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Research Article

Meromorphic Parabolic Starlike Functions Associated with q-Hypergeometric Series

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We introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving the q-hypergeometric functions. We obtained coefficient inequalities, growth and distortion inequalities, and closure results for functions $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

1. Introduction

Let ξ be a fixed point in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathscr{H}(\Delta)$ the class of functions which are regular and

$$\mathscr{A}(\xi) = \left\{f \in H(\Delta) : f(\xi) = f'(\xi) - 1 = 0\right\}. \quad (1)$$

Also denote by $\mathscr{S}_\xi = \{f \in \mathscr{A}(\xi) : f \text{ is univalent in } \Delta\}$, the subclass of $\mathscr{A}(\xi)$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n \quad (2)$$

which are analytic in Δ. Note that $\mathscr{S}_0 = \mathscr{S}$ is subclasses of $\mathscr{A}$ consisting of univalent functions in Δ. By $\mathscr{S}_w^*(\beta)$ and $\mathscr{K}_w(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions $\Re\{(z-\xi)f'(z)/f(z)\} > \beta$, and $\Re\{1 + ((z-\xi)f''(z)/f'(z))\} > \beta$, $(z - w) \in \Delta$ for $0 \leqq \beta < 1$ introduced and studied by Kanas and Ronning [1]. The class $\mathscr{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathscr{K}_\xi^*(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [2, 3] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathscr{K} = \mathscr{K}_0(0)$ and $\mathscr{S}_0^* = \mathscr{S}^*(0)$, respectively, are the well-known standard classes of convex and starlike functions.

Let Σ denote the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (3)$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Denote by Σ_ξ the subclass of Σ consisting of the functions of the form

$$f(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_n (z - \xi)^n, \quad a_n \geq 0;\ z \neq \xi. \quad (4)$$

A function f of the form (4) is in the class of *meromorphic starlike of order* γ $(0 \leq \gamma < 1)$ denoted by $\Sigma_\xi^*(\gamma)$, if

$$-\Re\left(\frac{(z-\xi) f'(z)}{f(z)}\right) > \gamma, \qquad z - \xi \in \Delta := \Delta^* \cup \{0\}, \quad (5)$$

and is in the class of *meromorphic convex of order* γ ($0 \le \gamma < 1$) denoted by $\Sigma_\xi^K(\gamma)$, if

$$-\Re\left(1 + \frac{(z-\xi) f''(z)}{f'(z)}\right) > \gamma, \quad z - \xi \in \Delta := \Delta^* \cup \{0\}. \tag{6}$$

For functions $f(z)$ given by (4) and $g(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} b_n (z-\xi)^n$, ($b_n \ge 0$) we define the Hadamard product or convolution of f and g by

$$(f * g)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_n b_n (z-\xi)^n. \tag{7}$$

More recently, Purohit and Raina [4] introduced a generalized q-Taylor's formula in fractional q-calculus and derived certain q-generating functions for q-hypergeometric functions. In this work we proceed to derive a generalized differential operator on meromorphic functions in $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving these functions and discuss some of their properties.

For complex parameters $a_1, \dots, a_l$ and $b_1, \dots, b_m$ ($b_j \ne 0, -1, \dots$; $j = 1, 2, \dots, m$) the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} {}_l\Psi_m(a_1, \dots a_l; b_1, \dots, b_m; q, z) &:= \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} \\ &\quad \times \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n, \end{aligned} \tag{8}$$

with $\binom{n}{2} = n(n-1)/2$ where $q \ne 0$ when $l > m+1$ ($l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}$).

The q-shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a; q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n \in \mathbb{N}, \end{cases} \tag{9}$$

and in terms of basic analogue of the gamma function

$$(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{10}$$

It is of interest to note that $\lim_{q \to 1^-}((q^a; q)_n/(1-q)^n) = (a)_n = a(a+1)\cdots(a+n-1)$ is the familiar Pochhammer symbol and

$${}_l\Psi_m(a_1, \dots, a_l; b_1, \dots, b_m; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \frac{z^n}{n!}. \tag{11}$$

Now for $z \in \mathbb{U}$, $0 < |q| < 1$, and $l = m+1$, the basic hypergeometric function defined in (8) takes the form

$$\begin{aligned} {}_l\Psi_m(a_1; \dots a_l; b_1, \dots, b_m; q, z) &= \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} z^n, \end{aligned} \tag{12}$$

which converges absolutely in the open unitdisk $\mathbb{U}$.

