METHOD OF FUNDAMENTAL SOLUTIONS FOR BIHARMONIC EQUATION BASED ON ALMANSI-TYPE DECOMPOSITION

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Abstract. The aim of this paper is to analyze mathematically the method of fundamental solutions applied to the biharmonic problem. The key idea is to use Almansi-type decomposition of biharmonic functions, which enables us to represent the biharmonic function in terms of two harmonic functions. Based on this decomposition, we prove that an approximate solution exists uniquely and that the approximation error decays exponentially with respect to the number of the singular points. We finally present results of numerical experiments, which verify the sharpness of our error estimate.

Keywords: method of fundamental solutions; biharmonic equation; Almansi-type decomposition

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1. Introduction and main results

Let Ω be a bounded simply connected region in the plane. We then consider the following boundary value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta^2 = \partial^4/\partial x^4 + \partial^4/\partial x^2 \partial y^2 + \partial^4/\partial y^4$ is the biharmonic operator in the plane, $\partial u/\partial \nu$ denotes the outward normal derivative of u on $\partial\Omega$, and f and g are given data defined on $\partial\Omega$.

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It is well-known that for a given biharmonic function u, there exist two holomorphic functions φ and ψ in Ω such that the following relation holds [9]:

$$u(X,Y) = \varphi(z) + \overline{\varphi}(z) + \overline{z}\psi(z) + z\overline{\psi}(z) = 2\Re(\varphi(z) + \overline{z}\psi(z)), \quad z = X + \mathrm{i}Y.$$

In particular, defining $\gamma(X,Y) = 2\Re\varphi(z)$, $\alpha(X,Y) = 2\Re\psi(z)$, and $\beta(X,Y) = 2\Im\psi(z)$, we have

$$u(X,Y) = \gamma(X,Y) + X\alpha(X,Y) + Y\beta(X,Y).$$

Namely, the above formula implies that any biharmonic function can be decomposed into three harmonic functions, two of which are conjugate harmonic. Moreover, Krakowski and Charnes [8], and Bock and Gürlebeck [2] showed that the number of harmonic functions is indeed equal to 2, that is, for a given biharmonic function u in Ω, there exist harmonic functions p, q, $\overline{p}$, $\overline{q}$, $\overline{\overline{p}}$, $\overline{\overline{q}}$ such that the following identities hold:

$$\begin{aligned} u(X,Y) &= p(X,Y) + (X^2+Y^2)q(X,Y), \\ u(X,Y) &= \overline{p}(X,Y) + X\overline{q}(X,Y), \\ u(X,Y) &= \overline{\overline{p}}(X,Y) + Y\overline{\overline{q}}(X,Y). \end{aligned} \tag{1.2}$$

Especially, (1.2) is a decomposition of Almansi type, which was first considered by Almansi [1]. Therefore, we only have to find suitable approximations of two harmonic functions. We hereafter consider the case of (1.2) restricted to the case, where Ω is a disk D_ϱ with radius ϱ having the origin as its center.

Based on the Almansi-type decomposition of biharmonic function, the following scheme for the method of fundamental solutions (MFS for short) can be obtained [5]. Choose the singular points $\{y_k\}_{k=1}^N$ as $y_k = R\omega^{k-1}$, $1 \leqslant k \leqslant N$, and construct approximations for p and q as follows:

$$p^{(N)}(x) = \sum_{k=1}^{N} Q_k^p E(x - y_k), \quad q^{(N)}(x) = \sum_{k=1}^{N} Q_k^q E(x - y_k),$$

where $E(x) = \frac{1}{2}\pi^{-1}\log|x|$ is the fundamental solution of the Laplace operator Δ, $R > \varrho$, and $\omega = \exp(2\pi\mathrm{i}/N)$. Namely, an approximation $u^{(N)}$ for the solution u of (1.1) is given by

$$u^{(N)}(x) = \sum_{k=1}^{N} (Q_k^p + |x|^2 Q_k^q) E(x - y_k). \tag{1.3}$$

R e m a r k 1.1. In the usual formulation of MFS, an approximate solution is given by

$$u^{(N)}(x)=\sum_{k=1}^{N}(Q_k^pE(x-y_k)+Q_k^qF(x-y_k)), \tag{1.4}$$

since the function $F(x)=\frac{1}{8}\pi^{-1}|x|^2\log|x|$ is the fundamental solution for the biharmonic operator Δ^2. MFS of the form (1.4) has been proposed first by Karageorghis and Fairweather [4] and used in the subsequent papers [11], [3], but so far there exists no mathematical result such as the unique existence and the exponential convergence of approximate solution.

The coefficients $\{Q_k^{p,q}\}_{k=1}^N$ are determined by the collocation method, that is, taking the collocation points $\{x_j\}_{j=1}^N$ as $x_j=\varrho\omega^{j-1}$, and imposing the boundary conditions

$$u^{(N)}(x_j)=f(x_j),\quad \frac{\partial u^{(N)}}{\partial\nu}(x_j)=g(x_j),\quad j=1,2,\ldots,N. \tag{1.5}$$

This type of MFS based on Almansi-type decomposition was investigated by Li et al. [10] and some mathematical analysis was done. However, they consider the Trefftz method rather than the collocation method. Thus, the aim of this paper is to establish mathematical theory of MFS based on Almansi-type decomposition (1.3) together with the collocation method when Ω is the disk D_ϱ as a first step for developing mathematical theory in arbitrary region.

We are now in a position to state the main theorems of this paper.

Theorem 1.1. *An approximate solution $u^{(N)}$ for* (1.1) *of the form* (1.3) *satisfying* (1.5) *exists uniquely if and only if $R^N-\varrho^N\neq 1$.*

Theorem 1.2. *Suppose that $R^N-\varrho^N\neq 1$ and $R\neq 1$ hold. Also suppose that the Fourier coefficients $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ of f and g can be estimated as*

$$|f_n|,\ |g_n|=O(b^{|n|})\quad \text{as } n\to\infty,$$

where $b\in\,]0,1[$. Then we have

$$\|u-u^{(N)}\|_{L^\infty(\Omega)}=\begin{cases} O\Big(N\Big(\dfrac{\varrho}{R}\Big)^N\Big), & b\Big(\dfrac{R}{\varrho}\Big)^2<1,\\ O\Big(N^2\Big(\dfrac{\varrho}{R}\Big)^N\Big), & b\Big(\dfrac{R}{\varrho}\Big)^2=1,\\ O(Nb^{N/2}), & b\Big(\dfrac{R}{\varrho}\Big)^2>1.\end{cases}$$

The content of this paper is as follows. In Section 2, we prove Theorem 1.1, which ensures the unique existence of an approximate solution. In Section 3, the exponential decay of the approximation error, that is, Theorem 1.2 is proved. In Section 4, we present several results of numerical experiments, which exemplify the sharpness of our error estimate. We also compare the conventional scheme (1.4) with the present scheme (1.3) based on Almansi-type decomposition. In Section 5, we summarize this paper and give some concluding remarks.

2. Unique existence

In this section, we establish the unique existence of an approximate solution $u^{(N)}$ for (1.1) of the form (1.3) satisfying (1.5) provided that the singular points $\{y_k\}_{k=1}^N$ and the collocation points $\{x_j\}_{j=1}^N$ are given as in the previous section. Hereafter, we identify $\mathbb{R}^2$ with $\mathbb{C}$, and often write the complex number x as the two-dimensional vector $x = (X_1, X_2)^{\mathrm{T}}$.

