METHOD OF THE GENERATING FUNCTIONAL

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This is a survey of the present state of the method of the generating functional which makes it possible to effectively study distributions of point random measures on a complete, separable metric space. The principal attention is devoted to the study of distributions of configurations of infinite systems of statistical physics — Gibbs distributions.

INTRODUCTION

For a long time classical equilibrium statistical physics was the consumer only of trivial ideas and methods of probability theory. The situation changed in principle when after the classical works of Onsager [43], Bogolyubov [5], Van Hove [45], Minlos [23], Dobrushin [12, 14], and Lanford and Ruelle [40] models of infinite systems entered statistical physics. The rigorous mathematical study of such systems led to the creation of a large class of substantive probabilistic models whose investigation required the creation of new mathematical methods or considerable development of those already available. At present methods of studying the mathematical models of statistical physics find ever greater application in probability theory, thus enriching it. The present survey is devoted to one of them — the method of the generating functional.

The generating functional (GF) was introduced by Bogolyubov [5] in classical statistical physics as a generalization of generating functions of discrete random variables. In this work infinite systems were considered (then formally), and various types of equations for the GF of such systems were obtained. Mathematical problems of this method were considered in [7]. The main substantive results were obtained for system of small density. A summary of these investigations is given in [6].

For a long time the method of the GF was used in statistical physics to obtain exact and approximation equations of various types for correlation functions [1, 2]. Such work continues at present [9, 31]. The regeneration of interest in mathematical problems of the method of the GF is connected with the active study of the thermodynamic limit transition (the limit transition to infinite systems). In a series of works [25, 26, 28] mathematical justification for the method of the GF is given, and the equivalence of this method to the presently widely used method of random Gibbs fields is proved [27]. The possibilities of the method of the GF for solving mathematical problems of statistical physics is demonstrated in [29].

Mathematical development of the method of the GF proceeded in parallel with the "physical" line of development. The object of the method in probability theory became point random processes (branching processes, infinitely divisible processes) [17, 35]. A summary of the development of the method of the GF for point processes is given in [46, 47]; there is also an extensive bibliography there. Considerable broadening of the range of application of the method was carried out in [41]. It is interesting that, proceeding from this form of development of the method of the GF, it was again introduced into statistical physics in the works of Ryazanov [33, 34].

Both lines of development of the method of the GF enriched it. The Bogolyubov equation is the main tool for investigating properties of measures in the method of the GF. Use of this equation makes it possible to invoke methods of functional analysis to solve traditional problems of probability theory. It is just this that makes the method of the GF effective.

In this work the method of the GF and its application in statistical physics are described. In Sec. 1 facts regarding measures on configuration space of systems of point particles are collected. Here a generally adopted terminology does not always exist, and we have

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taken it upon ourselves to introduce terms which have justified their necessity in probability theory or statistical physics. Section 2 is devoted to generating functionals. The definition and a characterization of GF as well as methods of recovering measures on the basis of the GF are presented. It should be noted that at present there is no remotely complete dictionary "properties of a measure – properties of its GF." In Sec. 3 configuration measures of finite systems of classical equilibrium statistical physics are considered. Here the Bogolyubov equation is first introduced, and it is proved that it completely characterizes the GF of the measures in question. Infinite systems of statistical physics are considered in Sec. 4. Here characteristics of configuration measures of these systems in terms of their GF are presented. The equivalence of various definitions of Gibbs distributions is proved. Section 5 is devoted to the problem of equivalence of ensembles in statistical physics and its solution by the method of the GF. Section 6 is devoted to the characterization of various conditions of weakening of correlations (conditions of regularity) in terms of continuous dependence of the GF on an external field.

There are also other examples of application of the method of the GF for solving specific problems of statistical physics: construction of various types of expansions of the GF in terms of parameters introduced into the Bogolyubov equation, the study of symmetry of thermodynamic states, etc. We shall concentrate attention on application of the method of the GF in equilibrium classical statistical physics, leaving aside its applications in quantum statistical physics and quantum field theory and also in nonequilibrium thermodynamics. A description of the entire range of application of the method of the GF should form the topic of a separate survey.

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1. Configuration Measures

Phase space of a single particle E is the basic building block for the mathematical model used both in the theory of stochastic processes and in statistical physics. E is a complete, separable metric space with the natural σ-algebra of Borel sets $\mathfrak{S}$, a measure μ, and a ring of bounded sets $\mathfrak{S}_0 \subset \mathfrak{S}$. (A more general situation is considered in [41].) Points $x \in E$ are interpreted as sites where events can occur; for simplicity an event is understood to mean finding the particle at the given point. If the state of a particle (event) is not completely characterized by its position, then a marking space S (or a space of spins as is said in statistical physics) is further introduced. An event in this case is characterized by an ordered pair (x, s), and phase space is the direct product $\tilde{E} = E \times S$. Assuming S to be a metric space (with distance ρ_2), $\tilde{E}$ can be metrized in the standard way,

$$\rho([x, s], [\tilde{x}, \tilde{s}]) = \rho_1(x, \tilde{x}) + \rho_2(s, \tilde{s}),$$

where ρ_1 is the metric in E. If S is a complete, separable space, then $\tilde{E}$ is also complete and separable. Moreover, by passing, if necessary, to the topologically equivalent metric $\rho_2(1 + \rho_2)^{-1}$, it may be assumed that the marking space is bounded. In this case a set $\tilde{M} \subset \tilde{E}$ is bounded if and only if its projection $M_E = \{x : (x, s) \in \tilde{M}\}$ is bounded in E. Thus, the very important property of boundedness of sets is determined completely by the space of positions E.

The most used phase spaces in the theory of point processes and statistical physics are $\mathbf{R}^\nu$ (continuous models) and $\mathbf{Z}^\nu$ (discrete and lattice models).

We shall consider purely point measures [30] ξ on E (by a measure we mean a nonnegative measure; exceptions will be specially mentioned). We eliminate pathological measures by requiring that ξ be bounded on $\mathfrak{S}_0$, and measures with this property we call locally finite. Let M be the set of integral, locally finite measures on $[E, \mathfrak{S}]$. The simplest representative of M is the Dirac measure $\delta_x : \forall X \in \mathfrak{S}\ \delta_x(X) = 1$, if $x \in X$, and is equal to zero otherwise. All measures in M are constructed from these simplest measures.

Definition 1.1. A subset $X \subset E$ is called locally finite if it has only finite intersection with each $Y \in \mathfrak{S}_0$.

It is obvious that a locally finite set is countable.

LEMMA 1.1 [17]. In order that $\xi \in M$ it is necessary and sufficient that there exist a locally finite set $X_\xi \subset E$ and functions $\varphi_\xi : X_\xi \to \mathbf{N}$ such that

$$\xi = \sum_{x \in X_\xi} \varphi_\xi(x)\,\delta_x. \tag{1.1}$$

Thus, each measure $\xi \in M$ can be thought of as a means of separating out points X_ξ of the set E and assigning to each such point a multiplicity $\varphi_\xi(x) = \xi(\{x\})$. Another interpretation of ξ, adopted not only in statistical physics, is that the measure ξ determines the configuration of a system (infinite) of identical particles situated at points of $X_\xi = \operatorname{supp}\varphi_\xi$, whereby $\varphi_\xi(x)$ is the number of particles at the point x (the occupation number). Measures $\xi \in M$, for which $0 \leqslant \varphi_\xi \leqslant 1$, are called simple. Such measures are uniquely characterized by the locally finite set X_ξ.

We denote by C_E the set of locally finite subsets of E and by M_1 the subset of simple measures in M. It is obvious that there is a one-to-one correspondence between these sets:

$$\xi = \sum_{x \in X_\xi} \delta_x. \tag{1.2}$$

Elements of C_E are naturally called simple, locally finite configurations. The form (1.2) can be given to relation (1.1) if a more general concept of a configuration is introduced. Using the fact that φ_ξ assumes only natural-number values, we rewrite the sum (1.1) in the form

$$\xi = \sum_{x \in X_\varphi} \delta_x, \tag{1.3}$$

where X_φ is the full "list" of elements of X_ξ, whereby each element is repeated in the list as many times as its multiplicity $\varphi_\xi(x)$. No structure for X_φ is assumed [this is not needed for (1.3)]. Thus, X_φ is a system of points in $X \subset C_E$, taken without particular order and with possible repetitions defined by the function φ. We call X_φ a locally finite configuration, while $\operatorname{supp}\varphi$ we call the support of the configuration. This definition is a natural generalization of a configuration of n particles which are considered a point of the space E^n, whereby in a system of identical particles there is no need to consider the order of their arrangement.

Relation (1.3) establishes a one-to-one correspondence between M and C_E^φ — the set of locally finite configurations. By historical tradition in statistical physics probabilistic models are constructed using C_E^φ (or C_E), while in the theory of point processes M is used. It is natural to use both these realizations, choosing in a specific problem the one which leads to the simpler solution. Where it causes no confusion we shall identify the measure ξ and the configuration corresponding to it.

The value of $\xi(V)$ for $V \in \mathfrak{S}_0$ is interpreted as the number of particles in the configuration X_φ, lying in V. For this reason elements of the set M are called counting measures [17]. A structure of a measurable space in M is introduced by means of cylinder sets of the form

$$C_\Delta^n = \{\xi \in M : \xi(\Delta) = n,\ \Delta \in \mathfrak{S}_0\},\quad n = 0, 1, 2.... \tag{1.4}$$

The set C_Δ^n consists of configurations containing exactly n particles in the set $\Delta \in E$. The minimal σ-algebra generated by all sets C_Δ^n we denote by $\mathfrak{M}$. A probability measure P on the measurable space $[M, \mathfrak{M}]$ is called a configuration measure with phase space E. Together with the sets (1.4) $\mathfrak{M}$ also contains sets

$$C_{\Delta_1,\Delta_2,\Delta_3,\ldots,\Delta_k}^{n_1,n_2,n_3,\ldots,n_k} = \{\xi \in M : \xi(\Delta_i) = n_i,\ \Delta_i \in \mathfrak{S}_0\},\ \Delta_i \cap \Delta_j = \varnothing \tag{1.5}$$

generating algebras of cylinder sets. It can be proved in the usual way that a measure P can be uniquely recovered from its restriction to a system of generators (1.5) [37, 47]. However, it is not always convenient to define a measure by its restriction to the system (1.5), since this system is too "coarse." A finer system of subsets can be constructed as follows.

We denote by $\mathfrak{M}_\Lambda$ the minimal σ-algebra of subsets of M generated by the system C_Δ^n with $\Delta \subset \Lambda \subset \mathfrak{S}_0$. A configuration measure P can be uniquely recovered from its restrictions to $\mathfrak{M}_\Lambda$.

THEOREM 1.1 [41]. Let $\Lambda_1, \Lambda_2, \ldots$ be an expanding sequence of sets $\Lambda_k \in \mathfrak{S}_0$ $\Lambda_k \uparrow E$ and suppose the probability measures P_k on $[M, \mathfrak{M}_{\Lambda_k}]$ (k = 1, 2,...) are such that for m > k the restriction of P_m to $\mathfrak{M}_{\Delta_k}$ is consistent (coincides) with P_k. Then there exists a measure P on $[M, \mathfrak{M}]$, whose restriction to $\mathfrak{M}_{\Lambda_{\mathfrak{S}}}$ is equal to P_k.