Corresponding to the function ${}_l\Psi_m(a_1; \dots a_l; b_1, \dots, b_m; q, z)$ recently for meromorphic functions $f \in \Sigma_0$ consisting functions of the form (3), Huda and Darus [5] introduce q-analogue of Liu-Srivastava operator as below:

$$\begin{aligned} {}_l\Psi_m(a_1; \dots a_l; b_1, \dots, b_m; q, z) * f(z) &= \frac{1}{z}\, {}_l\Psi_m(a_1; \dots a_l; b_1, \dots, b_m; q, z) * f(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m, q)_{n+1}} a_n z^n, \end{aligned} \tag{13}$$

where $z \in \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

In this paper for functions $f \in \Sigma_\xi$ and for real parameters $a_1, \dots, a_l$ and $b_1, \dots, b_m$ ($b_j \ne 0, -1, \dots$; $j = 1, 2, \dots, m$) we define the following new linear operator:

$$\mathcal{I}_m^l(a_1; \dots a_l; b_1, \dots, b_m; q, z-\xi) : \Sigma_\xi \longrightarrow \Sigma_\xi, \tag{14}$$

as

$$\begin{aligned} \mathcal{I}_m^l(a_1; \dots a_l; b_1, \dots, b_m; q, z-\xi) &= \frac{1}{z-\xi}\, {}_l\Psi_m(a_1; \dots a_l; b_1, \dots, b_m; q, z-\xi) \\ \mathcal{I}_m^l[a_1, q] &= \frac{1}{z-\xi} + \sum_{n=1}^{\infty} \Upsilon_n^{l,m}[a_1, q](z-\xi)^n, \end{aligned} \tag{15}$$

where

$$\Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m, q)_{n+1}}. \tag{16}$$

Throughout our study for $f \in \Sigma_\xi$, we let

$$\begin{aligned} \mathcal{I}_m^l f(z) &= \mathcal{I}_m^l[a_l, q] * f(z) \\ &= \frac{1}{z-\xi} + \sum_{n=1}^{\infty} \Upsilon_m^l(n)\, a_n (z-\xi)^n, \end{aligned} \tag{17}$$

$$\begin{aligned} \Upsilon_m^l(n) &= \Upsilon_n^{l,m}[a_1, q] \\ &= \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1} (b_1; q)_{n+1} \cdots (b_m, q)_{n+1}}, \end{aligned} \tag{18}$$

unless otherwise stated.

Motivated by earlier works on meromorphic functions by function theorists (see [6–14]), we define the following new subclass of functions in Σ_ξ by making use of the generalized operator $\mathcal{I}_m^l$.

For $0 \le \gamma < 1$ and $0 \le \lambda < 1/2$, we let $\mathcal{M}^l_m(\lambda, \beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}^l_m f(z)\right)' + \lambda(z-\xi)^2\left(\mathcal{I}^l_m f(z)\right)''}{(1-\lambda)\mathcal{I}^l_m f(z) + \lambda(z-\xi)\left(\mathcal{I}^l_m f(z)\right)'}\right) > \beta\left|\frac{(z-\xi)\left(\mathcal{I}^l_m f(z)\right)' + \lambda(z-\xi)^2\left(\mathcal{I}^l_m f(z)\right)''}{(1-\lambda)\mathcal{I}^l_m f(z) + \lambda(z-\xi)\left(\mathcal{I}^l_m f(z)\right)'} + 1\right| + \gamma, \tag{19}$$

where $\mathcal{I}^l_m f$ is given by (17).

Further, shortly we can state this condition by

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)} + 1\right| + \gamma, \tag{20}$$

where

$$G(z) = (1-\lambda)\mathcal{I}^l_m f(z) + \lambda(z-\xi)\left(\mathcal{I}^l_m f(z)\right)' = \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty}(n\lambda-\lambda+1)\Upsilon^l_m(n)a_n(z-\xi)^n, \quad a_n \ge 0. \tag{21}$$

It is of interest to note that, on specializing the parameters λ, β and l, m, we can define various new subclasses of Σ_ξ. We illustrate two important subclasses in the following examples.

Example 1. For $\lambda = 0$, we let $\mathcal{M}^l_m(0, \beta, \gamma) = \mathcal{M}^l_m(\beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}^l_m f(z)\right)'}{\mathcal{I}^l_m f(z)}\right) > \beta\left|\frac{(z-\xi)\left(\mathcal{I}^l_m f(z)\right)'}{\mathcal{I}^l_m f(z)} + 1\right| + \gamma, \tag{22}$$

where $\mathcal{I}^l_m f(z)$ is given by (17).

Example 2. For $\lambda = 0$, $\beta = 0$ we let $\mathcal{M}^l_m(0, 0, \gamma) = \mathcal{M}^l_m(\gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$-\Re\left(\frac{(z-\xi)\left(\mathcal{I}^l_m f(z)\right)'}{\mathcal{I}^l_m f(z)}\right) > \gamma, \tag{23}$$

where $\mathcal{I}^l_m f(z)$ is given by (17).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, and closure results for the function class $\mathcal{M}^l_m(\lambda, \beta, \gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathcal{M}^l_m(\lambda, \beta, \gamma)$ are also discussed.

2. Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function, $f \in \mathcal{M}^l_m(\lambda, \beta, \gamma)$, we recall the following lemmas.

Lemma 3. *If γ is a real number and w is a complex number, then $\Re(w) \ge \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \ge 0$.*

Lemma 4. *If w is a complex number and γ, k are real numbers, then*

$$\Re(w) \ge k|w-1| + \gamma \Longleftrightarrow \Re\left\{w\left(1+ke^{i\theta}\right) - ke^{i\theta}\right\} \ge \gamma, \quad -\pi \le \theta \le \pi. \tag{24}$$

Analogous to the lemma proved by Dziok et al. [8], we state the following lemma without proof.