Since we can compute the derivatives of $E(x - y_k)$ and $|x|^2 E(x - y_k)$ with respect to X_1 as

$$\begin{aligned}\frac{\partial}{\partial X_1} E(x - y_k) &= \frac{1}{2\pi} \frac{X_1 - y_{k1}}{|x - y_k|^2},\\ \frac{\partial}{\partial X_1}(|x|^2 E(x - y_k)) &= 2X_1 E(x - y_k) + |x|^2 \frac{1}{2\pi} \frac{X_1 - y_{k1}}{|x - y_k|^2},\end{aligned}$$

and the derivatives of them with respect to X_2 can be computed in a similar way, the normal derivative of $u^{(N)}$ at $x \in \partial\Omega$ can be computed as follows:

$$\begin{aligned}\frac{\partial u^{(N)}}{\partial \nu}(x) &= \sum_{k=1}^{N} \Bigg[\frac{Q_k^p}{2\pi|x - y_k|^2} \begin{pmatrix} X_1 - y_{k1} \\ X_2 - y_{k2} \end{pmatrix} \cdot \frac{1}{|x|} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &\quad + Q_k^q \Bigg\{ 2E(x - y_k) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \cdot \frac{1}{|x|} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &\quad + \frac{|x|^2}{2\pi|x - y_k|^2} \begin{pmatrix} X_1 - y_{k1} \\ X_2 - y_{k2} \end{pmatrix} \cdot \frac{1}{|x|} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Bigg\} \Bigg] \\ &= \sum_{k=1}^{N} \Big[\frac{Q_k^p}{2\pi} \frac{(x|x|^{-1}, x - y_k)}{|x - y_k|^2} + Q_k^q \Big\{ 2|x|E(x - y_k) + \frac{|x|^2}{2\pi} \frac{(x|x|^{-1}, x - y_k)}{|x - y_k|^2} \Big\} \Big] \\ &= \sum_{k=1}^{N} \Big[\frac{Q_k^p}{2\pi} \Re\Big(\frac{x|x|^{-1}}{x - y_k} \Big) + Q_k^q \Big\{ 2|x|E(x - y_k) + \frac{|x|^2}{2\pi} \Re\Big(\frac{x|x|^{-1}}{x - y_k} \Big) \Big\} \Big],\end{aligned}$$

where $(\cdot,\cdot)$ denotes the usual two-dimensional Euclidean inner product. In particular, at a collocation point $x_j = \varrho\omega^{j-1}$, we have

$$\frac{\partial u^{(N)}}{\partial \nu}(x_j) = \sum_{k=1}^{N}\Big[\frac{Q_k^p}{2\pi}\Re\Big(\frac{\omega^{j-1}}{x_j - y_k}\Big) + Q_k^q\Big\{2\varrho E(x_j - y_k) + \frac{\varrho^2}{2\pi}\Re\Big(\frac{\omega^{j-1}}{x_j - y_k}\Big)\Big\}\Big].$$

Therefore, the collocation equations (1.5) are equivalent to the system of $2N$ linear equations

$$G\boldsymbol{Q} = \boldsymbol{b},$$

where

$$\begin{aligned}
G &= \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \in \mathbb{R}^{2N\times 2N}, \\
G_{11} &= (E(x_j - y_k);\ j,k = 1,2,\ldots,N) \in \mathbb{R}^{N\times N}, \\
G_{12} &= (\varrho^2 E(x_j - y_k);\ j,k = 1,2,\ldots,N) = \varrho^2 G_{11} \in \mathbb{R}^{N\times N}, \\
G_{21} &= \Big(\frac{1}{2\pi}\Re\Big(\frac{\omega^{j-1}}{x_j - y_k}\Big);\ j,k = 1,2,\ldots,N\Big) \in \mathbb{R}^{N\times N}, \\
G_{22} &= \Big(2\varrho E(x_j - y_k) + \frac{\varrho^2}{2\pi}\Re\Big(\frac{\omega^{j-1}}{x_j - y_k}\Big);\ j,k = 1,2,\ldots,N\Big) \\
&= 2\varrho G_{11} + \varrho^2 G_{21} \in \mathbb{R}^{N\times N}, \\
\boldsymbol{Q} &= \begin{pmatrix} \boldsymbol{Q}^p \\ \boldsymbol{Q}^q \end{pmatrix} \in \mathbb{R}^{2N}, \quad \boldsymbol{Q}^{p,q} = (Q_1^{p,q}, Q_2^{p,q}, \ldots, Q_N^{p,q})^{\mathrm{T}} \in \mathbb{R}^N, \\
\boldsymbol{b} &= \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix} \in \mathbb{R}^{2N}, \quad \boldsymbol{f} = (f(x_1), f(x_2), \ldots, f(x_N))^{\mathrm{T}} \in \mathbb{R}^N, \\
\boldsymbol{g} &= (g(x_1), g(x_2), \ldots, g(x_N))^{\mathrm{T}} \in \mathbb{R}^N.
\end{aligned}$$

We note that each matrix $G_{\mu\nu}$ is circulant. Therefore, using the discrete Fourier transform, these matrices can be diagonalized as follows:

$$W^{-1} G_{\mu\nu} W = \mathrm{diag}(\gamma_{\mu\nu 0}, \gamma_{\mu\nu 1}, \ldots, \gamma_{\mu\nu, N-1}),$$

where $W = (N^{-1/2}\omega^{(j-1)(k-1)};\ j,k = 1,2,\ldots,N)$ denotes the discrete Fourier transform, and

$$\begin{aligned}
\gamma_{11l} &= \sum_{m=0}^{N-1} \omega^{ml}\frac{1}{2\pi}\log|\varrho - R\omega^m|, \quad \gamma_{12l} = \varrho^2\gamma_{11l}, \\
\gamma_{21l} &= \sum_{m=0}^{N-1} \omega^{ml}\frac{1}{2\pi}\Re\Big(\frac{1}{\varrho - R\omega^m}\Big), \quad \gamma_{22l} = 2\varrho\gamma_{11l} + \varrho^2\gamma_{21l}.
\end{aligned}$$

Defining two auxiliary functions φ_l and ψ_l as

$$\varphi_l(z) = \sum_{m=0}^{N-1} \omega^{ml} E(z - R\omega^m), \quad \psi_l(z) = \sum_{m=0}^{N-1} \frac{\omega^{ml}}{2\pi} \Re\Big(\frac{z|z|^{-1}}{z - R\omega^m}\Big)$$

for $l = 1, 2, \ldots, N$, we have

$$\gamma_{11l} = \varphi_l(\varrho), \quad \gamma_{12l} = \varrho^2 \varphi_l(\varrho), \quad \gamma_{21l} = \psi_l(\varrho), \quad \gamma_{22l} = 2\varrho\varphi_l(\varrho) + \varrho^2\psi_l(\varrho). \tag{2.1}$$

Consequently, we obtain

$$\widetilde{W}^{-1} G \widetilde{W} = \begin{pmatrix} \operatorname{diag}(\gamma_{110}, \gamma_{111}, \ldots, \gamma_{11,N-1}) & \operatorname{diag}(\gamma_{120}, \gamma_{121}, \ldots, \gamma_{12,N-1}) \\ \operatorname{diag}(\gamma_{210}, \gamma_{211}, \ldots, \gamma_{21,N-1}) & \operatorname{diag}(\gamma_{220}, \gamma_{221}, \ldots, \gamma_{22,N-1}) \end{pmatrix},$$

where

$$\widetilde{W} = \begin{pmatrix} W & O \\ O & W \end{pmatrix}.$$

Using the permutation matrix

$$P = (\, \boldsymbol{e}_1 \quad \boldsymbol{e}_{N+1} \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_{N+2} \quad \ldots \quad \boldsymbol{e}_N \quad \boldsymbol{e}_{2N}\,),$$

we have

$$P^{-1}\widetilde{W}^{-1} G \widetilde{W} P = \operatorname{diag}(\Phi_0, \Phi_1, \ldots, \Phi_{N-1}), \quad \Phi_l = \begin{pmatrix} \gamma_{11l} & \gamma_{12l} \\ \gamma_{21l} & \gamma_{22l} \end{pmatrix},$$

thus

$$\det G = \prod_{l=0}^{N-1} \det \Phi_l.$$

Therefore, G is nonsingular if and only if Φ_l are nonsingular. The determinant of Φ_l can be computed as

$$\det \Phi_l = \gamma_{11l}\gamma_{22l} - \gamma_{12l}\gamma_{21l} = \varphi_l(\varrho)(2\varrho\varphi_l(\varrho) + \varrho^2\psi_l(\varrho)) - \varrho^2\varphi_l(\varrho)\psi_l(\varrho) = 2\varrho\varphi_l(\varrho)^2.$$

Hence, G is nonsingular if and only if no $\varphi_l(\varrho)$ is equal to 0.