There is a constructive method of defining measures on $\mathfrak{M}_\Lambda$ which is constantly used in statistical physics. We denote by $\mathscr{K}(\Lambda)$ the set of finite configurations of c_0 particles in $\Lambda\in\mathfrak{S}_0$. $\mathscr{K}(\Lambda)$ is a subset of configurations C_E^φ with support in Λ. $\mathscr{K}(\Lambda)$ can be represented by a decomposition into nonintersecting sets $\mathscr{K}_m(\Lambda)$,

$$\mathscr{K}(\Lambda)=\bigcup_{m=0}^{\infty}\mathscr{K}_m(\Lambda),\quad \mathscr{K}_m(\Lambda)=\{c_0\in\mathscr{K}(\Lambda)\colon |c_0|=m\}, \tag{1.6}$$

where $|c_0|$ is the number of particles in configuration c_0. $\mathscr{K}_0(\Lambda)$ consists of one element — the empty set $\emptyset$. We denote $\mathscr{K}(E)$ by $\mathscr{K}$. Let $b_m\colon E^m\to\mathscr{K}_m$ be the mapping annihilating the order of arrangement of coordinates of points $(x_1, x_2,\ldots, x_m)\in E^m$. The mapping b_m carries the measurable structure of E^m to $\mathscr{K}_m$ and that of $\bigcup_{m=0}^{\infty}E^m$ to $\mathscr{K}$. We denote by $\mathfrak{B}_{\Lambda^m}$ the σ-algebra of measurable sets in $\mathscr{K}_m(\Lambda)$, by $\mathfrak{B}_\Lambda$ those in $\mathscr{K}(\Lambda)$, by $\mathfrak{B}_m$ those in $\mathscr{K}_m$, and by $\mathfrak{B}$ those in $\mathscr{K}$. Let μ be a measure on E, let μ_m be a measure on E_m, and let

$$\Omega=\bigcup_{m=0}^{\infty}\Omega_m\quad (\Omega_m\in\mathfrak{B}_{\Lambda^m}) \tag{1.7}$$

be a measurable set in $\mathscr{K}$. We obtain a measure λ on $[\mathscr{K}, \mathfrak{B}]$ by setting

$$\lambda(\Omega)=\sum_{m=0}\frac{1}{m!}\mu_m(b_m^{-1}(\Omega_m)), \tag{1.8}$$

whereby $\lambda(\emptyset)=1$ by definition. The same relation introduces a measure λ on $\mathscr{K}(\Lambda)$ (for $\Omega\in\mathfrak{B}_\Lambda$). In particular,

$$\lambda(\mathscr{K}_m(\Lambda))=\frac{\mu_m(\Lambda^m)}{m!}=\frac{\mu(\Lambda)^m}{m!},\quad \lambda(\mathscr{K}(\Lambda))=e^{\mu(\Lambda)}. \tag{1.9}$$

Thus, the measures λ obtained on $\mathscr{K}(\Lambda)$ are finite. Assigning to each configuration X_φ its part contained in $\Lambda\in\mathfrak{S}_0$ (it is defined by the restriction of φ to Λ), we obtain a mapping

$$\mathscr{P}_\Lambda\colon C_E^\varphi\to\mathscr{K}(\Lambda).$$

It is obvious that $\mathscr{P}(C_\Lambda^m)=\mathscr{K}_m(\Lambda)$, and hence

$$\mathscr{P}_\Lambda^{-1}(\mathfrak{B}_\Lambda)=\mathfrak{M}_\Lambda, \tag{1.10}$$

i.e., the measurable structures on $[M, \mathfrak{M}_\Lambda]$ and $[\mathscr{K}(\Lambda), \mathfrak{B}_\Lambda]$ are consistent; hence, any measure on $[\mathscr{K}(\Lambda), \mathfrak{B}_\Lambda]$ can be carried over to $[M, \mathfrak{M}_\Lambda]$. The σ-algebra $\mathfrak{M}_\Lambda$ consists of sets of the form $C_\Lambda^\Omega=\mathscr{P}_\Lambda^{-1}(\Omega)$ $(\Omega\in\mathfrak{B}_\Lambda)$, which are called cylinder sets [24]. A rather broad class of measures on $[\mathscr{K}(\Lambda), \mathfrak{B}_\Lambda]$ is given by measures which are absolutely continuous with respect to the measure λ. Each such measure σ_Λ is determined by nonnegative functions Λ^m which are integrable on $\{\sigma_\Lambda(x_1, x_2,\ldots,x_m)\}$ and are symmetric in all arguments,

$$\sigma_\Lambda(\Omega)=\int_\Omega\sigma_\Lambda(c_0)\,d\lambda(c_0)=\sum_{m=0}^{\infty}\frac{1}{m!}\int\sigma_\Lambda(x_1,\ldots, x_m)\,d\mu_m, \tag{1.11}$$

where $\Omega\in\mathfrak{B}_\Lambda$, and the collection of functions $\{\sigma_\Lambda(x_1, x_2,\ldots,x_m)\}$ is considered as a function $\sigma_\Lambda(c_0)$ on $\mathscr{K}(\Lambda)$ [this is possible, since $\sigma_\Lambda(x_m)$ are symmetric functions].

<u>Definition 1.2 [44].</u> The family of nonnegative symmetric functions $\{\sigma_\Lambda(x_1, x_2,\ldots, x_m)\}$ $(\Lambda\in\mathfrak{S}_0)$ is called a system of densities of a distribution if $\sigma_\Lambda(x_1,\ldots,x_m)$ are integrable on Λ^m and satisfy the conditions

$$\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\Lambda^m}\sigma_\Lambda(x_1,\ldots, x_m)\,d\mu_m=1, \tag{1.12}$$

$$\sigma_\Delta(x_1,\ldots, x_m)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\Lambda\setminus\Delta)^n}\sigma_\Lambda(x_1,\ldots, x_m, x_{m+1},\ldots, x_{n+m})\,d\mu_n(x_{m+1},\ldots, x_{n+m}) \tag{1.13}$$

$\forall\Lambda\in\mathfrak{S}_0$ such that $\Lambda\supset\Delta$.

It is natural to consider a system of densities of a distribution as a system of non-negative functions $\sigma_\Lambda(c_0)$ on $\mathscr{K}(\Lambda)$. Corresponding to this, conditions (1.12) and (1.13) can be rewritten in the form

$$\int_{\mathscr{K}(\Lambda)} \sigma_\Lambda(c_0)\, d\lambda(c_0) = 1, \tag{1.14}$$

$$\sigma_\Delta(c_0) = \int_{\mathscr{K}(\bar{\Lambda})} \sigma_\Lambda(c_0 \cup c_1)\, d\lambda_{\bar{\Delta}}(c_1), \tag{1.15}$$

where $\Delta \subset \Lambda$ and $\bar{\Delta} = \Lambda \setminus \Delta$.

THEOREM 1.2. A system of densities of a distribution defines a system of consistent probability measures P_Λ on $[M, \mathfrak{M}_\Lambda]$.

Proof. According to (1.11) the functions $\sigma_\Lambda(c_0)$ define measures σ_Λ on $\mathscr{K}(\Lambda)$. The measure σ_Λ is "carried over" to $\mathfrak{M}_\Lambda$ by the relation

$$P_\Lambda(C_\Lambda^\Omega) = \sigma_\Lambda(\Omega) = \int_\Omega \sigma_\Lambda(c_0)\, d\lambda(c_0). \tag{1.16}$$

Consistency of the measures σ_Λ determining the consistency of P_Λ follows from (1.13). Indeed, if $\Delta_1 \cap \Delta_2 = \varnothing$, then

$$[\mathscr{K}(\Delta_1 \cup \Delta_2), \mathfrak{B}_{\Delta_1 \cup \Delta_2}] = [\mathscr{K}(\Delta_1), \mathfrak{B}_{\Delta_1}] \times [\mathscr{K}(\Delta_2), \mathfrak{B}_{\Delta_2}],$$

i.e., the measurable space $[\mathscr{K}(\Delta_1 \cup \Delta_2), \mathfrak{B}_{\Delta_1 \cup \Delta_2}]$ is the product of the measurable spaces $[\mathscr{K}(\Delta_1), \mathfrak{B}_{\Delta_1}]$ and $[\mathscr{K}(\Delta_2), \mathfrak{B}_{\Delta_2}]$. In exactly the same way the measure $\lambda_{\Delta_1 \cup \Delta_2}$ is the product of measure λ_{Δ_1} and λ_{Δ_2}. It is now obvious that the restriction of the measure σ_Λ to $[\mathscr{K}(\Delta), \mathfrak{B}_\Delta]$ $C_\Delta C \Lambda$ is defined by the density

$$\tilde{\sigma}_\Delta(c_0) = \int_{\mathscr{K}(\Delta_1)} \sigma_\Lambda(c_0 \cup c_1)\, d\lambda_{\Delta_1}(c_1), \quad \Delta_1 = \Lambda \setminus \Delta,$$

which by (1.15) coincides with the density $\sigma_\Delta(c_0)$ defining the measure σ_Δ (the configuration in the argument of the function σ_Λ is broken into two parts: c_0 with supports in Δ and c_1 with support in Δ_1).

2. Generating Functionals of Configuration Measures

To simplify notation we shall denote by c locally finite configurations in the space C_E^φ. By introducing marks for each x in a configuration $c \in C_E^\varphi$ (for example, the index of a particle), the configuration c is converted into a set. The usual operations with sets can thus be extended to configurations c which also considerably simplifies the notation. This was actually already used in Sec. 1. In place of the decomposition (1.3) we now have

$$\xi = \sum_{x \in c} \delta_x. \tag{2.1}$$

Where convenient we shall assume a configuration measure to be defined on C_E^φ. For simplicity we assume that $E = R^\nu$. We consider a measure ξ as an element of the space of generalized functions $\mathscr{D}'$. The space of test functions $\mathscr{D}$ consists of regular, compactly supported functions [30]. Using the relation of duality between these spaces

$$\langle \psi, \xi \rangle = \int \psi(y)\, d\xi(y) = \int \psi(y) \sum_{x \in c} \delta(x-y)\, dy = \sum_{x \in c} \psi(x), \tag{2.2}$$

we write the characteristic functional [10] of the configuration measure P

$$\int \exp\{i \langle \psi, \xi \rangle\}\, dP(c) = \int \exp\left\{i \sum_{x \in c} \psi(x)\right\} dP(c) = \int \prod_{x \in c} \exp\{i\psi(x)\}\, dP(c).$$

For functions $\psi \in \mathscr{D}$ the sum (2.2) is finite.

In analogy with the characteristic functional, for a measure P with an arbitrary phase space E we consider the functional

$$B_P(u) = \int \prod_{x \in c} u(x)\, dP(c). \tag{2.3}$$

A functional of this type was first considered by Bogolyubov [5] who introduced for it the term "generating functional," since the functional (2.3) is the natural generalization of generating functions of discrete random variables. Later the functional B appeared in works on random point processes [42, 46, 47] and random measures [41]. We shall call the functional (2.3) a generating functional of the configuration measure P or, when it causes no confusion, a generating functional (GF).

We begin the study of a GF with the question of its domain $\mathcal{U}$. The set $\mathcal{U}$ should be large enough that the GF uniquely determines the measure P: coincidence on $\mathcal{U}$ of the generating functionals of two measures P_1 and P_2 implies that the measures P_1 and P_2 are identical. Extension of the domain $\mathcal{U}$, as a rule, simplifies reconstructing properties of the measure on the basis of properties of its GF (in any case this affords additional possibilities). An example of this are generating functions of discrete random variables which can be considered as the analytic continuation to the interior of the unit disk (with center at zero) in the complex plane of characteristic functions defined only on its boundary. In exactly the same way extension of the domain of a GF makes it possible to utilize more fully its analytic properties in investigating the configuration measures it represents.

The set of complex-valued Borel functions u(x) = 1 + t(x) on E for which the products

$$\prod_{x\in c} u(x)=\prod_{x\in c}(1+t(x))=(\Pi t)(c) \tag{2.4}$$

converge absolutely $\forall c\in C_E^{\mathfrak{P}}$ we denote by $\mathcal{U}(E)$. By (2.4) functions in $\mathcal{U}(E)$ define configurations $c\in C_E^{\mathfrak{P}}$. Absolute convergence of product (2.4) is equivalent to the conditions

$$\sum_{x\in c}|t(x)|=\int|t(x)|\,d\xi<+\infty,$$

i.e., the functions t(x) must be integrable with respect to all measures $\xi\in M$. The set of functions t with this property we denote by $\mathcal{T}(E)$. It is obvious that $\mathcal{U}(E)=1+\mathcal{T}(E)$. Let

$$\mathcal{T}_f(E)=\{t\in\mathcal{T}(E):\operatorname{supp} t\in\mathfrak{S}_0\}$$

be the set of compactly supported functions in $\mathcal{T}(E)$. It is easy to show that

$$\mathcal{T}(E)=\mathcal{T}_f(E).$$

Functions in $\mathcal{U}_f(E)=1+\mathcal{T}_f(E)$ form the basic reservoir from which various domains of GF are composed. This set can be broadened somewhat by admitting convergence of the product (2.4) to zero. Further extension of $\mathcal{U}(E)$ is possible only by considering specific properties of the measure P [for example, requiring the existence of functions (2.4) only almost everywhere on $C_E^{\mathfrak{P}}$ with respect to the measure P or considering only simple measures in M_1].

The domain of existence of the GF is one of the essential characteristics of a measure P, since it determines the store of functions of the form (2.4) which are integrable with respect to the measure P. The broader the domain of existence of the GF, the narrower the class of measures they determine, and the richer the properties of these measures.

$\mathcal{U}_f(E)$ contains the subset

$$\mathcal{U}_1^f(E)=\{u\in\mathcal{U}_f(E):|u(x)|\leqslant 1\;\forall x\in E\},$$

which by (2.4) defines functions on $C_E^{\mathfrak{P}}$ integrable with respect to any measure P. $\mathcal{U}_1^f(E)$ contains, in particular, the domain of the characteristic functional. From this it is clear that it suffices to uniquely determine the configuration measure P on the basis of the GF. For this purpose a smaller set $\mathcal{U}_{1,r}^f$ suffices which consists only of nonnegative Borel functions on E:

$$\mathcal{U}_{1,r}^f(E)=\{u\in\mathcal{U}_1^f(E):0\leqslant u(x)\leqslant 1,\;\forall x\in E\}.$$

THEOREM 2.1 [41]. If for configuration measures P_1, P_2 with phase space E

$$B_{P_1}(u)=B_{P_2}(u)\quad\forall u\in\mathcal{U}_{1,r}^f(E),$$

then $P_1 = P_2$.