Lemma 5. *Suppose that $\gamma \in [0,1)$, $r \in (0,1]$, and the function $H \in \Sigma_\xi(\gamma)$ is of the form $H(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} b_n(z-\xi)^n$, $0 < |z-\xi| < r < 1$, with $b_n \ge 0$, then*

$$\sum_{n=1}^{\infty}(n+\gamma)b_n r^{n+1} \le 1-\gamma. \tag{25}$$

Theorem 6. *Let $f \in \Sigma_\xi$ be given by (4). Then $f \in \mathcal{M}^l_m(\lambda, \beta, \gamma)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+\beta) + (\gamma+\beta)](1+n\lambda-\lambda)\Upsilon^l_m(n)a_n \le (1-2\lambda)(1-\gamma). \tag{26}$$

Proof. If $f \in \mathcal{M}^l_m(\lambda, \beta, \gamma)$, then by (20) we have

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)} + 1\right| + \gamma. \tag{27}$$

Making use of Lemma 4,

$$-\Re\left(\frac{(z-\xi)\left(1+\beta e^{i\theta}\right)G'(z) + \beta e^{i\theta}G(z)}{G(z)}\right) > \gamma, \tag{28}$$

where $G(z)$ is given by (21). Substituting $G(z)$, $G'(z)$ and letting $|z-\xi| < r \to 1^-$, we have

$$\left\{\left((1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}[n(1+\beta) + (\gamma+\beta)] \times (1+n\lambda-\lambda)\Upsilon^l_m(n)a_n\right) \times \left((1-2\lambda) - \sum_{n=1}^{\infty}(1+n\lambda-\lambda)\Upsilon^l_m(n)a_n\right)^{-1}\right\} > 0. \tag{29}$$

This shows that (26) holds.

Conversely, assume that (26) holds. Since $-\Re(w) > \gamma$, if and only if $|w+1| < |w-(1-2\gamma)|$, it is sufficient to show that

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| < 1, \qquad |w-(1-2\gamma)| \neq 0 \quad \text{for } |z-\xi| < r \leq 1,\ (z-\xi) \in \Delta. \tag{30}$$

Using (26) and taking $w(z) = ((z-\xi)(1+\beta e^{i\theta})G'(z) + \beta e^{i\theta}G(z))/G(z)$, we get

$$\begin{aligned}\left|\frac{w+1}{w-(1-2\gamma)}\right| &\leq \left(\left(\sum_{n=1}^{\infty}(1+n\lambda-\lambda)\left[(n+1)(1+\beta)\right]\Upsilon_m^l(n)\,a_n\right)\right.\\ &\quad\times\left(2(1-\gamma)(1-2\lambda)-\sum_{n=1}^{\infty}(1+n\lambda-\lambda)\right.\\ &\quad\left.\left.\times\left[n(1+\beta)+(\beta+2\gamma-1)\right]\Upsilon_m^l(n)\,a_n\right)^{-1}\right)\\ &\leq 1.\end{aligned} \tag{31}$$

Thus we have $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$. □

For the sake of brevity throughout this paper we let

$$\begin{aligned} d_n(\lambda,\beta,\gamma) &= \left[n(1+\beta)+(\gamma+\beta)\right](1+n\lambda-\lambda),\\ d_1(\lambda,\beta,\gamma) &= (1+\gamma+2\beta),\end{aligned} \tag{32}$$

unless otherwise stated.

Our next result gives the coefficient estimates for functions in $\mathcal{M}_m^l(\lambda,\beta,\gamma)$.

Theorem 7. *If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then*

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}, \qquad n = 1,2,3,\ldots. \tag{33}$$

The result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}(z-\xi)^n, \qquad n = 1,2,3,\ldots. \tag{34}$$

Proof. If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then we have, for each n,

$$\begin{aligned} d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)\,a_n &\leq \sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)\,a_n\\ &\leq (1-\gamma)(1-2\lambda).\end{aligned} \tag{35}$$

Therefore we have

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}. \tag{36}$$

Since

$$f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}(z-\xi)^n \tag{37}$$

satisfies the conditions of Theorem 6, $f_n(z) \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ and the equality is attained for this function. □

Theorem 8. *Suppose that there exists a positive number ν:*

$$\nu = \inf_{n\in\mathbb{N}}\left\{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)\right\}. \tag{38}$$

If $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, then

$$\begin{aligned}\left|\frac{1}{r}-\frac{(1-\gamma)(1-2\lambda)}{\nu}r\right| &\leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r, \quad (|z-\xi| = r),\\ \left|\frac{1}{r^2}-\frac{(1-\gamma)(1-2\lambda)}{\nu}\right| &\leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z-\xi| = r).\end{aligned} \tag{39}$$