In order to see the precise nature of φ_l and ψ_l, the following lemma can be applied.

Lemma 2.1. *For all $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r \in [0, R[$, $\theta \in \mathbb{R}$, we have*

$$\varphi_l(z) = \begin{cases} \dfrac{1}{2\pi} \log|z^N - R^N|, & l = 0, \\ -\dfrac{N}{4\pi} \displaystyle\sum_{n \equiv l} \frac{1}{|n|} \Big(\frac{r}{R}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta}, & l = 1, \ldots, N-1. \end{cases}$$

Also, for all $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r \in]0, R[$, $\theta \in \mathbb{R}$, we have

$$\psi_l(z) = -\frac{N}{4\pi r} \sum_{\substack{n \equiv l \\ n \in \mathbb{Z}\setminus\{0\}}} \Big(\frac{r}{R}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta}.$$

Here and hereafter, $m \equiv n$ always means that $m \equiv n \pmod{N}$.

P r o o f. The expression for φ_l can be obtained in the same way as in [7], Lemma 1. We here prove a formula for ψ_l. We can expand the kernel function as

$$\begin{aligned}
\Re\Bigl(\frac{z|z|^{-1}}{z-R\omega^m}\Bigr) &= \Re\Bigl(\frac{\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}-R\omega^m}\Bigr) = \Re\Bigl(-\frac{1}{R\omega^m}\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-r\mathrm{e}^{\mathrm{i}\theta}/R\omega^m}\Bigr)\\
&= \Re\Bigl(-\frac{\mathrm{e}^{\mathrm{i}\theta}}{R\omega^m}\sum_{n=0}^{\infty}\Bigl(\frac{r\mathrm{e}^{\mathrm{i}\theta}}{R\omega^m}\Bigr)^n\Bigr) = \Re\biggl(-\frac{1}{r}\sum_{n=1}^{\infty}\Bigl(\frac{r}{R}\Bigr)^n\mathrm{e}^{\mathrm{i}n\theta}\omega^{-mn}\biggr)\\
&= -\frac{1}{2r}\sum_{n\in\mathbb{Z}\setminus\{0\}}\Bigl(\frac{r}{R}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}\omega^{-mn}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\psi_l(z) &= \sum_{m=0}^{N-1}\frac{\omega^{ml}}{2\pi}\Bigl[-\frac{1}{2r}\sum_{n\in\mathbb{Z}\setminus\{0\}}\Bigl(\frac{r}{R}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}\omega^{-mn}\Bigr]\\
&= -\frac{1}{4\pi r}\sum_{n\in\mathbb{Z}\setminus\{0\}}\Bigl(\frac{r}{R}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}\sum_{m=0}^{N-1}\omega^{m(l-n)} = -\frac{N}{4\pi r}\sum_{\substack{n\equiv l\\ n\in\mathbb{Z}\setminus\{0\}}}\Bigl(\frac{r}{R}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}.
\end{aligned}$$

□

From this lemma, we have

$$\varphi_0(\varrho) = \frac{1}{2\pi}\log|\varrho^N - R^N|, \quad \varphi_l(\varrho) = -\frac{N}{4\pi}\sum_{n\equiv l}\frac{1}{|n|}\Bigl(\frac{\varrho}{R}\Bigr)^{|n|} < 0, \quad l = 1,\dots,N-1.$$

Hence, we can conclude that G is nonsingular if and only if $R^N - \varrho^N \neq 1$ holds, which proves Theorem 1.1.

3. Error analysis

In this section, we give an estimate for the approximation error, which shows that the approximation error decays exponentially with respect to N when the boundary data f and g are analytic.

3.1. Exact solution for (1.1)**.** We write down the exact solution u for (1.1) using the Fourier expansion. Since p and q are harmonic in the disk D_ϱ, they have the complex Fourier expansions

$$p(r,\theta) = \sum_{n\in\mathbb{Z}} a_n\Bigl(\frac{r}{\varrho}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}, \quad q(r,\theta) = \sum_{n\in\mathbb{Z}} b_n\Bigl(\frac{r}{\varrho}\Bigr)^{|n|}\mathrm{e}^{\mathrm{i}n\theta}$$

for $0 \leqslant r \leqslant \varrho$ and $\theta \in \mathbb{R}$. Then the exact solution has the form

$$u(r,\theta) = p(r,\theta) + r^2 q(r,\theta) = \sum_{n\in\mathbb{Z}} (a_n + r^2 b_n)\Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta}$$

for $0 \leqslant r \leqslant \varrho$ and $\theta \in \mathbb{R}$. The coefficients $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ are determined by the boundary conditions. From the Dirichlet boundary condition, we have

$$f(\varrho\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} (a_n + \varrho^2 b_n)\mathrm{e}^{\mathrm{i}n\theta},$$

that is,

$$a_n + \varrho^2 b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\varrho\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = f_n.$$

Concerning the Neumann boundary condition, since the normal derivative of u can be computed as

$$\begin{aligned}\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial r}\Big|_{r=\varrho} &= \frac{\partial}{\partial r}\bigg[a_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a_n}{\varrho^{|n|}} r^{|n|}\mathrm{e}^{\mathrm{i}n\theta} + \sum_{n\in\mathbb{Z}} \frac{b_n}{\varrho^{|n|}} r^{|n|+2}\mathrm{e}^{\mathrm{i}n\theta}\bigg]\bigg|_{r=\varrho} \\ &= \frac{1}{\varrho}\sum_{n\in\mathbb{Z}} a_n|n|\mathrm{e}^{\mathrm{i}n\theta} + \varrho\sum_{n\in\mathbb{Z}} b_n(|n|+2)\mathrm{e}^{\mathrm{i}n\theta},\end{aligned}$$

we obtain

$$\frac{a_n|n|}{\varrho} + \varrho b_n(|n|+2) = \frac{1}{2\pi}\int_0^{2\pi} g(\varrho\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = g_n, \quad n \in \mathbb{Z}.$$

Using the above relations, we can write down $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ explicitly in terms of the Fourier coefficients of f and g as follows:

$$a_n = \Big(1 + \frac{|n|}{2}\Big) f_n - \frac{\varrho}{2} g_n, \quad b_n = \frac{1}{2\varrho^2}(\varrho g_n - |n| f_n), \quad n \in \mathbb{Z}.$$

3.2. Explicit form of the approximate solution. Let $G' = P^{-1}GP$ and $W' = P^{-1}\widetilde{W}P$, that is,

$$W' = \begin{pmatrix} I & I & I & \dots & I \\ I & \omega I & \omega^2 I & \dots & \omega^{N-1} I \\ I & \omega^2 I & \omega^4 I & \dots & \omega^{2(N-1)} I \\ \vdots & \vdots & \vdots & & \vdots \\ I & \omega^{N-1} I & \omega^{2(N-1)} I & \dots & \omega^{(N-1)(N-1)} I \end{pmatrix},$$

where I denotes the 2×2 identity matrix. Then we have

$$P^{-1}\widetilde{W}^{-1}G\widetilde{W}P = (W')^{-1}G'W',$$

and therefore we have

$$(W')^{-1}G'W' = \mathrm{diag}(\Phi_0, \Phi_1, \ldots, \Phi_{N-1}). \tag{3.1}$$

Using these matrices, the linear system $G\boldsymbol{Q} = \boldsymbol{b}$ can be transformed into

$$G'\boldsymbol{Q}' = \boldsymbol{b}',$$

where

$$\begin{aligned}\boldsymbol{Q}' &= P^{-1}\boldsymbol{Q} = (Q_1^p, Q_1^q, Q_2^p, Q_2^q, \ldots, Q_N^p, Q_N^q)^{\mathrm{T}},\\ \boldsymbol{b}' &= P^{-1}\boldsymbol{b} = (f(x_1), g(x_1), f(x_2), g(x_2), \ldots, f(x_N), g(x_N))^{\mathrm{T}}.\end{aligned}$$