Proof. For any k pairwise nonintersecting sets $\Delta_1,\ldots,\Delta_k\in\mathfrak{S}_0$

$$u_\alpha=\alpha_1\chi_{\Delta_1}+\alpha_2\chi_{\Delta_2}+\ldots+\alpha_k\chi_{\Delta_k}+\chi_{\bar{\Delta}}\in\mathcal{U}_{1,r}^f, \tag{2.5}$$

where $0 \leqslant \alpha_1,\ldots,\alpha_k \leqslant 1$, χ_{Δ_i} are the indicators of the set Δ_i, and $\bar{\Delta}=E\setminus\bigcup_{l=1}^{k}\Delta_l$. Substitution of u_α into (2.4) gives

$$B_P(u_\alpha)=\sum_{n_1,\ldots,n_k}^{\infty}\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_n^{n_k}P\left(C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}\right), \tag{2.6}$$

where $C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}$ are the cylinder sets (1.5). From the coincidence of the functionals (2.6) of the two measures P_1 and P_2 it obviously follows that

$$P_1\left(c^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}\right)=P_2\left(C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}\right),$$

which guarantees that the measures P_1 and P_2 coincide.

The GF Bp on functions (2.5) defines the generating function of the random vector ($\xi\times(\Delta_1)$, $\xi(\Delta_2),\ldots,\xi(\Delta_k)$). Indeed, by (1.19)

$$\xi(\Delta_l)=\sum_{x\in c}\chi_{\Delta_l}(x).$$

The generating function of the vector $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ is the mathematical expectation of the random variable

$$\prod_{l=1}^{k}\alpha_l^{\xi(\Delta_l)}=\prod_{x\in c}\prod_{l=1}^{k}\alpha_l^{\chi_{\Delta_l}(x)}. \tag{2.7}$$

For a vector $\alpha=(\alpha_1,\ldots,\alpha_k)$ with projections $0\leqslant\alpha_i\leqslant 1$ the function

$$\prod_{l=1}^{k}\alpha_l^{\chi_{\Delta_l}(x)} \tag{2.8}$$

coincides with the function u_α of (2.5). Thus, (2.6) defines the generating function of the vector $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ for real α_i. Since the coefficients of the expansion (2.6) do not exceed one, by extending this expression to the interior of the unit polydisk (with center at zero) of the complex plane $\mathbf{C}^k$, we obtain the generating function

$$B_P\begin{pmatrix}\alpha_1,&\ldots,&\alpha_k\\ \Delta_1,&\ldots,&\Delta_k\end{pmatrix}\equiv B_p(u_\alpha). \tag{2.9}$$

The distributions of the vectors $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ are finite-dimensional distributions of the measure P. They can be recovered on the basis of the functional B_p

$$P\left(C^{n_1,\ldots,n_k}_{\Delta_1,\ldots,\Delta_k}\right)=\prod_{l=1}^{k}(n_l!)^{-1}\frac{\partial^{n_1+\ldots+n_k}}{\partial\alpha_1^{n_1}\partial\alpha_2^{n_2}\ldots\partial\alpha_k^{n_k}}B_p\begin{pmatrix}\alpha_1&\ldots&\alpha_k\\ \Delta_1&\ldots&\Delta_k\end{pmatrix}\Bigg|_0.$$

This implies that the domain of the functional B_p can be restricted to the set of functions (2.5).

In [41] characteristic properties of GF with domain $\mathcal{U}^f_{1,r}$ are found. To formulate the corresponding theorem we define the set

$$\mathcal{U}_0^f(E)=\{u\in\mathcal{U}_1(E):\operatorname{supp}u\in\mathfrak{S}_0\}=1-\mathcal{U}^f_{1,r}(E) \tag{2.10}$$

and the operator of first difference $\Delta[h]$

$$(\Delta[h]\psi)(x)=\psi(x+h)-\psi(x), \tag{2.11}$$

where $\psi:X\to Y$, X, Y are linear spaces.

THEOREM 2.2. In order that a functional B(u) defined on $\mathcal{U}_{1,r}^f(E)$ be the GF of a configuration measure P with phase space E it is necessary and sufficient that

1) for a nondecreasing sequence $u_n\in\mathcal{U}^f_{1,r}(E)$ converging to one

$$\lim_{n\to\infty}B(u_n)=1. \tag{2.12}$$

2) $\forall h_1,\ldots,h_k\in\mathcal{U}_0^f(E)$ and $\forall u\in\mathcal{U}^f_{1,r}(E)$ such that $u+h_1+\ldots+h_k\in\mathcal{U}^f_{1,r}(E)$

$$(\Delta[h_1]\Delta[h_2]\ldots\Delta[h_k]B)(u)\geqslant 0. \tag{2.13}$$

The domain of the GF can be extended [41] to the set

$$\mathcal{U}_{1,r}=\{u\in\mathcal{U}(E):0\leqslant u\leqslant 1\}\supset\mathcal{U}_{1,r}^{f}.$$

In this case to the characteristic properties (2.12) and (2.13) there is added a further one: for any nonincreasing sequence $u_n\in\mathcal{U}_{1,r}(E)$ converging pointwise to $u\in\mathcal{U}_{1,r}(E)$,

$$\lim_{n\to\infty} B(u_n)=B(u). \tag{2.14}$$

The generating functions (2.9) of finite-dimensional distributions of the random measure P are constructed on the basis of the GF on $\mathcal{U}_{1,r}$. In order that these functions be obtained directly from the functionals B, its domain should be extended to $\mathcal{U}_{1,r}(E)\cup\mathcal{U}_1'(E)$. But this is little. The method of the generating functional will be effective if it is possible to use in it the well developed machinery of analytic mappings [36]. A minimal necessary condition for this is that the domain of the GF be a finitely open set of some linear space. $\mathcal{T}_f(E)$ is a linear space, but in it $\mathcal{T}_1'(E)=\mathcal{U}_1'(E)-1$ is not finitely open. Considering these circumstances, we take as the domain of the GF the entire space $\mathcal{T}_f(E)$ and the GF corresponding to this we denote by $B_P(t)$.

Extension of the domain of the GF to $\mathcal{T}_f(E)$ restricts the class of random measures for which $B_P(t)$ exists. Nevertheless, this class is not empty, since it contains, for example, random measures describing mathematical models of statistical physics. For the sequel we shall require some concepts and results from the theory of analytic mappings. We present them here in the form they will be used in the present work. General definitions and results can be found in the classical monograph [36].

Let X and Y be Banach spaces over the field of complex numbers with norms $\|\ \|_1$ and $\|\ \|_2$.

Definition 2.1. We call a mapping $f:X\to Y$ G-differentiable (Gateaux differentiable) if $\forall x,\ h\in X$ the abstract function $f(x+\alpha h)$ of the complex variable α is defined which at the point $\alpha=0$ has the derivative

$$\frac{d}{d\alpha}f(x+\alpha h)|_{\alpha=0}=\delta f(x;\ h). \tag{2.15}$$

$\delta f(x;\ h)$ for fixed x is a linear mapping in h from X to Y called the first variation (the Gateaux differential).

THEOREM 2.3. A G-differentiable mapping f has variations (differentials)

$$\delta^n f(x;\ h_1,h_2,\ldots,h_n)=\frac{\partial^n}{\partial\alpha_1\ldots\partial\alpha_n}f(x+\alpha_1h_1+\ldots+\alpha_nh_n)|_{\alpha_i=0} \tag{2.16}$$

of all orders which are symmetric, n-linear forms in $h_1,\ldots,h_n$ G-differentiable with respect to x for fixed $h_1,\ldots,h_n$. A G-differentiable mapping f can be expanded in the Taylor series

$$f(x+h)=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^n f(x,\ h), \tag{2.17}$$

which converges $\forall x,\ h\in X$ in the norm $\|\ \|_2$ of the space Y; here

$$\delta^n f(x;\ h)=\delta^n f(x;\ h_1,\ldots,h_n)|_{h_i=h}. \tag{2.18}$$

Definition 2.2. A mapping f is called locally bounded if $\forall x_0\in X$ there exists a number $r(x_0)>0$, $q(x_0)>0$ such that $\|f(x)\|_2\leqslant q(x_0)$ whenever $\|x-x_0\|_1\leqslant r(x_0)$.

Definition 2.3. A G-differentiable and locally bounded mapping f is called an analytic mapping.

THEOREM 2.4. An analytic mapping f is strongly (Frechet) differentiable and its variations of n-th order are continuous, n-linear forms, whereby

$$\|\delta^n f(x;\ h_1,\ldots,h_n)\|\leqslant n!q(x)\|h_1\|\ldots\|h_n\|(r(x)^{-n}). \tag{2.19}$$

For fixed $h_1,\ldots,h_n$, $\delta^n f(x,\ h_1,\ldots,h_n)$ is an analytic mapping in x and

$$\delta(\delta^n f(x;\ h_1,\ldots,h_n),\ h_{n+1})=\delta^{n+1}f(x;\ h_1,\ldots,h_{n+1}). \tag{2.20}$$

THEOREM 2.5. Let $\{f_k(x)\}$ be a sequence of analytic and locally uniformly bounded mappings of X into Y. If in some sphere $\|x-x_0\|_1\leqslant r$ there exists the pointwise $\lim_{k\to\infty} f_k(x)$ (in the norm of the space Y), then this limit f(x) exists everywhere in X and is an analytic mapping, whereby

$$\delta^n f(x;h)=\lim_{k\to\infty}\delta^n f_k(x;h). \tag{2.21}$$

All definitions and formulations of theorems are presented for the space X in precisely the form they are required. In order to better understand what analytic properties are to be required of a GF, we consider the class of configuration measures concentrated on the space M_1 — simple measures on $\mathbf{Z}^\nu$. A measure concentrated on M_1 we call a simple configuration measure P.

THEOREM 2.6. The GF B_P of simple configuration measures with phase space $\mathbf{Z}^\nu$ are analytic functionals on $L'(\mathbf{Z}^\nu)$.

Proof. For simple measures ξ on $\mathbf{Z}^\nu$

$$\int |t(x)|\,d\xi=\sum_{x\in c}|t(x)|\leqslant\sum_{x\in\mathbf{Z}^\nu}|t(x)|=\|t\|,$$

where $\|t\|$ is the norm of t in the space $L'(\mathbf{Z}^\nu)$ of functions absolutely summable on $\mathbf{Z}^\nu$. Thus, $\mathcal{T}(\mathbf{Z}^\nu)=L'(\mathbf{Z}^\nu)$. Since

$$\left|\prod_{x\in c}(1+t(x))\right|\leqslant e^{\sum_{x\in c}|t(x)|}\leqslant e^{\|t\|}, \tag{2.22}$$

by (2.4) a mapping Π is defined acting from $L'(\mathbf{Z}^\nu)$ into the space of functions on M_1 (or C_E) which are integrable with respect to a simple configuration measure P. This explains why $L'(\mathbf{Z}^\nu)$ is taken as the domain of the GF. It follows from (2.22) that

$$|B_p(t)|=\left|\int\left(\prod t\right)(c)\,dP(c)\right|\leqslant\exp\{\|t\|\}, \tag{2.23}$$

i.e., the GF is locally bounded on $L'(\mathbf{Z}^\nu)$. To prove analyticity of $B_P(t)$ it remains to prove analyticity in the whole complex plane of the function $B_p(t+\alpha h)$, $\forall t,h\in L'(\mathbf{Z}^\nu)$. This latter follows trivially from the fact that the series

$$\prod_{x\in c}(1+t(x)+\alpha h(x))=\sum_{|c_0|=0}^{\infty}\alpha^{|c_0|}\sum_{x\in c_0}h(x)\prod_{x\in c\setminus c_0}t(x)$$

is majorized by the series

$$e^{\|t\|}\sum_{|c_0|=0}^{\infty}|\alpha|^{|c_0|}(|c_0|!)^{-1}\|h\|^{|c_0|}=\exp\{\|t\|+|\alpha|\,\|h\|\}.$$

The properties of GF of simple configuration measures with phase space $\mathbf{Z}^\nu$ serve as a standard making it possible to distinguish an entire class of measures which should be called analytic — these are configuration measures with GF analytic on $L'(E,\mu)$. Since the overwhelming majority of results in the method of the GF have been obtained for these measures, we shall concentrate attention just on them. Having in mind applications to statistical physics, we consider two classes of measures: with phase space $\mathbf{Z}^\nu$ (discrete models) and with phase space R^ν and Lebesgue measure μ (continuous models). Practically all results obtained for configuration measures of continuous models carry over automatically to configuration measures of discrete models. To be specific, we therefore consider configuration measures with phase space R^ν.

THEOREM 2.7 [28]. A functional A(t) analytic on $L'(R^\nu)$ has the form

$$A(t)=\sum_{s=0}^{\infty}\frac{1}{s!}\int a(x)_s\prod_{i=1}^{s}t(x_i)\,d(x)_s, \tag{2.24}$$

where $a(x)_s=a(x_1,\ldots,x_s)\in L^\infty(R^{s\nu})$, $t(x)\in L'(R^\nu)$, $d(x)_s$ denotes integration with respect to Lebesgue measure on $R^{\nu s}$.