If $\nu = d_1(\lambda,\beta,\gamma)\Upsilon_m^l(1) = (1+\gamma+2\beta)\Upsilon_m^l(1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)r^2\Upsilon_m^l(1)}(z-\xi). \tag{40}$$

Proof. Let $f \in \sum_{\xi}$ and be given by (4)

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty}a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty}a_n. \tag{41}$$

Since $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, and by Theorem 6,

$$\sum_{n=1}^{\infty}a_n \leq \frac{(1-\gamma)(1-2\lambda)}{\nu}. \tag{42}$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r. \tag{43}$$

Similarly

$$|f(z)| \geq \left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right|. \tag{44}$$

The result is sharp for function (40) with

$$\nu = d_1(\lambda,\beta,\gamma)\Upsilon_m^l(1) = (1+\gamma+2\beta)\Upsilon_m^l(1). \tag{45}$$

Similarly we can prove the otherinequality $|f'(z)|$. □

3. Order of Starlikeness

In the following theorem we obtain the order of starlikeness for the class $\mathscr{M}^l_m(\lambda,\beta,\gamma)$. We say that f given by (4) is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|z-\xi| < r$ when it satisfies condition (5) in $|z-\xi| < r$.

Theorem 9. *Let the function f given by* (4) *be in the class $\mathscr{M}^l_m(\lambda,\beta,\gamma)$. Then, if there exists*

$$r = r_1(\lambda,\gamma,\rho) = \inf_{n\ge 1}\left[\frac{(1-\rho)d_n(\lambda,\beta,\gamma)\Upsilon^l_m(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\right]^{1/(n+1)} \tag{46}$$

and it is positive, then f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$.

Proof. Let the function $f \in \mathscr{M}^l_m(\lambda,\beta,\gamma)$ be of the form (4). If $0 < r \le r_1(\lambda,\gamma,\rho)$, then by (46)

$$r^{n+1} \le \frac{(1-\rho)\,d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}, \tag{47}$$

for all $n \in \mathbb{N}$. From (47) we get

$$\frac{n+\rho}{1-\rho}r^{n+1} \le \frac{d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}{(1-\gamma)(1-2\lambda)}, \tag{48}$$

for all $n \in \mathbb{N}$, and thus

$$\sum_{n=1}^{\infty}\frac{n+\rho}{1-\rho}a_n r^{n+1} \le \sum_{n=1}^{\infty}\frac{d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}{(1-\gamma)(1-2\lambda)}a_n \le 1, \tag{49}$$

because of (26). Hence, from (49) and (25), f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho) = r$. □

Suppose that there exists a number $\tilde{r}$, $\tilde{r} > r_1(\lambda,\gamma,\rho)$, such that each $f \in \mathscr{M}^l_m(\lambda,\beta,\gamma)$ is meromorphically starlike of order ρ in $|z-\xi| < \tilde{r} \le 1$. The function

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}(z-\xi)^n \tag{50}$$

is in the class $\mathscr{M}^l_m(\lambda,\beta,\gamma)$; thus it should satisfy (25) with $\tilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)\,a_n\tilde{r}^{n+1} \le 1-\rho, \tag{51}$$

while the left–hand side of (51) becomes

$$\begin{aligned}
&(n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}\tilde{r}^{n+1}\\
&\quad > (n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}\frac{(1-\rho)\,d_n(\lambda,\beta,\gamma)\,\Upsilon^l_m(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\\
&\quad = 1-\rho,
\end{aligned} \tag{52}$$

which contradicts (51). Therefore the number $r_1(\lambda,\gamma,\rho)$ in Theorem 9 cannot be replaced with a greater number. This means that $r_1(\lambda,\gamma,\rho)$ is called radius of meromorphically starlikeness of order ρ for the class $\mathscr{M}^l_m(\lambda,\beta,\gamma)$.

4. Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,j}(z-\xi)^n, \quad a_{n,j} \ge 0, \tag{53}$$

we define the Hadamard product or convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,1}a_{n,2}(z-\xi)^n. \tag{54}$$

Let

$$\Psi(n,\lambda) = \frac{(n\lambda-\lambda+1)}{(1-2\lambda)}\Upsilon^l_m(n). \tag{55}$$

Theorem 10. *For functions f_j $(j = 1,2)$ defined by* (53)*, let $f_1 \in \mathscr{M}^l_m(\lambda,\beta,\gamma)$ and $f_2 \in \mathscr{M}^l_m(\lambda,\beta,\delta)$. Then $f_1 * f_2 \in \mathscr{M}^l_m(\lambda,\beta,\eta)$ where*

$$\begin{aligned}
&\eta\\
&= 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)},
\end{aligned} \tag{56}$$

and $\Psi(1,\lambda) = \Upsilon^l_m(1)/(1-2\lambda)$. The results are the best possible for

$$\begin{aligned}
f_1(z) &= \frac{1}{z-\xi} + \frac{1-\gamma}{(1+\gamma+2\beta)\Psi(1,\lambda)}(z-\xi),\\
f_2(z) &= \frac{1}{z-\xi} + \frac{1-\delta}{(1+\delta+2\beta)\Psi(1,\lambda)}(z-\xi),
\end{aligned} \tag{57}$$

where $\Psi(1,\lambda) = \Upsilon^l_m(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it suffices to prove that

$$\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)\,a_{n,1}a_{n,2} \le 1, \tag{58}$$

where η is defined by (56) under the hypothesis. It follows from (26) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\\
&\qquad\times\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \le 1.
\end{aligned} \tag{59}$$

