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Metastable Stationary Solutions of the radial d dimensional Sine-Gordon Model

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ABSTRACT

We show that the radial d-dimensional Sine-Gordon equation has localised metastable breather-like solutions. We analyse some properties of these solutions concentrating on their relative stability.

1. Introduction.

Some solutions of the d-dimensional sine-Gordon equation

$$\partial_\mu \partial^\mu f + \mu^2 \sin(f) = 0, \tag{1.1}$$

were studied in [1-3]. The authors of these papers demonstrated the existence of time dependent solutions similar to breathers which they called pulsons. The pulsons were not stable as they radiated away their energy and slowly died out. Geicke [4] also showed that a pulson can be thought of as a single radial 2π kink bouncing against the boundary at the origin. During the bouncing process some energy was radiated leading to the slow death of the soliton.

Recently [5], while studying the existence of non-topological solutions in the two dimensional baby-skyrmion model, we have observed a formation of a metastable time dependent localised field configuration, which appeared to be similar to a radially symmetric solution of the $(d+1)$ dimensional sine-Gordon equation:

$$f_{tt} - f_{rr} - \frac{d-1}{r} f_r + \mu^2 \sin(f) = 0, \tag{1.2}$$

with $d = 2$ in the baby-skyrmion model case and where $f_x = \frac{\partial f}{\partial x}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$.

We discoved it by observing that although our original configuration radiated away relatively quickly some energy, it then settled to a nontrivial configuration which resembled a breather-like solution of the sine-Gordon model.

When an appropriate initial condition was chosen for f the field oscillated with frequency T slightly larger than 2π and radiated a small amount of energy to gradually become a solution periodic in time and localised at the origin.

In [5] we have decided to call this metastable solution a pseudo-breather because of its similarity with the sine-Gordon breather solution.

2. Two-dimensional sine-Gordon model.

All our attempts to describe the pseudo-breather analytically have failed. Nevertheless, for large values of r, f is very small, in which case (1.2) can be approximated by

$$f_{tt} - f_{rr} - \frac{d-1}{r} f_r + \mu^2 f = 0, \tag{2.1}$$

and for $d = 2$ we thus have

$$f \sim sin(\omega t) J_0((\omega^2 - \mu^2)^{1/2} r)$$

if $\omega^2 > \mu^2$ or

$$f \sim sin(\omega t) K_0((\mu^2 - \omega^2)^{1/2} r)$$

otherwise.

The non-linearity of (1.2) makes it impossible to use a product ansatz to look for periodic solutions, but when we looked at the time evolution of f obtained numerically, we noticed that it oscillates in phase everywhere on the radial axis. This observation made us devise the following method to derive an approximation for a pseudo-breather profile. This method is loosely inspired by the geodesic approximation used to study the scattering of two or three dimensional solitons.

The numerical solutions we have obtained suggest that the pseudo-breather are quite close to the form

$$f(r,t) \sim \sin(\omega t) g(r). \tag{2.2}$$

Inserting this expression into (1.2) does not lead to an ordinary differential equation for $g(r)$. On the other hand, if we insert (2.2) into the action from which (1.2) is derived and perform the time integration over one period of oscillation T:

$$\begin{aligned} S &= \int_0^T dt \int_0^\infty r^{d-1}\, dr\, [f_t^2 - f_r^2 - 2(1 - \cos(f))] \\ &= \int_0^T dt \int_0^\infty r^{d-1}\, dr\, [\cos^2(\omega t)\omega^2 g^2 - \sin^2(\omega t)\partial_r g \partial_r g - 2(1 - \cos(g \sin(\omega t)))] \\ &= -\frac{T}{2} \int_0^\infty [g_r^2 - \omega^2 g^2 + 4(1 - J_0(g))] r^{d-1} dr \end{aligned} \tag{2.3}$$

we obtain a time independent action for g, where $\omega = 2\pi/T$ and J_0 is the 0-th Bessel function. Using (2.3) we can now derive the following equation for g

$$g_{rr} + (d-1)\frac{g_r}{r} + \frac{4\pi^2}{T^2} - 2J_1(g) = 0. \tag{2.4}$$

The profile of the pseudo-breather is thus approximately described by an ordinary differential equation which can be solved numerically. Unfortunately, (2.4) depends on the period of oscillation T which we have to choose before solving the equation. We have found that we can find a solution of (2.4) for any value of T larger than 2π, but none when $T \leq 2\pi$.