Proof. By Theorem 2.3 A(t) can be represented by the expansion (2.17)

$$A(t)=\sum_{s=0}^{\infty}\frac{1}{s!}\delta^s A(0;t),$$

where $\delta^s A(0; t)$ are continuous (bounded) s-linear forms on $L'(R^\nu)$. For s = 1 $\delta^1 A(0; t)$ is a continuous linear functional on $L'(R^\nu)$ which, as is known, has the form

$$\delta^1 A(0; t) = \int a(x) t(x) dx,$$

where $a(x) \in L^\infty(R^\nu)$. Let $\mathscr{L}_2(L^1, C)$ be the space of bounded, bilinear forms on $L'(R^\nu) \times L'(R^\nu)$. Using the construction of the tensor product, each bilinear form on $\mathscr{L}_2(L', C)$ can be identified with a continuous linear functional in $\mathscr{L}(L'\hat{\otimes}L', C)$, where $L'\hat{\otimes}L'$ is the completion of $L'(R^\nu)\otimes L'(R^\nu)$ in the strong cross norm [19]. Since $L'(R^\nu)\hat{\otimes}L'(R^\nu) = L'(R^{2\nu})$, it follows that

$$\mathscr{L}(L'\hat{\otimes}L', C) = \mathscr{L}(L'(R^{2\nu}), C) = L'(R^{2\nu})^* = L^\infty(R^{2\nu}).$$

From this it follows that each form in $\mathscr{L}_2(L', C)$ has integral form. This goes also for the quadratic form corresponding to it. Thus,

$$\delta^2 A(0; t) = \int a(x, y) t(x) t(y) dxdy.$$

Similarly, $\mathscr{L}_s(L', C)$ — the space of bounded s-linear forms — can be identified with $L^\infty(R^{\nu s})$, and each s-linear, bounded form on $L'(R^\nu)$ has integral form, whereby for symmetric, polylinear forms the kernels $a(x)_s$ are symmetric functions. The proof of the theorem is complete.

Thus, each functional analytic on $L'(R^\nu)$ generates ("produces") a collection of functions $\{a(x)_s\}_1^\infty$. If the point of the expansion is not zero, then in place of (2.24) we have

$$A(t+h) = \sum \frac{1}{s!} \int a(t; (x)_s) \prod_{i=1}^{s} h(x_i) d(x)_s. \tag{2.25}$$

Definition 2.4 [28]. The kernel $a(t; (x)_s)$ defining the differential of s-th order of a functional A(t) analytic on $L'(R^\nu)$

$$\delta^s A(t; h) = \int a(t; (x)_s) \prod_{i=1}^{s} h(x_i) d(x)_s, \tag{2.26}$$

is called the functional derivative of A(t) of s-th order (at the point t) and is written

$$a(t; (x)_s) = \mathscr{D}(x)_s A(t) = \mathscr{D}(x_1, \ldots, x_s) A(t). \tag{2.27}$$

We agree to write $\mathscr{D}(x)_0 A(t) \equiv A(t)$. The collection of functional derivatives of a functional A(t) analytic on $L'(R^\nu)$ can be considered a function $\mathscr{D}(c_0) A(t)$ on the space of finite configurations $\mathscr{K}$. With consideration of this the expansion (2.25) can be written in the form

$$A(t+h) = \int_{\mathscr{K}} \mathscr{D}(c_0) A(t) \prod_{x \in c_0} h(x) d\lambda(c_0). \tag{2.28}$$

By Theorem 2.4 each functional derivative $\mathscr{D}(x)_s A(t)$ is an analytic mapping from $L'(R^\nu)$ into $L^\infty(R^{\nu s})$, and for it there is the expansion

$$\mathscr{D}(c_0) A(t+h) = \int_{\mathscr{K}} \mathscr{D}(c_0 \cup c_1) \prod_{x \in c_1} h(x) d\lambda(c_1). \tag{2.29}$$

The next theorem gives the most important property of analytic configuration measures.

THEOREM 2.8. On each σ-algebra $\mathfrak{M}_\Lambda (\Lambda \in \mathfrak{S}_0)$ a configuration measure P with a GF $B_p(t)$ analytic on $L'(R^\nu)$ is given by the system of the densities

$$\sigma_\Lambda(c_0) = \mathscr{D}(c_0) B_p(-\chi_\Lambda), \tag{2.30}$$

where χ_Λ is the indicator of the set $\Lambda \in \mathfrak{S}_0$.

Proof. For the proof it suffices to consider the measure P on the generators (1.5) of the algebra $\mathfrak{M}_\Lambda$. On these sets by (2.9) the measure is determined by the restriction of the GF to functions (2.5). Since

$$\alpha^{\chi_\Delta(x)} = 1 + (\alpha - 1)\chi_\Delta(x), \tag{2.31}$$

the functions of (2.5) u_α can be written in the form

$$u_\alpha(x)=\prod_{l=1}^{k}[1+(\alpha_l-1)\chi_{\Delta_l}(x)] \tag{2.32}$$

and

$$B_p(u_\alpha)=B_p\left(\prod_{l=1}^{k}[1+(\alpha_l-1)\chi_{\Delta_l}]-1\right). \tag{2.33}$$

Computing the derivatives with respect to α_i of this expression, we obtain by (2.9)

$$P\left(C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}\right)=\int_{\Lambda^n}\mathcal{D}(x)_n B_p(-\chi_\Lambda)\prod_{j=1}^{k}\prod_{l=n_{j-1}}^{n_j}\chi_{\Delta_j}(x_l)\,d(x)_n, \tag{2.34}$$

where $\Delta_l\subset\Lambda$ and $\Delta_i\cap\Delta_l=\varnothing$, $i\neq l$, $n_0=0$, $n=\sum_{l=1}^{k}n_l$, which proves that on $\mathfrak{M}_\Lambda$ the measure is given by the density (2.30). From (2.34) by the nonnegativity of the measure P it follows that

$$\mathcal{D}(c_0)B_p(-\chi_\Lambda)\geqslant 0,\quad \forall\Lambda\in\mathfrak{S}_0,\quad \forall C_0\in\mathcal{K}. \tag{2.35}$$

Remark. It is obvious that a configuration measure with phase space R^ν (and Lebesgue measure μ) whose restriction to $\mathfrak{M}_\Lambda$ is given by densities $\sigma_\Lambda(C_0)$ is simple. Hence, analytic configuration measures with phase space R^ν (and Lebesgue measure μ) are simple configuration measures.

The property of the GF of an analytic configuration measure of determining its densities is characteristic.

THEOREM 2.9 [27]. In order that a functional B(t) analytic on $L'(R^\nu)$ be the GF of a configuration measure P with phase space R^ν it is necessary and sufficient that the following conditions be satisfied:

$$B(0)=1, \tag{2.36}$$

$$\mathcal{D}(c_0)B(-\chi_\Lambda)\geqslant 0,\quad \forall\Lambda\in\mathfrak{S}_0,\quad \forall c_0\in\mathcal{K}. \tag{2.37}$$

Proof. The necessity of (2.37) has already been proved. (2.36) follows trivially from definition (2.3) with consideration of the representation (2.4). Conversely, if conditions (2.36) and (2.37) are satisfied, then we define the nonnegative functions on $\mathcal{K}$

$$\sigma_\Lambda(c_0)=\mathcal{D}(c_0)B(-\chi_\Lambda),\quad \forall\Lambda\in\mathfrak{S}_0, \tag{2.38}$$

and show that they satisfy the conditions of consistency and normalization (1.14), (1.15), i.e., they are densities of the distribution

$$\int_{\mathcal{K}(\Lambda)}\sigma_\Lambda(c_0)\,d\lambda(c_0)=\int_{\mathcal{K}}\mathcal{D}(c_0)B(-\chi_\Lambda)\prod_{x\in c_0}\chi_\Lambda(x)\,d\lambda(c_0)=B(0)=1.$$

Further, for $\Delta\subset\Lambda$ and $\bar\Delta=\Lambda\setminus\Delta$ we have by (2.29)

$$\sigma_\Delta(c_0)=\mathcal{D}(c_0)B(-\chi_\Delta)=\mathcal{D}(c_0)B(-\chi_\Lambda+\chi_{\bar\Delta})$$

$$=\int_{\mathcal{K}}\mathcal{D}(c_0\cup c_1)B(-\chi_\Lambda)\prod_{x\in c(0)}\chi_{\bar\Delta}(x)\,d\lambda(c_1)=\int_{\mathcal{K}(\bar\Delta)}\sigma_\Lambda(c_0\cup c_1)\,d\lambda(c_1).$$

By Theorem 1.2 the system of densities of a distribution defines some configuration measure P. It remains to prove that the measure P and the functional B(t) are connected by the relation

$$B(t)=\int\prod_{x\in c}(1+t(x))\,dP(c). \tag{2.39}$$

First of all, it is necessary to prove the existence (P-almost everywhere) and summability of the function $(\Pi t)(c)$ on M. We choose a function $t\in L'(R^\nu)$ (more precisely, a representative of the corresponding equivalence class) bounded on R^ν. With each increasing sequence $\Lambda_n\in\mathfrak{S}_0\left(\bigcup_{n=1}^{\infty}\Lambda_n=R^\nu\right)$ there is associated a particular method of finding $(\Pi|t|)(c)$ for which the partial products are the functions

$$\psi_n(c)=\prod_{x\in c}(1+\chi_{\Lambda_n}(x)|t(x)|)=(\Pi\chi_{\Lambda_n}|t|)(c). \tag{2.40}$$

The functions $\psi_n(c)$ depend only on the part of the configuration c contained in Λ_n: $\psi_n(c)=\psi_n(c\cap\Lambda_n)$. Therefore,

$$\int_{C_E}\psi_n(c)\,dP(c)=\int_{C^{\mathscr{K}(\Lambda_n)}_{\Lambda_n}}\psi_n(c_0)\,d\overline{P}(c_0)$$

$$=\int_{\mathscr{K}(\Lambda_n)}\prod_{x\in c_0}(1+\chi_{\Lambda_n}(x)|t(x)|)\,\mathscr{D}(c_0)\,B(-\chi_{\Lambda_n})\,d\lambda(c_0)=B(\chi_{\Lambda_1}|t|), \tag{2.41}$$

where $\overline{P}$ denotes the restriction of the measure P to the algebra $\mathfrak{M}_\Lambda$ which is given by the system of densities of the distribution (2.38). As $n\to\infty$, $\chi_{\Lambda_n}tt\to$ in $L'(R^\nu)$; therefore, by the continuity of B(t) the right side of (2.41) is bounded uniformly with respect to n. The existence and integrability of $(\Pi|t|(c)$ [and with it of $(\Pi t)(c)$] follows from the theorem of Levy [18]. Replacing in (2.41) $|t|$ by t and passing to the limit, we obtain (2.39). Since the right side in this relation does not depend on the choice of representative $t\in L'(R^\nu)$, the left side also does not depend on it. The relation (2.39) extends to bounded functions $t\in L'(R^\nu)$ trivially by passing to the limit.

Condition (2.35) is only a weak version of conditions (2.13). The following conditions are a complete analogue of (2.13).

LEMMA 2.1 [27]. An analytic GF $B_p(t)$ on the set

$$\mathscr{I}_1=\{t\in L'(R^\nu):1+t(x)\geqslant 0 \text{ п. в. в } R^\nu\} \tag{2.42}$$

satisfies the condition

$$\mathscr{D}(c_0)\,B_p(t)\geqslant 0,\quad \forall t\in\mathscr{I}_1,\quad \forall c_0\in\mathscr{K}. \tag{2.43}$$

Proof. $\forall\Delta\in\mathfrak{S}_0$ and $t\in\mathscr{I}_1$

$$\mathscr{D}(c_0)\,B_p(\chi_\Delta t)=\int_{\mathscr{K}(\Delta)}\mathscr{D}(c_0\cup c_1)\,B(-\chi_\Delta)\prod_{x\in c_1}[\chi_\Delta(x)(1+t(x))]\,d\lambda(c_1)\geqslant 0,$$

since nonnegative expressions stand under the integral sign. Letting $\Delta\to R^\nu$, by the analyticity of functional derivatives we obtain (2.43).

It is obvious that $1+\mathscr{I}_1\supset\mathscr{U}_1^f(R^\nu)$; nevertheless, $\mathscr{I}_1$ "occupies little space" in $L'(R^\nu)$.

LEMMA 2.2 [27]. The set $\mathscr{I}_1$ is nowhere dense in $L'(R^\nu)$.

