Meromorphic $N=2$ Wess–Zumino supersymmetric quantum mechanics

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The ordinary (holomorphic) $N=2$ Wess–Zumino model in supersymmetric quantum mechanics is extended to the case where the superpotential $V(z)$ is a meromorphic function on $\mathbb{C}\cup\{\infty\}$. The extended model is analyzed in a mathematically rigorous way. Self-adjoint extensions and the essential self-adjointness of the supercharges are discussed. The supersymmetric Hamiltonian defined by one of the self-adjoint extensions of the supercharges has no fermionic zero-energy states ("vanishing theorem"). It is proven that if $V(z)$ has only one pole at $z=0$ in $\mathbb{C}$, then the supercharges are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2\setminus\{0\};\mathbb{C}^4)$. The special case where $V(z)=\lambda z^{-p}(p\in\mathbb{N},\lambda\in\mathbb{C}\setminus\{0\})$ is analyzed in detail to prove the following facts: (i) the number of the bosonic zero-energy ground state(s) is equal to $p-1$; (ii) the supercharges are not Fredholm.

I. INTRODUCTION

In the ordinary (holomorphic) $N=2$ Wess–Zumino (WZ) model in supersymmetric quantum mechanics (SSQM),[1-4] which describes the interaction between a complex bosonic degree of freedom, denoted by $z\in\mathbb{C}$, and two fermionic degrees of freedom, the superpotential $V(z)$ is a polynomial of z. It has been proven[1] on this model that there exist no fermionic zero-energy states ("vanishing theorem") and the number of the bosonic zero-energy ground state(s) is equal to deg $V-1$. Moreover, in the case where deg $V\geqslant 3$, the structure of the degenerate ground states has been discussed.[3,4] In Ref. 2 the $N=2$ WZ model has been extended to the case where the superpotential $V(z)$ is a nonpolynomial holomorphic function; in particular, it has been shown that in the case $V(z)=\lambda e^{\alpha z}(\lambda\in\mathbb{C}\setminus\{0\},\alpha>0\text{:const})$, there exist infinitely many bosonic zero-energy ground states.

It is interesting (at least from a mathematical point of view) to see what happens if the superpotential $V(z)$ is a meromorphic function. This is the basic motivation of this paper. This namely leads us to consider the $N=2$ WZ model with a meromorphic superpotential, which may be called the meromorphic WZ model. Generally speaking, in constructing a SSQM model in a mathematically rigorous way, one first defines the supercharges of the model on a suitable dense domain in the Hilbert space of state vectors for the model and then has to prove the (essential) self-adjointness of them. This can be easily done in the case of the aforementioned holomorphic WZ model.[1,2] In the case of the meromorphic WZ model, however, this is not so obvious, because the Dirac type operators representing the supercharges of the model are singular in the sense that their potentials have singularities and hence one must be careful about defining them properly; it is nontrivial whether the supercharges are essentially self-adjoint on suitable regular domains. This requires us to consider the problem on self-adjoint extensions of the supercharges.

The outline of the present paper is as follows. In Sec. II we define the $N=2$ WZ model with a meromorphic superpotential $V(z)$ on $\mathbb{C}\cup\{\infty\}$. The Hilbert space of the state vectors for the model is realized as $L^2(\mathbb{R}^2;\mathbb{C}^4)$, the Hilbert space of $\mathbb{C}^4$-valued square integrable functions on $\mathbb{R}^2$. Let $\mathbb{P}$ be the set of the poles of $V(z)$ in $\mathbb{C}$. We first construct two self-adjoint extentions of one of the supercharges restricted to $C_0^\infty(\mathbb{C}\setminus\mathbb{P};\mathbb{C}^4)$, the space of $\mathbb{C}^4$-valued C^∞-functions with compact support in $\mathbb{C}\setminus\mathbb{P}$. We show that the vanishing theorem holds for the supersymmetric (SUSY) Hamiltonian defined by one of the self-adjoint supercharges. Then we discuss the problem of the essential self-adjointness of the SUSY Hamiltonian and the supercharges. In particular, we prove that if $V(z)$ has only one pole at $z=0$ in $\mathbb{C}$ [$z=\infty$ may be a pole of $V(z)$], then the SUSY Hamiltonian and the supercharges are essentially self-adjoint on $C_0^\infty(\mathbb{C}\setminus\{0\};\mathbb{C}^4)$.

In Sec. III we analyze in detail the meromorphic WZ model with $V(z)=\lambda/z^p(\lambda\in\mathbb{C}\setminus\{0\},p\in\mathbb{N})$. We show that, in this case, a symmetry group acts on the quantum system under consideration. A structure similar to this appears in the case of the holomorphic WZ model with $V(z)=\lambda z^p$ (see Ref. 3). We prove that the number of the bosonic zero-energy ground state(s) is equal to $p-1$ and clarify the structure of the ground state(s).

It is known that the supercharges of the holomorphic WZ model with a polynomial superpotential are Fredholm.[1] In the last section we examine whether the meromorphic WZ model has this property. But the result is negative. We prove that the supercharges of the WZ model discussed in Sec. III are not Fredholm.

II. THE $N=2$ WZ MODEL WITH A MEROMORPHIC SUPERPOTENTIAL

In this section we define the $N=2$ WZ model with a meromorphic superpotential and discuss general aspects of it. The Hilbert space of state vectors for the model is given by

$$\mathscr{H}=L^2(\mathbb{C};\mathbb{C}^4)=L^2(\mathbb{R}^2;\mathbb{C}^4)$$

$$=\left\{\begin{pmatrix}f_1\\f_2\\f_3\\f_4\end{pmatrix}\middle| f_j\in L^2(\mathbb{R}^2),\ j=1,2,3,4\right\},$$

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which is decomposed as

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-,$$

with

$$\mathscr{H}_+ = \left\{ \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix} \Big| f,g \in L^2(\mathbb{R}^2) \right\} \cong L^2(\mathbb{R}^2;\mathbb{C}^2),$$

called the space of *bosonic states*, and

$$\mathscr{H}_- = \left\{ \begin{pmatrix} 0 \\ 0 \\ f \\ g \end{pmatrix} \Big| f,g \in L^2(\mathbb{R}^2) \right\} \cong L^2(\mathbb{R}^2;\mathbb{C}^2),$$

called the space of *fermionic states.*

Let

$$\psi_1 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_0 + \sigma_3 \\ \sigma_0 - \sigma_3 & 0, \end{pmatrix},$$

$$\psi_2 = \frac{1}{2}\begin{pmatrix} 0 & i\sigma_1 + \sigma_2 \\ -i\sigma_1 - \sigma_2 & 0, \end{pmatrix},$$

with $\{\sigma_j\}_{j=1}^3$ and σ_0 being the Pauli matrices and the 2×2 identity matrix, respectively.