We can represent $(G')^{-1}$ explicitly from (3.1) as follows:

$$\begin{aligned}(G')^{-1} &= ([(G')^{-1}]_{kj}\,;\ k, j = 1, 2, \ldots, N),\\ [(G')^{-1}]_{kj} &= \frac{1}{N}\sum_{l=1}^{N}\omega^{(k-j)(l-1)}\Phi_{l-1}^{-1} \in \mathbb{R}^{2\times 2}.\end{aligned}$$

If we write the boundary data f and g in the form of Fourier series expansions

$$f(\varrho \mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}n\theta}, \quad g(\varrho \mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} g_n \mathrm{e}^{\mathrm{i}n\theta},$$

then the coefficients $\{Q_k^{p,q}\}_{k=1}^N$ are given as follows:

$$\begin{aligned}\begin{pmatrix} Q_k^p \\ Q_k^q \end{pmatrix} &= \sum_{j=1}^{N}\left(\frac{1}{N}\sum_{l=1}^{N}\omega^{(k-j)(l-1)}\Phi_{l-1}^{-1}\right)\sum_{n\in\mathbb{Z}}\begin{pmatrix} f_n \\ g_n \end{pmatrix}\omega^{(j-1)n}\\ &= \sum_{n\in\mathbb{Z}}\omega^{(k-1)n}\Phi_n^{-1}\begin{pmatrix} f_n \\ g_n \end{pmatrix}.\end{aligned}$$

Therefore, the approximate solution can be written as

$$\begin{aligned}u^{(N)}(x) &= \sum_{k=1}^{N}(Q_k^p + |x|^2 Q_k^q)E(x - y_k) = \sum_{k=1}^{N}E(x - y_k)\begin{pmatrix} 1 & |x|^2 \end{pmatrix}\begin{pmatrix} Q_k^p \\ Q_k^q \end{pmatrix}\\ &= \sum_{k=1}^{N}E(x - y_k)\begin{pmatrix} 1 & |x|^2 \end{pmatrix}\sum_{n\in\mathbb{Z}}\omega^{(k-1)n}\Phi_n^{-1}\begin{pmatrix} f_n \\ g_n \end{pmatrix}\\ &= \sum_{n\in\mathbb{Z}}\varphi_n(x)\begin{pmatrix} 1 & |x|^2 \end{pmatrix}\Phi_n^{-1}\begin{pmatrix} f_n \\ g_n \end{pmatrix}\\ &= \sum_{n\in\mathbb{Z}}\frac{\varphi_n(x)}{\det \Phi_n}[\gamma_{22n}f_n - \gamma_{12n}g_n + |x|^2(-\gamma_{21n}f_n + \gamma_{11n}g_n)].\end{aligned}$$

Thus, we can evaluate the error $\|u-u^{(N)}\|_{L^\infty(\Omega)}$ as

$$
\begin{aligned}
\|u-u^{(N)}\|_{L^\infty(\Omega)} &= \sup_{\substack{0\leqslant r<\varrho\\ \theta\in\mathbb{R}}} |u(r\mathrm{e}^{\mathrm{i}\theta}) - u^{(N)}(r\mathrm{e}^{\mathrm{i}\theta})| \\
&= \sup_{\substack{0\leqslant r<\varrho\\ \theta\in\mathbb{R}}} \Big| \sum_{n\in\mathbb{Z}} \Big(a_n \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} - (\gamma_{22n} f_n - \gamma_{12n} g_n) \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \\
&\quad + r^2 \sum_{n\in\mathbb{Z}} \Big(b_n \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} - (-\gamma_{21n} f_n + \gamma_{11n} g_n) \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \Big| \\
&= \sup_{\substack{0\leqslant r<\varrho\\ \theta\in\mathbb{R}}} \Big| \sum_{n\in\mathbb{Z}} f_n \Big(\Big(1+\frac{|n|}{2}\Big) \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} - \gamma_{22n} \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \\
&\quad + \sum_{n\in\mathbb{Z}} g_n \Big(-\frac{\varrho}{2} \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} + \gamma_{12n} \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \\
&\quad + r^2 \sum_{n\in\mathbb{Z}} f_n \Big(-\frac{|n|}{2\varrho^2} \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} + \gamma_{21n} \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \\
&\quad + r^2 \sum_{n\in\mathbb{Z}} g_n \Big(\frac{1}{2\varrho} \Big(\frac{r}{\varrho}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\theta} - \gamma_{11n} \frac{\varphi_n(r\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big) \Big|.
\end{aligned}
$$

Since the 1st and 2nd terms are harmonic, and the 3rd and 4th terms are of the form $r^2 \times$ harmonic, the above expression can be bounded by

$$
\sum_{n\in\mathbb{Z}} (|f_n| e_{1n} + |g_n| e_{2n} + \varrho^2 |f_n| e_{3n} + \varrho^2 |g_n| e_{4n}), \tag{3.2}
$$

where $e_{jn} = \sup_{\theta\in\mathbb{R}} \alpha_{jn}(\theta)$ and

$$
\begin{aligned}
\alpha_{1n}(\theta) &= \Big| \Big(1+\frac{|n|}{2}\Big) \mathrm{e}^{\mathrm{i}n\theta} - \gamma_{22n} \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big|, \quad & \alpha_{2n}(\theta) &= \Big| -\frac{\varrho}{2} \mathrm{e}^{\mathrm{i}n\theta} + \gamma_{12n} \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big|, \\
\alpha_{3n}(\theta) &= \Big| -\frac{|n|}{2\varrho^2} \mathrm{e}^{\mathrm{i}n\theta} + \gamma_{21n} \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big|, & \alpha_{4n}(\theta) &= \Big| \frac{1}{2\varrho} \mathrm{e}^{\mathrm{i}n\theta} - \gamma_{11n} \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\det \Phi_n} \Big|.
\end{aligned}
$$

First, we give global estimates on e_{jn}.

Lemma 3.1. *There exists a positive constant $C = C(\varrho, R)$ such that*

$$
e_{1n}, e_{3n} \leqslant C(1+|n|), \quad e_{2n}, e_{4n} \leqslant C
$$

hold for all $n \in \mathbb{Z}$.

P r o o f. Using the triangle inequality, each $\alpha_{jn}(\theta)$ can be bounded as follows:

$$\alpha_{1n}(\theta) \leqslant 1 + \frac{|n|}{2} + \Big|\gamma_{22n}\frac{\varphi_n(\varrho e^{i\theta})}{\det \Phi_n}\Big|, \quad \alpha_{2n}(\theta) \leqslant \frac{\varrho}{2} + \Big|\gamma_{12n}\frac{\varphi_n(\varrho e^{i\theta})}{\det \Phi_n}\Big|,$$
$$\alpha_{3n}(\theta) \leqslant \frac{|n|}{2\varrho^2} + \Big|\gamma_{21n}\frac{\varphi_n(\varrho e^{i\theta})}{\det \Phi_n}\Big|, \qquad \alpha_{4n}(\theta) \leqslant \frac{1}{2\varrho} + \Big|\gamma_{11n}\frac{\varphi_n(\varrho e^{i\theta})}{\det \Phi_n}\Big|.$$

Since $\gamma_{\mu\nu n}$, φ_n, and $\det \Phi_n$ are periodic with respect to n with period N, it suffices to establish estimates on e_{jn} only for the case when $n \in \Lambda'_N := \{p \in \mathbb{Z};\ -\frac{1}{2}N < p \leqslant \frac{1}{2}N\}$.

