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MESH GRADATION CONTROL

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ABSTRACT

This paper gives a procedure to control the size variation in a mesh adaption scheme where the size specification (the so-called control space) is used to govern the mesh generation stage. The method consists in replacing the initial control space by a reduced one by means of size or metric. It allows to improve, *a priori*, the quality of the adapted mesh and to speed up the adaption procedure. Several numerical examples show the efficiency of the reduction scheme. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: mesh gradation; mesh adaption; surface mesh generation; anisotropic mesh

1. INTRODUCTION

The first stage in an adaptive finite element scheme (cf. References 1 and 2) consists in creating an initial mesh of a given domain Ω, which is used to perform an initial computation (for example a flow solver). A size specification field is deduced (e.g. at the vicinity of each mesh vertex, the desired mesh size is specified), based on the numerical results. If the mesh does not satisfy the size specification field, then a new constrained mesh, governed by this field, is constructed (see, e.g. References 3–5). The size specification field is usually obtained via an error estimate.[6,7] Actually, the estimation gives a discrete size specification field. Using an adequate size interpolation over the mesh elements, a continuous field is then obtained. In the case of surface meshes, a geometric error (apart from the computation) which indicates the gap between the facetization and the real surface, is considered. The larger the rate of the mesh size variation, the worse is the shape quality of the adapted mesh. *The problem that we face is how to control the mesh size variation or gradation*.

Metrics are commonly used to normalize the mesh size specification to one in any direction (cf. Reference 8), and are defined as symmetric positive-definite matrices associated to any point of the domain. According to this generalized size specification, the mesh gradation can be obviously controlled by changing the mesh size specification rather than by modifying the mesh generator itself. This can be achieved by applying a metric smoothing method witch consists of replacing a metric by an average of neighbouring metrics (cf. Reference 4). Notice that, this method does not control explicitly the mesh gradation. Moreover, if this latter is not satisfactory, the smoothing procedure can be iterated.

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It is obvious that the mesh gradation can be improved if we reduce the mesh size in any direction, and possibly by preserving the metric shape (e.g. the metric is locally proportional to the initial one). Hence, the mesh gradation is controlled by analysing all mesh edges, replacing, if necessary, the size specification at the endpoints and iterate if a size modification occurred. In the isotropic case where the size specification is defined as a function h, the mesh gradation problem is to bound, in the new mesh,

1. either the norm of the gradient of the function h,
2. or the ratio of the Euclidean length of two adjacent edges.

The same procedures can be applied in the generalized anisotropic case.

This paper discusses these two approaches to control the mesh gradation and can be considered as a complement of the papers in Reference 5 or 9 related to mesh gradation control. In Section 3, some definitions related to mesh adaption and control space are given. In Section 4, the notions of H-variation and H-shock associated to a control space are introduced and two correction procedures, the so-called H-corrections, are presented. In Section 5, several numerical examples of mesh adpation without and with H-correction are given. Section 6 is devoted to a brief conclusion.

2. MESH ADAPTED TO A CONTROL SPACE

In this section we recall the notion of mesh *adapted* to a control space or a metric map. Let Ω be a domain of R^d ($d=2$ or 3) and $\mathcal{M}_d(\Omega)$ be a *continuous* field of metrics associated with the points of Ω. The metric at a point P of Ω characterizes the desired edge- or element size in the vicinity of P. By normalizing the desired size to *one*, the metric at P can be defined as a symmetric positive definite matrix of order d, denoted as $\mathcal{M}_d(P)$, and is given in two dimensions by

$$\mathcal{M}_2(P)=\begin{pmatrix} a(P) & b(P) \\ b(P) & c(P) \end{pmatrix} \tag{1}$$

with $a(P)>0$ and $a(P)c(P)-b^2(P)>0$ and in three dimensions by

$$\mathcal{M}_3(P)=\begin{pmatrix} a(P) & b(P) & c(P) \\ b(P) & d(P) & e(P) \\ c(P) & e(P) & f(P) \end{pmatrix} \tag{2}$$

with $a(P)>0$, $a(P)d(P)-b^2(P)>0$ and $\mathrm{Det}(\mathcal{M}_3(P))>0$, Det being the determinant. Under this metric definition, the size requirement in the vicinity of P can be expressed as

$${}^t\overrightarrow{PX}\,\mathcal{M}_d(P)\overrightarrow{PX}=1, \tag{3}$$

where PX represents any edge adjacent to P. The geometric locus of all points X verifying this equation is usually an ellipsoid, denoted as $\overline{\mathcal{M}_3(P)}$, which represents an approximation† of the unit ball at P. The pair $(\Omega, \mathcal{M}_d(\Omega))$ defines a *Riemannian* structure over Ω and is a so-called *continuous control space*.

Actually, the continuous field $\mathcal{M}_d(\Omega)$ of metrics is defined via an interpolation over a *discrete control space*. The latter is usually composed by a mesh $T(\Omega)$ of Ω, the so-called *background*

†The metric is assumed to be constant in an adequate neighbourhood of P

mesh, and a *discrete* field of metrics $\mathcal{M}_d(T(\Omega))$ associated with the vertices of $T(\Omega)$ and denoted as $(T(\Omega), \mathcal{M}_d(T(\Omega)))$. The background mesh, which is a simplicial decomposition of the domain, allows, by interpolation (linear, geometric or other), to construct the continuous field of metrics from the discrete one.

A mesh of Ω adapted to a control space $(\Omega, \mathcal{M}_d(\Omega))$ is a mesh having all edges of unit length with respect to the associated Riemannian structure. This mesh is also called *unit mesh* of Ω via the structure. Recall that the length of a mesh edge $PQ = (P + t\overrightarrow{PQ})_{0 \leqslant t \leqslant 1}$ is given by

$$l(PQ) = \int_0^1 \sqrt{{}^t\overrightarrow{PQ}\, \mathcal{M}_d(P + t\overrightarrow{PQ})\, \overrightarrow{PQ}}\, \mathrm{d}t \tag{4}$$

where $\mathcal{M}_d(P + t\overrightarrow{PQ})$ is the metric at the point $P + t\overrightarrow{PQ}$ along PQ.

By definition, a unit mesh has a good *size quality* with respect to the associated control space. This mesh does not necessarily satisfy the shape quality requirement. Indeed, the latter depends on the edge lengths and the volumes of the elements (cf. Reference 5). An *equilateral mesh* is a unit mesh having a good *shape quality*.

3. H-VARIATION, H-SHOCK AND H-CORRECTION

Let us consider the discrete control space $(T(\Omega), \mathcal{M}_d(T(\Omega)))$; we are interested in determining locally the corresponding size variation in Ω. Obviously, an equilateral mesh cannot be obtained if the size variation is too important. Indeed, if the size changes dramatically in the vicinity of a point, the degree of this point[‡] increases, thus leading to small volumes for the connected elements (as compared to the ideal volume of an element) and then, the shape quality of the mesh is slightly degraded at this point. In this section, we show how we can approximate locally the H-variation and the H-shock associated to the control space $(T(\Omega), \mathcal{M}_d(T(\Omega)))$, and we propose a *reasonable* correction (e.g. not too perturbing) of $\mathcal{M}_d(T(\Omega))$ to reduce the corresponding size variation.

3.1. Size variation

The size variation can be defined in two different ways. The first measures the gradient of the size function, and the second indicates the ratio of the Euclidean length of two adjacent edges. These measures are first defined in the isotropic context, and then generalized to the anisotropic case.

In the case where the metric field is isotropic, the metric at a point P can be expressed by $\mathcal{M}_d(P) = h^{-2}(P) I_d$, where $I_d(P)$ is the identity matrix and $h(P)$ is the desired size at P. Indeed, equation (3) is equivalent to $\|\overrightarrow{PX}\| = h(P)$. The length of any edge PQ of the background mesh can be then written as

$$l(PQ) = \|\overrightarrow{PQ}\| \int_0^1 \frac{1}{h(t)}\, \mathrm{d}t \tag{5}$$

where $h(t)$ is a monotonic interpolation function such that $h(0) = h(P)$ and $h(1) = h(Q)$. This function permits to extend the discrete control space to a continuous one. To clarify this notion,

[‡]The number of elements or edges sharing this point

let us consider the size interpolation corresponding to a geometric progression as defined by

$$h(t) = h(P)\left(\frac{h(Q)}{h(P)}\right)^t \tag{6}$$

thus, we obtain

$$l(PQ) = \|\overrightarrow{PQ}\| \frac{h(Q) - h(P)}{h(P)h(Q)\log(h(Q)/h(P))} \tag{7}$$

if $h(P) \neq h(Q)$ and if $h(P) = h(Q)$

$$l(PQ) = \frac{\|\overrightarrow{PQ}\|}{h(P)}. \tag{8}$$

Definition. The H-variation $v(PQ)$ related to the edge PQ of $\mathscr{T}(\Omega)$ is the value

$$v(PQ) = \frac{h(Q) - h(P)}{\|\overrightarrow{PQ}\|} \tag{9}$$

This definition is identical to the gradient of the function h, if Q tends towards P, and thus represents a discrete approximation of the gradient.