Thus we need to find the largest η such that

$$\sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)a_{n,1}a_{n,2} \leq \sum_{n=1}^{\infty}\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \times\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \leq 1. \quad (60)$$

By virtue of (59) it is sufficient to find the largest η, such that

$$\frac{\sqrt{(1-\gamma)(1-\delta)}}{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}\Psi(n,\lambda)} \leq \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \times\frac{1-\eta}{[n(1+\beta)+(\eta+\beta)]}, \quad (61)$$

which yields

$$\eta\leq 1-\Big(((1-\gamma)(1-\delta)(2n+1+\beta)) \times([n(1+\beta)+(\gamma+\beta)][n(1+\beta)+(\delta+\beta)] \times\Psi(n,\lambda)-(1-\gamma)(1-\delta)(n+1))^{-1}\Big), \quad (62)$$

for $n\geq 1$ where $\Psi(n,\lambda)$ is given by (55) and, since $\Psi(n,\lambda)$ is a decreasing function of n $(n\geq 1)$, we have

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda)-2(1-\gamma)(1-\delta)}, \quad (63)$$

and $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$, which completes the proof. □

Theorem 11. *Let the functions f_j, $(j=1,2)$, defined by* (53) *be in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then $(f_1 * f_2)(z)\in\mathcal{M}_m^l(\lambda,\beta,\eta)$ where*

$$\eta = 1-\frac{(1-\gamma)^2(3+\beta)}{(1+\gamma+2\beta)^2\Psi(1,\lambda)-2(1-\gamma)^2} \quad (64)$$

with $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$.

Proof. By taking $\delta=\gamma$ in the above theorem, the results follow. □

For functions in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$, we can prove the following inclusion property.

Theorem 12. *Let the functions f_j $(j=1,2)$ defined by* (53) *be in the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then the function h, defined by*

$$h(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)(z-\xi)^n, \quad (65)$$

is in the class $\mathcal{M}_m^l(\lambda,\beta,\delta)$ where

$$\delta\leq 1-\frac{4(1-\gamma)^2(1+\beta)}{[1+\gamma+2\beta]^2\Psi(1,\lambda)+2(1-\gamma)^2}, \quad (66)$$

and $\Psi(1,\lambda)=\Upsilon_m^l(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{(1-\delta)}\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1, \quad (67)$$

where $f_j\in\mathcal{M}_m^l(\lambda,\beta,\gamma)$ $(j=1,2)$; we find from (53) and Theorem 6 that

$$\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 a_{n,j}^2 \leq \sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}a_{n,j}\right]^2\leq 1, \quad (68)$$

which would yield

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1. \quad (69)$$

On comparing (67) and (69) it can be seen that inequality (66) will be satisfied if

$$\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{1-\delta}\left(a_{n,1}^2+a_{n,2}^2\right) \leq\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 \times\left(a_{n,1}^2+a_{n,2}^2\right). \quad (70)$$

That is, if

$$\delta\leq 1-\frac{2(1-\gamma)^2[(n+1)(1+\beta)]}{[n(1+\beta)+(\gamma+\beta)]^2\Psi(n,\lambda)+2(1-\gamma)^2}, \quad (71)$$

where $\Psi(n,\lambda)$ is given by (55) and $\Psi(n,\lambda)$ is a decreasing function of n $(n\geq 1)$, we get (66), which completes the proof. □

5. Closure Theorems

We state the following closure theorems for $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ without proof (see [8–10]).

Theorem 13. *Let the function* $f_k(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} a_{n,k}(z-\xi)^n$ *be in the class* $\mathcal{M}_m^l(\lambda,\beta,\gamma)$ *for every* $k = 1,2,\ldots,m$. *Then the function* f *defined by*

$$f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_{n,k}(z-\xi)^n, \quad (a_{n,k} \geq 0) \tag{72}$$

belongs to the class $\mathcal{M}_m^l(\lambda,\beta,\gamma)$, *where* $a_{n,k} = (1/m)\sum_{k=1}^{m} a_{n,k}$, $(n = 1,2,\ldots)$.

Theorem 14. *Let* $f_0(z) = 1/(z-\xi)$ *and* $f_n(z) = (1/(z-\xi)) + ((1-\gamma)(1-2\lambda)/d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n))(z-\xi)^n$ *for* $n = 1,2,\ldots$. *Then* $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ *if and only if* f *can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z)$ *where* $\eta_n \geq 0$ *and* $\sum_{n=0}^{\infty} \eta_n = 1$.

Theorem 15. *The class* $\mathcal{M}_m^l(\lambda,\beta,\gamma)$ *is closed under convex linear combination.*

Now, we prove that the class is $\mathcal{M}_m^l(\lambda,\beta,\gamma)$ closed under integral transforms.