We first show the estimate of e_{1n}. Since $\gamma_{22n} = 2\varrho\varphi_n(\varrho) + \varrho^2\psi_n(\varrho)$ and $\det \Phi_n = 2\varrho\varphi_n(\varrho)^2$, we have

$$\Big|\gamma_{22n}\frac{\varphi_n(\varrho e^{i\theta})}{\det \Phi_n}\Big| \leqslant \Big(1 + \frac{\varrho}{2}\frac{|\psi_n(\varrho)|}{|\varphi_n(\varrho)|}\Big)\frac{|\varphi_n(\varrho e^{i\theta})|}{|\varphi_n(\varrho)|}.$$

When $n = 0$, we evaluate $|\varphi_0(\varrho)|$, $|\psi_0(\varrho)|$, $|\varphi_0(\varrho e^{i\theta})|$ as follows. As for $|\varphi_0(\varrho)|$, we know that $|\varphi_0(\varrho)| \neq 0$ and

$$|\varphi_0(\varrho)| = \Big|\frac{1}{2\pi}\log(\varrho^N - R^N)\Big| = \frac{N}{2\pi}\Big|\log R + \frac{1}{N}\log\Big(1 - \Big(\frac{\varrho}{R}\Big)^N\Big)\Big|,$$
$$\log R + \frac{1}{N}\log\Big(1 - \Big(\frac{\varrho}{R}\Big)^N\Big) \to \log R \quad \text{as} \quad N \to \infty,$$

which implies that there exists a positive constant C' such that $|\varphi_0(\varrho)| \geqslant C'N$ holds. Next, $|\psi_0(\varrho)|$ can be bounded in a straightforward way as

$$|\psi_0(\varrho)| = \Big|-\frac{N}{4\pi\varrho}\cdot 2\sum_{l=1}^{\infty}\Big(\frac{\varrho}{R}\Big)^{lN}\Big| = \Big|\frac{N}{2\pi\varrho}\frac{(\varrho R^{-1})^N}{1-(\varrho R^{-1})^N}\Big| \leqslant CN\Big(\frac{\varrho}{R}\Big)^N.$$

Finally, concerning $|\varphi_0(\varrho e^{i\theta})|$, we have

$$|\varphi_0(\varrho e^{i\theta})| = \Big|\frac{1}{2\pi}\log|(\varrho e^{i\theta})^N - R^N|\Big| \leqslant \frac{1}{2\pi}\max\{|\log(R^N - \varrho^N)|, |\log(R^N + \varrho^N)|\}.$$

These estimates lead us to

$$\alpha_{10}(\theta) \leqslant 1 + \Big(1 + CN\Big(\frac{\varrho}{R}\Big)^N\Big)\max\Big\{1, \frac{|\log(R^N + \varrho^N)|}{|\log(R^N - \varrho^N)|}\Big\} \leqslant C.$$

When $n \in \Lambda'_N \setminus \{0\}$, we have

$$|\psi_n(\varrho)| = \Big|-\frac{N}{4\pi\varrho}\sum_{m\equiv n}\Big(\frac{\varrho}{R}\Big)^{|m|}\Big| \leqslant CN\Big(\frac{\varrho}{R}\Big)^{|n|},$$
$$|\varphi_n(\varrho)| = \Big|-\frac{N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\Big| \geqslant \frac{N}{4\pi|n|}\Big(\frac{\varrho}{R}\Big)^{|n|},$$
$$|\varphi_n(\varrho e^{i\theta})| = \Big|-\frac{N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}e^{im\theta}\Big| \leqslant \frac{N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|} = |\varphi_n(\varrho)|.$$

Therefore, we obtain

$$\alpha_{1n}(\theta) \leqslant 1 + \frac{|n|}{2} + (1 + C|n|) \leqslant C(1 + |n|),$$

which implies that $e_{1n} \leqslant C(1+|n|)$.

Using the above estimates, we obtain bounds for e_{2n}, e_{3n}, and e_{4n}. Hence, we have the desired global bounds. $\square$

We next give more precise estimates on e_{jn} for $0 \leqslant n \leqslant \frac{1}{2}N$.

Lemma 3.2. *There exists a positive constant $C = C(\varrho, R)$ such that*

$$e_{10}, e_{30} \leqslant C\Big(\frac{\varrho}{R}\Big)^N, \quad e_{20}, e_{40} \leqslant CN^{-1}\Big(\frac{\varrho}{R}\Big)^N,$$

and

$$e_{1n}, e_{3n} \leqslant Cn\Big(\frac{\varrho}{R}\Big)^{N-2n}, \quad e_{2n}, e_{4n} \leqslant \frac{Cn}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-2n}$$

hold for $1 \leqslant n \leqslant \frac{1}{2}N$ with sufficiently large N.

Proof. We first note that if N is sufficiently large then $|\varphi_0(\varrho)|^{-1} \leqslant C/(N|\log R|)$ holds.

First, we show estimates on e_{1n}. When $n = 0$, we have

$$\begin{aligned}
|\alpha_{10}(\theta)| &= \Big|1 - \gamma_{220}\frac{\varphi_0(\varrho e^{i\theta})}{\det \Phi_0}\Big| = \frac{|2\varrho\varphi_0(\varrho)^2 - (2\varrho\varphi_0(\varrho) + \varrho^2\psi_0(\varrho))\varphi_0(\varrho e^{i\theta})|}{2\varrho\varphi_0(\varrho)^2} \\
&= \frac{|2\varrho\varphi_0(\varrho)(\varphi_0(\varrho) - \varphi_0(\varrho e^{i\theta})) - \varrho^2\psi_0(\varrho)\varphi_0(\varrho e^{i\theta})|}{2\varrho\varphi_0(\varrho)^2} \\
&\leqslant \frac{|\varphi_0(\varrho) - \varphi_0(\varrho e^{i\theta})|}{|\varphi_0(\varrho)|} + \frac{\varrho}{2}\frac{|\psi_0(\varrho)|}{|\varphi_0(\varrho)|}\frac{|\varphi_0(\varrho e^{i\theta})|}{|\varphi_0(\varrho)|}.
\end{aligned}$$

The previous estimates yield the following bound for the 2nd term:

$$\frac{\varrho}{2}\frac{|\psi_0(\varrho)|}{|\varphi_0(\varrho)|}\frac{|\varphi_0(\varrho e^{i\theta})|}{|\varphi_0(\varrho)|} \leqslant C\Big(\frac{\varrho}{R}\Big)^N.$$

As for the 1st term, we have

$$\begin{aligned}
|\varphi_0(\varrho) - \varphi_0(\varrho e^{i\theta})| &= \Big|\frac{1}{2\pi}\log|\varrho^N - R^N| - \frac{1}{2\pi}\log|(\varrho e^{i\theta})^N - R^N|\Big| \\
&= \frac{1}{2\pi}\Big|-\sum_{l=1}^{\infty}\frac{1}{l}\Big(\frac{\varrho}{R}\Big)^{lN} + \sum_{l=1}^{\infty}\frac{1}{l}\Big(\frac{\varrho}{R}\Big)^{lN}\Re(e^{ilN\theta})\Big| \\
&\leqslant \frac{1}{\pi}\sum_{l=1}^{\infty}\frac{1}{l}\Big(\frac{\varrho}{R}\Big)^{lN} \leqslant C\Big(\frac{\varrho}{R}\Big)^N.
\end{aligned}$$

Therefore, we obtain

$$|\alpha_{10}(\theta)| \leqslant C\Big(\frac{\varrho}{R}\Big)^N, \quad \text{or} \quad e_{10} \leqslant C\Big(\frac{\varrho}{R}\Big)^N.$$

When $1 \leqslant n \leqslant \frac{1}{2}N$, since $\gamma_{22n} = 2\varrho\varphi_n(\varrho) + \varrho^2\psi_n(\varrho)$ and $\det\Phi_n = 2\varrho\varphi_n(\varrho)^2$, we have

$$\begin{aligned}
&\Big(1+\frac{|n|}{2}\Big)\mathrm{e}^{in\theta} - \gamma_{22n}\frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\det\Phi_n}\\
&\quad = \frac{1}{2\varrho\varphi_n(\varrho)^2}\Big[2\varrho\varphi_n(\varrho)^2\Big(1+\frac{|n|}{2}\Big)\mathrm{e}^{in\theta} - (2\varrho\varphi_n(\varrho)+\varrho^2\psi_n(\varrho))\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})\Big]\\
&\quad = \frac{1}{2\varrho\varphi_n(\varrho)^2}\Big[\underbrace{2\varrho\varphi_n(\varrho)\Big(\varphi_n(\varrho)\Big(1+\frac{|n|}{2}\Big)\mathrm{e}^{in\theta} - \varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})\Big) - \varrho^2\psi_n(\varrho)\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}_{=:(*)}\Big].
\end{aligned}$$