In Figure 1 we exhibit a few profiles g of the solutions of (2.4) , as well as the value of the energy as a function of T.

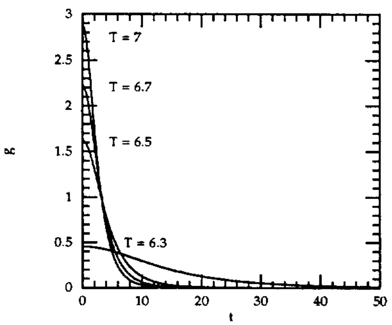

Figure 1.a : Solution Profile $g(r)$ for $T = 7$, 6.7, 6.5 and 6.3.

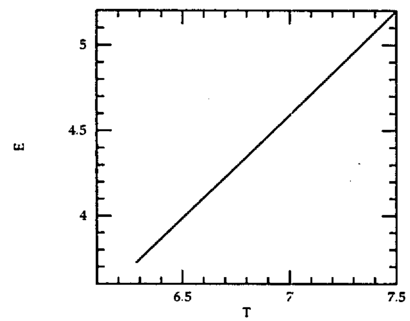

Figure 1.b : Energy as a function of T.

We see that as the period comes closer to the lower value $T = 2\pi$, the amplitude goes asymptotically to zero while the energy remains finite. The support of the solution actually increases as the energy decreases. The energy on the other hand is a linear function of T.

To find solutions of (1.2) we have then used the solutions of (2.4) as the initial conditions: $f(r) = g(r)$; $\partial f/\partial t = 0$, and we performed the time integration numerically. We have found that the profiles obtained by solving (1.2) do indeed lead to nearly periodic solutions. The profiles do oscillate with their time dependence well approximated by (2.2) . Moreover, the period of oscillation of the solution is given, at least during the first few cycles, by the parameter T used to compute the profile. As shown in Figure 2, when $T \leq 6.5$, the solutions are very stable. Even after 150000 cycles the solutions have not spread out nor radiated any noticeable amount of energy. When T is larger, the solutions of (2.4) give expressions which radiate away some energy and slowly spread out but eventually settle to stable configurations.

In Figure 2 we show the time evolution of the energy and of the amplitude of oscillation at the origin, defined as $h(t) = max_{t-T<\tau<t} f(r = 0, \tau)$, for 4 different initial conditions. To compute the time evolution of the energy, the waves radiated by the pseudo-breathers were absorbed at the edge of the grid corresponding to large values

of r. The amplitude of oscillation at the origin shows how, as time increases, the pseudo-breathers slowly spread out.

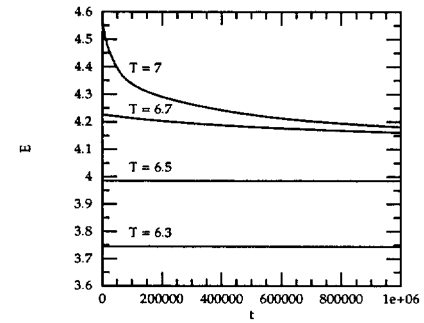

Figure 2.a : Time evolution for the Pseudo-breathers energy.

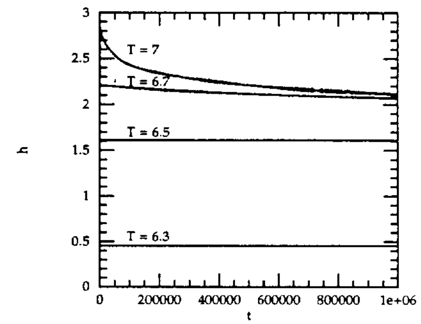

Figure 2.b : Time evolution for the Pseudo-breather amplitude

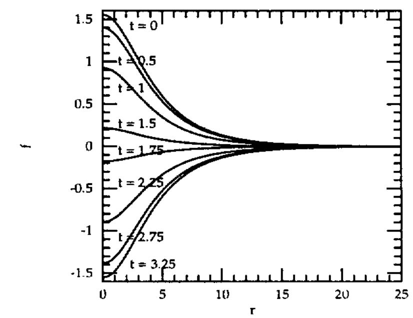

Figure 3.a : Profiles for the pseudo-breather.