Let $V(z)$ $(z\in\mathbb{C})$ be a meromorphic function on $\mathbb{C}\cup\{\infty\}$ and $\{a_1,...,a_N\}$ $(N<\infty)$ be its poles in $\mathbb{C}[z=\infty$ may be a pole of $V(z)]$. Then one of the supercharges of the WZ model with the superpotential $V(z)$ is given by the following operator in $\mathscr{H}$:

$$\hat{Q} = i\psi_2\bar{\partial} + i\psi_2^*\partial + i\psi_1(\partial V) - i\psi_1^*(\partial V)^*$$
$$= \begin{pmatrix} 0 & 0 & i\partial V & -\partial \\ 0 & 0 & -\bar{\partial} & -i(\partial V)^* \\ -i(\partial V)^* & \partial & 0 & 0 \\ \bar{\partial} & i\partial V & 0 & 0 \end{pmatrix}, \quad (2.1)$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial z^*$. We remark that the meromorphic WZ model under consideration also has two supercharges as in the case of the holomorphic WZ model,[1-4] but, in the present paper, we consider only one of them. Arguments similar to those given below apply to the other supercharge as well.

In general, the supercharge(s) of a model in SSQM should be self-adjoint.[5-7] As is seen from (2.1), $\hat{Q}$ is an operator of Dirac type with a singular potential. This makes it nontrivial whether $\hat{Q}$ is (essentially) self-adjoint. A natural regular domain for $\hat{Q}$ is $C_0^\infty(\Omega;\mathbb{C}^4)$, where

$$\Omega = \mathbb{C}\setminus\{a_n\}_{n=1}^N.$$

We first construct two self-adjoint extensions of the operator

$$Q_0 = \hat{Q} \restriction C_0^\infty(\Omega;\mathbb{C}^4). \quad (2.2)$$

The problem of the essential self-adjointness of Q_0 will be discussed in Sec. II C. We shall denote by $D(A)$ the domain of the operator A.

A. Self-adjoint extensions of Q_0

We introduce generalized derivatives associated with the differential operators ∂ and $\bar{\partial}$: let $f\in L^1_{\text{loc}}(\Omega)$ and $m,n\in\mathbb{N}\cup\{0\}$. If there exists a function $f_{m,n}\in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega f(x,y)\partial^m\bar{\partial}^n u(x,y)dx\,dy$$
$$= (-1)^{m+n}\int_\Omega f_{m,n}(x,y)u(x,y)dx\,dy, \quad u\in C_0^\infty(\Omega),$$

then we say that $f_{m,n}$ is the generalized $\{m,n\}$-derivative of f and write as

$$f_{m,n} = D^m\bar{D}^n f.$$

Let

$$Q_- = -\frac{1}{2}(\sigma_1 + i\sigma_2)\partial - \frac{1}{2}(\sigma_1 - i\sigma_2)\bar{\partial}$$
$$+ \frac{i}{2}(\sigma_0+\sigma_3)\partial V - \frac{i}{2}(\sigma_0 - \sigma_3)(\partial V)^*$$
$$= \begin{pmatrix} i\partial V & -\partial \\ -\bar{\partial} & -i(\partial V)^* \end{pmatrix}, \quad (2.3)$$

with

$$D(Q_-) = C_0^\infty(\Omega;\mathbb{C}^2). \quad (2.4)$$

In general, we shall denote by $\langle\cdot,\cdot\rangle$ inner product (linear in the second variable) and by $\|\cdot\|$ norm.

Lemma 2.1: The adjoint Q_-^* of Q_- is given as follows:

$$D(Q_-^*) = \{(\begin{smallmatrix} f \\ g \end{smallmatrix})\in L^2(\mathbb{R}^2;\mathbb{C}^2) \,|\, \exists\bar{D}f,$$
$$\exists Dg \text{ satisfying } Dg - i(\partial V)^*f\in L^2(\mathbb{R}^2),$$
$$\bar{D}f + i(\partial V)g\in L^2(\mathbb{R}^2)\}, \quad (2.5)$$

$$Q_-^*\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} Dg - i(\partial V)^*f \\ \bar{D}f + i(\partial V)g \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix}\in D(Q_-^*). \quad (2.6)$$

Proof: Let M be the set given by the right-hand side of (2.5). Let $(f,g)\in D(Q_-^*)$. Then there exists a vector $(\eta,\xi)\in L^2(\mathbb{R}^2;\mathbb{C}^2)$ such that for all $u,v\in C_0^\infty(\Omega)$,

$$\left\langle \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, Q_- \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle, \quad (2.7)$$

which is equivalent to the equation

$$\langle\eta,u\rangle + \langle\xi,v\rangle = \langle f, i(\partial V)u - \partial v\rangle$$
$$- \langle g, \bar{\partial}u + i(\partial V)^*v\rangle, u,v\in C_0^\infty(\Omega). \quad (2.8)$$

Taking $v=0$, we have $(z = x+iy, x,y\in\mathbb{R})$

$$\langle g,\bar{\partial}u\rangle = \langle f, i(\partial V)u\rangle - \langle\eta,u\rangle$$
$$= \int_\Omega [if(x,y)^*(\partial V)(z) - \eta(x,y)^*]u(x,y)dx\,dy.$$

It is easy to see that $i(\partial V)f^* - \eta^*\in L^1_{\text{loc}}(\Omega)$. Hence, $\bar{D}g^*$ exists and

$$-\bar{D}g^* = i(\partial V)f^* - \eta^*.$$

Taking the complex conjugate of this equation, we obtain

$$Dg - i(\partial V)^*f = \eta\in L^2(\mathbb{R}^2).$$

Similarly, taking $u=0$ in (2.8) implies that $\bar{D}f$ exists and

$$\bar{D}f + i(\partial V)g = \xi\in L^2(\mathbb{R}^2).$$

Thus $D(Q_-^*)\subset M$.

Let $(f,g)\in M$ and set

$$\eta = Dg - i(\partial V)^* f, \quad \xi = \overline{D}f + i(\partial V)g.$$

Then, $\eta,\xi\in L^2(\mathbb{R}^2)$ and (2.7) holds for all $u,v\in C_0^\infty(\Omega)$. This implies that $(f,g)\in D(Q_-^*)$. ∎

It follows from Lemma 2.1 that

$$C_0^\infty(\Omega;\mathbb{C}^2)\subset D(Q_-^*)$$

and hence $D(Q_-^*)$ is dense in $L^2(\mathbb{R}^2)$. Therefore, Q_- is closable. We denote its closure by $\overline{Q}_-$. Let

$$Q=\begin{pmatrix} 0 & \overline{Q}_- \\ Q_-^* & 0 \end{pmatrix} \tag{2.9}$$

with $D(Q)=D(Q_-^*)\oplus D(\overline{Q}_-)$.