A proof of the lemma is presented in [27]. Functions on $\mathscr{K}$

$$\rho(c_0)\equiv\mathscr{D}(c_0)\,B_p(0) \tag{2.44}$$

have special significance in statistical physics; they are called correlation functions. There is a connection between the functions $\rho(c_0)$ and $\sigma_\Delta(c_0)$ [23]. It can be established in an elementary way in terms of the generating functional $B_p(t)$:

$$\sigma_\Delta(c_0)=\mathscr{D}(c_0)\,B_p(-\chi_\Delta)=\int_{\mathscr{K}(\Delta)}\mathscr{D}(c_0\cup c_1)\,B_p(0)\,(-1)^{|c_1|}\prod_{x\in c_1}\chi_\Delta(x)\,d\lambda(c_1)$$

$$=\int_{\mathscr{K}(\Delta)}\rho(c_0\cup c_1\,(-1)^{|c|}\prod_{x\in c_1}\chi_\Delta(x)\,d\lambda(c_1), \tag{2.45}$$

$$\rho(c_0)=\mathscr{D}(c_0)\,B_p(0)$$

$$=\int_{\mathscr{K}(\Delta)}\mathscr{D}(c_0\cup c_1)\,B_p(-\chi_\Delta)\prod_{x\in c_1}\chi_\Delta(x)\,d\lambda(c_1)=\int_{\mathscr{K}(\Delta)}\sigma_\Delta(c_0\cup c_1)\prod_{x\in c_1}\chi_\Delta(x)\,d\lambda(c_1). \tag{2.46}$$

The mathematical expectations of variables of summation type can be expressed in terms of the functions $\rho(c_0)$ [24]. Let $\psi_n(x_1,\ldots,x_n)$ be a symmetric function. We denote by c_n the configuration in which $|c_n|=n$, i.e., $c_n=\{x_1,\ldots,x_n\}$ is a collection of n points $x_i\in R^\nu$. We define, formally for the time being, a function on C_E

$$\psi^{(n)}(c) = \sum_{c_n \subset c} \psi_n(c_n) \tag{2.47}$$

and call it the summation function of n-th order.

LEMMA 2.3. Let P be a configuration measure with analytic GF $B_P(t)$. $\forall \psi_n(c_n) \in L'(R^{n\nu})$ the functions (2.47) are summable on P and

$$\int \psi^{(n)}(c)\, dP(c) = (n!)^{-1} \int \rho(x_1, \ldots, x_n)\, \psi_n(x_1, \ldots, x_n)\, d(x)_n. \tag{2.48}$$

Proof. Exactly in the same way as in Theorem 2.9, it suffices to prove that $\forall \Delta \in \mathfrak{S}_0$ the functions

$$\psi_\Delta^{(n)}(c) = \sum_{c_n \subset c} \psi_n(c)\, \chi_\Delta(c_n)$$

are integrable, where $\chi_\Delta(c_n)$ is the indicator of Δ^n.

$$\int \psi_\Delta^{(n)}(c)\, dP(c) = \int_{\mathcal{K}(\Delta)} \psi_\Delta^{(n)}(c)\, \sigma_\Delta(c)\, d\lambda(c)$$

$$= \sum_{m \geq n} \frac{1}{n!\,(m-n)!} \int_{\Delta^m} \psi_n(x_1, \ldots, x_n)\, \chi_{\Delta^n}(x_1, \ldots, x_n)\, \sigma_\Delta(x_1, \ldots, x_m)\, d(x)_m$$

$$= \frac{1}{n!} \int \rho(x_1, \ldots, x_n)\, \psi_n(x_1, \ldots, x_n)\, \chi_{\Delta^n}(x_1, \ldots, x_n)\, d(x_n),$$

and (2.48) is obtained from this by passing to the limit $\Delta \to R^\nu$. With this the general study of properties of GF is completed. In the following sections solution of some problems of statistical physics by the method of GF will be demonstrated. From a general point of view this reveals the possibilities of the method in specific applications.

3. Gibbs Grand Canonical Ensemble

The simplest model of a physical system in equilibrium with the surrounding medium is a system of identical point particles occupying a bounded region $\Delta \subset R^\nu$, and capable of exchanging energy and particles with the medium. A mathematical model of such a system is the probability space $[\mathcal{K}, \mathfrak{B}, P_\Delta]$, where $\Delta \in \mathfrak{S}_c$, and the measure P_Δ is given by a density $p_\Delta(c)$ with respect to the measure λ in $\mathcal{K}(\Delta)$

$$p_\Delta(c) = \Xi_\Delta^{-1} z^{|c|} \exp\{-\beta H(c)\}\, \chi_\Delta(c), \tag{3.1}$$

where $\chi_\Delta(c) = \prod_{x \in c} \chi_\Delta(x)$, and χ_Δ is the indicator of Δ; thus, the measure P_Δ is concentrated on $\mathcal{K}(\Delta)$. The quantity $\beta > 0$ is inversely proportional to the temperature and characterizes the intensity of the thermal interaction of the system with the medium. The intensity of material interaction (the transport of energy by particles) is characterized by the activity $z > 0$. The Hamiltonian $H(c)$ is determined by the potentials of the external field $\Phi_1(x)$ and the binary interaction $\Phi_2(x)$

$$H(c) = \sum_{x \in c} \Phi_1(x) + \sum_{\{x,y\} \subset c} \Phi_2(x-y). \tag{3.2}$$

where Φ_1 and Φ_2 are measurable, essentially lower semibounded functions on R^ν, and Φ_2 is an even function

$$\Phi_2(x) = \Phi_2(-x). \tag{3.3}$$

Values assumed by the potentials on a set of measure zero are inconsequential for the density $p_\Delta(c)$; therefore, potentials differing on a set of measure zero are identified.

$$\Xi_\Delta = \int_{\mathcal{K}} z^{|c|} \exp\{-\beta H(c_0)\}\, d\lambda(c) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Delta^n} \exp\{-\beta H(x)_n\}\, d(x)_n \tag{3.4}$$

is the statistical sum. A sufficient condition on the Hamiltonian ensuring its existence $\forall \Delta \in \mathfrak{S}_0$, is the stability condition: $\exists b > 0$ such that

$$H(c) \geq -b\,|c| \text{ (a.e. in } \mathcal{K}). \tag{3.5}$$

The measure P_Δ defined by the density (3.1) is called the Gibbs grand canonical distribution, while the mathematical model $[\mathscr{K}, \mathfrak{B}, P_\Delta]$ is called the Gibbs grand canonical ensemble (GCE). The measure P_Δ can be considered defined on $[M, \mathfrak{M}]$ by the system of densities of the distribution

$$\sigma_\Lambda^\Delta(c) = \mathscr{D}(c) B_\Delta(-\chi_\Lambda), \tag{3.6}$$

constructed on the basis of the GF

$$B_\Delta(t) = \int_{\mathscr{K}(\Delta)} \prod_{x \in c} (1 + t(x)) p_\Delta(c) d\lambda(c). \tag{3.7}$$

For stable Hamiltonians analyticity of the functional (3.7) on $L'(R^\nu)$ is easily established. From the set of configuration measures on $[M, \mathfrak{M}]$ the Gibbs distribution is distinguished by the equation which its GF satisfies. We require additionally that $\forall\beta>0$,

$$\exp\{-\beta\Phi_2(x)\} - 1 = f(x) \in L'(R^\nu) \cap L^\infty(R^\nu). \tag{3.8}$$

A binary potential with this property we call regular.

LEMMA 3.1 [25]. The GF of GCE with stable Hamiltonian H and regular binary potential Φ_2 satisfies the Bogolyubov equation

$$\mathscr{D}(x) B_\Delta(t) = z_\Delta(x) B_\Delta(t + f[x](t+1)), \tag{3.9}$$

where

$$z_\Delta(x) = \chi_\Delta(x) z \exp\{-\beta\Phi_1(x)\} \tag{3.10}$$

and $f[x](y) = f(x - y)$. The Bogolyubov equation is characteristic for the GF of GCE. To establish this some of its simplest properties are required. We rewrite (3.9) in a somewhat more general form for some functional A(t)

$$\mathscr{D}(x) A(t) = z(x) A(t + f[x](t+1)), \tag{3.11}$$

where

$$z(x) \in L^\infty(R^\nu), \quad f(x) = f(-x) \in L'(R^\nu) \cap L^\infty(R^\nu). \tag{3.12}$$

(3.12) constitutes minimal conditions on the (generally speaking, complex-valued) functions z(x) and f(x) in order that Eq. (3.11) be meaningful; z and f are parameters of Eq. (3.11). The Bogolyubov equation (3.11) is an equation with shifted argument, and the shift f[x](t + 1) depends on t. For this equation there is only the one additional condition

$$A(0) = 1 \tag{3.13}$$

— the normalization condition. This "point" condition can single out a unique solution of Eq. (3.11) if the latter is considered on the entire space $L'(R^\nu)$. In this case solutions of the equation must be analytic functionals on $L'(R^\nu)$, since they have a Frechet derivative on $L'(R^\nu)$. A natural class of functionals is thus distinguished which contains solutions of the Bogolyubov equation — functionals analytic on $L'(R^\nu)$. We denote it by $\mathscr{H}$.

LEMMA 3.2 [27]. Each solution $A \in \mathscr{H}$ of the Bogolyubov equation satisfies the relation

$$A(t) = A(\chi_\Delta t), \tag{3.14}$$

where χ_Δ is the indicator of Δ = supp z(x).

Proof. We represent $t \in L'(R^\nu)$ in the form

$$t(x) = \chi_\Delta(x) t(x) + (1 - \chi_\Delta(x)) t(x) = t_1(x) + t_2(x).$$

For a solution A(t) of Eq. (3.11) we have

$$A(t) = A(t_1 + t_2) = A(t_1) + \int_0^1 \int \mathscr{D}(x) A(t_1 + \alpha t_2) t_2(x) dx d\alpha = A(t_1) = A(\chi_\Delta t),$$

since by (3.11) $\mathscr{D}(x) A(t) \sim z(x)$, and $z(x)(1 - \chi_\Delta(x)) t(x) \equiv 0$.

LEMMA 3.3. If $\Delta = \operatorname{supp} z(x) \in \mathfrak{S}_0$, then the general solution of the Bogolyubov equation is

$$A(t) = A_0 \int_{\mathscr{K}(\Delta)} \prod_{\{x,y\} \subset c} (1 + f(x-y)) \prod_{x \in c} z(x)(1 + t(x)) d\lambda(c), \tag{3.15}$$

if it belongs to $\mathscr{H}$ and only the trivial solution $A \equiv 0$ otherwise; A_0 = const.

Proof. Successively computing the functional derivatives in (3.11), we obtain

$$\mathscr{D}(c)\, A(t) = \prod_{\tilde{x} \subset c} z(\tilde{x}) \prod_{\{x, y\} \subset c} (1 + f(x-y))\, A(t + f[c](t+1)), \tag{3.16}$$

where

$$f[c](y) = \prod_{x \in c} (1 + f(x-y)) - 1. \tag{3.17}$$

By Lemma 3.2

$$\mathscr{D}(c)\, A(-\chi_\Delta) = \prod_{x \in c} z(x) \prod_{\{x, y\}} (1 + f(x-y))\, A(-\chi_\Delta),$$

whereby $A(-\chi_\Delta)$ is defined, since by hypothesis $\chi_\Delta \in L'(R^\nu)$. Thus, all derivatives of the functional A(t) at one point have been found. On the basis of them it remains to construct the series (3.15). The constant A_0 is obviously equal to $A(-\chi_\Delta)$. If the series (3.15) does not define an analytic functional on $L'(R^\nu)$, then there remains only the one trivial solution which obviously exists, since Eq. (3.11) is linear and homogeneous.

If we substitute f and z defined by (3.8) and (3.10), respectively, into (3.15), then the functional A(t) hereby obtained differs from $B_\Delta(t)$ — the GF of the GCE — only by a factor.

THEOREM 3.1. The Bogolyubov equation with parameters z of (3.10) and f of (3.8), where Φ_1 and Φ_2 are defined by (3.2) — a stable Hamiltonian, Φ_2 is a regular binary potential, and $\Delta \in \mathfrak{S}_0$, has a unique solution satisfying the normalization condition (3.13).

Proof. By the estimate (3.5) the series (3.15) converges $\forall t \in L'(R^\nu)$. It remains to normalize it:

$$A(0) = 1 = A_0 \int \exp\{-\beta H(c)\}\, d\lambda(c) = A_0 \Xi_\Delta.$$

For real z(x) and f(x) $\Xi \neq 0$, and the constant A_0 is uniquely determined: $A_0 = \Xi_\Delta^{-1}$. If this is satisfied the functional (3.15) coincides with the GF of the GCE.

The next result is a corollary of Lemma 3.1 and Theorem 3.1.

THEOREM 3.2. A configuration measure with phase space R^ν is a Gibbs distribution with stable potential and regular binary potential if and only if its GF satisfies the Bogolyubov equation.

The Bogolyubov equation together with the normalization condition (3.13) distinguishes the GF of the GCE from the class of analytic functionals and hence can be set as the foundation for statistical physics. Equilibrium classical statistical physics hereby acquires the canonical form of a physical theory: the basic equation whose solutions determine the states of the physical system is introduced axiomatically. In this case the generating functional is converted from a convenient auxiliary device into a characteristic of the state of the system. We shall treat GF in just this way. We shall demonstrate advantages of this formulation of statistical mechanics in the next section.