Definition. The H-shock $c(PQ)$ related to the edge PQ of $\mathscr{T}(\Omega)$ is the value

$$c(PQ) = \max\left(\frac{h(Q)}{h(P)}, \frac{h(P)}{h(Q)}\right)^{1/l(PQ)} \tag{10}$$

This value represents the mesh gradation along the edge PQ and measures the distorsion of the interpolation function h along PQ.

The H-variation (resp. H-shock) varies in the interval $[0, \infty[$(resp. $[1, \infty[$). These measures can be defined at the mesh vertices of $\mathscr{T}(\Omega)$, by considering the measures related to the adjacent edges. More precisely, the H-variation $v(P)$ (resp. the H-shock $c(P)$) at the vertex P of $\mathscr{T}(\Omega)$ is the maximal value of the H-variations (resp. H-shocks) related to the adjacent edges,

$$v(P) = \max_Q v(PQ) \quad \text{and} \quad c(P) = \max_Q c(PQ) \tag{11}$$

where Q belongs to the set of the endpoints of all edges adjacent to P.

Hence, we obtain a map of the measures associated with the vertices of the background mesh. Using an interpolation, it is then possible to deduce, from this discrete map, a size variation at any point of the domain Ω. By definition, the H-variation (resp. H-shock) of the control space $(\mathscr{T}(\Omega), \mathscr{M}_d(\mathscr{T}(\Omega)))$ is the maximal H-variation (resp. H-shock) value at the mesh vertices.

Let us consider the general case where an anisotropic metric field is specified. In order to get an estimation of the size variation along an edge PQ of $\mathscr{T}(\Omega)$, we consider the size specification in one direction only; more precisely, along the direction PQ. This enables us to retrieve the isotropic case, by means of a rough approximation. Therefore, mesh sizes are associated with P and Q, and the related isotropic formula are applied. This consists in finding the point P_1 (resp. Q_1) on the supporting line of PQ belonging to $\overline{\mathscr{M}_d(P)}$ (resp. $\overline{\mathscr{M}_d(Q)}$) thus verifying $P_1 = PQ \cap \overline{\mathscr{M}_d(P)}$ (resp.

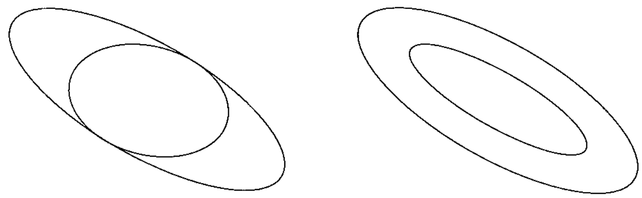

Figure 1. Unit ball correction

$Q_1 = QP \cap \overline{\mathscr{M}_d(Q)})$ with $h(0) = \|\overrightarrow{PP_1}\|$ and $h(1) = \|\overrightarrow{QQ_1}\|$. The H-variation is quite similar to the isotropic case, in using the following approximation of the edge length $l(PQ)$:

$$l(PQ) = \|\overrightarrow{PQ}\| \int_0^1 \frac{1}{h(t)}\,\mathrm{d}t \tag{12}$$

where $h(t)$ is a monotonic (size interpolation) function such that $h(0) = \|\overrightarrow{PP_1}\|$ and $h(1) = \|\overrightarrow{QQ_1}\|$, and if we consider a geometric size interpolation function along PQ, we retrieve the isotropic H-shock expression.

3.2. *Size correction*

In this section, we propose two approaches to bound the size variation in a control space, related to a specific measure (H-variation or H-shock).

3.2.1. *Hv-correction*

The first approach is based on the H-variation measure and consists in replacing a size specification by a reduced one in all directions. In the isotropic case, the shape of the unit ball is preserved, as its size becomes smaller. In the anisotropic case, the unit ball changes while being still included in the initial ball (Figure 1, left-hand side).

In the isotropic case, the problem is to bound the H-variation $v(PQ)$ along an edge PQ by a given threshold value α

$$v(PQ) = \frac{|h(Q) - h(P)|}{\|\overrightarrow{PQ}\|} \leqslant \alpha \tag{13}$$

while modifying the size specifications[§] $h(P)$ and $h(Q)$. The new size specifications are then given by

$$h(P) = \min(h(P), h(Q) + \alpha\|\overrightarrow{PQ}\|)$$

and

$$h(Q) = \min(h(Q), h(P) + \alpha\|\overrightarrow{PQ}\|) \tag{14}$$

[§]Notice that only one size specification is modified

In the general anisotropic case, the operator min related to the sizes is replaced by an operator related to the metrics. In Reference 5, the operator $\cap$ is the intersection operator, denoted as $(.,.)$, representing a metric whose size specification satisfies at best both those of the operands.

Notice that

$$\|\overrightarrow{PQ}\| = l_P(PQ)h(P) = l_Q(PQ)h(Q) \tag{15}$$

where $l_P(PQ)$ (resp. $l_Q(PQ)$) is the length of PQ in the metric $\mathcal{M}_d(P)$ (resp. $\mathcal{M}_d(Q)$), then equation (14) can be written as

$$h(P) = \min(h(P), h(Q)(1 + \alpha l_Q(PQ)))$$

and

$$h(Q) = \min(h(Q), h(P)(1 + \alpha l_P(PQ))) \tag{16}$$

and, as observing that a size scaling of value γ is equivalent to a metric scaling of value γ^{-2}, we obtain

$$\mathcal{M}_d(P) = (\mathcal{M}_d(P), \mathcal{M}_d(Q)(1 + \alpha l_P(PQ))^{-2})$$

and

$$\mathcal{M}_d(Q) = (\mathcal{M}_d(Q), \mathcal{M}_d(P)(1 + \alpha l_Q(PQ))^{-2}). \tag{17}$$

Remark. The correction related to a mesh edge does not take the metric interpolation along the edge into account. Moreover, the metric intersection changes the shape of the corresponding unit balls.

Now, we propose an algorithm called *Hv-correction* which consists in applying iteratively the reduction (if necessary) to the edges of the mesh $T(\Omega)$.

Hv-correction

- Loop while the metric at a point is modified
  - Loop over the edges of $T(\Omega)$
    - Let PQ be the current edge
    - Compute $l_P(PQ)$ (resp. $l_Q(PQ)$) the length of PQ in the metric $\mathcal{M}_d(P)$ (resp. $\mathcal{M}_d(Q)$)
    - Compute $\gamma_P = (1 + \alpha l_P(PQ))^{-2}$ and $\gamma_Q = (1 + \alpha l_Q(PQ))^{-2}$
    - Replace $\mathcal{M}_d(P)$ (resp. $\mathcal{M}_d(Q)$) by $(\mathcal{M}_d(P), \gamma_P \mathcal{M}_d(Q))$ (resp. $(\mathcal{M}_d(Q), \gamma_Q \mathcal{M}_d(P))$).

3.2.2. Hc-correction

The second approach is based on the H-shock measure and consists in replacing a size specification by a reduced one in all directions, as preserving the shape of the corresponding unit ball (Figure 1, right-hand side). Before describing the proposed correction procedure, let us first introduce several definitions related to the control spaces.

Definitions. Let $(T(\Omega), \mathcal{M}_d(T(\Omega)))$ and $(T(\Omega), \mathcal{M}'_d(T(\Omega)))$ be two discrete control space associated to the same mesh $T(\Omega)$ of Ω. These two control spaces are called proportional if the corresponding metrics are proportional, e.g. if

$$\forall X \text{ vertex of } T(\Omega), \quad \exists \alpha_X \in R, \quad \mathcal{M}'_d(X) = \alpha_X \mathcal{M}_d(X) \tag{18}$$

As, for all X, $\alpha_X \leqslant 1$, the control space $(T(\Omega), \mathscr{M}'_d(T(\Omega)))$ is called *reduction* of the control space $(T(\Omega), \mathscr{M}_d(T(\Omega)))$.

This allows to define an order property over the control spaces, denoted $<$, and to rewrite the previous equation as

$$(T(\Omega), \mathscr{M}'_d(T(\Omega))) < (T(\Omega), \mathscr{M}_d(T(\Omega))) \tag{19}$$

The problem can be then reformulated in order to bound the H-shock in terms of proportional control spaces.

Problem. Let $(T(\Omega), \mathscr{M}_d(T(\Omega)))$ be a given discrete control space; the problem that we face is to construct a *minimal* reduction (which is, otherwise, maximal for the associated order property) of $(T(\Omega), \mathscr{M}_d(T(\Omega)))$ with an associated H-shock bounded by a given threshold β.

Let us consider, for sake of simplification, the case where the metric field of the control space is isotropic. Let PQ be an edge of $T(\Omega)$ with a H-shock, $c(PQ)$, bigger than the threshold β and suppose $h(P) \leqslant h(Q)$. The problem is then to 'reduce' $h(Q)$ such that the new H-shock correponding to the edge PQ is equal to β. Let $h'(Q)$ be the new value of $h(Q)$ after reduction. Let r be a real verifying $h'(Q) = rh(P)$; then r must satisfy the equation

$$\frac{h(P)}{\|\overrightarrow{PQ}\|} \frac{r \log^2(r)}{r-1} = \log(\beta) \tag{20}$$

Hence, we obtain a non-trivial equation, to determine r. Therefore, we propose an approximate solution.