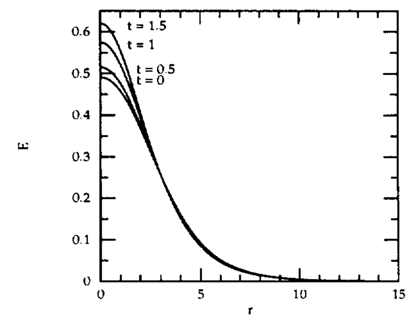

Figure 3.b : Pseudo-breather Energy density profile

In Figure 3. we present snapshots of the profiles $f(r)$ and of the energy profiles for a pseudo-breather solution at various times during the first half of the cycle. To obtain these graphics we have integrated (1.2) numerically starting from an inital condition given by the solution of (2.4) with $T = 6.4863$. The amplitude at the origin was then $\pi/2$ and the energy $E = 3.969$.

Looking at Figure 3.a, we see that the amplitude of the solution can be characterised by the amplitude of oscillation at the origin (the largest value of f at the origin during a full cycle). From now on we will use this parameter to study the time evolution of pseudo-breathers.

Next we have analysed the stability of our periodic field configurations. For this we have taken the profile corresponding to $T = 6.5$ and have run a few simulations with this profile slightly perturbed: $f(r) = g_{6.5}(0)$; $\partial f(r)/\partial t = K\, exp(-c(r-R)^2)$, where K, c and R are parameters describing the perturbation. This initial condition (for nonzero value of K) adds to the profile a kinetic contribution in the shape of a lump located close to the pseudo-breather. We have run a few simulations for some values of the parameters. In every case the wave created by the perturbation interacted with the pseudo-breather, but it quickly dissipated and eventually left, leaving behind the configuration in an exited mode: the amplitude at the origin was not constant but it oscillated in time. Nevertheless, this excitation did not affect the stability of the pseudo-breather.

In Figure 4 we present the time evolution of the amplitute of an excited pseudo-breather, after the wave has left it. In this example, we chose $K = 0.4, c = 3.16$ and $R = 25$. The perturbation increased the energy of the profile by 15%. When comparing Figure 4 with Figure 3.b we clearly see the modulation of the amplitude introduced by the perturbation.

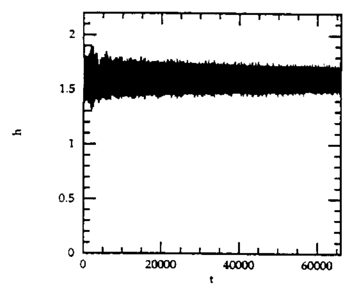

Figure 4 : Perturbed pseudo-breather:
T=6.5; K=0.4; c=3.16; R=25. E=4.69

Finally, by trial and error, we have found that the expression

$$f(r,0) = 4\text{atan}(C\exp(-\frac{2r}{K\pi}\text{atan}(r/K))) \qquad (2.5)$$

with $K = 10^{1/2}$ and $C = \tan(\pi/8)$, constitutes a good initial condition to describe our metastable field configuration: it has roughly the correct energy and it settles reasonably quickly to the solution.

In Figure 5 we present the time evolution of the energy and the amplitude of oscillation of the field configuration over 1000000 units of time (which corresponds to over 150000 cycles) for the initial condition (2.5) for some values of C and $K = 10^{1/2}$. In all cases the configuration corresponds to a localised lump at the origin and the amplitude of oscillation, shown in Figure 5.b, is modulated (hence the dark bands on the figure) unless we take $C \sim 0.4143$.

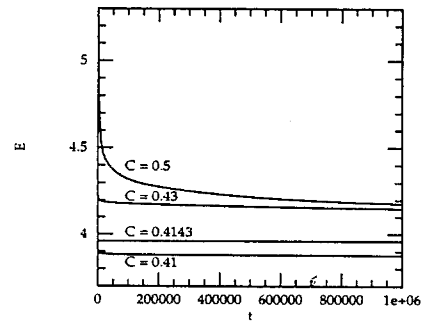

Figure 5.a : Energy for the Pseudo Breather.