The expression $(*)$ within the brackets can be computed as follows:

$$\begin{aligned}
(*) &= 2\varrho\cdot\frac{-N}{4\pi}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\\
&\quad\times\Big[-\frac{N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\Big(1+\frac{|n|}{2}\Big)\mathrm{e}^{in\theta} - \frac{-N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{im\theta}\Big]\\
&\quad - \varrho^2\frac{-N}{4\pi\varrho}\sum_{l\equiv n}\Big(\frac{\varrho}{R}\Big)^{|l|}\frac{-N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{im\theta}\\
&= \frac{\varrho N^2}{8\pi^2}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}(\mathrm{e}^{in\theta}-\mathrm{e}^{im\theta})\\
&\quad + \frac{\varrho N^2}{16\pi^2}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\sum_{m\equiv n}\frac{|n|}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{in\theta}\\
&\quad - \frac{\varrho N^2}{16\pi^2}\sum_{l\equiv n}\Big(\frac{\varrho}{R}\Big)^{|l|}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{im\theta}
\end{aligned}$$

$$= \frac{\varrho N^2}{8\pi^2}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}(\mathrm{e}^{in\theta}-\mathrm{e}^{im\theta}) \tag{3.3a}$$

$$+ \frac{\varrho N^2}{16\pi^2}\sum_{\substack{l,m\equiv n\\(l,m)\neq(n,n)}}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|l|+|m|}\Big[\frac{|n|}{|l|}\mathrm{e}^{in\theta}-\mathrm{e}^{im\theta}\Big]. \tag{3.3b}$$

The sum (3.3a) can be bounded as follows:

$$
\begin{aligned}
|(3.3\mathrm{a})| &\leqslant \frac{\varrho N^2}{8\pi^2}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\Big|\sum_{m\in I(n)}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}m\theta})\Big| \\
&\leqslant C\frac{N^2}{n}\Big(\frac{\varrho}{R}\Big)^{n}\Big|\sum_{t=1}^{\infty}\Big\{\frac{1}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}(tN+n)\theta})\Big\} \\
&\quad+\frac{1}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}(n-tN)\theta})\Big| \\
&\leqslant C\frac{N^2}{n}\Big(\frac{\varrho}{R}\Big)^{n}\sum_{t=1}^{\infty}\Big\{\frac{1}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}+\frac{1}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}\Big\} \\
&\leqslant C\frac{N^2}{n(N-n)}\Big(\frac{\varrho}{R}\Big)^{N}\Big(1+\Big(\frac{\varrho}{R}\Big)^{2n}\Big)\sum_{t=0}^{\infty}\Big(\frac{\varrho}{R}\Big)^{tN}\leqslant C\frac{N^2}{n(N-n)}\Big(\frac{\varrho}{R}\Big)^{N},
\end{aligned}
$$

where $I(n)=\{m\in\mathbb{Z};\ m\equiv n,\ m\neq n\}$. The sum (3.3b) is decomposed into 3 parts as follows:

$$
\begin{aligned}
&(3.4\mathrm{a}) \qquad (3.3\mathrm{b}) = \frac{\varrho N^2}{16\pi^2}\Big[\Big(\frac{\varrho}{R}\Big)^{n}\sum_{m\in I(n)}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}m\theta}) \\
&(3.4\mathrm{b}) \qquad\qquad +\frac{1}{|n|}\Big(\frac{\varrho}{R}\Big)^{|n|}\sum_{l\in I(n)}\Big(\frac{|n|}{|l|}-1\Big)\Big(\frac{\varrho}{R}\Big)^{|l|}\mathrm{e}^{\mathrm{i}n\theta} \\
&(3.4\mathrm{c}) \qquad\qquad +\sum_{l\in I(n)}\Big(\frac{\varrho}{R}\Big)^{|l|}\sum_{m\in I(n)}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\Big[\frac{|n|}{|l|}\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}m\theta}\Big]\Big].
\end{aligned}
$$

Each term can be estimated as

$$
\begin{aligned}
|(3.4\mathrm{a})| &= \Big(\frac{\varrho}{R}\Big)^{n}\Big|\sum_{t=1}^{\infty}\Big\{\frac{1}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}(tN+n)\theta}) \\
&\quad+\frac{1}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}(\mathrm{e}^{\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}(n-tN)\theta})\Big\}\Big| \\
&\leqslant 2\Big(\frac{\varrho}{R}\Big)^{n}\sum_{t=1}^{\infty}\Big[\frac{1}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}+\frac{1}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}\Big] \\
&= \frac{2}{N-n}\Big(\frac{\varrho}{R}\Big)^{n}\sum_{t=1}^{\infty}\Big[\frac{N-n}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}+\frac{N-n}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}\Big] \\
&\leqslant \frac{2}{N-n}\Big(\frac{\varrho}{R}\Big)^{N}\Big(1+\Big(\frac{\varrho}{R}\Big)^{2n}\Big)\sum_{t=1}^{\infty}\Big(\frac{\varrho}{R}\Big)^{(t-1)N}\leqslant\frac{C_1}{N-n}\Big(\frac{\varrho}{R}\Big)^{N},
\end{aligned}
$$

$$
|(3.4\mathrm{b})| = \frac{1}{n}\Big(\frac{\varrho}{R}\Big)^{n}\Big|\sum_{s=1}^{\infty}\Big\{\Big(\frac{n}{sN+n}-1\Big)\Big(\frac{\varrho}{R}\Big)^{sN+n}+\Big(\frac{n}{sN-n}-1\Big)\Big(\frac{\varrho}{R}\Big)^{sN-n}\Big\}\Big|
$$

$$\leqslant \frac{1}{n}\Big(\frac{\varrho}{R}\Big)^n \sum_{s=1}^{\infty}\Big[\Big(\frac{\varrho}{R}\Big)^{sN+n} + \Big(\frac{\varrho}{R}\Big)^{sN-n}\Big] \leqslant \frac{C_2}{n}\Big(\frac{\varrho}{R}\Big)^N,$$

$$|(3.4\text{c})| = \Big|\sum_{l\in I(n)}\Big(\frac{\varrho}{R}\Big)^{|l|}\Big[\sum_{t=1}^{\infty}\Big\{\frac{1}{tN+n}\Big(\frac{\varrho}{R}\Big)^{tN+n}\Big(\frac{|n|}{|l|}\mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{\mathrm{i}(tN+n)\theta}\Big)$$

$$+ \frac{1}{tN-n}\Big(\frac{\varrho}{R}\Big)^{tN-n}\Big(\frac{|n|}{|l|}\mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{\mathrm{i}(n-tN)\theta}\Big)\Big\}\Big]\Big|$$

$$\leqslant \frac{1}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-n}\sum_{l\in I(n)}\Big(\frac{\varrho}{R}\Big)^{|l|}\Big(\frac{|n|}{|l|}+1\Big)\Big\{\Big(\frac{\varrho}{R}\Big)^{2n}+1\Big\}\sum_{t=1}^{\infty}\Big(\frac{\varrho}{R}\Big)^{(t-1)N}$$

$$\leqslant \frac{C_3}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-n}\sum_{l\in I(n)}\Big(\frac{\varrho}{R}\Big)^{|l|} \leqslant \frac{C_3}{N-n}\Big(\frac{\varrho}{R}\Big)^{2(N-n)}.$$

Therefore, we have

$$|(3.3\text{a})| + |(3.3\text{b})| \leqslant C\Big[\frac{N^2}{n(N-n)}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{N-n}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{n}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{N-n}\Big(\frac{\varrho}{R}\Big)^{2(N-n)}\Big].$$