Proposition 2.2: The operator Q is a self-adjoint extension of Q_0.

Proof: The self-adjointness of Q is obvious. We see from (2.1)–(2.3) and (2.6) that $Q=Q_0$ on $C_0^\infty(\Omega;\mathbb{C}^4)$, which implies that Q is an extension of Q_0. ∎

The SUSY Hamiltonian H of the WZ model with the supercharge Q is given by

$$H\equiv Q^2=\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{2.10}$$

with

$$H_+=\overline{Q}_- Q_-^*, \tag{2.11}$$

the *bosonic Hamiltonian*, and

$$H_-=Q_-^*\overline{Q}_-, \tag{2.12}$$

the *fermionic Hamiltonian*. Since $H=Q_0^2$ on $C_0^\infty(\Omega;\mathbb{C}^4)$, we have

$$H_+=H_-+\begin{pmatrix} 0 & -i\partial^2 V \\ i(\partial^2 V)^* & 0 \end{pmatrix}, \tag{2.13}$$

$$H_-=(-\overline{\partial}\partial+|\partial V|^2)\sigma_0, \tag{2.14}$$

on $C_0^\infty(\Omega;\mathbb{C}^2)$. Therefore, $H_\pm$ are a self-adjoint extension of a two-dimensional Schrödinger operator with a matrix-valued singular potential, respectively.

We can construct another self-adjoint extension of Q_0. Let

$$\begin{aligned} Q_+ &= \frac{1}{2}(\sigma_1+i\sigma_2)\partial+\frac{1}{2}(\sigma_1-i\sigma_2)\overline{\partial} \\ &\quad +\frac{i}{2}(\sigma_0-\sigma_3)\partial V-\frac{i}{2}(\sigma_0+\sigma_3)(\partial V)^* \\ &= \begin{pmatrix} -i(\partial V)^* & \partial \\ \overline{\partial} & i(\partial V) \end{pmatrix}, \end{aligned} \tag{2.15}$$

with

$$D(Q_+)=C_0^\infty(\Omega;\mathbb{C}^2). \tag{2.16}$$

Then, in the same way as in the proof of Lemma 2.1, we can prove the following lemma.

Lemma 2.3: The operator Q_+^* is given as follows:

$$\begin{aligned} D(Q_+^*)=\{\tbinom{f}{g}\in L^2(\mathbb{R}^2)\,|\,&\exists Dg, \\ &\exists \overline{D}f \text{ satisfying } -Dg+i(\partial V)f\in L^2(\mathbb{R}^2), \\ &\overline{D}f+i(\partial V)^* g\in L^2(\mathbb{R}^2)\}, \end{aligned}$$

$$Q_+^*\begin{pmatrix} f \\ g \end{pmatrix}=\begin{pmatrix} -Dg+i(\partial V)f \\ -\overline{D}f-i(\partial V)^* g \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix}\in D(Q_+^*).$$

Lemma 2.3 implies that Q_+ is closable. Then, in the same way as in the proof of Proposition 2.2, we have the following fact.

Proposition 2.4: The operator

$$\widetilde{Q}=\begin{pmatrix} 0 & Q_+^* \\ \overline{Q}_+ & 0 \end{pmatrix}, \tag{2.17}$$

with $D(\widetilde{Q})=D(\overline{Q}_+)\oplus D(Q_+^*)$ is a self-adjoint extension of Q_0.

Obviously we have

$$\overline{Q}_+\subset Q_-^*, \quad \overline{Q}_-\subset Q_+^*.$$

B. Vanishing theorem and equations for bosonic zero-energy ground states

In this subsection we consider only the WZ model with the supercharge Q. We are interested in the zero-energy ground state(s) of the model. The "vanishing theorem" holds also in the present case.

Theorem 2.5:

$$\operatorname{Ker} H_-=\operatorname{Ker}\overline{Q}_-=\{0\}. \tag{2.18}$$

Proof: The first equality in (2.18) follows from (2.12). We prove the second equality in (2.18). By integration by parts, we have for all $u,v\in C_0^\infty(\Omega)$

$$\left\|Q_-\begin{pmatrix} u \\ v \end{pmatrix}\right\|^2=\|(\partial V)u\|^2+\|(\partial V)^* v\|^2+\|\partial v\|^2+\|\overline{\partial}u\|^2. \tag{2.19}$$

Let $(f,g)\in\operatorname{Ker}\overline{Q}_-$. Then there exist sequences $\{f_n\}_{n=1}^\infty,\{g_n\}_{n=1}^\infty\subset C_0^\infty(\Omega)$ such that $f_n\to f$, $g_n\to g$ $(n\to\infty)$ and

$$Q_-\begin{pmatrix} f_n \\ g_n \end{pmatrix}\to 0.$$

It follows from (2.19) that

$$\|\partial g_n\|^2\to 0, \quad \|\overline{\partial}f_n\|^2\to 0.$$

Note that

$$\|\partial g_n\|^2=-\langle g_n,\overline{\partial}\partial g_n\rangle=\tfrac{1}{4}\{\|\partial_x g_n\|^2+\|\partial_y g_n\|^2\}.$$

Hence, it follows that $g\in D(D_x)\cap D(D_y)$ and

$$D_x g=D_y g=0,$$

where D_x (resp. D_y) is the generalized derivative in direction x (resp. y). This, together with the fact $g\in L^2(\mathbb{R}^2)$, implies that $g=0$. Similarly we can show that $f=0$. Thus (2.18) follows. ∎

We next consider the zero-energy ground state(s) of H_+. It follows from (2.11) that

$$\operatorname{Ker} H_+=\operatorname{Ker} Q_-^*. \tag{2.20}$$

We derive partial differential equations for any zero-energy ground state of H_+.

Lemma 2.6: (i) The vector $(f,-ig)\in L^2(\mathbb{R}^2;\mathbb{C}^2)$ is in $\operatorname{Ker} H_+$ if and only if Dg and $\overline{D}f$ exist satisfying

$$Dg+(\partial V)^* f=0, \quad \overline{D}f+(\partial V)g=0. \tag{2.21}$$

(ii) Every $(f,-ig)\in\operatorname{Ker} H_+$ is in $C^\infty(\Omega;\mathbb{C}^2)$ and satisfies

$$(-\partial\bar\partial+|\partial V|^2)f+(\partial^2 V)(\partial V)^{-1}\bar\partial f=0, \qquad (2.22)$$

$$(-\partial\bar\partial+|\partial V|^2)g+(\bar\partial^2 V^*)(\bar\partial V^*)^{-1}\partial g=0, \quad (2.23)$$

(iii) For every solution $g\in C^\infty(\Omega)$ to Eq. (2.23), the pair $(f,-ig)$ with $f=-(\bar\partial V^*)^{-1}Dg$ is a solution to Eq. (2.21).