As $h'(Q) \leqslant h(Q)$, $l(PQ) \leqslant l'(PQ)$, where $l'(PQ)$ is the length of PQ computed with the new value $h'(Q)$. Let $c'(PQ)$ be the new H-shock of PQ (taking into account $h'(Q)$). We have

$$c'(PQ) = \left(\frac{h'(Q)}{h(P)}\right)^{1/l'(PQ)} \leqslant \left(\frac{h'(Q)}{h(P)}\right)^{1/l(PQ)} = c(PQ)\eta^{1/l(PQ)} \tag{21}$$

where $\eta = h'(Q)/h(Q)$. To have $c'(PQ) \leqslant \beta$, it is sufficient to satisfy the inequation

$$c(PQ)\eta^{1/l(PQ)} \leqslant \beta \tag{22}$$

which yields

$$\eta \leqslant \left(\frac{\beta}{c(PQ)}\right)^{l(PQ)} \tag{23}$$

In summary, assuming that $h(P) \leqslant h(Q)$, the value of $h(Q)$ must be reduced by a factor¶ η lesser than $(\beta/c(PQ))^{l(PQ)}$, so that the H-shock associated with PQ is lesser than β. By taking η equal to the limit value, the corresponding reduction approximates the minimal reduction.

¶Notice that this bound for η is not optimal

The same approach can be applied to the general anisotropic case, if we consider a rough computation of the edge length. Let $\mathcal{M}_d(P)$ (resp. $\mathcal{M}_d(Q)$) be the metric at P (resp. Q) and $h(P)$ (resp. $h(Q)$) representing the unit length in the direction $\overrightarrow{PQ}$ in the vicinity of P (resp. Q). Suppose $h(P) \leqslant h(Q)$ and that the H-shock associated with PQ is bigger than the threshold value β. In this case, the reduction consists in replacing $\mathcal{M}_d(Q)$ by $\mathcal{M}_d(Q)/\eta^2$, where η is the size reduction factor found in the isotropic case.

Now, we propose an algorithm called *Hv-correction* which consists in applying iteratively the reduction (if necessary) to the edges of the mesh $T(\Omega)$.

Hc-correction

- While the H-shock along an edge is $\geqslant \beta$
  - Loop over the edges of $T(\Omega)$
    - * Let PQ be the current edge
    - * Compute $h(P)$ (resp. $h(Q)$) the unit length in the vicinity of P (resp. Q) in the direction $\overrightarrow{PQ}$ (suppose $h(Q) \geqslant h(P)$)
    - * Compute $l(PQ)$, the length of PQ
    - * Compute $c(PQ)$, the H-shock on PQ
    - * If $c(PQ) \geqslant \beta$ then replace $\mathcal{M}_d(Q)$ by $\mathcal{M}_d(Q)/\eta^2$, where $\eta = \left(\frac{\beta}{c(PQ)}\right)^{l(PQ)}$.

Remark. In the Hv or Hc-correction procedure, after the first iteration, an edge is examined if the metric at one of its end points is modified. We cannot determine, *a priori*, the number of iteration during the process. Thus, to accelerate the edge treatment, we form a dynamic list of edges (to be examined).

4. EXAMPLES

In this section, we show several examples of adapted meshes to a control space, as well as H-corrections related to different specified threshold values. These examples show, in particular, the effect of the H-correction on the resulting mesh quality after adaption, as well as the speed up of the convergence in the mesh adaption loop. Notice that the adaption process is iterated until a mesh is obtained which respects (*a priori*) the requirement associated with the control space related to the next iteration.

4.1. Planar mesh examples

In the following paragraphs, we consider four examples. In the first two examples, the control space is analytically defined, as for the last two examples, the control space is provided by an error estimate from the solution of a numerical viscous flow computation.

4.1.1. Example 1

Let us consider a square of side length 4, centred at the origin. We would like to obtain a mesh adapted to the isotropic discontinuous control space defined as

$$h(x,y) = \begin{cases} 0{\cdot}005 & \text{if } 0{\cdot}99 \leqslant r^2(x,y) \leqslant 1{\cdot}01 \\ 1{\cdot}0 & \text{if } 1{\cdot}01 < r^2(x,y) \\ 0{\cdot}1 & \text{else} \end{cases} \tag{24}$$

where $h(x, y)$ represents the size associated with any point (x, y) of the domain and where $r^2(x, y) = x^2 + y^2$. This field defines a constant size in three regions, a disk and two rings. The size variation is constant almost everywhere except on the interface between the regions where the variation is infinite.

The adaption procedure is iterated, starting from a rough mesh of the square, until the corresponding mesh satisfies the specified size field, within a given tolerance. The control space associated with each iteration is composed by the mesh generated at the previous iteration and the size map associated with the mesh vertices.

As the control space is discontinuous and as the initial mesh is pretty coarse, 20 iterations have been necessary to obtain an adapted mesh (cf. Figure 2, top left). The mesh contains 10 976 triangles. The worst (resp. average) quality[‖] is 0·008 (resp. 0·8). A zoom of a critical area (where the size specification is discontinuous) is illustrated in Figure 2 (bottom left). The two parts of Figure 3 (top-left and bottom-left) show a conversion of the previous mesh elements into quadrilaterals (cf. Reference 9) and the zoom on the critical region.

Hereafter, we consider for each iteration a reduction of the control space with a H-shock $\beta = 2$. In this case, only 8 adaption iterations have been necessary to obtain an adapted mesh (cf. Figure 2 top right). This mesh contains 28 844 elements. The worst (resp. average) quality is 0·5 (resp. 0·9). A zoom on the same critical region is illustrated in Figure 2 (bottom right). The mesh quality is much better and the adaption scheme has converged more quickly. Notice that a side effect of the reduction is that the new mesh contains more elements, the edge lengths being usually smaller than one. The two parts of Figure 3 (top-right and bottom-right) show the conversion of the adapted mesh into quadrilaterals. As expected, the quality is better again.

4.1.2. Example 2

We consider the same example as in the previous section, and we would like to obtain a mesh adapted to the control space defined as

$$\mathcal{M}(x, y) = \mathcal{R}(\theta(x, y)) \begin{pmatrix} \dfrac{1}{h_1^2(x, y)} & 0 \\ 0 & \dfrac{1}{h_2^2(x, y)} \end{pmatrix} \mathcal{R}^{-1}(\theta(x, y)) \tag{25}$$

where $\mathcal{M}(x, y)$ is the metric associated with any point of the domain and $\mathcal{R}(\theta(x, y))$ is the rotation matrix of angle $\theta(x, y)$

$$\mathcal{R}(\theta(x, y)) = \begin{pmatrix} \cos\theta(x, y) & -\sin\theta(x, y) \\ \sin\theta(x, y) & \cos\theta(x, y) \end{pmatrix} \tag{26}$$

and

$$\theta(x, y) = \arctan\frac{y}{x}$$
$$h_1(x, y) = 0{\cdot}0005 + 1{\cdot}5\,|1 - r(x, y)|$$

‖The mesh quality ranges from 0 (worst) to 1 (best)

$$h_2(x, y) = 0{\cdot}1\, r(x, y) + 1{\cdot}5|1 - r(x, y)| \tag{27}$$

This metric field specifies constant sizes of 0·1 along the tangents to the circle $r(x, y) = 1$ centred at the origin and sizes of 0·0005 along the orthogonal directions, with a variation towards an isotropic metric (as we move away from the circle). The maximal stretching ratio is 200 (close to the circle).

As previously, the adaption procedure is iterated, starting from a coarse mesh, until the corresponding mesh satisfies the specified size field, within a given tolerance.

As the metric variation in the control space is important and because the initial mesh is coarse, 10 iterations of the mesh adaption have been necessary to obtain an adapted mesh (cf. Figure 4, top left). This mesh contains 1312 elements. The worst (resp. average) quality is 0·015 (resp. 0·5). An enlargement of the area of large metric variation is illustrated in Figure 4 (bottom left). The two parts of Figure 5 (top and bottom left) show a conversion of the mesh into quadrilaterals.

At each iteration, the control space is replaced by a H-shock reduction of value $\beta = 2$. In this case, five iterations of the adaption procedure have been necessary to obtain an adapted mesh (cf. Figure 4, top right). This mesh contains 3082 elements. The worst (resp. average) quality is 0·1 (resp. 0·75). An enlargement of the critical area is illustrated in Figure 4 (bottom right). As expected, the quality of the new adapted mesh is better than the previous one and the adaption scheme converged faster. Figure 5 (top and bottom right) shows the conversion of this mesh into quadrilaterals. The quadrilateral mesh quality is better as well.

4.1.3. Hypersonic flow around NACA0012

We consider a classical compressible Navier–Stokes flow configuration around a NACA0012 aerofoil at Mach 2 with Reynolds number 10 000. The finite element solution of this problem is obtained using the NSC2KE solver[10] based on a finite-volume Galerkin technique and on a Runge–Kutta time-step integration scheme with 4 steps.