Hence, we obtain

$$\alpha_{1n}(\theta) \leqslant C\Big(2\varrho\cdot\frac{N^2}{16n^2\pi^2}\Big(\frac{\varrho}{R}\Big)^{2n}\Big)^{-1}\Big[\frac{N^2}{n(N-n)}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{N-n}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{n}\Big(\frac{\varrho}{R}\Big)^N + \frac{N^2}{N-n}\Big(\frac{\varrho}{R}\Big)^{2(N-n)}\Big]$$

$$\leqslant C\Big[\frac{n^2}{n(N-n)}\Big(\frac{\varrho}{R}\Big)^{N-2n} + \frac{n^2}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-2n} + n\Big(\frac{\varrho}{R}\Big)^{N-2n} + \frac{n^2}{N-n}\Big(\frac{\varrho}{R}\Big)^{2(N-n)}\Big]$$

$$= C\Big[\frac{n^2}{n(N-n)} + \frac{n^2}{N-n} + n + \frac{n^2}{N-n}\Big(\frac{\varrho}{R}\Big)^N\Big]\Big(\frac{\varrho}{R}\Big)^{N-2n}$$

$$\leqslant Cn\Big(\frac{\varrho}{R}\Big)^{N-2n},$$

that is, we have shown that

$$e_{1n} = \sup_{\theta\in\mathbb{R}}\alpha_{1n}(\theta) \leqslant Cn\Big(\frac{\varrho}{R}\Big)^{N-2n}.$$

We next show the estimates on e_{2n}. By definition, we have

$$\alpha_{2n}(\theta) = \Big|-\frac{\varrho}{2}\mathrm{e}^{\mathrm{i}n\theta} + \varrho^2\varphi_n(\varrho)\frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{2\varrho\varphi_n(\varrho)^2}\Big| = \frac{\varrho}{2}\Big|\mathrm{e}^{\mathrm{i}n\theta} - \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\varphi_n(\varrho)}\Big|$$

$$= \frac{\varrho}{2}\frac{|\varphi_n(\varrho)\mathrm{e}^{\mathrm{i}n\theta} - \varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})|}{|\varphi_n(\varrho)|}.$$

Thus from [7], Lemma 2, we find the estimate

$$e_{20} \leqslant CN^{-1}\Big(\frac{\varrho}{R}\Big)^{N}, \quad e_{2n} \leqslant \frac{Cn}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-2n}, \quad 1 \leqslant n \leqslant \frac{N}{2}.$$

By definition, we have

$$\begin{aligned}\alpha_{3n}(\theta) &= \Big|-\frac{|n|}{2\varrho^2}\mathrm{e}^{in\theta} + \psi_n(\varrho)\frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{2\varrho\varphi_n(\varrho)^2}\Big| \\ &= \frac{1}{2\varrho^2\varphi_n(\varrho)^2}\big||n|\varphi_n(\varrho)^2\mathrm{e}^{\mathrm{i}n\theta} - \varrho\psi_n(\varrho)\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})\big|.\end{aligned}$$

When $n = 0$, we have

$$\alpha_{30}(\theta) = \frac{1}{2\varrho^2\varphi_0(\varrho)^2}\cdot\varrho|\psi_0(\varrho)||\varphi_0(\varrho\mathrm{e}^{\mathrm{i}\theta})| \leqslant C\Big(\frac{\varrho}{R}\Big)^{N}.$$

When $1 \leqslant n \leqslant \frac{1}{2}N$, we have

$$\begin{aligned}&|n|\varphi_n(\varrho)^2\mathrm{e}^{\mathrm{i}n\theta} - \varrho\psi_n(\varrho)\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta}) \\ &\quad= |n|\frac{-N}{4\pi}\sum_{l\equiv n}\frac{1}{|l|}\Big(\frac{\varrho}{R}\Big)^{|l|}\frac{-N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{in\theta} \\ &\qquad - \varrho\frac{-N}{4\pi\varrho}\sum_{l\equiv n}\Big(\frac{\varrho}{R}\Big)^{|l|}\frac{-N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|m|}\mathrm{e}^{im\theta} \\ &\quad= \frac{N^2}{16\pi^2}\sum_{l,m\equiv n}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|l|+|m|}\Big[\frac{|n|}{|l|}\mathrm{e}^{in\theta} - \mathrm{e}^{im\theta}\Big] \\ &\quad= \frac{N^2}{16\pi^2}\sum_{\substack{l,m\equiv n\\(l,m)\neq(n,n)}}\frac{1}{|m|}\Big(\frac{\varrho}{R}\Big)^{|l|+|m|}\Big[\frac{|n|}{|l|}\mathrm{e}^{in\theta} - \mathrm{e}^{im\theta}\Big].\end{aligned}$$

Therefore, we can estimate $\alpha_{3n}(\theta)$ in the same way as $\alpha_{1n}(\theta)$:

$$\alpha_{3n}(\theta) \leqslant Cn\Big(\frac{\varrho}{R}\Big)^{N-2n}, \quad \text{or} \quad e_{3n} \leqslant Cn\Big(\frac{\varrho}{R}\Big)^{N-2n}.$$

We finally show the estimates on e_{4n}. By definition, we have

$$\alpha_{4n}(\theta) = \Big|\frac{1}{2\varrho}\mathrm{e}^{in\theta} - \varphi_n(\varrho)\frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{2\varrho\varphi_n(\varrho)^2}\Big| = \frac{1}{2\varrho}\Big|\mathrm{e}^{in\theta} - \frac{\varphi_n(\varrho\mathrm{e}^{\mathrm{i}\theta})}{\varphi_n(\varrho)}\Big|,$$

which can be bounded as $\alpha_{2n}(\theta)$, that is,

$$\alpha_{40}(\theta) \leqslant CN^{-1}\Big(\frac{\varrho}{R}\Big)^{N}, \quad \alpha_{4n}(\theta) \leqslant \frac{Cn}{N-n}\Big(\frac{\varrho}{R}\Big)^{N-2n}, \quad 1 \leqslant n \leqslant \frac{N}{2}.$$

□

We now give the error estimate. Using the symmetry $e_{jn} = e_{j,-n}$, we have by (3.2) that

$$\|u - u^{(N)}\|_{L^\infty(\Omega)} \leqslant \sum_{n\in\mathbb{Z}}(|f_n|e_{1n} + |g_n|e_{2n} + \varrho^2|f_n|e_{3n} + \varrho^2|g_n|e_{4n})$$

$$= 2(|f_0|e_{10} + |g_0|e_{20} + \varrho^2|f_0|e_{30} + \varrho^2|g_0|e_{40}) \tag{3.5a}$$

$$+ 2\sum_{n=1}^{[N/2]}(|f_n|e_{1n} + |g_n|e_{2n} + \varrho^2|f_n|e_{3n} + \varrho^2|g_n|f_{4n}) \tag{3.5b}$$

$$+ 2\sum_{n=[N/2]+1}^{\infty}(|f_n|e_{1n} + |g_n|e_{2n} + \varrho^2|f_n|e_{3n} + \varrho^2|g_n|e_{4n}). \tag{3.5c}$$

The term (3.5a) can be estimated as

$$(3.5\text{a}) \leqslant C\Big(\frac{\varrho}{R}\Big)^N$$

and (3.5b) is estimated as

$$(3.5\text{b}) \leqslant C\sum_{n=1}^{[N/2]} b^n n\Big(\frac{\varrho}{R}\Big)^{N-2n} \leqslant CN\Big(\frac{\varrho}{R}\Big)^N \sum_{n=1}^{[N/2]}\Big(b\Big(\frac{R}{\varrho}\Big)^2\Big)^n.$$

Using

$$\sum_{n=1}^{m}\tau^n \leqslant \begin{cases} \dfrac{\tau^{m+1}}{\tau-1} & \text{if } \tau > 1, \\ m & \text{if } \tau = 1, \\ \dfrac{1}{1-\tau} & \text{if } 0 < \tau < 1, \end{cases}$$

for $\tau = b(R/\varrho)^2$ we have

$$(3.5\text{b}) \leqslant \begin{cases} CNb^{N/2}, & b\Big(\dfrac{R}{\varrho}\Big)^2 > 1, \\ CN^2\Big(\dfrac{\varrho}{R}\Big)^N, & b\Big(\dfrac{R}{\varrho}\Big)^2 = 1, \\ CN\Big(\dfrac{\varrho}{R}\Big)^N, & b\Big(\dfrac{R}{\varrho}\Big)^2 < 1. \end{cases}$$

Finally, the last term (3.5c) can be estimated as

$$(3.5\text{c}) \leqslant C\sum_{n=[N/2]+1}^{\infty} nb^n \leqslant CNb^{N/2}.$$

Hence, we have proved Theorem 1.2.