Proof: (i) This follows from (2.20) and Lemma 2.1.

(ii) Let $(f,-ig)\in\operatorname{Ker} H_+$. Then, by part (i), (2.21) holds. Since $\partial V\in C^\infty(\Omega)$, it follows that $D[(\partial V)g]$ exists and

$$D[(\partial V)g]=(\partial^2 V)g+(\partial V)Dg,$$

which, together with (2.21), implies that $D\bar D f$ exists and

$$(-D\bar D+|(\partial V)|^2)f+(\partial^2 V)(\partial V)^{-1}\bar D f=0.$$

Similarly, we can show that $\bar D D g$ exists and

$$(-\bar D D+|(\partial V)|^2)g+(\bar\partial^2 V^*)(\bar\partial V^*)^{-1}Dg=0.$$

Since $|\partial V|^2$, $(\partial^2 V)(\partial V)^{-1}$, and $(\bar\partial^2 V^*)(\bar\partial V^*)^{-1}$ are in $C^\infty(\Omega)$, it follows from elliptic regularity (e.g., Sec. IX.6 in Ref. 8) that f and g are in $C^\infty(\Omega)$. Thus the desired results follow.

(iii) This follows from direct computations. ■

C. Essential self-adjointness of Q_0

In this subsection we give a sufficient condition for Q_0 to be essentially self-adjoint. Let

$$L=-\partial\bar\partial+|\partial V|^2, \qquad (2.24)$$

with $D(L)=C_0^\infty(\Omega)$. We prepare some lemmas.

Lemma 2.7: The following inequality holds:

$$\|\partial\bar\partial u\|^2+\||\partial V|^2u\|^2-\|(\partial^2V)u\|^2 \leqslant \|Lu\|^2,\quad u\in C_0^\infty(\Omega). \qquad (2.25)$$

Proof: See the proof of Lemma 2.7 in Ref. 2. ■

Lemma 2.8: For every $\epsilon>0$, there exists a constant $c_\epsilon>0$ such that

$$|\partial^2 V(z)|^2\leqslant\epsilon|\partial V(z)|^4+c_\epsilon,\quad z\in\Omega. \qquad (2.26)$$

Proof: Let $k_n\geqslant 1$ and $m\geqslant 0$ be the order of $V(z)$ at $z=a_n$ and $z=\infty$, respectively. We can write $V(z)$ as

$$V(z)=b_0+b_1z+b_2z^2+\cdots+b_mz^m + \sum_{n=1}^{N}\left\{\frac{c_{n,1}}{z-a_n}+\frac{c_{n,2}}{(z-a_n)^2}+\cdots+\frac{c_{n,k_n}}{(z-a_n)^{k_n}}\right\},$$

where b_r and $c_{n,j}$ are constants ($c_{n,k_n}\neq 0, n=1,\dots,N$). We have for $m\geqslant 2$

$$|\partial^2V(z)|\sim|b_m|m(m-1)|z|^{m-2},$$
$$|\partial V(z)|\sim|b_m|m|z|^{m-1},$$

as $|z|\to\infty$. Hence, for every $\epsilon>0$, there exists a constant $R_\epsilon>\max\{|a_1|,\dots,|a_N|\}$ such that

$$|\partial^2V(z)|^2\leqslant\epsilon|\partial V(z)|^4,\quad |z|>R_\epsilon. \qquad (2.27)$$

This inequality obviously holds for the case $m=0,1$. On the other hand, we have for each $n=1,\dots,N$,

$$|\partial^2V(z)|\sim k_n(k_n+1)|c_{n,k_n}|/|z-a_n|^{k_n+2},$$
$$|\partial V(z)|^2\sim k_n^2|c_{n,k_n}|^2/|z-a_n|^{2(k_n+1)},$$

as $z\to a_n$. Hence, for every $\epsilon>0$, there exists a constant δ_ϵ such that

$$|\partial^2V(z)|^2\leqslant\epsilon|\partial V(z)|^4,$$
$$z\in S_\epsilon\equiv\cup_{n=1}^N\{z\in\mathbb{C}|\ 0<|z-a_n|<\delta_\epsilon\}. \qquad (2.28)$$

We can take R_ϵ such that

$$S_\epsilon\subset B_\epsilon\equiv\{z\in\mathbb{C}|\ |z|\leqslant R_\epsilon\}.$$

Since $\partial^2V(z)$ is holomorphic in $B_\epsilon\setminus S_\epsilon$, the quantity

$$c_\epsilon=\sup_{|z|\in B_\epsilon\setminus S_\epsilon}|\partial^2V(z)|^2$$

is finite. Then (2.26) follows from (2.27) and (2.28). ■

Lemma 2.9: Suppose that L is essentially self-adjoint on $C_0^\infty(\Omega)$. Then

$$\bar L=-\tfrac14\Delta_\Omega+|\partial V|^2, \qquad (2.29)$$

with $D(\bar L)=D(\Delta_\Omega)\cap D(|\partial V|^2)$, where Δ_Ω is the closure of the two-dimensional Laplacian Δ restricted to $C_0^\infty(\Omega)$.

Proof: Let $0<\epsilon<1$. By (2.25) and (2.26), we have for all $u\in C_0^\infty(\Omega)$:

$$\tfrac14\|\Delta_\Omega u\|^2+(1-\epsilon)\||\partial V|^2u\|^2\leqslant\|\bar Lu\|^2+c_\epsilon\|u\|^2. \qquad (2.30)$$

By the assumption, $C_0^\infty(\Omega)$ is a core for $\bar L$. It is easy to see that $|\partial V|^2$ is essentially self-adjoint on $C_0^\infty(\Omega)$. Hence, via a limiting argument, (2.30) extends to all $u\in D(\bar L)$ and, at the same time, we have $D(\bar L)\subset D(\Delta_\Omega)\cap D(|\partial V|^2)$. Let $\hat L=-\Delta_\Omega/4+|\partial V|^2$ with $D(\hat L)=D(\Delta_\Omega)\cap D(|\partial V|^2)$. Then $\hat L$ is symmetric and $\bar L\subset\hat L$. Since $\bar L$ is self-adjoint, it follows that $\bar L=\hat L$. ■

Lemma 2.10: Let T be a symmetric operator in a Hilbert space. Suppose that there exists a dense subspace $D\subset D(T^2)$ and T^2 is essentially self-adjoint on D. Then T is essentially self-adjoint on every core of the closure of $T^2\restriction D$.

Proof: This can be easily proven by applying a Glimm–Jaffe–Nelson type theorem on essential self-adjointness of symmetric operators (e.g., Theorem X.37 in Ref. 8). ■

We are now ready to state and prove the main result in this subsection.