4. Gibbs Distributions of Infinite Systems

Although Gibbs GCE are defined $\forall \Delta \in \mathfrak{S}_0$, the behavior of real systems is described by ensembles for which $\Delta \subset R^\nu$ is "sufficiently large"; this is the macroscopic condition of systems of statistical physics. In correspondence with this an analysis must be made of the asymptotic (for Δ expanding to all of R^ν) properties of Gibbs distributions. The GCE induces a measure P_Δ on $[M, \mathfrak{M}]$. This makes it possible to construct a limit Gibbs distribution P ($P_\Delta \to P$ as $\Delta \to R^\nu$) which approximates the properties of the GCE for large Δ [23]. The limiting procedure used to find the limit Gibbs distribution — the thermodynamic limiting procedure (TLP) is the mathematically precisely formulated macroscopic condition for systems studied in statistical physics. Consideration of TLP in the method of GF leads to the problem of characterizing convergence of configuration measures in terms of convergence of their GF. In order to formulate the fundamental theorem in this direction, we introduce the following definition.

Definition 4.1 [47]. A sequence of configuration measures P_n with phase space E is called convergent in distribution to a measure P if it converges to P on the system of generators $C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}$ of the algebra of cylinder sets in $\mathfrak{M}$.

In other words, P_n converges in distribution to P if the sequence of distributions of the vectors $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ induced by the measures P_n converges to the distribution induced by the measure P.

THEOREM 4.1 [46, 47]. A sequence of configuration measures P_n converges in distribution to a measure P if and only if the sequence of GF $B_{P_n}(u)$ converges pointwise to $B_P(u)$ on $\mathcal{U}^f_{1,r} \times (E)$.

Proof. If $B_{P_n}(u)\to B_P(u)$ $\forall u\in\mathcal{U}^f_{1,r}(E)$, then the generating functions $B_{P_n}(u_\alpha)$ of the vectors $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ defined by (2.6) converge to $B_P(u_\alpha)$. This is sufficient for convergence of the corresponding distributions. Conversely, convergence of the distributions of the vectors $(\xi(\Delta_1),\ldots,\xi(\Delta_k))$ implies convergence of the GF $B_{P_n}(u)$ to $B_P(u)$ on simple functions $u\in\mathcal{U}^f_{1,r}(E)$. For an arbitrary $u\in\mathcal{U}^f_{1,r}(E)$ there exist two monotone sequence $u_m \leqslant u \leqslant u'_m$ approximating it from above and below. By (2.13) for any m, n we have

$$B_{P_n}(u_m)\leqslant B_{P_n}(u)\leqslant B_{P_n}(u_m'). \tag{4.1}$$

Hence for $n \to \infty$

$$B_P(u_m)\leqslant \varliminf_{n\to\infty} B_{P_n}(u)\leqslant \varlimsup_{n\to\infty} B_{P_n}(u)\leqslant B_P(u_m').$$

It remains to use the monotone convergence theorem.

For those phase spaces for which the system of cylinder sets $C^{n_1\ldots n_k}_{\Delta_1\ldots\Delta_k}$ is the class defining convergence [4], convergence of the configuration measures in distribution is equivalent to weak convergence of these measures. This is the situation with the phase space R^ν or Z^ν.

THEOREM 4.2. For weak convergence of measures P_n with phase space R^ν or Z^ν to a measure P, it is necessary and sufficient that the GF $B_{P_n}(u)$ converge to the GF $B_P(u)$ $\forall u\in\mathcal{U}^f_{1,r}$.

This theorem is a simple corollary of Theorem 4.1 and the theorem on coincidence of convergence in distribution and weak convergence of measures on R^∞ proved in [4].

The problem of characterizing relative compact sets of configuration measures in terms of properties of the GF corresponding to them has not been solved. For analytic measures we shall present an effective sufficient criterion for relative compactness. With a view to further applications, we consider this criterion in the space of $\mathcal{H}$ of analytic functionals on $L'(R^\nu)$. Weak convergence of measures leads to pointwise convergence of their GF; in $\mathcal{H}$ we therefore introduce the topology of pointwise convergence (we call it weak convergence).

LEMMA 4.1. A weakly closed, locally uniformly bounded set $\mathcal{H}_0\subset\mathcal{H}$ is weakly compact.

Proof. By Theorem 2.4 local uniform boundedness of $\mathcal{H}_0$ implies local uniform boundedness of the generating functionals $A\in\mathcal{H}_0$, and this, in turn, implies equicontinuity of $\mathcal{H}_0$. By the second theorem of Ascoli [38] the weak topology in $\mathcal{H}_0$ coincides with the topology of pointwise convergence on a dense set in $L'(R^\nu)$. Since $L'(R^\nu)$ is separable, this topology is metrizable. By Theorem 2.5 $\mathcal{H}_0$ is closed in the space of functionals continuous on $L' \times (R^\nu)$ with the weak topology, and it is compact in it by the third Ascoli theorem [38].

We denote by $\mathcal{H}_p\subset\mathcal{H}$ the set of generating functionals, i.e., functionals satisfying conditions (2.36), (2.37). The next lemma distinguishes compact sets in this set.

LEMMA 4.2. Let $\mathcal{H}_0\subset\mathcal{H}$ be a closed, locally uniformly bounded set. Then $\mathcal{H}_{p,0}=\mathcal{H}_p\cap\mathcal{H}_0$ is compact.

The definition of compactness of a set of configuration measures in terms of compactness in $\mathcal{H}$ has the advantage that it preserves the property of analyticity of a measure for the limit points of these sets. The criterion of compactness formulated in Lemma 4.2 is sufficient to prove the existence of a limit Gibbs distribution. To clarify characteristic properties of these distributions it is necessary to further study the Bogolyubov equation (3.11).

LEMMA 4.3. Let $A_n(t)$ be a sequence of solutions of the Bogolyubov equation with parameters (z_n, f_n). If $\hat{A}_n\in\mathcal{H}_0$ and converges weakly to A, $z_n \to z$ in the weak-* topology of

$L^\infty(R^\nu)$, $f_n \to f$ in $L'(R^\nu)$, and $\|f_n\|_\infty < a < +\infty$, then A is a solution of the Bogolyubov equation with parameters (z, f).

To prove the lemma it is necessary to write the Bogolyubov equation for A_n and pass to the limit in it. On the left side this is possible by Theorem 2.5, while on the right it is possible by the equicontinuity of functionals in $\mathscr{H}_0$ and convergence in $L'(R^\nu)$ of the argument of the functional

$$\|t+f_n[x](t+1)-(t+f[x](t+1)\|_1\to 0 \ \forall x\in R^\nu.$$

Remark. The assertion of Lemma 4.3 remains in force if $\mathscr{H}_0$ is replaced by $\mathscr{H}_{p,0}$. In this case also $A\in\mathscr{H}_{p,0}$.

THEOREM 4.3. Suppose Φ_1 and Φ_2 define a stable Hamiltonian, Φ_2 is regular, and for given z, β there exists $\mathscr{H}_{p,0}$ such that the GF of the GCE $B_\Delta(t)\in\mathscr{H}_{p,0}\forall\Delta\in\mathfrak{S}_0$. Then for the given z, β there exists at least one limit Gibbs distribution whose GF satisfies the Bogolyubov equation with parameters

$$z(x)=z\exp\{-\beta\Phi_1(x)\},\quad f(x)=\exp\{-\beta\Phi_2(x)\}-1 \tag{4.2}$$

Proof. From the stability condition it follows that $\|z\|_\infty$ and $\|f\|_\infty$ are definite. On the basis of an increasing sequence $\Delta_n\in\mathfrak{S}_0$ such that $\bigcup\Delta_n=R^\nu$ we construct a set of GF of the GCE $B_{\Delta_n}(t)$ (with the given z, β). By hypothesis this set is relatively compact. It remains to select from it a convergence subsequence. By Lemma 4.3 its limit B(t) will satisfy the Bogolyubov equation with parameters (z, f) of (4.2), since $B_{\Delta_n}(t)$ satisfies the Bogolyubov equation (3.9), and $z_{\Delta_n}(x) = \chi_{\Delta_n}(x)z(x) \to z(x)$ as $n \to \infty$.

The condition of Theorem 4.3 encompasses the broadest class of interactions with a regular, binary potential for which infinite models were considered in classical statistical physics (it includes the class of potentials investigated in [12, 14, 22, 28, 44]). Of course, this is a condition difficult to verify and its description in terms of other concepts admitting effective utilization is a current problem. Theorems 3.2 and 4.3 make it possible to formulate the following definition.

Definition 4.2. A Gibbs distribution with parameters (z, f) is a configuration measure with phase space R^ν (or $\mathbf{Z}^\nu$) whose GF satisfies the Bogolyubov equation with parameters (z, f).

Among solutions of the Bogolyubov equation the GF of Gibbs distributions are distinguished by the following property.

THEOREM 4.4. A solution A(t) of the Bogolyubov equation with parameters (z, f) defines the GF of a Gibbs distribution if and only if

$$A(t)>0,\quad \forall t\in\mathscr{I}_1. \tag{4.3}$$

Proof. If A(t) is the GF of a Gibbs distribution, then by definition of the GF (2.39) and of the set $\mathscr{I}_1$ (2.42) $A(t) \geqslant 0$. To prove positivity of A(t) on $\mathscr{I}_1$ we suppose otherwise: $\exists t_0\in\mathscr{I}_1$, for which $A(t_0) = 0$. We choose a function $\psi\in\mathscr{I}_1$ of constant sign such that $\|\psi\|_\infty<+\infty$. Then for a real variable α such that $0\leqslant\alpha\leqslant\|\psi\|_\infty^{-1}$, we have $-\alpha|\psi|\in\mathscr{I}_1$. By definition (2.39) of GF we have

$$0\leqslant A(t_0-\alpha|\psi|(t_0+1))\leqslant A(t_0)=0\quad \forall\alpha\in[0,\|\psi\|_\infty^{-1}]. \tag{4.4}$$

Now $A(t_0-\alpha|\psi|(t_0+1))$ for fixed t and ψ is an entire function of the complex variable α. It follows from (4.4) that it is identically equal to zero. In particular, at the point α = 1 (or α = −1)

$$A(t_0+\psi(t_0+1))=0. \tag{4.5}$$

By means of the representation $\psi = \psi_+ + \psi_-$, where $\psi_+ \geqslant 0$, $\psi_- \leqslant 0$ and their supports do not intersect, this equality extends to $\forall\psi\in\mathscr{I}_1\cap L^\infty(R^\nu)$. From the Bogolyubov equation we obtain the system (3.16) which shows that $\mathscr{D}(c)A(t_0)=0$, since f[c](y) defined by (3.17) belong to $\mathscr{I}_1\cap L^\infty(R^\nu)$. Vanishing of A together with all its derivatives obviously contradicts the normalization condition (2.36). Conversely, suppose A(t) is a solution of the Bogolyubov equation. From the system (3.16) and condition (4.3) it then follows that

$$\mathscr{D}(c_0)A(t_\Delta)\geqslant 0\ \forall t\in\mathscr{I}_1 \text{ and } c_0\in\mathscr{K},$$

i.e., A is a GF. We fix a function f and the normalized solution of the Bogolyubov equation with a given function z we denote by A(t; z).

LEMMA 4.4. Suppose for $\psi\in L^1(R^\nu)\cap L^\infty(R^\nu)$ $A(\psi; z)\neq 0$. Then

$$A(t+\psi(t+1); z)/A(\Psi; z)=A(t; (1+\psi)z). \tag{4.6}$$

Proof. By direct substitution into the Bogolyubov equation it is easy to see that for given $\psi A(t + \psi(t + 1); z)$ satisfies the Bogolyubov equation with parameters $(1 + \psi)z$, f. Normalizing this functional, we arrive at (4.6). The property (4.6) of solutions of the Bogolyubov equation is called the property of multiplicity. This name is connected with the fact that

$$A(t+\psi(t+1); z)=A(\psi; z)\,A(t; (1+\psi)z). \tag{4.7}$$

For fixed f, z we consider the parametric set $A(t + \psi(t + 1); z)$. We demonstrate the fundamental importance of the multiplicative property for simple lattice systems.

LEMMA 4.5. The set of GF of limit Gibbs distributions with parameters (3.12) of simple lattice systems lies in the closure of the set $A(t; (1 + \psi)z_0$, where $\psi\in\mathcal{Y}_1$.

Proof. By the estimate (2.23) this set is locally uniformly bounded and hence relatively compact. For all z such that $\operatorname{supp} z\subset\operatorname{supp} z_0$ there exists a sequence ψ_n such that $(1 + \psi_n)z_0 \to z$ in the weak-* topology of the space $L^\infty(Z^\nu)$. By the relative compactness of the set $A(t; (1 + \psi)z_0)$, from the sequence $A(t; (1 + \psi_n)z_0)$ it is possible to extract a convergent subsequence. On the basis of Lemma 4.3 its limit is $A(t; z)$, i.e., it satisfies the Bogolyubov equation with parameters (z, f). In particular, in this manner it is possible to construct the GF of the GCE of finite systems $A(t; z_\Delta)$ $\Delta\subset\operatorname{supp} z_0$. In the closure of the last set lie the GF of limit Gibbs distributions.