4. Numerical experiments

We present some results of numerical experiments.

4.1. Ω: disk. We first consider the case where Ω is a disk D_ϱ with $\varrho = 1$. We adopt the following polynomials as the boundary conditions:

$$f(x) = x_1^4 - x_2^4 \ (x = (x_1, x_2)^{\mathrm{T}}), \quad g(x) = 4(x_1^3, -x_2^3)^{\mathrm{T}} \cdot \nu. \tag{4.1}$$

Then it can be easily checked that $u(x) = x_1^4 - x_2^4$ is the exact solution for (1.1). We choose R equal to 2, and the result is depicted in Figure 1.

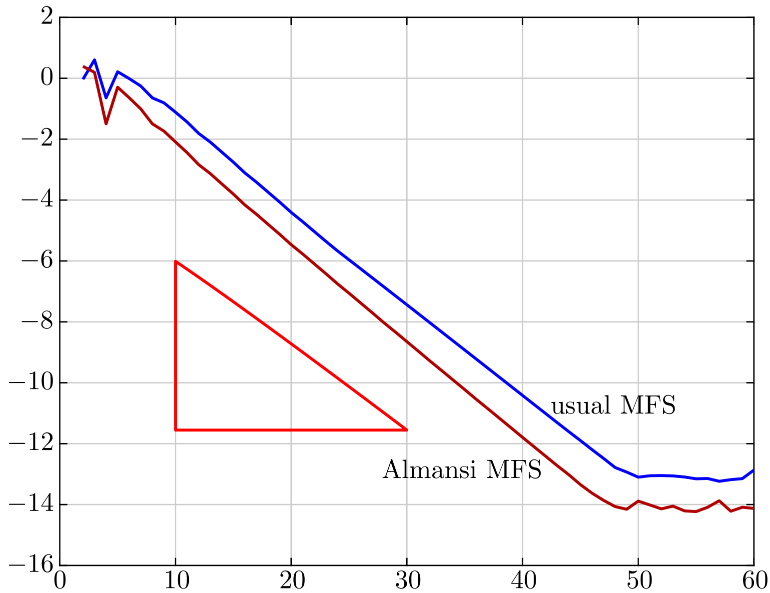

Figure 1. Numerical experiment with boundary data f and g given by (4.1), and the parameter $R = 2$. The upper blue colored line and the lower orange colored line represent numerical solutions obtained by the conventional scheme (1.4) and the present scheme (1.3), respectively. The slope of the hypotenuse of the red colored triangle represents the theoretical order of convergence.

In Figures 1 and 2, the horizontal and vertical axes represent N and the common logarithms of errors, respectively. It can be observed that our error estimate grasps the behavior of the approximation error very well. Moreover, it can be found that the order of convergence for the present scheme is higher than that for the conventional scheme, which causes a difference in accuracy. It is expected that a backward error analysis will play an important role in investigating the cause of this phenomenon.

4.2. Ω: interior simply connected region surrounded by polynomial curve. We next consider the case where the boundary Γ is given by a polynomial curve $\Psi_{l,r}(\{|z|=1\})$, where $l=4$, $r=8$, and $\Psi_{l,r}$ is defined as

$$\Psi_{l,r}(z) = z + \frac{z^l}{r},$$

which is a conformal mapping in $D_{\sqrt[l-1]{r/l}}$. Since MFS and DSM, which is a variant of MFS, for a potential problem in a Jordan region have been studied mathematically in [6], [12] and so on, it is natural to expect that theoretical error analysis could be done for the present scheme similarly. Therefore, we here present a numerical experiment for the case where Ω is the interior simply connected region surrounded by a polynomial curve, and verify numerically that the same error estimate could be obtained. The singular points $\{y_k\}_{k=1}^N$ and the collocation points $\{x_j\}_{j=1}^N$ are arranged by

$$\begin{cases} y_k = \Psi_{4,8}(R\omega^{k-1}), & k = 1, 2, \ldots, N, \\ x_j = \Psi_{4,8}(\omega^{j-1}), & j = 1, 2, \ldots, N, \end{cases}$$

where R is taken to be equal to 1.2. The boundary conditions are given by (4.1). The result of the numerical experiment is depicted in Figure 2.

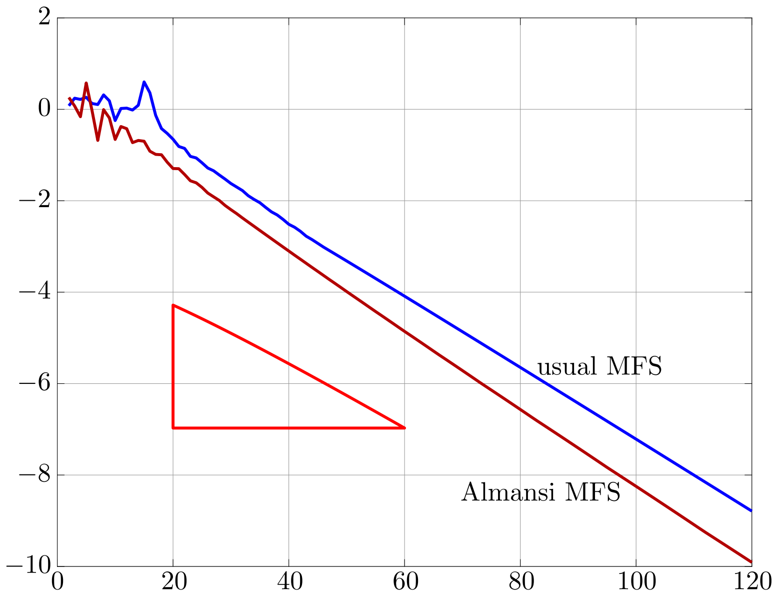

Figure 2. Numerical experiment with boundary data f and g given by (4.1), and the parameter $R = 1.2$. The upper blue colored line and the lower orange colored line represent numerical solutions obtained by the conventional scheme (1.4) and the present scheme (1.3), respectively. The slope of the hypotenuse of the red colored triangle represents the expected order of convergence.

We can again observe in this situation that the accuracy of the approximate solution by the present scheme is better than that by the conventional scheme. Moreover, its convergence rate is what we can expect from the theoretical analysis of MFS and DSM in [6], [12]. Therefore, it should be expected that the theoretical convergence analysis could be done in the case where Ω is bounded by a regular analytic Jordan curve.

5. Concluding remarks

In this paper, we have considered a typical boundary value problem for the biharmonic equation, and its approximate solution by MFS based on the Almansi-type decomposition of the biharmonic function. As a result, we have proved that approximate solution actually exists uniquely except for at most one value of N, and the approximation error decays exponentially with respect to N. Numerical results support our error analysis.

We note that Almansi-type decomposition also holds for polyharmonic functions, therefore, our approach in this paper can be applied to boundary value problems for the polyharmonic equations. Possible directions of future research are the following. The first topic is to extend the results in this paper to general Jordan regions. Numerical results in Section 4 strongly suggest that a theoretical error estimate such as [6], [12] also holds for the present scheme. The second topic is to compute numerically the stream function for the Stokes flow. Since the stream function for the Stokes flow satisfies the biharmonic equation, there is a possibility to apply the present scheme to investigating various aspects of the Stokes fluid.

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