Theorem 16. *Let the function* $f(z)$ *given by* (4) *be in* $\mathcal{M}_m^l(\lambda,\beta,\gamma)$. *Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)\,du \quad (0 < u \leq 1, 0 < c < \infty) \tag{73}$$

is in $\mathcal{M}_m^l(\lambda,\beta,\delta)$, *where*

$$\begin{aligned}\delta \leq \ & \left(n^2(1+\beta) + n\left[(\gamma+\beta) + (1+\beta)(1+c\gamma)\right]\right.\\ & \left. + (c+1)(\gamma+\beta) + c\beta(1-\gamma)\right)\\ & \times\left(n^2(1+\beta) + n\left[(\gamma+\beta) + (1+c)(1+\beta)\right]\right.\\ & \left. + (1+c)(\gamma+\beta) + c(1-\gamma)\right)^{-1}.\end{aligned} \tag{74}$$

The result is sharp for the function $f(z) = (1/(z-\xi)) + ((1-\gamma)(1-2\lambda)/(1+\gamma+2\beta)\Upsilon_m^l(1))(z-\xi)$.

Proof. Let $f(z) \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$. Then

$$F(z) = c\int_0^1 u^c f(uz)\,du = \frac{1}{z-w} + \sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n(z-\xi)^n. \tag{75}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{cd_n(\lambda,\beta,\delta)\Upsilon_m^l(n)}{(c+n+1)(1-\delta)}a_n \leq 1. \tag{76}$$

Since $f \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$, we have

$$\sum_{n=1}^{\infty}\frac{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)}a_n \leq 1. \tag{77}$$

Note that (76) is satisfied if

$$\frac{cd_n(\lambda,\beta,\delta)\Upsilon_m^l(n)}{(c+n+1)(1-\delta)} \leq \frac{d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)}{(1-\gamma)(1-2\lambda)}. \tag{78}$$

From (78), we have

$$\begin{aligned}\delta \leq \ & \left(\left(n^2(1+\beta) + n\left[(\gamma+\beta) + (1+\beta)(1+c\gamma)\right]\right.\right.\\ & \left. + (c+1)(\gamma+\beta) + c\beta(1-\gamma)\right)\\ & \times\left(n^2(1+\beta) + n\left[(\gamma+\beta) + (1+c)(1+\beta)\right]\right.\\ & \left.\left. + (1+c)(\gamma+\beta) + c(1-\gamma)\right)^{-1}\right) = \Phi(n).\end{aligned} \tag{79}$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follow. □

6. Partial Sums

Silverman [15] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested in searching results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [15] and Cho and Owa [16], we will investigate the ratio of a function of the form (4) to its sequence of partial sums. Consider

$$f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n(z-\xi)^n, \tag{80}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma)\Upsilon_m^l(n)a_n \leq (1-\gamma)(1-2\lambda). \tag{81}$$

More precisely we will determine sharp lower bounds for $\Re(f(z)/f_k(z))$ and $\Re(f_k(z)/f(z))$. In this connection we make use of the well-known results that $\Re((1+w(z))/(1-w(z))) > 0$, $(z-\xi \in \Delta)$, if and only if $w(z) = \sum_{n=1}^{\infty} c_n(z-\xi)^n$ satisfies the inequality $|w(z)| \leq |z-\xi|$.

Unless otherwise stated, we will assume that f is of the form (4) and its sequence of partial sums is denoted by (80).

Theorem 17. *Let* $f(z) \in \mathcal{M}_m^l(\lambda,\beta,\gamma)$ *be given by* (4) *which satisfies condition* (26) *and suppose that all of its partial sums* (80) *do not vanish in* Δ. *Moreover, suppose that*

$$2 - 2\sum_{n=1}^{k}|a_n| - \frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| > 0, \quad \forall k \in \mathbb{N}. \tag{82}$$

Then,

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)} \quad (z-\xi \in \Delta), \tag{83}$$

where

$$d_n(\lambda,\beta,\gamma) \geq \begin{cases} (1-\gamma)(1-2\lambda), & \text{if } n=1,2,3,\dots,k \\ d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1), & \text{if } n=k+1,k+2,\dots. \end{cases} \tag{84}$$

The result (83) *is sharp with the function given by*

$$f(z)=\frac{1}{z-\xi}+\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}(z-\xi)^{k+1}. \tag{85}$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \times\left[\frac{f(z)}{f_k(z)}-\left(1-\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}\right)\right] \\ &= 1+\left(\left(\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right)\times\left((1-\gamma)(1-2\lambda)\right)^{-1}\times\sum_{n=k+1}^{\infty}a_n(z-\xi)^{n+1}\right)\times\left(1+\sum_{n=1}^{k}a_n(z-\xi)^{n+1}\right)^{-1}\right). \end{aligned} \tag{86}$$

It suffices to show that $\Re(w(z))>0$; hence we find that

$$\begin{aligned} \left|\frac{1+w(z)}{1-w(z)}\right| &\leq \left(\left(\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right)\times\left((1-\gamma)(1-2\lambda)\right)^{-1}\times\sum_{n=k+1}^{\infty}|a_n|\right) \right. \\ &\quad \times\left(2-2\sum_{n=1}^{k}|a_n| - \left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)\right)\times\left((1-\gamma)(1-2\lambda)\right)^{-1} \left.\times\sum_{n=k+1}^{\infty}|a_n|\right)^{-1}\right)\leq 1. \end{aligned} \tag{87}$$

From condition (26), it readily yields the assertion (83) of Theorem 17.