We note that actually somewhat more has been proved: in the closure of $A(t; (1 + \psi)z_0)$ lie the GF of limit Gibbs distributions with parameters (z, f) where z is such that $\operatorname{supp} z \subset \operatorname{supp} z_0$. If $\operatorname{supp} z_0 = Z^\nu$, then all limit Gibbs distributions with given function f can be recovered from one solution of the Bogolyubov equation with parameters (z_0, f). A characterization of Gibbs distributions is possible in terms of the canonical form of the conditional probabilities. This approach forms the basis for the definition of the Gibbs distributions of Dobrushin [12, 14] and is most popular at present.

Let P be a configuration measure with phase space R^ν. Each configuration $\tilde{c}$ of C_{R^ν} can be represented in the form (c_0, c), where $c_0=\tilde{c}\cap\Delta$, $\Delta\in\mathfrak{S}_0$, $c\in C_{\bar{\Delta}}$, $\bar{\Delta}=R^\nu\setminus\Delta$, and can define a conditional probability $P_\Delta(\cdot|c)$ (relative to the measure P) under the condition that $c\in C_{\bar{\Delta}}$ is fixed.

Definition 4.3. A configuration measure P with phase space R^ν is called a Gibbs distribution if $\forall\Delta\in\mathfrak{S}_0$ and any configuration $c\in C_{\bar{\Delta}}$ outside Δ the conditional probability $P_\Delta(\cdot|c)$ is given by the density

$$p_\Delta(c_0|c)=\Xi_\Delta^{-1}(c)\,z^{|c_0|}\exp\{-\beta H(c_0, c)\}\,\chi_\Delta(c_0) \tag{4.8}$$

relative to a measure in $\mathcal{K}(\Delta)$, where

$$H(c_0, c)=H(c_0)+\sum_{\substack{x\in c_0\\ y\in c}}\Phi_2(x-y), \tag{4.9}$$

$$\Xi_\Delta(c)=\int_{\mathcal{K}(\Delta)} z^{|c_0|}\exp\{-\beta H(c_0, c)\}\,d\lambda(c_0). \tag{4.10}$$

Thus, for a Gibbs distribution the conditional probability is determined by the Gibbs distribution of the GCE in Δ. The configuration of particles c outside Δ creates an external field. This configuration is called the boundary conditions. The same idea regarding the canonical form of conditional probabilities can be expressed in a somewhat different form [40] by using conditional mathematical expectations.

Definition 4.4. A configuration measure P is called a Gibbs distribution if it satisfies the Dobrushin–Lanford–Ruelle (D–L–R) equation

$$\int_{C_{R^\nu}}\varphi(c)\,dP(c)=\int_{\mathcal{K}(\Delta)} d\lambda(c_0)\int_{C_{\bar{\Delta}}}\varphi(c_0\cup c)\,z^{|c_0|}\exp\{-\beta H(c_0)+H(c_0, c)\}\,dP(c), \tag{4.11}$$

where $\varphi(c)$ is an arbitrary function on C_{R^ν} which is summable with respect to P and $H(c_0, c)$ is defined by (4.9).

The equivalence of both definitions is obvious. From the viewpoint of Definition 4.3, finding Gibbs distributions is a rather traditional problem of probability theory: recovery of the distribution on the basis of a given collection of conditional probabilities. Having proved equivalence of Definitions 4.3 and 4.2, we shall demonstrate the usefulness of the method of the GF for solving such problems of probability theory.

It is convenient to compare Definitions 4.2 and 4.4. We note, first of all, that existence (P-almost everywhere) and summability of the functions

$$\exp\{-\beta H(c_0, c)\}=\prod_{y\in c}(1+f[c_0](y)) \tag{4.12}$$

are contained in the conditions on the measure P ($f[c_0](y)$ defined in (3.17). Since $f\in L'(R^\nu)\cap L^\infty(R^\nu)$, it is natural to assume that all functions of the form

$$\prod_{x\in c}(1+t(x)),\quad t\in L'(R^\nu) \tag{4.13}$$

are integrable with respect to a measure P satisfying the D–L–R equation. Exactly the same sort of considerations – the necessity of writing the Bogolyubov equation characterizing Gibbs distributions – dictates the choice of $L'(R^\nu)$ as the domain of the GF. Thus, comparison of the D–L–R and Bogolyubov equations should be carried out for the class of analytic measures. The class of measures on which Eq. (4.11) is considered can be extended if the requirement of positivity of the measures is abandoned [44]. Such an extension is also possible in the method of the generating functional. The densities $\sigma_\Delta(c_0)$ defined by an analytic, normalized [condition (3.11)] functional A(t),

$$\sigma_\Delta(c_0)=\mathscr{D}(c_0)A(-\chi_\Delta), \tag{4.14}$$

in analogy to the densities (2.38) satisfy the conditions of consistency (1.15) and normalization (1.16). Hence, they define some measure on C_{R^ν} (not necessarily positive) [11].

Equations (4.11) and (3.11) are thus considered on the class of measures generated by analytic functions $A\in\mathscr{H}$. A distinguishing feature of such measures is that the class of functions integrable with respect to such measures contains functions of the form (4.13). Moreover, the measure is completely determined by the integrals of these functions. Considering this circumstance, in (4.11) we set $\varphi(c)=\prod_{x\in c}(1+t(x))$, and after elementary transformations we obtain an equation equivalent to Eq. (4.11)

$$A(t)=\int_{\mathscr{K}(\Delta)}d\lambda(c_0)\prod_{x\in c_0}(1+t(x))z^{|c_0|}\exp\{-\beta H(c_0)\}A((t+f[c_0])(1+t)(1-\chi_\Delta)-1), \tag{4.15}$$

where

$$A(t)=\int\prod_{x\in c}(1+t(x))\,dP(c) \tag{4.16}$$

is the functional in $\mathscr{H}$ defining the measure P.

THEOREM 4.5 [27]. For parameters (z, f) satisfying condition (3.12) Eqs. (4.15) and (3.11) on $\mathscr{H}$ are equivalent.

Proof. If A satisfies (3.11), then it also satisfies the system (3.16). Replacement on the right side of (4.15) of A of a complex argument according to (3.16) by the derivative $\mathscr{D}(c_0)A((1+t)(1-\chi_\Delta)-1)$ converts (4.15) into an identity. Conversely, suppose that A satisfies Eq. (4.15). Differentiating (4.15) at the point $t-\chi_\Delta(1+t)$, we obtain

$$\chi_\Delta(x)\mathscr{D}(x)A(t-\chi_\Delta(1+t))=\chi_\Delta(x)z(x)A((1+f[x])(1+t)(1-\chi_\Delta)-1). \tag{4.17}$$

If we remove the functions $\chi_\Delta(x)$, then Eq. (4.17) is precisely the Bogolyubov equation (3.11) written at the point $t-\chi_\Delta(t+1)$. It is easy to show that a functional not satisfying (4.17) does also not satisfy the Bogolyubov equation (for the details see [27]). The equivalence of Definitions 4.2 and 4.4 reveals additional properties of measures defined by positive solutions [those satisfying condition (4.3)] of the Bogolyubov equation. The next result gives a characteristic of the set of positive solutions of the Bogolyubov equation.

LEMMA 4.6 [27]. For given (z, f) the set of positive solutions of the Bogolyubov equation lying in the compact set $\mathscr{H}_0$, is closed and convex.

Since positive solutions define the GF of Gibbs distributions, Lemma 4.6 characterizes the set of Gibbs distributions with given parameters (z, f).

5. Equivalence of Ensembles

In addition to the GCE, in statistical physics the Gibbs canonical ensemble CE is considered; this is the probability space $[\mathcal{K}, \mathfrak{B}, P_\Delta^{(N)}]$, where $\Delta \in \mathfrak{S}_0$, the measure $P_\Delta^{(N)}$ is given by a density $p_\Delta^{(N)}(c)$ relative to the measure λ in $\mathcal{K}$,

$$p_\Delta^{(N)}(c) = Q^{-1}(N, \Delta, v)\exp\{-\beta H(c)\}\chi_\Delta(c), \tag{5.1}$$

if $|c| = N$ and $p_\Delta^{(N)}(c) = 0$ otherwise. The Hamiltonian H(c) is defined by (3.2), $v(x) = \exp\{-\beta\Phi_1(x)\}$,

$$Q(N, \Delta, v) = \int_{\mathcal{K}_N} \exp\{-\beta H(c)\}\chi_\Delta(c)\, d\lambda_N(c) \tag{5.2}$$

is the configuration integral, and

$$\lambda_N(\Omega_N) = (N!)^{-1}\mu_N(b_N^{-1}(\Omega_N)) \tag{5.3}$$

is the restriction of the measure λ to $\mathcal{K}_N$, $\Omega_N \in \mathfrak{B}_{\Delta^N}$. The Gibbs CE describes the state of a system with a fixed number of particles N. The measure $P_\Delta^{(N)}$ is called the canonical Gibbs distribution. Just as the grand canonical distribution, it may be assumed given on $[M, \mathfrak{M}]$ by a system of distribution densities

$$\sigma_\Lambda^{(\Delta)}(c_0) = \mathcal{D}(c_0)\mathcal{L}_\Delta^{(N)}(-\chi_\Lambda), \tag{5.4}$$

constructed on the basis of its GF

$$\mathcal{L}_\Delta^{(N)}(t) = \int_{\mathcal{K}} \prod_{x \in c}(1 + t(x))\, p_\Delta^{(N)}(c)\, d\lambda(c). \tag{5.5}$$

Here we have used the traditional notation [5] for the GF of a CE $\mathcal{L}_\Delta^{(N)}$. It is easy to see that the GF $\mathcal{L}_\Delta^{(N)}(t)$ of the CE is the ratio of the configuration integrals Q(N, Δ, v) of a system in a given external field Φ_1 and in one changed by $\Delta\Phi_1 = -\beta^{-1}\ln(1 + t(x))$

$$\mathcal{L}_\Delta^{(N)}(t) = Q(N, \Delta, (1+t)v)/Q(N, \Delta, v). \tag{5.6}$$

For regular potentials Φ_2 $\mathcal{L}_\Delta^{(N)}(t)$ satisfies the recursion relations [5]

$$\mathcal{D}(x)\mathcal{L}_\Delta^{(N)}(t) = v_\Delta(x)\frac{NQ(N-1, \Delta, v)}{Q(N, \Delta, v)}\mathcal{L}_\Delta^{(N-1)}(t + f[x](t+1)), \tag{5.7}$$

$v_\Delta = \chi_\Delta v$ and χ_Δ is the indicator of $\Delta \subset R^\nu$. We call (5.7) the Bogolyubov equation of the CE. The relation (5.7) differs in an essential way from the Bogolyubov equation which the GF of a GCE satisfies. It is natural to suppose that after the TLP the GF of the CE will also satisfy the Bogolyubov equation. For systems with a solid core this is proved in [8]; for lattice systems it is proved in [21]. Here we present constructions for continuous systems which are somewhat more general than those in [8]. Consideration of the TLP in a CE requires more stringent conditions on the binary potential.

Definition 5.1. A binary potential Φ_2 belongs to the class (A_{2-3}, B_2) if there exist constants $C > 0$, $\gamma > \nu$, $0 < d_1 < d_2 < +\infty$ such that the condition

$$B_2: |\Phi_2(x)| \leqslant C|x|^{-\gamma}, \quad |x| \geqslant d_2, \tag{5.8}$$

is satisfied as well as one of the conditions

$$A_2: |\Phi_2(x)| \geqslant C|x|^{\gamma}, \quad |x| \leqslant d_1, \tag{5.9}$$

or

$$A_3: \Phi_2(x) = +\infty, \quad |x| \leqslant d_1. \tag{5.10}$$

The class of such potentials is defined in [15]. For CE the correlation functions $\rho_\Delta^{(N)}(c)$ are defined in the standard way:

$$\rho_\Delta^{(N)}(c) = \mathcal{D}(c)\mathcal{L}_\Delta^{(N)}(0).$$

We denote by $|\Delta|$ the Lebesgue measure of the set $\Delta \in \mathfrak{S}_0$, ρ_0 is the largest density $\rho = N|\Delta|^{-1}$ in a system of solid spheres.

LEMMA 5.1. The correlation functions of CE $\rho_{\Delta}^{(N)}(c)$ for systems with potentials of the type (A_{2-3}, B_2) satisfy for sufficiently large $|\Delta|$ the estimates

$$\rho_{\Delta}^{(N)}(c) \leqslant \alpha^{|c|} \tag{5.11}$$

uniformly with respect to $\rho = N|\Delta|^{-1}$ and β belonging to any closed, finite subregion S lying inside the region $\beta > 0$, $\rho < \rho_0$ (for systems without solid core $\rho_0 = +\infty$).