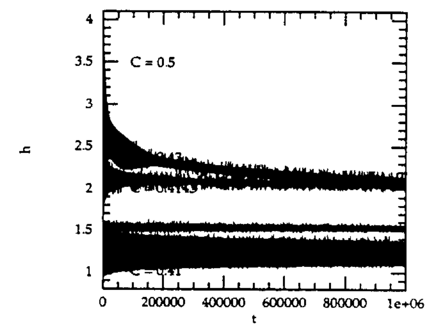

Figure 5.b : Amplitude of oscillation for the Pseudo Breather.

We can thus conclude that the ansatze (2.5) describe exited pseudo-breathers (except when $C = 0.4143$) which very slowly radiate their energies away to settle down to stable pseudo-breathers.

To conclude this chapter let us summarise what we have done; we have shown that the two-dimensional sine-Gordon equation has radially symmetric solutions which are periodic or nearly periodic in time. It is impossible to prove numerically that the solutions do not slowly die out. However, we have shown that over very long periods of time, corresponding to more than 150000 cycles, the solutions are amazingly stable.

The main difference between the pseudo-breathers and the pulsons is the stability of the pseudo-breathers as they do not die out. Moreover, the pseudo-breathers can be of various size and their profile, modulo the time dependance (2.2) , does not change with time.

3. Three and Four dimensional sine-Gordon models.

Having demonstrated the existence of periodic solutions of the two-dimensional sine-Gordon equation (1.2) , one may wonder whether such solutions exist also in higher dimensions. The only difference in (1.2) for the three dimensional and four dimensional equations is that d is now equal to 3 or 4, respectively. This small difference gives us hope of finding similar solutions.

As for the two-dimensional equation we start by assuming that a periodic solution will be close to the form (2.2) . Inserting this expression in the action for the sine-Gordon equation we can repeat our previous calculations (2.3) and derive the equation (2.4) for the profile g where we have now $d = 3$ or $d = 4$.

Equation (2.4) can be solved numerically and one can find a non trivial solution for any $T > 2\pi$. In Figure 6 we present the profiles obtained for a few values of T and the energies of the configuration as a function of T. We notice immediately that the energy of these configurations is much bigger than in the two dimensional case. Moreover, the energy is not, anymore, a linear function of T and the energy has a minimum value around $T = 6.7$ when $d = 3$ and close to $T = 6.5$ when $d = 4$. The amplitude of oscillation also goes to 0 as T goes to 2π.

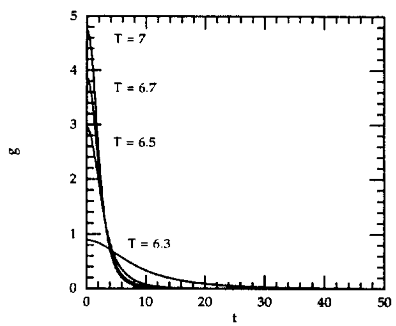

Figure 6.a : $d = 3$: solution profile g for $T = 7,\ 6.7,\ 6.5$ and 6.3.

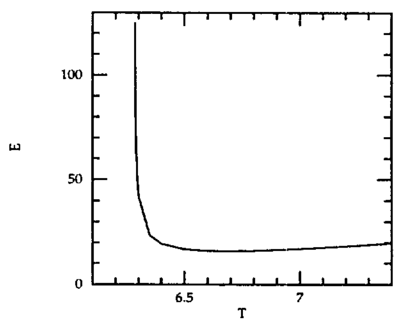

Figure 6.b : $d = 3$: Energy as a function of T.

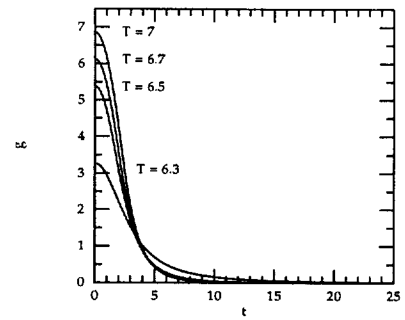

Figure 6.c : $d = 4$: solution profile g for $T = 7,\ 6.7,\ 6.5$ and 6.3.