The number of iterations is fixed to 6000, and the mesh is adapted every 500 iterations. Figure 6 shows the last adapted mesh (step 6 of the mesh adaption) containing 58 362 elements. The worst quality (resp. mean quality) is 0·13 (resp. 0·92). Figure 7 shows at the same step the adapted mesh for which a H-shock $\beta = 2$ reduction has been applied, leading to 68 484 elements. The worst quality (resp. mean quality) is 0·54 (resp. 0·94). Notice that a quality improvement of order 4 is achieved with 1·7 times more elements.

If the solver can handle large size variations, it is preferable not to modify the space control (to avoid extra-computation). Conversely, the correction allows to improve the mesh quality, thus leading to decrease the vertex degrees (related to the computational matrix band-width).

4.1.4. Transonic unsteady flow around NACA0012

The second example is a prediction of a transonic unsteady flow computation around a NACA0012 aerofoil with buffeting.[11] The flow parameters are, respectively, the Mach number 0·775, the angle of attack 4 degrees and the Reynolds number 10^7. The number of iterations is fixed to 200 000, and the mesh is adapted every 1000 iterations. In this case, the main difficulties are related to the possible lack of information during the interpolation and the moving shock wave capture. In this case, the H-correction is required, especially to refine the mesh in the shock and boundary layer regions.

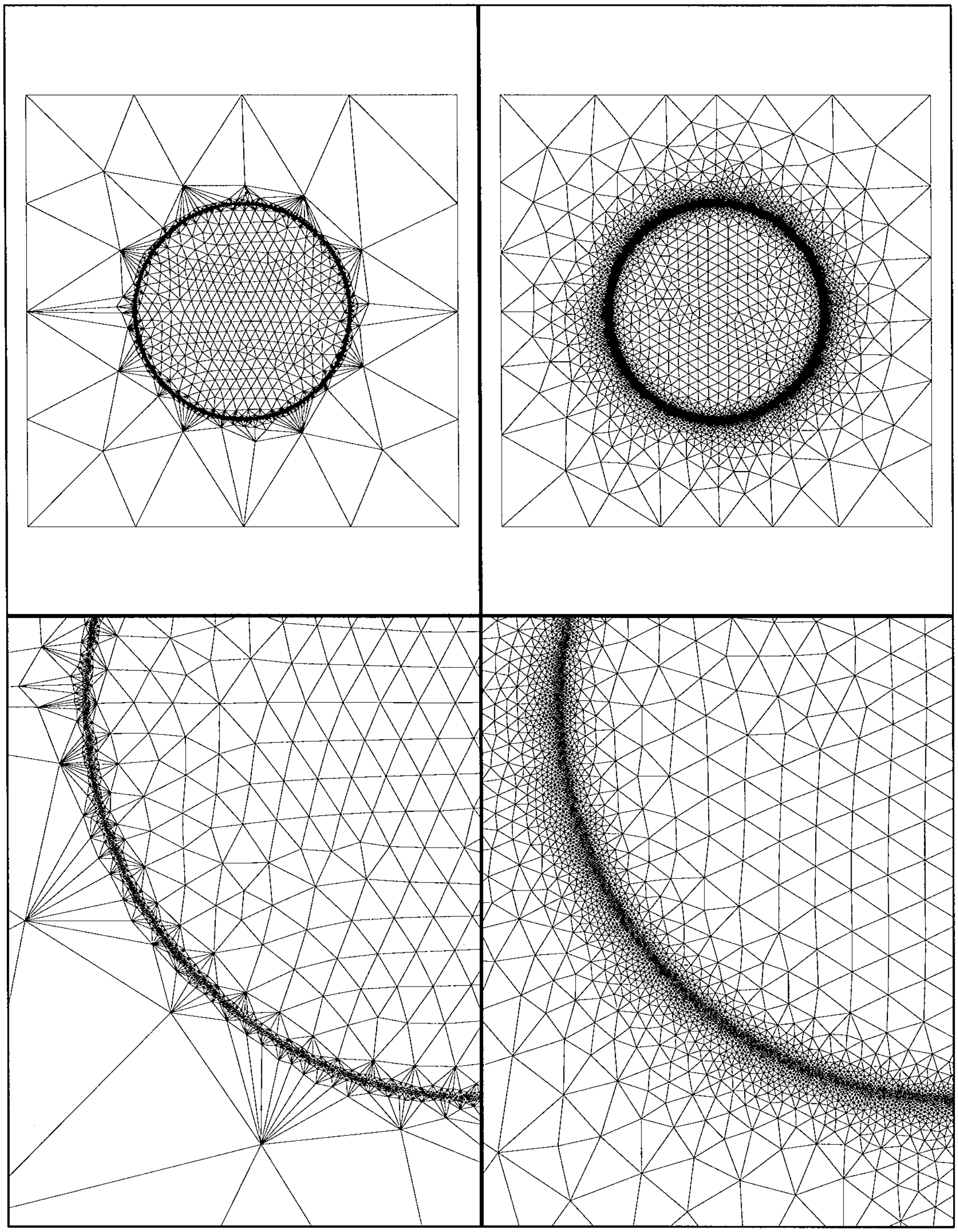

Figure 2. Triangular adapted (isotropic) mesh of a square without and with H-correction

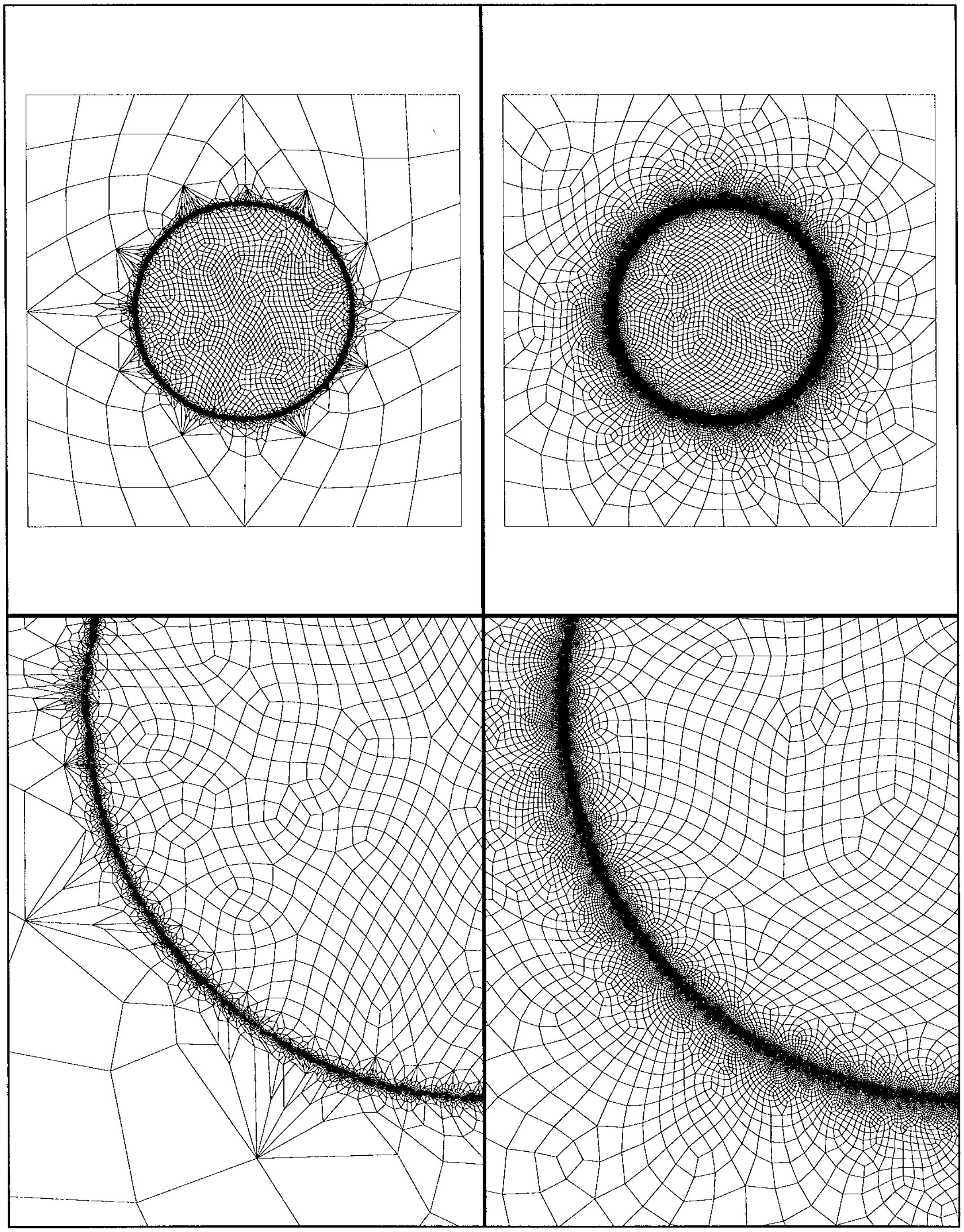

Figure 3. Quad adapted (isotropic) mesh of a square without and with H-correction

