, then $J(f)$ is residual in C.*

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After proving Theorem 1 in § 2, we show that if in addition the set of points on C at which f has an asymptotic value is of measure 2π on C, then $J(f)$ has measure 2π on C also. It is important to note that these results apply to the Tsuji functions, a well-studied class of meromorphic functions. Following the notation in [5], let $f^*(z) = |f'(z)|/(1 + |f(z)|^2)$ be the spherical derivative of f in D; f is a *Tsuji function* if for some finite constant $l > 0$, $\sup_{r<1}\left[\int_0^{2\pi} f^*(re^{i\theta})\,r d\theta\right] < l$. It is a consequence of [5, Theorem 6] that each Tsuji function is in class A_m.

2. Proof of Theorem 1. We begin with a few lemmas. For any $\gamma \in C$ and any $\beta \in (-\pi/2, \pi/2)$, let $T(\gamma, \beta)$ denote the chord in D ending at γ and making angle β with the radius to γ.

LEMMA 1. *Let f be meromorphic in D and suppose for some $\gamma \in C$, $\beta \in (-\pi/2, \pi/2)$, that $T(\gamma, \beta)$ is not a Julia segment for f. Then there exists $\varepsilon > 0$ such that $T(\gamma, \alpha)$ is not a Julia segment for f if $\alpha \in (-\pi/2, \pi/2)$ and $|\alpha - \beta| < \varepsilon$.*

Proof. Obvious.

For each $\beta \in (-\pi/2, \pi/2)$, let $E(\beta) = \{\gamma \in C\colon T(\gamma, \beta)$ is not a Julia segment for $f\}$.

LEMMA 2. *Let f be meromorphic in D and E be the set of points on C which are not Julia points for f. Then $E = \bigcup_\beta E(\beta)$, where β is rational and $\beta \in (-\pi/2, \pi/2)$.*

Proof. If $\gamma \in E$, from Lemma 1 it follows that $\gamma \in E(\beta)$ for some rational $\beta \in (-\pi/2, \pi/2)$. And it is obvious that for each rational $\beta \in (-\pi/2, \pi/2), E(\beta) \subset E$.

For each chord $T(\gamma, \beta)$ at $\gamma \in C$, we let $\Delta(\gamma, \beta, \alpha)$ denote the Stolz angle in D at γ which is symmetric about $T(\gamma, \beta)$ and has vertex angle α. (We presume here that $0 < \alpha < \pi/2 - |\beta|$.)

LEMMA 3. *Let β, α be fixed, where $\beta \in (-\pi/2, \pi/2)$ and $0 < \alpha < \pi/2 - |\beta|$. If $M = \tanh^{-1}\{\sin(\alpha/2)/[4 + \sin(\alpha/2)]\}$ and $z \in T(1, \beta)$, then $\{w \in D\colon \rho(w, z) < M\} \subset \Delta(1, \beta, \alpha)$.*

Proof. From a lemma of P. Lappan [7, Lemma 2], if $\rho(z, w) < M$, then $|w - z|/(1 - |z|) \leq 2\tanh M/(1 - \tanh M) = (1/2)\sin(\alpha/2) < \sin(\alpha/2)$. Thus $|w - z| < (1 - |z|)\sin(\alpha/2) \leq |1 - z|\sin(\alpha/2)$, and $w \in \Delta(1, \beta, \alpha)$.

Now suppose $f \in A_m$ is of the first kind but $J(f)$ is not residual. Then $C - J(f)$ is of second category on C, and Lemma 2 implies that for some rational $\beta \in (-\pi/2, \pi/2)$, $E(\beta)$ is of second category on C.

For each positive integer n, let

$$E_n(\beta) = \{\gamma \in E(\beta) : \exists \alpha \geq 1/n \ni f \text{ omits at least three values} \atop \text{in } \Delta(\gamma, \beta, \alpha)\}.$$

We see that $E_{n+1}(\beta) \supset E_n(\beta)$, and $E(\beta) = \bigcup_n E_n(\beta)$. Thus for some integer N, $E_N(\beta)$ is of second category on C.

Since, for each $\gamma \in E_N(\beta)$, f omits at least three values in $\Delta(\gamma, \beta, 1/N)$, there exists $d(\gamma) > 0$ with this property: for any two sets A, B on W whose union contains the values omitted by f in $\Delta(\gamma, \beta, 1/N)$, either $\operatorname{diam} A \geq d(\gamma)$, or $\operatorname{diam} B \geq d(\gamma)$. For each positive integer j, let $E_{N,j}(\beta) = \{\gamma \in E_N(\beta) : d(\gamma) \geq 1/j\}$. Clearly $E_{N,j+1}(\beta) \supset E_{N,j}(\beta)$, and $E_N(\beta) = \bigcup_j E_{N,j}(\beta)$. Hence, for some integer $J > 0$, $E_{NJ}(\beta)$ is of second category on C.