Theorem 2.11: Suppose that L is essentially self-adjoint. Then: (i) The operators $H_\pm$ are essentially self-adjoint on $C_0^\infty(\Omega;\mathbb{C}^2)$ and self-adjoint with $D(H_+)=D(H_-)=D(\Delta_\Omega)\cap D(|\partial V|^2)$. Moreover, the following operator equalities hold:

$$H_-=(-\tfrac14\Delta_\Omega+|\partial V|^2)\sigma_0, \qquad (2.31)$$

$$H_+=H_-+\begin{pmatrix}0 & -i\partial^2V\\ i(\partial^2V)^* & 0\end{pmatrix}. \qquad (2.32)$$

(ii) The operator Q_0 is essentially self-adjoint on every core of H.

Proof: (i) The assertion about H_- is easily proven. Let

$$H_I=\begin{pmatrix}0 & -i\partial^2 V\\ i(\partial^2 V)^* & 0\end{pmatrix}.$$

Then we have from (2.13)

$$H_+=H_-+H_I$$

on $C_0^\infty(\Omega;\mathbb{C}^2)$. By (2.26) and (2.30) we can show that for all $\psi\in C_0^\infty(\Omega;\mathbb{C}^2)$ and all $\epsilon\in(0,1)$,

$$\|H_I\psi\|^2\leqslant\frac{\epsilon}{1-\epsilon}\|H_-\psi\|^2+\frac{c_\epsilon}{1-\epsilon}\|\psi\|^2.$$

Since H_- is essentially self-adjoint on $C_0^\infty(\Omega)$, this inequality extends to all $\psi\in D(H_-)$. Take $\epsilon<1/2$. Then we have

$$\epsilon/(1-\epsilon)<1.$$

Therefore, by the Kato–Rellich theorem (e.g., Sec. X.2 in Ref. 8 or Chap. V Sec. 4 in Ref. 9), H_+ is self-adjoint with $D(H_+)=D(H_-)$ and essentially self-adjoint on every core of H_-.

(ii) Since we have

$$H=Q_0^2$$

on $C_0^\infty(\Omega;\mathbb{C}^4)$ and H is essentially self-adjoint on $C_0^\infty(\Omega;\mathbb{C}^4)$ by part (i), we can apply Lemma 2.10 with $T=Q_0$ and $D=C_0^\infty(\Omega;\mathbb{C}^4)$ to obtain the desired result. ■

When is the assumption of Theorem 2.11 satisfied? In the present paper we give an answer to the question only in a special case.

Theorem 2.12: Consider the case where $V(z)$ has only one pole at $z=0$ in $\mathbb{C}$ [$z=\infty$ may be a pole of $V(z)$]. Then L is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2\setminus\{0\})$ and the conclusion of Theorem 2.11 holds.

Proof: We have

$$4L=-\Delta+4|\partial V|^2$$

on $C_0^\infty(\mathbb{R}^2\setminus\{0\})$. In the same way as in the proof of Lemma 2.8, we can show that for every $\epsilon>0$, there exists a constant $b_\epsilon>0$ such that

$$1/|z|^2\leqslant\epsilon|4\partial V(z)|^2+b_\epsilon,\quad z\in\mathbb{C}\setminus\{0\}.$$

Take $\epsilon<1$. Then we can apply the Kalf–Walter–Schmincke–Simon theorem (Theorem X.30 in Ref. 8) to the Schrödinger operator $4L$ to conclude that $4L$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2\setminus\{0\})$ and so is L. ■

III. GROUND STATE(S) OF THE $N=2$ WZ MODEL WITH $V(z)=\lambda/z^p$

In this section we analyze the ground state structure of the $N=2$ WZ model with the superpotential

$$V(z)=\lambda/z^p,\tag{3.1}$$

where $\lambda\in\mathbb{C}\setminus\{0\}$ and $p\geqslant1$. In this case, the supercharge Q and the SUSY Hamiltonian H (resp. $H_\pm$) are essentially self-adjoint on $C_0^\infty(\Omega;\mathbb{C}^4)$ [resp. $C_0^\infty(\Omega;\mathbb{C}^2)$] (Theorem 2.12).

Let $z=re^{i\theta}$ (the polar coordinate: $r>0,\theta\in[0,2\pi]$) and

$$M=i\frac{\partial}{\partial\theta}-\frac{(p+2)}{2}\sigma_3.\tag{3.2}$$

Then, in the same way as in the proof of Lemma 3.1 in Ref. 3, we can prove the following lemma.

Lemma 3.1: For all $t,s\in\mathbb{R}$,

$$e^{itM}e^{isH_+}=e^{isH_+}e^{itM}.$$

This lemma implies that H_+ and M commute in the proper sense (e.g., Sec. VIII.5 in Ref. 10) and that the unitary group $\{\exp itM\,|\,t\in\mathbb{R}\}$ is a symmetry group of the quantum system under consideration. Note that the coefficient $-(p+2)/2$ of σ_3 in the generator M is different from that in the generator L of the symmetry group given in Ref. 3 to analyze the structure of the degenerate ground state of the $N=2$ WZ model with $V(z)=\lambda z^p$ (M is just equal to L with p replaced by $-p$).

The spectrum $\sigma(M)$ of M is purely discrete and is given as

$$\sigma(M)=\{n-(p/2)\,|\,n\in\mathbb{Z}\}.$$

The eigenspace of M with eigenvalue $1-n+(p/2)$ is given by

$$\mathscr{H}_n=\left\{\begin{pmatrix}u(r)e^{-i(p+2-n)\theta}\\ v(r)e^{in\theta}\end{pmatrix}\Big|\,u,v\in L^2(\mathbb{R}_+,r\,dr)\right\},\tag{3.3}$$

where $\mathbb{R}_+=(0,\infty)$, and $\mathscr{H}_+$ is decomposed as

$$\mathscr{H}_+=\bigoplus_{n=-\infty}^{\infty}\mathscr{H}_n.\tag{3.4}$$

By Lemma 3.1, H_+ is reduced by each $\mathscr{H}_n$. We denote by $H_{+,n}$ the reduced part of H_+ to $\mathscr{H}_n$.

Let K_ν be the modified Bessel function of the third kind (e.g., Ref. 11) and set

$$g_n(x,y)=\frac{1}{r^{p+1}}K_{(p+1-n)/p}\times\left(\frac{2|\lambda|}{r^p}\right)e^{in\theta},\quad n\in\mathbb{Z},x+iy=re^{i\theta}.\tag{3.5}$$

Let

$$\Psi_n=\begin{pmatrix}e^{i\arg\lambda}g^*_{p+2-n}\\ -ig_n\end{pmatrix}.\tag{3.6}$$

We prove the following theorem.