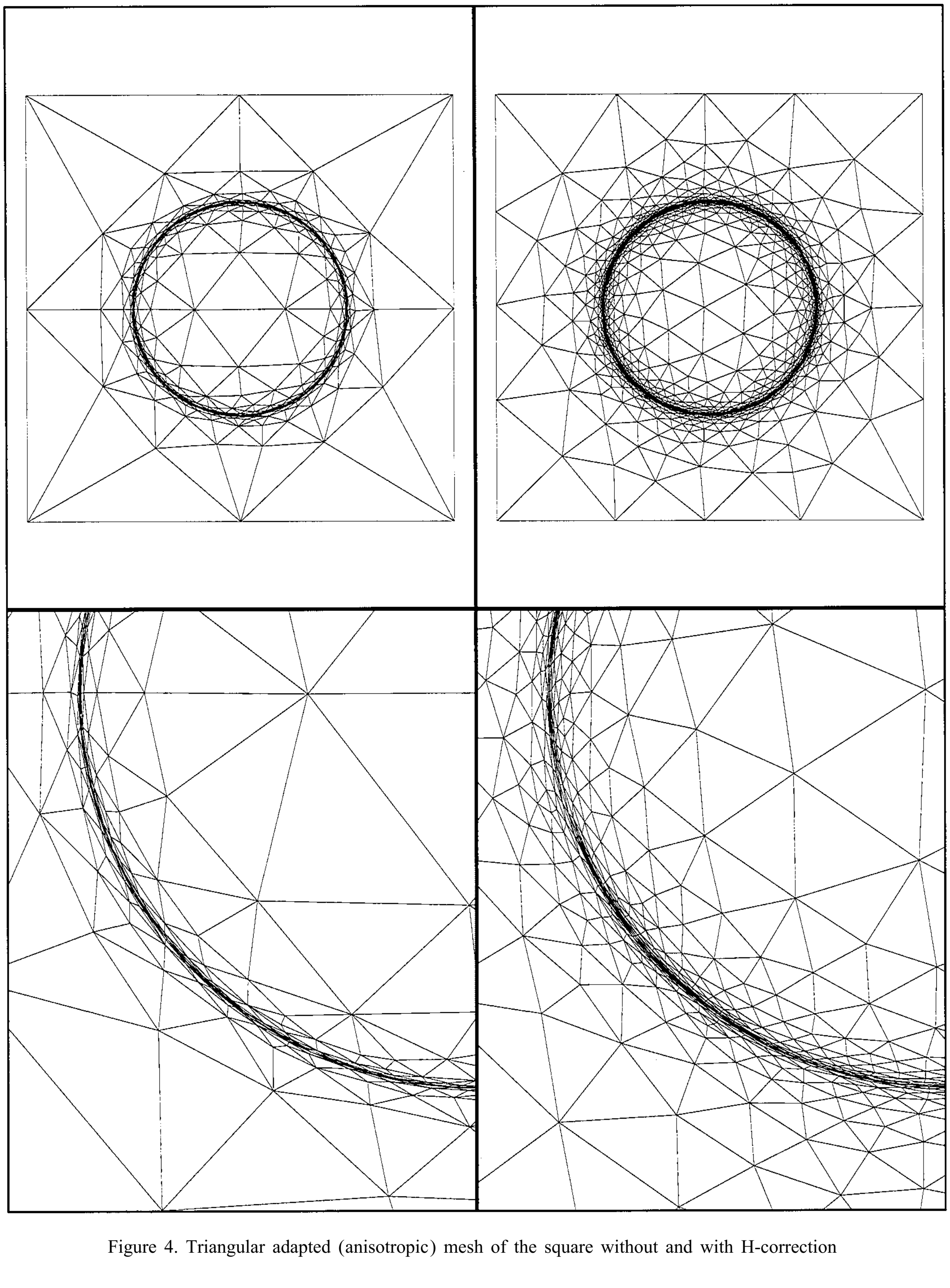

Figure 4. Triangular adapted (anisotropic) mesh of the square without and with H-correction

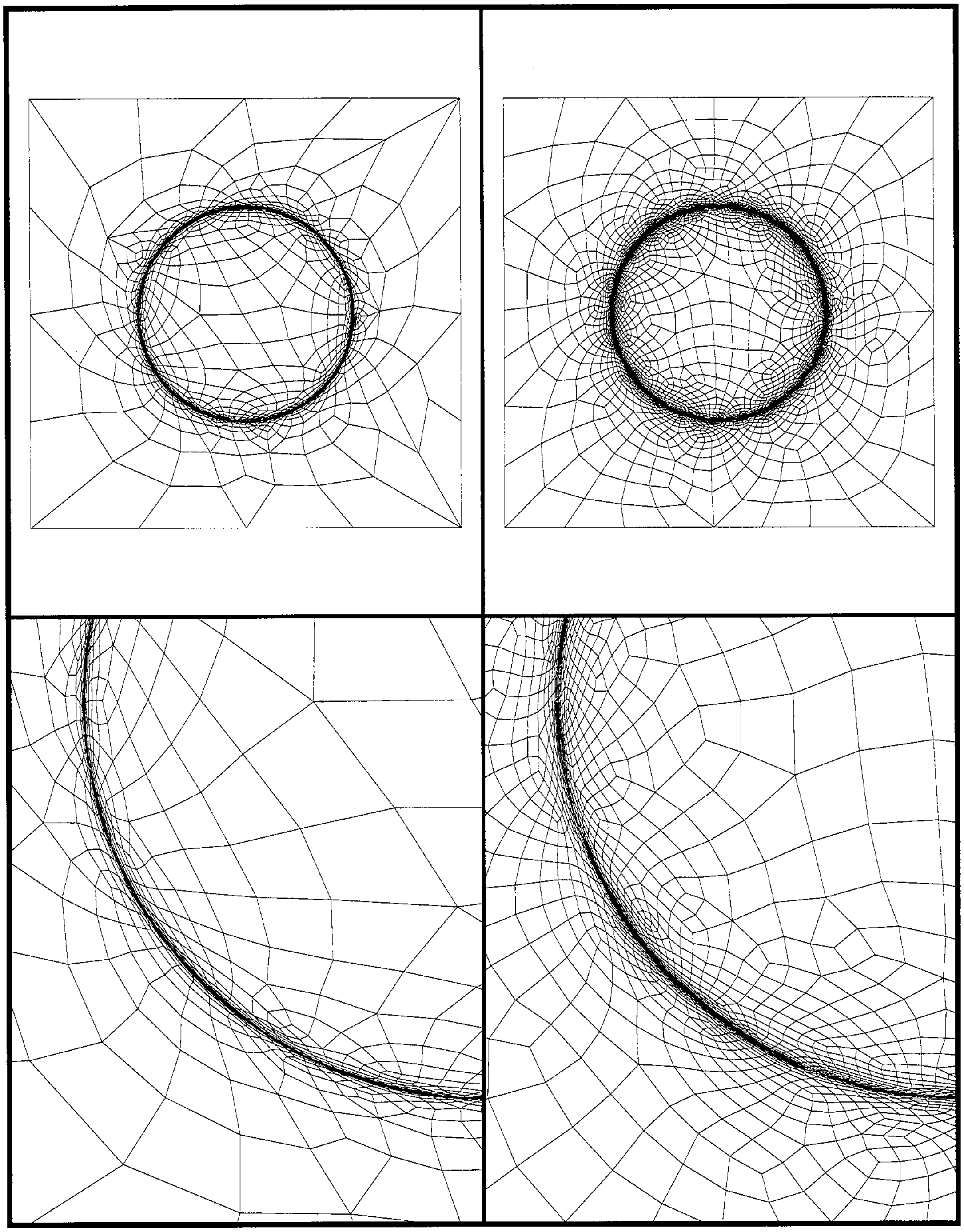

Figure 5. Quad adapted (anisotropic) mesh of the square without and with H-correction