If $\gamma \in E_{NJ}(\beta)$, there exists $\mu > 0$ such that f never assumes some three distinct values in the region $\Delta(\gamma, \beta, 1/N) \cap \{|z - \gamma| < \mu\}$. For each positive integer k, let $E_{NJ,k}(\beta) = \{\gamma \in E_{NJ}(\beta) : \mu \geq 1/k\}$. Since $E_{NJ,k+1}(\beta) \supset E_{NJ,k}(\beta)$ and $E_{NJ}(\beta) = \bigcup_k E_{NJ,k}(\beta)$, there exists integer $K > 0$ such that $E_{NJK}(\beta)$ is of second category on C. For brevity let us denote $E_{NJK}(\beta)$ by E^*.

There exists an arc A on C such that E^* is dense in A. Since $f \in A_m$, we can choose an interior point $\lambda \in A$ at which there ends a curve Γ in D along which f has a limit. Let Γ' be the "last part" of Γ in $\{|z - \lambda| < 1/K\}$ and $\{w_m\}$ be any sequence on Γ' converging to λ. For each m, let $\varepsilon(m) = 2^{-m}$, and let $\delta(m) > 0$ be chosen so that $X[f(w), f(w_m)] < \varepsilon(m)$ whenever $\rho(w, w_m) < \delta(m)$. For m sufficiently large there exists $\gamma_m \in E^*$ for which the chord $T(\gamma_m, \beta)$ intersects the neighborhood $\{w \in D : \rho(w, w_m) < \delta(m)\}$. We select a point $v_m \in T(\gamma_m, \beta)$ (which may be w_m itself) such that $|v_m - \gamma_m| < 1/K$ and $\rho(v_m, w_m) < \delta(m)$. Within each such neighborhood $\{w \in D : \rho(w, w_m) < \delta(m)\}$ we deform Γ' to make it pass through v_m. The resulting curve we call Γ^*, and we see that f has a limit as $z \to \lambda$ along Γ^*.

Since f is of the first kind, $\{v_m\}$ is a ρ-sequence [7, Corollary 3.1]. Thus for each $r > 0$, [4, Theorem 2] implies that there exist sets $G(m, r)$, $H(m, r)$ on W with chordal diameter less than or equal to r, and integer $M(r) > 0$ such that for $m > M(r)$

$$W - [G(m, r) \cup H(m, r)] \subset f[\{w \in D\colon \rho(w, v_m) < r\}] \,.$$

We choose r to be less than the smaller of $1/J$ and $\tanh^{-1}\{\sin(1/2N)/[4 + \sin(1/2N)]\}$. For this choice of r and every $m > M(r)$ we have:

(i) $\{w \in D\colon \rho(w, v_m) < r\} \subset \Delta(\gamma_m, \beta, 1/N)$ by Lemma 3;

(ii) $f[\{w \in D\colon \rho(w, v_m) < r\}]$ omits at least three values, all of which lie in $G(m, r) \cup H(m, r)$.

But since $\gamma_m \in E^* \subset E_{NJ}(\beta)$, either diam $G(m, r) \geq 1/J > r$, or diam $H(m, r) \geq 1/J > r$. This is a contradiction. Hence $J(f)$ is residual on C.

3. Further results. Now we consider functions meromorphic in D which have asymptotic values at almost every point of C.

THEOREM 2. *Suppose f is meromorphic in D and has an asymptotic value at each point of a set of measure 2π on C. If f is of the first kind, then, $J(f)$ is residual and of measure 2π on C.*

Proof. That $J(f)$ is residual follows from Theorem 1. If $C - J(f)$ has positive measure on C, we can easily alter the selection process in the proof of Theorem 1 so that the resulting set E^* has positive measure on C. And at every point of some subset of E^* of positive measure f has an asymptotic value.

Let $\lambda \in E^*$ be a two-sided accumulation point of E^* at which f has an asymptotic value. The remainder of the argument proceeds as in the proof of Theorem 1: we construct a curve ending at γ along which f has a limit, and which intersects a sequence of chords $\{T(\gamma_m, \beta)\}$, where $\gamma_m \in E^*$ and $\gamma_m \to \lambda$.

Since Tsuji functions of the first kind satisfy the hypotheses of both Theorems 1 and 2, we have a corollary.

COROLLARY 1. *If f is a Tsuji function of the first kind, $J(f)$ is residual and of measure 2π on C.*

For each $\alpha \in D$, let $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. In [3] Collingwood and Piranian defined the *Tsuji set* of a meromorphic function f to be the set of points $\alpha \in D$ such that $f \circ \phi_\alpha$ is a Tsuji function. The following lemma, whose proof we omit, permits a slight extension of Corollary 1.

