Message-Passing Estimation from Quantized Samples

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***Abstract*—Recently, relaxed belief propagation and approximate message passing have been extended to apply to problems with general separable output channels rather than only to problems with additive Gaussian noise. We apply these to estimation of signals from quantized samples with minimum mean-squared error. This provides a remarkably effective estimation technique in three settings: an oversampled dense signal; an undersampled sparse signal; and any signal when the quantizer is not regular. The error performance can be accurately predicted and tracked through the state evolution formalism. We use state evolution to optimize quantizers and discuss several empirical properties of the optimal quantizers.**

I. Overview

Estimation of a signal from quantized samples arises both from the discretization in digital acquisition devices and the quantization performed for compression. An example in which treating quantization with care is warranted is analog-to-digital conversion, where the advantage from oversampling is increased by replacing conventional linear estimation with nonlinear estimation procedures [1]–[3]. Sophisticated approaches are also helpful when using sparsity or compressibility to reconstruct an undersampled signal [4]–[6].

A rather general abstraction is to consider $y = Q(Ax)$, where $x \in \mathbb{R}^n$ is a signal of interest, $A \in \mathbb{R}^{m \times n}$ is a linear mixing matrix, and $Q : \mathbb{R}^m \to \mathbb{R}^m$ is a quantizer. We will limit our attention here to scalar quantizers, meaning that Q is separable into m scalar quantizers $q_i : \mathbb{R} \to \mathcal{Y} \subset \mathbb{R}$ with $\mathcal{Y}$ countable.

Implementation of belief propagation (BP) for estimation of a continuous-valued quantity requires discretization of densities; this is inexact and leads to high computational complexity. To handle quantization without any heuristic additive noise model and with low complexity, we use a recently-developed Gaussian-approximated BP algorithm, called *relaxed belief propagation* [7], [8], which extends earlier methods [9], [10] to nonlinear output channels.

Our first main contribution is to demonstrate that relaxed BP provides significantly-improved performance over traditional methods for estimating from quantized samples. Gaussian approximations of BP have previously been shown to be effective in a range of applications; the extension to general output channels [7], [8] is essential to our application.

Our second main contribution concerns the quantizer design. When quantizer outputs are used as an input to a nonlinear estimation algorithm, minimizing the mean-squared error (MSE) between quantizer input and output is not necessarily equivalent to minimizing the MSE of the final reconstruction. We use the fact that the MSE under large random mixing matrices A can be predicted accurately from a set of simple state evolution (SE) equations [8], [11]. Then, by modeling the quantizer as a part of the measurement channel, we use the SE formalism to optimize the quantizer to asymptotically minimize distortions after the reconstruction by relaxed BP.

II. Simulation Example

Form A from i.i.d. Gaussian random variables, i.e., $A_{ai} \sim \mathcal{N}(0, 1/m)$; and assume i.i.d. Gaussian noise with variance $\sigma^2 = 10^{-5}$ perturbs measurements before quantization. The signal $\mathbf{x}$ is generated with i.i.d. elements from the Gauss-Bernoulli distribution

$$\mathbf{x}_i \sim \begin{cases} \mathcal{N}(0, 10), & \text{with probability } 0.1; \\ 0, & \text{with probability } 0.9. \end{cases}$$

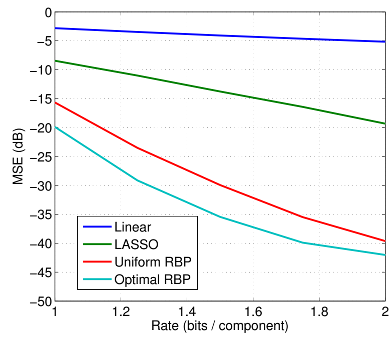

Fig. 1: Performance comparison.

Figure 1 presents a comparison of reconstruction distortions and confirms (a) the advantage of relaxed BP estimation; and (b) the advantage of optimizing quantizers using the SE equations. The quantization rate is varied from 1 to 2 bits per component of $\mathbf{x}$, and for each quantization rate, we optimize quantizers for the MSE of the *measurements* (labeled "Uniform RBP") and for MSE of the *reconstruction via relaxed BP* (labeled "Optimal RBP"). The figure also plots the MSE for linear MMSE estimation and lasso, both assuming the uniform quantizer that minimizes MSE of the measurements. Lasso performance was predicted by state evolution equations in [8], with the regularization parameter optimized. Relaxed BP offers dramatically better performance—more than 10 dB improvement at low rates. At higher rates, relaxed BP performance saturates due to the Gaussian noise at the quantizer input. Furthermore, optimizing the quantizer for the relaxed BP reconstruction improves performance by more than 4 dB for many rates. See also [12].

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