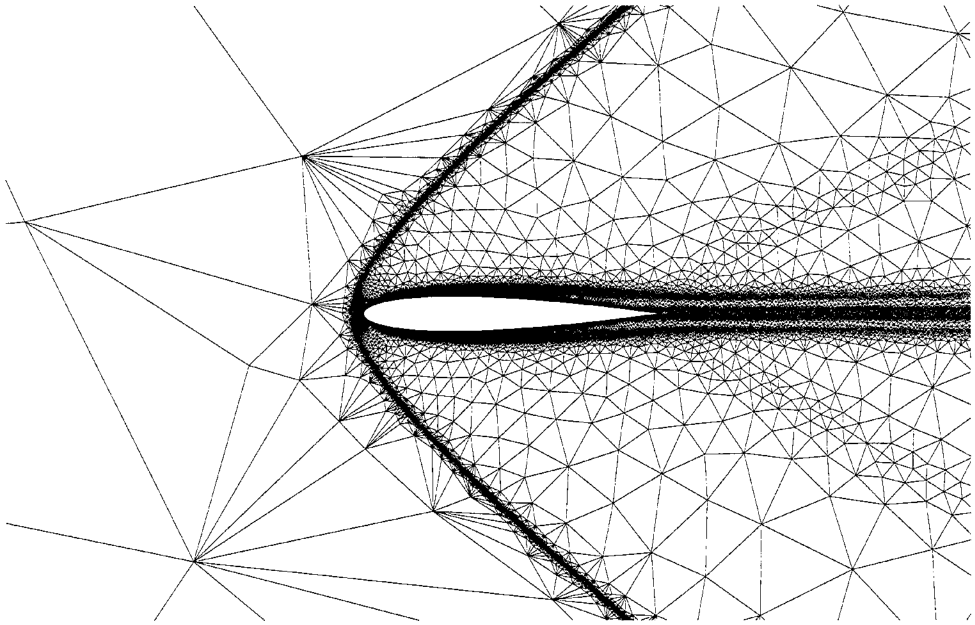

Figure 6. Adapted mesh without H-correction

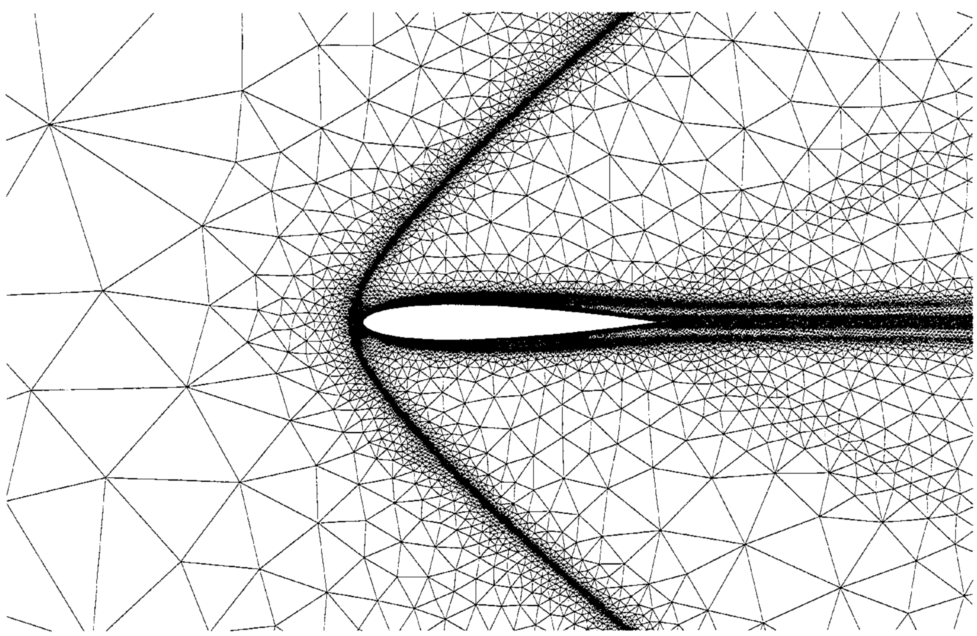

Figure 7. Adapted mesh with H-correction

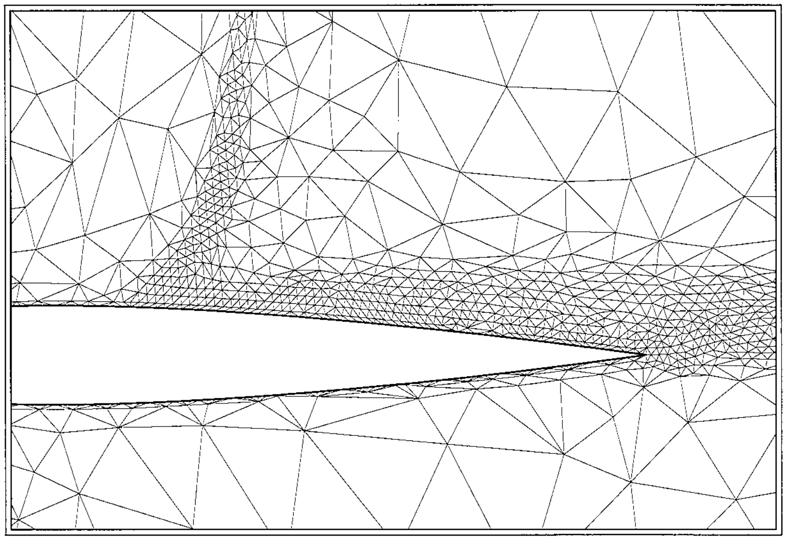

Figure 8. NACA0012 mesh without mesh graduation

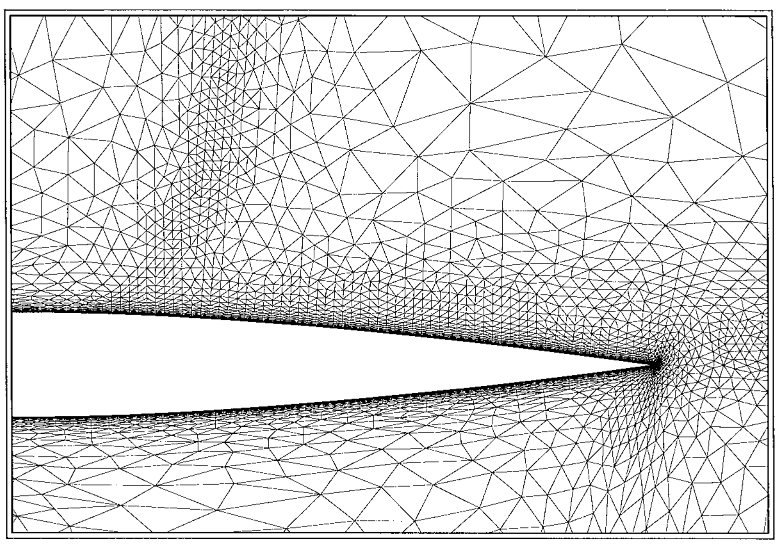

Figure 9. NACA0012 mesh with mesh graduation

Figures 8 and 9 show the adapted mesh without and with a H-variation $\alpha = 1{\cdot}3$. Figures 10–13 show the meshes and the iso-mach contours at steps 150 (lift min) and 190 (lift max) of mesh adaption, and Figure 14 illustrates the chronology of the lift.

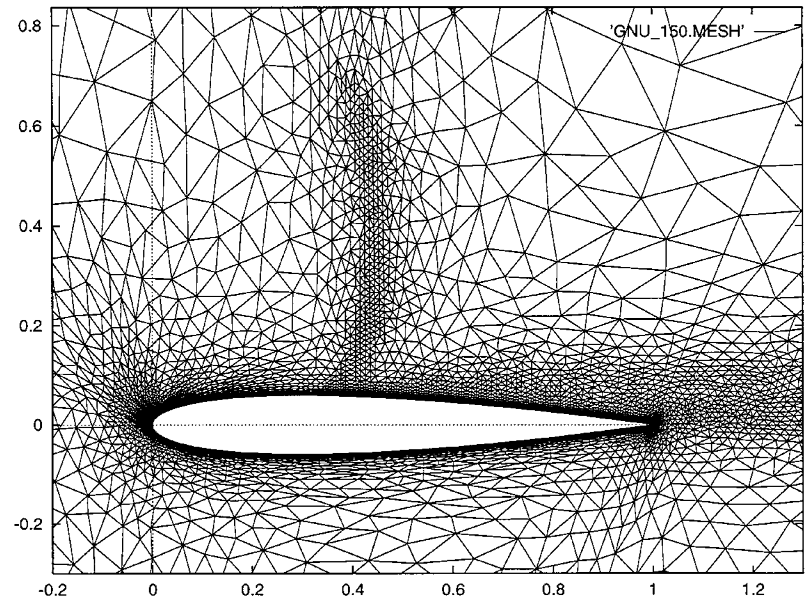

Figure 10. Mesh at time 0 (lift min)

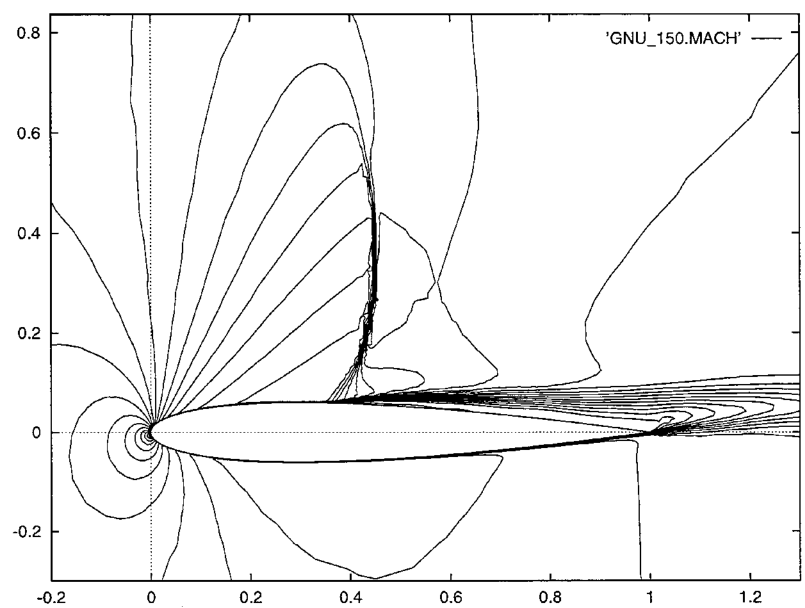

Figure 11. Iso-mach at time 0 (lift min)