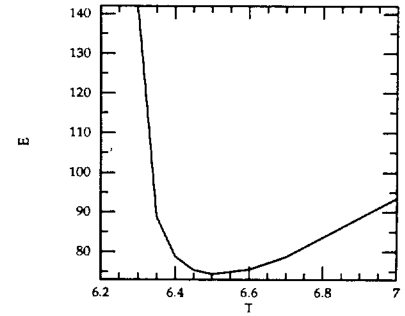

Figure 6.d : $d = 4$: Energy as a function of T.

The configurations given by (2.4) when $d = 3$ and $d = 4$ also lead to quasi periodic solutions when T is just slightly larger than 2π but they are much less stable than in the two dimensional case. The profile oscillates a few hundred or a few thousand times but then it suddenly collapses and quickly decays into pure waves.

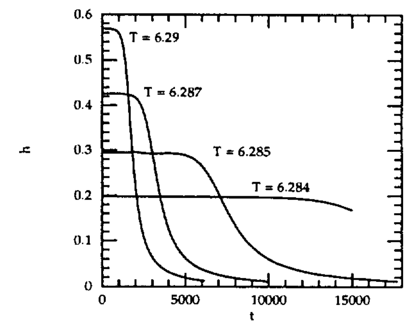

Figure 7.a : Time evolution for the pseudo-breather amplitude for 4 profiles ($d = 3$).

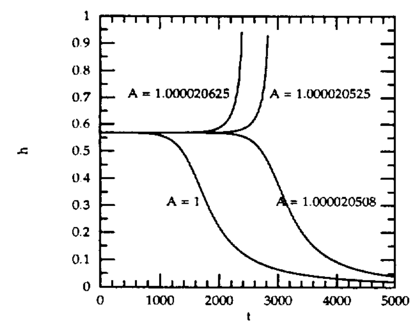

Figure 7.b : Time evolution for the the amplitude of 4 dilated pseudo-breathers profiles ($d = 3$).

In Figure 7.a we plot the time evolution of the amplitude of oscillation for 4 different profiles in the three dimensional case. It is clear that the field configurations live longer as the parameter T gets closer to 2π. In Figure 6.b we present the plot of the time evolution (in 3 dimensions) of the amplitude of profiles $Ag(r)$ obtained by multiplying the solution g of (2.4) (with $T = 6.29$) by a constant A. It shows that by multiplying a solution of (2.4) by an appropriate coefficient it is possible to extend the life time of the resultant field configuration. When A is too large the dilation introduces a perturbation producing an excitation at the origin which makes the amplitude to increase.

All this suggests that the equation (1.2) has periodic solutions also in three dimensions but that these solutions are unstable. The solutions of (2.4) are good initial conditions for seeking periodic solutions (when T is close enough to 2π).

The results in four dimensions were not that different from what we observed in three dimensions. Again, we have found many quasistable field configurations which were, however, even less stable. Like in the three dimensional case we can find the profiles by solving (2.4) (also only for $T > 2\pi$) and then study their stability. We have found that after relatively short period of time ($t \sim 200$) the amplitude decreases and the field dissipates. Like in three dimension we can increase the lifetime of our structures by multiplying the profiles by an appropriate value of A. In figure 8 we present the plots of the time evolution of the amplitude of our field configuration (with the profile for $T = 6.284389$ multiplied by four values of A). We note the strong dependence on A (the tuning requires the accuracy to more decimal points than in $d = 3$ case.

Hence we conclude that in contradistinction to the $d = 2$ case the pseudobreathers in $d = 3, 4$ are unstable.

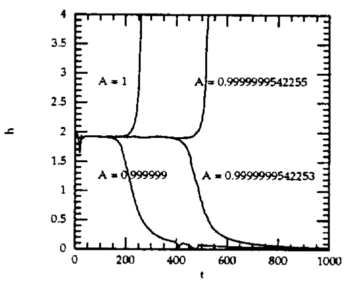

Figure 8 : Time evolution for the amplitude of 4 dilated pseudo-breathers profiles ($d = 4$).

4. Conclusions

We have shown that the sine-Gordon equation in 2, 3 and 4 dimensions has localised radially symmetric solutions which are periodic in time. These solutions are fairly stable for the two dimensional equation but, on the other hand, they are unstable in the three and four dimensional case. When the two dimensional solutions are perturbed with a radially symmetric perturbations they exhibit interesting exited modes of oscillation.

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