Theorem 3.2: Let $V(z)$ be given by (3.1). Then the bosonic Hamiltonian H_+ has exactly $p-1$ zero-energy ground state(s), i.e.,

$$\dim\operatorname{Ker}H_+=\dim\operatorname{Ker}Q^*_-=p-1,$$

and, if $p\geqslant2$, then an orthogonal basis of $\operatorname{Ker}H_+$ $(=\operatorname{Ker}Q^*_-)$ is given by $\{\Psi_n\}_{n=2}^p$; more precisely, for $n=2,\dots,p$, $H_{+,n}$ has a unique zero-energy ground state (up to constant multiples), which is given by Ψ_n.

Proof: Let $p\geqslant2$. Then, by direct computations using the recursion relation

$$zK'_\nu(z)+\nu K_\nu(z)=-zK_{\nu-1}(z)\tag{3.7}$$

and the fact that

$$K_\nu(z)=K_{-\nu}(z)\tag{3.8}$$

(e.g., Ref. 11), one can easily check that each Ψ_n with $2\leqslant n\leqslant p$ is in $D(Q^*_-)$ and satisfies (2.21). Hence, $\{\Psi_n\}_{n=2}^p\subset\operatorname{Ker}Q^*_-=\operatorname{Ker}H_+$. Thus to prove Theorem 3.2 we must show that no zero-energy ground states of H_+ exist other than linear combinations of Ψ_n's with $2\leqslant n\leqslant p$.

The following discussions are devoted to the proof of this fact.

By the reducibility of H_+ to $\mathscr{H}_n$, we have

$$\operatorname{Ker} H_+ = \bigoplus_{n=-\infty}^{\infty} \operatorname{Ker} H_{+,n}.$$

Hence, we need only to know what $\operatorname{Ker} H_{+,n}$ is like. Let $(f, -ig) \in \operatorname{Ker} H_{+,n}$ with

$$f = u(r) e^{-i(p+2-n)}, \quad g = v(r) e^{in\theta}.$$

Then, by Lemma 2.6(ii), we have that $u,v \in C^\infty(\mathbb{R}_+)$ and we see that v satisfies the differential equation

$$-v''(r) - \frac{2p+3}{r} v'(r) + \left\{ \frac{n(n-2p-2)}{r^2} + \frac{4p^2|\lambda|^2}{r^{2(p+1)}} \right\} \times v(r) = 0.$$

Let $x = 2|\lambda| r^p$ and define

$$w(x) = (1/r^{p+1}) v(1/r).$$

Then w satisfies

$$w''(x) + (1/x) w'(x) - \{1 + \nu_n^2/x^2\} w = 0, \tag{3.9}$$

where

$$\nu_n = |(p+1-n)/p|.$$

Since $v \in L^2(\mathbb{R}_+, r\,dr)$, the function w must satisfy

$$\int_0^\infty |w(x)|^2 x\, dx < \infty. \tag{3.10}$$

One easily notices that (3.9) is the modified Bessel equation.[11] It is well known that the modified Bessel functions $I_{\nu_n}(x)$ and $K_{\nu_n}(x)$ form a fundamental system of (3.9). Since

$$I_{\nu_n}(x) \sim e^x/\sqrt{2\pi x}$$

and

$$K_{\nu_n}(x) \sim \sqrt{\pi/2x}\, e^{-x}$$

as $x \to \infty$, each solution $w(x)$ to (3.9) satisfying (3.10) must be a constant multiple of $K_{\nu_n}(x)$. Moreover, taking into account the asymptotic behavior

$$K_{\nu_n}(x) \sim (1 - \delta_{0,\nu_n}) \frac{\text{const}}{x^{\nu_n}} + \delta_{0,\nu_n} \text{const} \log x$$

as $x \to +0$, we see that $w(x)$ with condition (3.10) is a solution to (3.9) if and only if it is a constant multiple of $K_{\nu_n}(x)$ with

$$0 \leqslant \nu_n < 1,$$

i.e.,

$$2 \leqslant n \leqslant 2p. \tag{3.11}$$

Thus we obtain $g = g_n$ (up to constant multiples) with the restriction (3.11). This shows also that, if $p = 1$, then $\operatorname{Ker} H_+ = \{0\}$, so that Theorem 3.2 with $p=1$ is proven. By Lemma 2.6(i), Dg must exist and

$$f = -(\partial V)^{*-1} D g_n.$$

By direct computations using (3.7), we can see that

$$Dg_n = \left\{ \frac{|p+1-n| + n - p - 1}{2r^{p+2}} K_{\nu_n}\left(\frac{2|\lambda|}{r^p}\right) + \frac{|\lambda| p}{r^{2(p+1)}} K_{\nu_n - 1}\left(\frac{2|\lambda|}{r^p}\right) \right\} e^{i(n-1)\theta}.$$

It follows from this expression and the asymptotics of $K_\nu(z)$ at $z=0$ and $z=\infty$ that f is in $L^2(\mathbb{R}^2)$ if and only if $2 \leqslant n \leqslant p$ and in this case, we have

$$f = e^{i \arg \lambda} g^*_{p+2-n}.$$

Thus we have proven that, for $2 \leqslant n \leqslant p$, the vector Ψ_n given by (3.6) is the only (normalizable) zero-energy state of $H_{+,n}$. This completes the proof of Theorem 3.2. ∎

Remarks: (i) If $n \in \mathbb{Z} \setminus \{2,\dots,p\}$, then Ψ_n is not in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, but, it satisfies the *partial differential equation*

$$H_+ \Psi_n = 0.$$

Thus each Ψ_n with $n \notin \{2,\dots,p\}$ is a generalized eigenfunction of H_+ with eigenvalue zero. This shows that there exist infinitely many generalized zero-energy eigenstates of H_+. This kind of phenomenon, which may be interesting, appears in the $N=2$ WZ model with $V(z) = \lambda z^p$ (see Ref. 3).

(ii) Let

$$\Phi_n = \begin{pmatrix} e^{-i\arg\lambda} K_{n/p}(2|\lambda|/r^p) e^{in\theta} \\ -iK_{(p-n)/p}(2|\lambda|/r^p) e^{i(n-p)} \end{pmatrix}. \tag{3.12}$$

Then one can easily check that for all $n \in \mathbb{Z}$,

$$Q_- \Phi_n = 0, \quad H_- \Phi_n = 0$$

as *partial differential equations in* $\mathbb{R}^2 \setminus \{0\}$. But, for all $n \in \mathbb{Z}$, $\Phi_n \notin L^2(\mathbb{R}^2;\mathbb{C}^2)$. Hence, each Φ_n is a generalized eigenfunction of H_- with eigenvalue 0. Thus H_- also has infinitely many generalized zero-energy eigenstates.