LEMMA 4. *Let f be meromorphic in D and $\alpha \in D$. Then f has a Julia point at $\gamma \in C$ if and only if $f \circ \phi_\alpha$ has a Julia point at $\phi_\alpha^{-1}(\gamma)$.*

COROLLARY 2. *If f is a meromorphic function of the first kind with nonempty Tsuji set, then $J(f)$ is residual and of measure 2π on C.*

4. Some examples. The author is grateful to Professor *K-F*. Tse for Example 1, which exhibits a function satisfying the hypothesis of the theorems and corollaries above.

EXAMPLE 1. There exists a Tsuji function of the first kind. Let Λ be a monotone spiral in D with the property that for any $\theta \in [0, 2\pi]$, if $\{z_n(\theta)\}_{n=1}^{\infty} = \Lambda \cap \{z \in D: \arg z = \theta\}$ then $\rho[z_n(\theta), z_{n+1}(\theta)] \to 0$ as $n \to \infty$. Select the monotone sequence $\{w_n\}_{n=0}^{\infty}$ from Λ with $w_0 = 0$ and $\rho(w_n, w_{n+1}) = 1/n$ for $n \geq 1$. Let $\{r_n\}$ be a sequence of positive numbers such that $r_n + r_{n+1} < |w_{n+1}| - |w_n|$ for each n, and $r_n = 0(1 - |w_n|)$ as $n \to \infty$.

If $\{a_n\}$ is a sequence of positive numbers such that $a_n < r_n^3$, it is shown in [3] that $f(z) = \sum_n [a_n/(z - w_n)]$ is a Tsuji function with these properties: (i) if $D_n = \{z: |z - w_n| < r_n\}$, the series for f converges uniformly in the plane less $\bigcup_n D_n$; (ii) if $\{w_k\}$ is a subsequence of $\{w_n\}$ such that $w_k \to \gamma \in C$, for k sufficiently large the values f omits in D_k lie in arbitrarily small neighborhoods of $f(\gamma)$.

Now let $\lambda \in C$, let $\{\xi_n\}$ be a sequence in D with $\xi_n \to \lambda$, and let $\delta > 0$ be fixed. For each integer k, if $z \in D_k$, $\sigma(z, w_k) \leq r_k/(1 - |w_k|)$, so $\sigma(z, w_k) \to 0$ and $\rho(z, w_k) \to 0$ as $k \to \infty$. Hence $\mathscr{D} = \bigcup_n \{z \in D: \rho(\xi_n, z) < \delta\}$ contains infinitely many disks D_k, and f cannot tend to a constant limit as $|z| \to 1$ in $\mathscr{D}$. Therefore f is of the first kind.

EXAMPLE 2. There exists a Tsuji function f of the second kind such that $J(f) = C$. This example is due to Collingwood and Piranian [3, Theorem 1]; here we show additionally that the function is of the second kind.

Let $z_n = (1 - n^{-1/2}) \exp(i \log n)$, $n = 2, 3, 4, \cdots$, and let $\{r_n\}_{n=2}^{\infty}, \{a_n\}_{n=2}^{\infty}$ be sequences of positive numbers such that $r_n + r_{n+1} < |z_{n+1}| - |z_n|$, $0 < a_n < r_n^3$. Then ([3, Theorem 1]) $f(z) = \sum_{n=2}^{\infty} [a_n/(z - z_n)]$ is a Tsuji function such that $J(f) = C$.

The sequence $\{z_n\}$ lies on the monotone spiral

$$\Gamma: \{z(t) = (1 - t^{-1/2}) \exp(i \log t): 1 \leq t < \infty\}.$$

For any $\theta \in [0, 2\pi]$, if $\{w_k(\theta)\}_{k=1}^{\infty} = \Gamma \cap \{z: \arg z = \theta\}$ is arranged in order of increasing modulus, direct calculation shows that $\lim_{k\to\infty} \rho[w_k(\theta), w_{k+1}(\theta)] = \pi/2$.

Choose a monotone sequence $\{\xi_k\}_{k=1}^{\infty}$ on the radius to $e^{i\theta}$ such that ξ_k is midway between $w_k(\theta)$ and $w_{k+1}(\theta)$, so that $\xi_k \to e^{i\theta}$. We can choose a number δ, $0 < \delta < \pi/2$, and positive integer K such that

$$\mathscr{D}(\delta, K) = \bigcup_{k \geq K} \{z \in D : \rho(z, \xi_k) < \delta\}$$

is disjoint from all the disks $\{z \in D : \rho(z, z_n) < r_n\}$. The details of [3, Theorem 1] show that f converges uniformly to a constant as $|z| \to 1$ in $\mathscr{D}(\delta, K)$. Thus f is of the second kind.

If E is a subset of C which is residual but of measure 0 on C, there exists a Tsuji function g of bounded characteristic such that $E \subset J(g)$ [3, Theorem 3]. Thus $J(g)$ is residual and of measure 0 on C. And there exists a Tsuji function h for which $J(h)$ is of measure 2π but of first category on C [2, pp. 199–200]. Both g and h are necessarily of the second kind.

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