To see that the function given by (85) gives the sharp result, we observe that for $z=re^{i\pi/(k+2)}$

$$\begin{aligned} \frac{f(z)}{f_k(z)} &= 1+\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}(z-\xi)^n \\ &\longrightarrow 1-\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}, \end{aligned} \tag{88}$$

when $r\to 1^-$ which shows that the bound (83) is the best possible for each $k\in\mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 18. *Under the assumptions of Theorem 17, we have*

$$\Re\left(\frac{f_k(z)}{f(z)}\right)\geq\frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)+(1-\gamma)(1-2\lambda)} \quad (z-w\in\Delta), \tag{89}$$

The result (89) *is sharp with the function given by* (85).

Proof. By setting

$$w(z) = \left(1+\frac{d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)}\right)\times\left[\frac{f_k(z)}{f(z)} - \frac{\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)/(1-\gamma)(1-2\lambda)\right)}{1+\left(d_{k+1}(\lambda,\beta,\gamma)\Upsilon_m^l(k+1)/(1-\gamma)(1-2\lambda)\right)}\right] \tag{90}$$

and proceeding as in Theorem 17, we get the desired result and so we omit the details. □

Concluding Remark. We observe that, if we specialize the parameters λ and β as mentioned in Examples 1 and 2, we obtain the analogous results for the classes $\mathscr{M}_m^l(\beta,\gamma)$ and $\mathscr{M}_m^l(\gamma)$. Further specializing the parameters l, m various other interesting results (as in Theorems 6–18) can be derived easily for the function class based on interesting differential operators as illustrated below.

(1) For $a_i=q^{a_i}$, $b_j=q^{b_j}$, $a_i>0$, $b_j>0$, $(i=1,\dots,l;\ j=1,\dots,m,\ l=m+1)$, $q\to 1$, the operator $\mathscr{I}_m^l f(z)=\mathscr{H}_m^l[a_1]f(z)$ defined by Liu and Srivastava [10].

(2) For $l=2$, $m=1$, $a_2=q$, $q\to 1$, the operator $\mathscr{L}_1^2[a_1,q,b_1,q]f(z)=\mathscr{L}[a_1;b_1]f(z)$ was introduced and studied by Liu and Srivastava [9].

(3) For $l=1$, $m=0$, $a_1=\delta+1$, $q\to 1$, the operator $\mathscr{L}[a_1;b_1]f(z)=D^{\delta}f(z)=(1/z(1-z)^{\delta+1})*f(z)$, $(\delta>-1)$

where D^{δ} is the differential operator which was introduced by Ganigi and Uralegaddi [17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Kanas and F. Ronning, "Uniformaly starlike and convex function and other related classes of univalent functions," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 53, pp. 95–105, 1991.

[2] A. W. Goodman, "On uniformly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 155, no. 2, pp. 364–370, 1991.

[3] A. W. Goodman, "On uniformly convex functions," *Annales Polonici Mathematici*, vol. 56, no. 1, pp. 87–92, 1991.

[4] S. D. Purohit and R. K. Raina, "Certain subclasses of analytic functions associated with fractional q-calculus operators," *Mathematica Scandinavica*, vol. 109, no. 1, pp. 55–70, 2011.

[5] A. Huda and M. Darus, "Integral operator defined by q-analogue of Liu-Srivastava operator," *Studia Universitatis Babeş-Bolyai—Series Mathematica*, vol. 58, no. 4, pp. 529–537, 2013.

[6] M. K. Aouf, "On a certain class of meromorphic univalent functions with positive coeffcients," *Rendiconti di Matematica e delle sue Applicazioni*, vol. 7, no. 11, pp. 209–219, 1991.

[7] M. K. Aouf and G. Murugusundaramoorthy, "On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator," *Australian Journal of Mathematical Analysis and Applications*, vol. 5, no. 1, article 3, pp. 1–17, 2008.

[8] J. Dziok, G. Murugusundaramoorthy, and J. Sokół, "On certain class of meromorphic functions with positive coeffcients," *Acta Mathematica Scientia B*, vol. 32, no. 4, pp. 1–16, 2012.

[9] J.-L. Liu and H. M. Srivastava, "Subclasses of meromorphically multivalent functions associated with a certain linear operator," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 35–44, 2004.

[10] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[11] M. L. Mogra, T. R. Reddy, and O. P. Juneja, "Meromorphic univalent functions with positive coeffcients," *Bulletin of the Australian Mathematical Society*, vol. 32, pp. 161–176, 1985.