IV. NON-FREDHOLMNESS OF THE SUPERCHARGE

For a densely defined closed linear operator A from a Hilbert space to another Hilbert space, the analytic index $\operatorname{index}(A)$ of A is defined by

$$\operatorname{index}(A) = \dim \operatorname{Ker} A - \dim \operatorname{Ker} A^*.$$

It is known that the number of the zero-energy ground state(s) of an SSQM model is related to the analytic index of the supercharges, restricted to the bosonic states or the fermionic ones, of the model. In the case of the meromorphic WZ model discussed in Sec. II, we have, from Theorem 2.5,

$$\begin{aligned} \dim \operatorname{Ker} H_+ &= \dim \operatorname{Ker} Q^*_- \\ &= \dim \operatorname{Ker} Q^*_- - \dim \operatorname{Ker} \overline{Q}_- \\ &= -\operatorname{index}(\overline{Q}_-). \end{aligned}$$

It is well known (e.g., Chap. IV Sec. 5.3 in Ref. 9) that if A is Fredholm, then index (A) is invariant under compact perturbations relative to A. Therefore, it is interesting to examine whether $\overline{Q}_-$ is Fredholm or not. We prove that, in the case where $V(z)$ is given by (3.1), $\overline{Q}_-$ is not Fredholm.

Theorem 4.1: Let Q_- be defined by (2.3) with $V(z)$ given by (3.1). Then $\overline{Q}_-$ is not Fredholm.

We shall prove that $\operatorname{Ran} \overline{Q}_-$ is not closed. Then Theorem 4.1 follows.

Since $\operatorname{Ker} \overline{Q}_- = \{0\}$ as already proved (Theorem 2.5), $\operatorname{Ran} \overline{Q}_-$ is closed if and only if

$$\|\overline{Q}_- \psi\| \geqslant c\|\psi\|, \quad \psi \in D(\overline{Q}_-),$$

with some constant $c>0$. Hence, to prove the nonclosedness of $\operatorname{Ran} \overline{Q}_-$, we need only to show that there exists a sequence $\{\Omega_n\}_n \subset D(\overline{Q}_-)$ $(\Omega_n \neq 0)$ such that

$$\|\overline{Q}_- \Omega_n\| / \|\Omega_n\| \to 0. \tag{4.1}$$

The idea to do that is to note that Φ_0 defined by (3.12) is a generalized eigenfunction with eigenvalue zero. We put

$$v_j(r) = K_j(2|\lambda|/r^p), \quad j=0,1.$$

Let $\rho \in C_0^\infty(\mathbb{R})$ satisfying

$$\rho(x) \geqslant 0, \quad \rho(x) = \rho(-x), \quad \text{for all } x \in \mathbb{R},$$
$$\rho(x) = 0 \text{ for } |x| \geqslant 1,$$
$$\int \rho(x) dx = 1,$$

and, for $\epsilon > 0$, define ρ_ϵ by

$$\rho_\epsilon(x) = (1/\epsilon)\rho(x/\epsilon).$$

Let $0 < \epsilon \ll c < \delta$ and introduce the following functions:

$$\phi_{\epsilon\delta}(r) = e^{-i\arg\lambda}\rho_\epsilon * (\chi_{[c,\delta]} v_0)(r),$$
$$\psi_{\epsilon\delta}(r) = \rho_\epsilon * (\chi_{[c,\delta]} v_1)(r),$$
$$g_{\epsilon\delta}(r,\theta) = \phi_{\epsilon\delta}(r)$$
$$f_{\epsilon\delta}(r,\theta) = \psi_{\epsilon\delta}(r) e^{-ip\theta},$$

where $*$ denotes convolution and $\chi_{[c,\delta]}$ is the characteristic function of $[c,\delta]$. Obviously $f_{\epsilon\delta}$ and $g_{\epsilon\delta}$ are in $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. Let

$$\Omega_{\epsilon\delta} = \begin{pmatrix} g_{\epsilon\delta} \\ -if_{\epsilon\delta} \end{pmatrix}.$$

Lemma 4.2: There exists a constant $C>0$ such that

$$C\delta^2 (\log\delta)^2 \leqslant \|g_{\epsilon\delta}\|^2, \tag{4.2}$$
$$C\delta^{2p+2} \leqslant \|f_{\epsilon\delta}\|^2, \tag{4.3}$$

for all sufficiently large δ and small ϵ.

Proof: We prove only (4.2). Estimate (4.3) can be proven similarly. By using the property that $\operatorname{supp} \phi_{\epsilon\delta} \subset [c-\epsilon, \delta+\epsilon]$ and the Schwarz inequality, we have

$$\|g_{\epsilon\delta}\|^2 = 2\pi \int_{c-\epsilon}^{\delta+\epsilon} |\phi_{\epsilon\delta}(r)|^2 r\, dr$$
$$\geqslant \frac{4\pi}{(\delta+c)(\delta-c+2\epsilon)} \left[\int_{c-\epsilon}^{\delta+\epsilon} r\, dr\, \phi_{\epsilon\delta}(r) \times 1\right]^2.$$

The integral part can be written as

$$\int_{c-\epsilon}^{\delta+\epsilon} r\, dr\, \phi_{\epsilon\delta}(r) = \int_c^\delta dx\, v_0(x) \int_{c-\epsilon}^{\delta+\epsilon} r\, dr\, \rho_\epsilon(r-x)$$
$$= \int_c^\delta x v_0(x) dx,$$

where in the last equality we have used the fact $\rho(x) = \rho(-x)$. By the asymptotic behavior $v_0(x) \sim \text{const} \log x$ as $x \to \infty$, which follows from the asymptotics of $K_0(z)$ at $z=0$, we have

$$\int_c^\delta v_0(x) x\, dx \geqslant \text{const } \delta^2 \log\delta$$

for large δ. Hence (4.2) follows. ■

Lemma 4.3: There exists a constant $C>0$ such that

$$\|\bar\partial g_{\epsilon\delta} + (\partial V)^* f_{\epsilon\delta}\|^2 \leqslant C\{\delta(\log\delta)^2/\epsilon + \epsilon\delta^{2p+2}\}, \tag{4.4}$$
$$\|\partial f_{\epsilon\delta} + (\partial V) g_{\epsilon\delta}\|^2 \leqslant C\{\delta^{2p+1}/\epsilon + \epsilon\delta^{2p+2}\}, \tag{4.5}$$

for all sufficiently large δ and small ϵ.