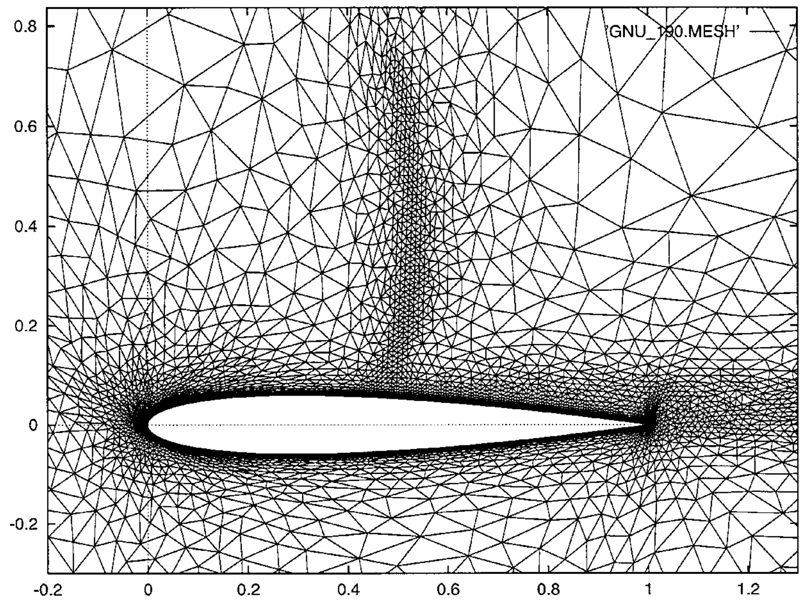

Figure 12. Mesh at time 8·78201 (lift max)

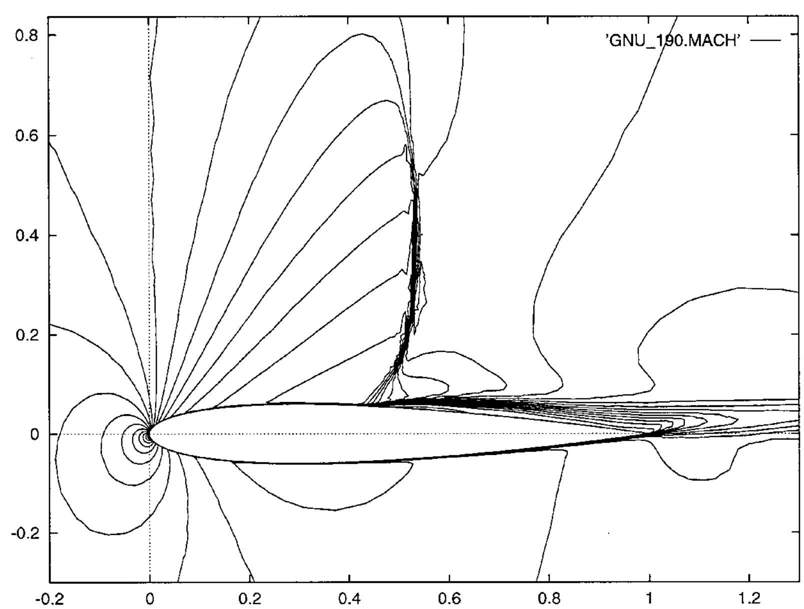

Figure 13. Iso-mach at time 8·78201 (lift max)

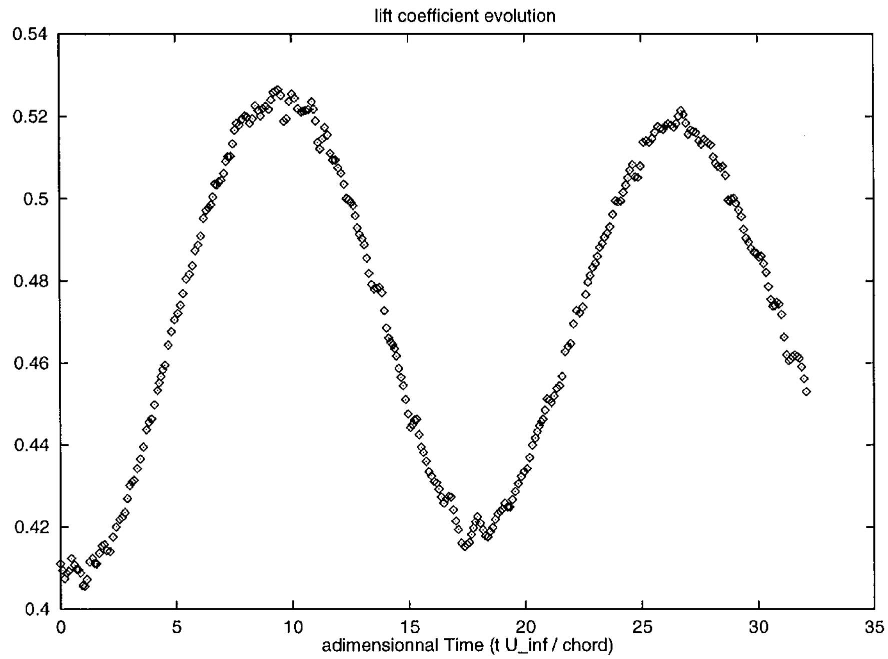

Figure 14. Chronology of the lift coefficient

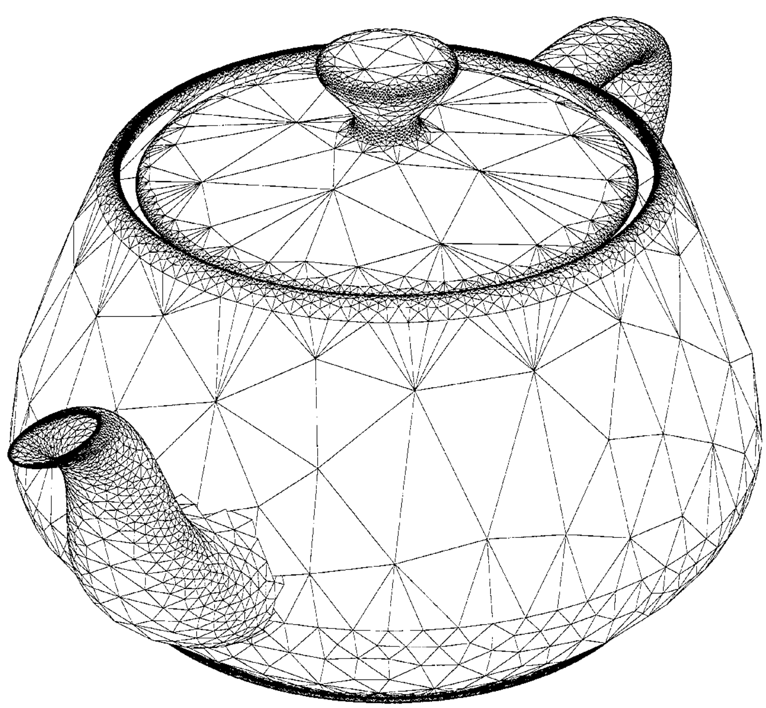

Figure 15. Utah teapot geometric surface meshes, without H-correction

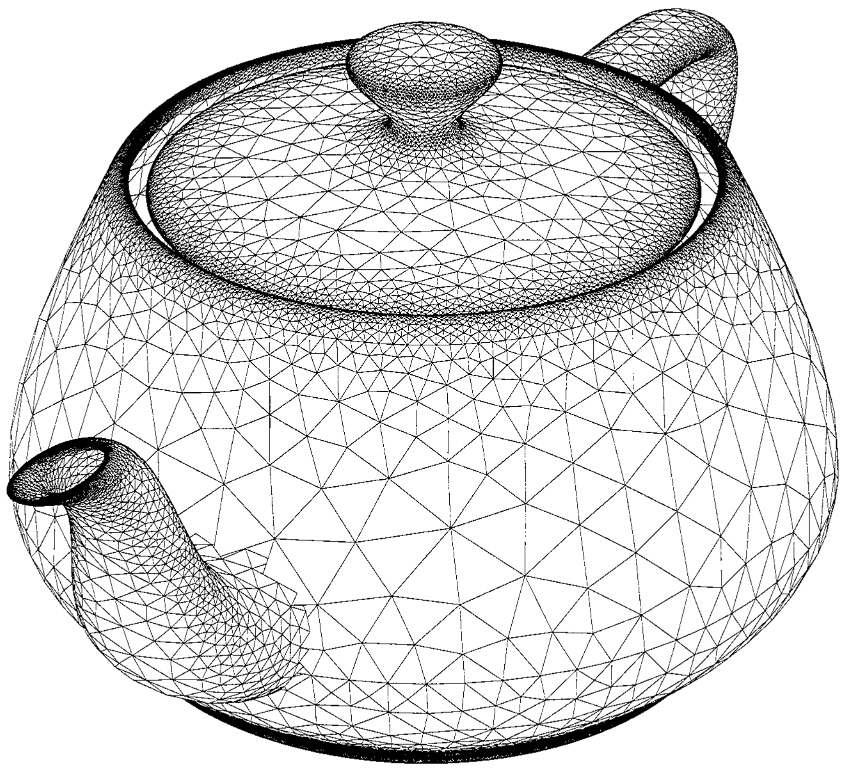

Figure 16. Utah teapot geometric surface meshes, with H-correction

4.2. Surface mesh example

We consider a piecewise parametric surface with four connected components (the so-called Utah Teapot). We are interested in constructing a *geometric* isotropic mesh of this surface. In Reference 12, we have shown that such a mesh must be adapted to the minimal radii of curvature at any point on the surface.