[12] S. Owa and N. N. Pascu, "Coeffcient inequalities for certain classes of meromorphically starlike and meromorphically convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 17, pp. 1–6, 2003.

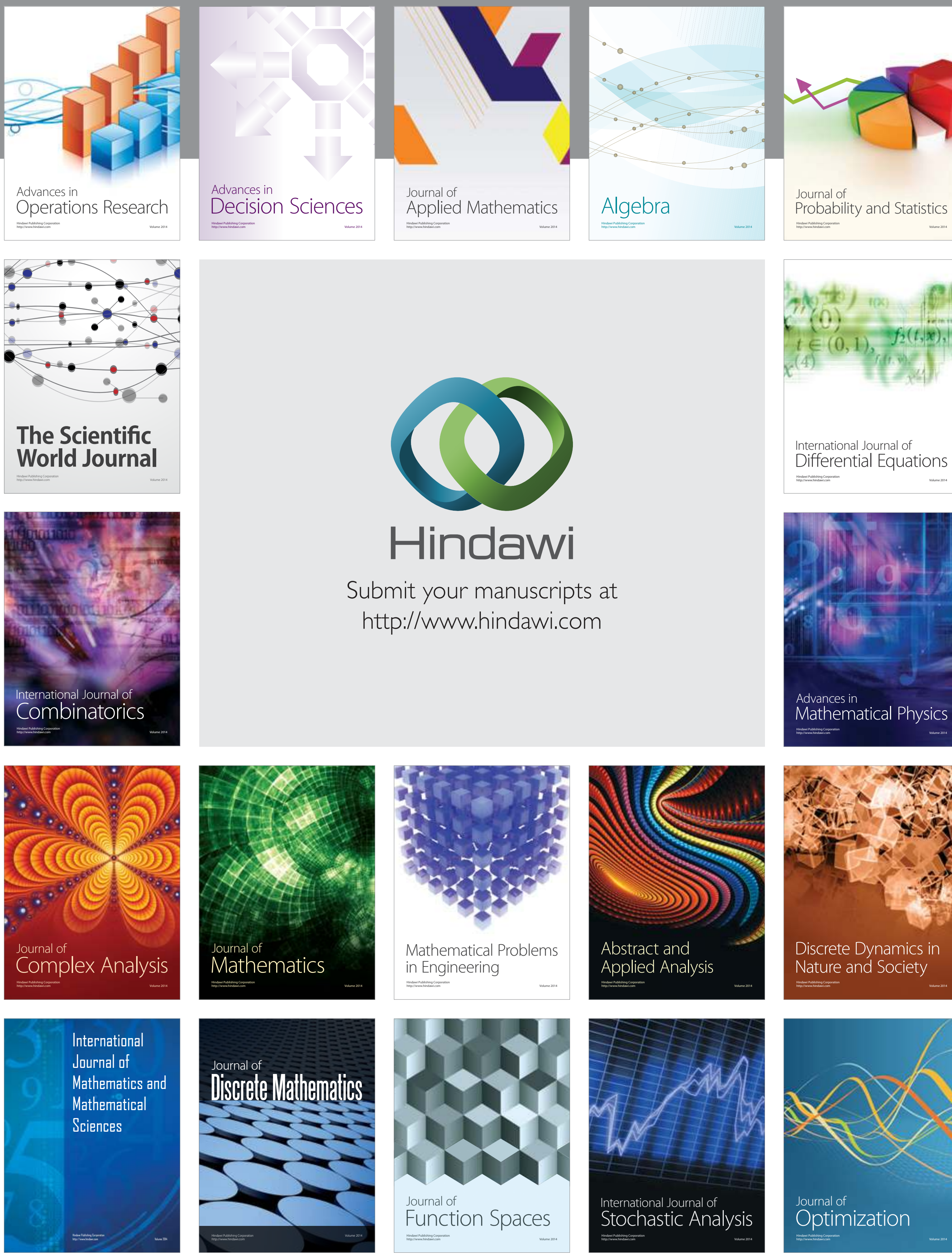
[13] C. Pommerenke, "On meromorphic starlike functions," *Pacific Journal of Mathematics*, vol. 13, pp. 221–235, 1963.

[14] B. A. Uralegaddi and C. Somanatha, "Certain differential operators for meromorphic functions," *Houston Journal of Mathematics*, vol. 17, no. 2, pp. 279–284, 1991.

[15] H. Silverman, "Partial sums of starlike and convex functions," *Journal of Mathematical Analysis and Applications*, vol. 209, no. 1, pp. 221–227, 1997.

[16] N. E. Cho and S. Owa, "Partial sums of certain meromorphic functions," *Journal of Inequalities in Pure & Applied Mathematics*, vol. 5, no. 2, article 30, 2004.

[17] M. R. Ganigi and B. A. Uralegaddi, "New criteria for meromorphic univalent functions," *Bulletin Mathematique de la Societe des Sciences Mathematiques de la Republique Socialiste de Roumanie*, vol. 33, no. 1, pp. 9–13, 1989.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