Proof: We have

$$\|\bar\partial g_{\epsilon\delta} + (\partial V)^* f_{\epsilon\delta}\|^2_{L^2(\mathbb{C};dz)} = \frac{\pi}{2} \int_{c-\epsilon}^{\delta+\epsilon} r\, dr \times \left| \frac{\partial}{\partial r} \phi_{\epsilon\delta} - \frac{2p\lambda^*}{r^{p+1}} \psi_{\epsilon\delta} \right|^2.$$

By integration by parts, we obtain

$$\frac{\partial}{\partial r} \phi_{\epsilon\delta} = e^{-i\arg\lambda} \left\{ [-\rho_\epsilon(x-r) v_0(x)]_c^\delta + \int_c^\delta dx\, \rho_\epsilon(x-r) \frac{\partial}{\partial x} v_0(x) \right\}.$$

Using the recursion formula (3.7) of $K_\nu(z)$ and (3.8), we see that

$$\frac{\partial}{\partial x} v_0(x) = \frac{2|\lambda| p}{x^{p+1}} v_1(x).$$

Hence, we have

$$\frac{\partial}{\partial r} \phi_{\epsilon\delta} - \frac{2p\lambda^*}{r^{p+1}} \psi_{\epsilon\delta}$$
$$= e^{-i\arg\lambda} [-\rho_\epsilon(x-r) v_0(x)]_c^\delta + 2p\lambda^* \int_c^\delta dx\, \rho_\epsilon(x-r) \left(\frac{1}{x^{p+1}} - \frac{1}{r^{p+1}}\right) v_1(x).$$

Therefore, we obtain

$$\|\bar\partial g_{\epsilon\delta} + (\partial V)^* f_{\epsilon\delta}\|^2 \leqslant A_{\epsilon\delta} + B_{\epsilon\delta},$$

with

$$A_{\epsilon\delta} = \frac{3\pi C_\rho}{2\epsilon} (v_0(\delta)^2 \delta + v_0(c)^2 c),$$
$$B_{\epsilon\delta} = 6\pi p^2 |\lambda|^2 \int_{c-\epsilon}^{\delta+\epsilon} r\, dr \times \left| \int_c^\delta \rho_\epsilon(x-r) \left(\frac{1}{x^{p+1}} - \frac{1}{r^{p+1}}\right) v_1(x) \right|^2,$$

and

$$C_\rho = \int_{-1}^1 dx |\rho(x)|^2.$$

It is easy to see that

$$A_{\epsilon\delta} \leqslant \text{const } \delta(\log\delta)^2/\epsilon$$

for large δ. Using the support property of ρ and the Schwarz inequality, we have

$$\frac{B_{\epsilon\delta}}{6\pi p^2 \lambda^2} \leqslant \int_{c-\epsilon}^{\delta+\epsilon} r\, dr \left[\int_{r-\epsilon}^{r+\epsilon} dx |\rho_\epsilon(x-r)|^2\right] \times \left[\int_{r-\epsilon}^{r+\epsilon} dx \left(\frac{1}{r^{p+1}} - \frac{1}{x^{p+1}}\right)^2 v_1(x)^2\right].$$

Note that

$$\int_{r-\epsilon}^{r+\epsilon} dx|\rho_\epsilon(x-r)|^2 = \frac{1}{\epsilon} C_\rho$$

and

$$\left(\frac{1}{r^{p+1}} - \frac{1}{(r-\epsilon)^{p+1}}\right)^2 - \left(\frac{1}{r^{p+1}} - \frac{1}{(r+\epsilon)^{p+1}}\right)^2 \geqslant 0.$$

Hence we obtain

$$\frac{B_{\epsilon\delta}}{6\pi p^2\lambda^2} \leqslant \frac{C_\rho}{\epsilon}\int_{c-\epsilon}^{\delta+\epsilon} r\,dr \times\left(\frac{1}{r^{p+1}} - \frac{1}{(r-\epsilon)^{p+1}}\right)^2 \int_{r-\epsilon}^{r+\epsilon} dx|v_1(x)|^2$$
$$\leqslant \frac{C_\rho}{\epsilon}\left(\frac{1}{(c-2\epsilon)^{p+1}} - \frac{1}{(c-\epsilon)^{p+1}}\right)^2 \times\int_{c-\epsilon}^{\delta+\epsilon} r\,dr\int_{r-\epsilon}^{r+\epsilon} dx|v_1(x)|^2.$$

Since $v_1(x)\sim\text{const } x^p$ as $x\to\infty$, we have $|v_1(x)|\leqslant\text{const } x^p$ for large x. Hence, for all sufficiently large r, we have

$$\int_{r-\epsilon}^{r+\epsilon} |v_1(x)|^2 dx \leqslant \text{const } \epsilon r^{2p}.$$

Consequently, fixing $\epsilon_0 > \epsilon$, we obtain

$$\int_{c-\epsilon}^{\delta+\epsilon} r\,dr\int_{r-\epsilon}^{r+\epsilon} |v_1(x)|^2 dx \leqslant \text{const } \delta^{2p+2},$$

for large δ. Since

$$\left(\frac{1}{(c-\epsilon)^{p+1}} - \frac{1}{(c-2\epsilon)^{p+1}}\right)^2 \sim \text{const } \epsilon^2 \quad \text{as } \epsilon\to 0,$$

we obtain

$$B_{\epsilon\delta} \leqslant \text{const } \epsilon\delta^{2p+2},$$

for large δ. Thus (4.4) follows. Similarly we can prove (4.5). ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: By Lemmas 4.2–4.3, there exist positive constants C_1 and C_2 such that

$$\|Q_-\Omega_{\epsilon\delta}\|^2 = \|\partial f_{\epsilon\delta} + (\partial V)g_{\epsilon\delta}\|^2 + \|\bar\partial g_{\epsilon\delta} + (\partial V)^* f_{\epsilon\delta}\|^2$$
$$\leqslant C_1\{(1/\epsilon)\delta^{2p+1} + \epsilon\delta^{2p+2}\}$$

and

$$\|\Omega_{\epsilon\delta}\|^2 = \|f_{\epsilon\delta}\|^2 + \|g_{\epsilon\delta}\|^2$$
$$\geqslant C_2(\delta^{2p+2} + \delta^2(\log\delta)^2),$$

for all sufficiently large δ and small ϵ. Therefore, we obtain

$$\|Q_-\Omega_{\epsilon\delta}\|^2/\|\Omega_{\epsilon\delta}\|^2 \leqslant \text{const}(1/\epsilon\delta + \epsilon),$$

for all sufficiently large δ and small ϵ. Take sequences $\{\epsilon_n\}_n, \{\delta_n\}_n$ such that $\epsilon_n\to 0$, $\delta_n\to\infty$, and $\epsilon_n\delta_n\to\infty$ as $n\to\infty$, and define $\Omega_n = \Omega_{\epsilon_n\delta_n}$. Then $\{\Omega_n\}_n$ satisfies (4.1). This completes the proof of Theorem 4.1. ∎

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