Thus, we define a discrete control space, composed of an initial arbitrary surface mesh and of the minimal radii of curvature size map associated with the mesh vertices. Figure 15 shows the adapted mesh to the above control space. The worst (resp. mean) quality of this mesh is 0·2 (resp. 0·85). Figure 16 shows the adapted mesh to a H-shock $\beta = 1{\cdot}5$ reduction of the initial space. The worst (resp. mean) quality of this mesh is 0·6 (resp. 0·94). This result clearly emphasizes the efficiency of the correction procedure on the resulting mesh quality. Figures 17 (top) and 18 (top) show enlargements on regions where the size variation is important. Figures 17 (bottom) and 18 (bottom) show enlargements of the same regions after H-correction.

Remark. These meshes are governed by the intrinsic geometric properties of the surface and are first generated in the corresponding parametric spaces and then mapped onto the surface. One should notice that the isotropic nature of the resulting mesh is related to the shape of the unit balls associated with the control space. Hence, the H-correction must be a Hc-correction.

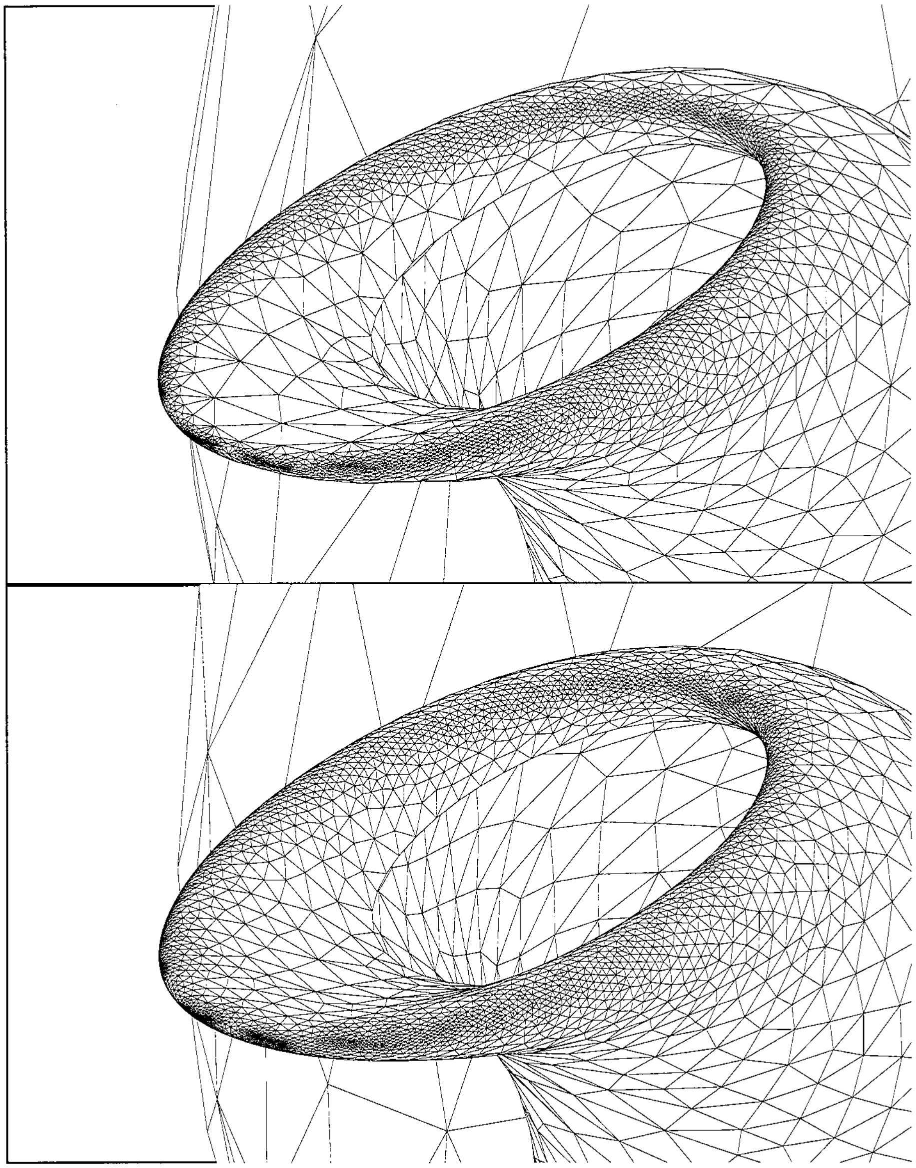

Figure 17. Enlargement of the teapot

5. CONCLUSION

In this paper, we have proposed a correction procedure to control the size variation of a control space which leads to improve (*a priori*) the resulting mesh quality in an adaption scheme. Several examples assessed the efficiency and the relevance of the method. A possible extension consists of determining explicitly the relationship between the size variation and the adapted mesh quality.

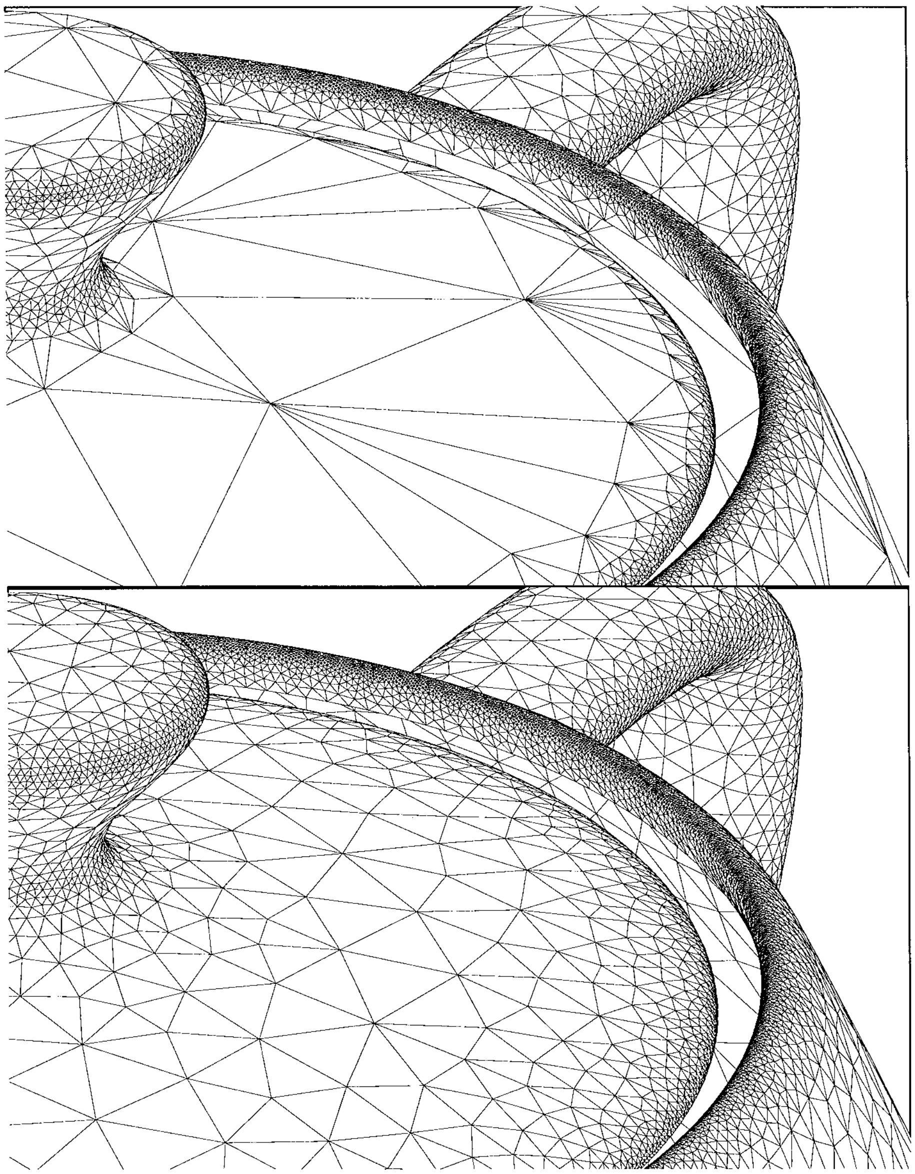

Figure 18. Enlargement of the teapot

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