For systems with solid core Lemma 5.1 is proved in [8]; $\alpha > 0$ contained in the estimate (5.11) is determined constructively. We denote by

$$z_{\Delta}^{(N)} = NQ(N-1, \Delta, v)\,Q^{-1}(N, \Delta, v) \tag{5.12}$$

the activity in the CE. The estimates (5.11) show that the set of GF of CE $\{\mathscr{L}_{\Delta}^{(N)}\}$ is weakly compact in $\mathscr{H}$. This solves the question of existence of thermodynamic limits of functionals of CE. In the Bogolyubov equation (5.7) it is possible to carry out the TLP (first setting v = 1, of course) by precisely the scheme considered in Sec. 4. However, in order that the limit functional satisfy the Bogolyubov equation it is necessary that

$$\lim_{k\to\infty} \mathscr{L}_{\Delta_k}^{(N_k-1)}(t) = \lim_{k\to\infty} \mathscr{L}_{\Delta_k}^{(N_k)}(t) \tag{5.13}$$

for those sequences of pairs (N_k, Δ_k) for which both limits exist and $N_k|\Delta_k|^{-1} \to \rho$. Condition (5.13) can be written in a different, equivalent form. In correspondence with the definition of the GF (5.5) and the activity (5.12) we have

$$\mathscr{L}_{\Delta}^{(N-1)}(t)/\mathscr{L}_{\Delta}^{(N)}(t) = z_{\Delta}^{(N)}(t)/z_{\Delta}^{(N)}, \tag{5.14}$$

where $z^{(N)}(t) = NQ(N-1, \Delta, 1+t)/Q(N, \Delta, 1+t)$ is the activity of the system in the field

$$\Phi_1(x) = -\beta^{-1}\ln(1+t(x)), \tag{5.15}$$

of course, under the condition $1 + t(x) > 0$. Thus, in place of (5.13) we can write

$$\lim_{k\to\infty} z_{\Delta_k}^{(N_k)}(t)/z_{\Delta_k}^{(N_k)} = 1. \tag{5.16}$$

This forces us to turn to the study of thermodynamic quantities of systems in the field (5.15). We first specify the set from which t(x) is to be taken so that (5.15) is meaningful. We denote by $C_\gamma(R^\infty)$ the subspace of functions $t(x) \in L'(R^\nu)$ satisfying the inequality

$$|t(x)| \leqslant b(1+|x|)^{-\gamma} \quad \text{a.e. in } R^\nu,$$

where $b > 0$, $\gamma > \nu$. We introduce for $t \in C_\gamma(R^\nu)$ the norm

$$\|t\|_\gamma = \operatorname{ess\,sup}(1+|x|)^\gamma |t(x)|. \tag{5.17}$$

Obviously, the topology introduced in $C_\gamma(R^\nu)$ by the norm (5.17) is stronger than the topology induced from $L'(R^\nu)$, since

$$\|t\| \leqslant \|t\|_\gamma \int (1+|x|)^{-\gamma} dx = a\|t\|_\gamma. \tag{5.18}$$

For potentials (A_{2-3}, B_2) in [15] existence is proved of the limit of the specific free energy

$$\lim_{N\to\infty,\, N|\Delta|^{-1}\to\rho} |\Delta|^{-1}\ln Q(N, \Delta) \equiv g(\rho, \beta). \tag{5.19}$$

This makes it possible to prove the following lemma.

LEMMA 5.2. For systems with potentials (A_{2-3}, B_2) the specific free energy and activity do not depend on an external field of the form

$$\Phi_1(x) = -\beta^{-1}\ln(1+t(x)); \quad t \in S_\delta \subset C_\gamma(R^\nu),\ \delta > 0, \tag{5.20}$$

where S_δ is the ball of radius $\delta > 0$ with center at zero defined by the norm (5.17).

The means of finding δ is indicated in the course of the proof of the lemma. The proof of the first part of the lemma is based on representing the specific free energy in the field (5.20) in the form

$$|\Delta|^{-1}\ln Q(N, \Delta, 1+t) = |\Delta|^{-1}\ln Q(N, \Delta) + |\Delta|^{-1}\ln \mathscr{L}_{\Delta}^{(N)}(t). \tag{5.21}$$

Because of estimates (5.11), the set $\{\mathscr{L}_{\Delta}^{(N)}(t)\}$ is locally uniformly bounded and hence equicontinuous. Equicontinuity of this family together with the normalization condition $\mathscr{L}_{\Delta}^{(N)}(0)=1$ demonstrate that $\mathscr{L}_{\Delta}^{(N)}(t)$ is uniformly bounded away from zero in some neighborhood of the point $t \equiv 0$. Passage to the limit in (5.21) gives

$$g(\rho, \beta, t)=\lim_{N\to\infty, N|\Delta|^{-1}\to\rho} |\Delta|^{-1} \ln Q(N, \Delta, 1+t)=g(\rho, \beta).$$

This proves the first part of the lemma.

The proof of the second part of the lemma is based on using the relations

$$\lim_{N\to\infty, N|\Delta|^{-1}\to\rho} z_{\Delta}^{(N)}=z=\exp\left\{-\frac{\partial g(\rho, \beta)}{\partial\rho}\right\},$$

obtained in the work [15]. The theorem on equivalence of ensembles is proved on the basis of Lemmas 5.1 and 5.2.

THEOREM 5.1. For systems with potentials (A_{2-3}, B_2) from each sequence $\mathscr{L}_{\Delta_k}^{N_k}(t)$ of GF of CE, where Δ_k is a chain of expanding, measurable, bounded sets such that $\bigcup\Delta_k=R^{\nu}$ and $\lim N_k|\Delta_k|^{-1}=\rho$, it is possible to extract a weakly convergent subsequence in $\mathscr{H}$ whose limit point satisfies the Bogolyubov equation.

For systems with potentials (A_{2-3}, B_2) by Lemma 5.1 the GF of CE lie in the compact set $\mathscr{H}_0$. Let $\mathscr{L}_{\Delta_k}^{(N_k)}(t)$ be a subsequence converging weakly to $\mathscr{L}(t)$. Each of its terms satisfies the Bogolyubov equation

$$\mathscr{D}(x)\mathscr{L}_{\Delta_k}^{(N_k)}(t)=z_{\Delta_k}^{(N_k)}\chi_{\Delta}(x)\mathscr{L}_{\Delta_k}^{(N_k-1)}(t+f[x](t+1)). \tag{5.22}$$

By refining the sequence $\mathscr{L}_{\Delta_k}^{(N_k)}(t)$, if necessary, we can arrange that together with it the sequence $\mathscr{L}_{\Delta_k}^{(N_k-1)}(t)$ also converges. Let $\tilde{\mathscr{L}}(t)$ be its limit. By Lemma 5.2 on S_δ

$$\mathscr{L}(t)=\lim_{k\to\infty}\mathscr{L}_{\Delta_k}^{(N_k)}(t)=\lim_{k\to\infty}\mathscr{L}_{\Delta_k}^{(N_k-1)}(t)=\tilde{\mathscr{L}}(t),$$

since $\mathscr{L}_{\Delta_k}^{(N_k)}(t)/\mathscr{L}_{\Delta_k}^{(N_k-1)}(t)\to 1$ on S_δ. $\mathscr{L}, \tilde{\mathscr{L}}$ are analytic functionals on $L'(R^\nu)$. We consider their restrictions to $C_\gamma(R^\nu)$ with norm (5.17). Since this norm is stronger than the norm induced from $L'(R^\nu)$, $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are analytic (in the norm $\|\ \|_\gamma$) on $C_\gamma(R^\nu)$. However, by coinciding on S_δ, they thus coincide on all of $C_\gamma(R^\nu)$, and, since $C_\gamma(R^\nu)$ is dense in $L'(R^\nu)$, they coincide on $L'(R^\nu)$, i.e., $\mathscr{L}=\tilde{\mathscr{L}}$. To complete the proof it remains to use Lemma 4.3 and pass to the limit in Eq. (5.22).

We have thus established that for the distinguished class of potentials the limit functionals of both ensembles satisfy the same equation (3.11) (with a consistent choice of the activity z), and in this sense the ensembles are equivalent. By Theorem 4.4 the equivalence of ensembles just proved means essentially that the thermodynamic limits of GF of CE define Gibbs measures. Complete equivalence of the Gibbs ensembles, i.e., coincidence of the limit functionals of both ensembles, holds for those pairs (z, β) and pairs (ρ, β) corresponding to them for which the Bogolyubov equation has a unique solution satisfying the normalization condition $\mathscr{L}(0)=1$. We encounter this situation, for example, for small z (or ρ).

6. Stability of Gibbs Distributions

Distinguishing parameters (z, f) for which there exists a unique Gibbs distribution is an important problem closely related to the problem of phase transitions in statistical physics. As Lemma 4.6 shows, the structure of the set of Gibbs distributions for fixed (z, f) is rather simple: it is a convex set which is completely characterized by its extremal points. These points are interpreted in statistical physics as states corresponding to the pure phases. It is assumed a priori that such states are relatively stable under weak perturbations of the system, for example, by an external field. From the viewpoint of probability theory the extremal points of the set of Gibbs distributions for given (z, f) must possess special properties of regularity. Thus, in principle there must be a connection between these two means of characterizing Gibbs distributions. It will be the purpose of this section to explicitly establish this connection for simple lattice systems.

At the beginning of the section a criterion for uniqueness of the Gibbs distribution of simple lattice systems is established: continuous dependence of the GF on the external field. This criterion is a standard for the connection of the stability of a Gibbs distribution with the condition of its regularity. After considering various regularity conditions it is established that for Gibbs distributions only two are essential: the condition of external uniform regularity and the condition of regularity which correspond to two forms of stability of Gibbs distributions.

The phase space for lattice systems is Z^{ν}. A lattice system whose Gibbs distributions are concentrated on simple configurations is called a simple lattice system. As is evident from the system (3.16), simple lattice systems are characterized by the condition $f(0) = -1$.

The uniqueness criterion for simple lattice systems is based on the property of multiplicativity (4.6) which we present here for convenience, changing only the notation for the functional A to $\mathscr{L}$

$$\mathscr{L}(t;(1+\psi)z)=\mathscr{L}(t+\psi(t+1);z)/\mathscr{L}(\psi;z),\quad \psi\in\mathscr{I}_1. \tag{6.1}$$

$\mathscr{L}(t;z)$ is a positive, normalized solution of the Bogolyubov equation — the GF of the Gibbs distribution for given (z, f). In all further considerations f is fixed and is not indicated in the notation. For infinite systems the parameter z is defined in analogy to (3.10):

$$z(x)=\exp\{-\beta\Phi_1(x)\}. \tag{6.2}$$

Replacing it by $(1 + \psi)z$ corresponds to replacing the potential of the external field by

$$\Delta\Phi_1=-\beta^{-1}\ln(1+\psi(x)). \tag{6.3}$$

For statistical physics it is important to know the dependence of the Gibbs distribution on the parameters (z, f). In the region of the parameters (z, f) where the Bogolyubov equation has a unique, normalized, positive solution, the dependence of the generating functional $\mathscr{L}$ on these parameters is determined "automatically." Where unique solvability of the equation does not hold, a special, physically justified procedure is needed to distinguish individual single-valued branches determining the change of state of the system as the parameterz (z, f) vary. The multiplicative condition (6.1) makes it possible to construct for each state a branch determining its variation under the action of the external field. We emphasize that the branch (6.1) is constructed independently of whether there exists one or several [for given (z, f)] solutions of the Bogolyubov equation.

Construction of the branch (6.1) can also be justified on the TLP. Let $\mathscr{L}(t;z_{\Delta_n})$ be a sequence of GF of the Gibbs GCE converging weakly to $\mathscr{L}(t;z)$. As a solution of the Bogolyubov equation (3.11), $\mathscr{L}(t;z_{\Delta_n})$ satisfies the multiplicative condition

$$\mathscr{L}(t;(1+\psi)z_{\Delta_n})=\mathscr{L}(t+\psi(1+t);z_{\Delta_n})/\mathscr{L}(\psi;z_{\Delta_n}). \tag{6.4}$$

Passing to the limit, in (6.4) on the left we obtain the functional determining the state with parameters $(1 + \psi)z$, f, while on the right we obtain precisely the right side of (6.1) which corroborates the correctness of the interpretation of the multiplicative condition.

The multiplicative condition defines a mapping $F:\mathscr{I}_1\to\mathscr{H}$. Stability of the Gibbs distribution with GF $\mathscr{L}(t;z)$ is naturally treated as continuity of F with a suitable choice of topologies in $\mathscr{I}_1$ and $\mathscr{H}$. In $\mathscr{H}$ we introduce the weak topology τ_f and in $\mathscr{I}_1$ the weak-* topology τ_* induced from $L^{\infty}(R^{\nu})$. This choice of topology is also motivated by topological properties of the Bogolyubov equation (see Lemma 4.3).

LEMMA 6.1. The Gibbs distribution of a simple lattice system for given (z, f) is unique if and only if there exists at least one positive solution $\mathscr{L}(t;z)$ of the Bogolyubov equation which defines by (6.1) a mapping $F:(\mathscr{I}_1,\tau_*)\to(\mathscr{H},\tau)$ sequentially continuous at zero.

Proof. Necessity. Let $\mathscr{L}(t;z)$ be the GF of the unique Gibbs distribution, and suppose $\mathscr{I}_1\ni\psi_n\to 0$. The sequence $\mathscr{L}(t;(1+\psi_n)z)$ constructed on the basis of (6.1) is relatively compact. By Lemma 4.3 all its convergent subsequences have the same limit $\mathscr{L}(t;z)$ — the unique Gibbs distribution (for given z, f). Thus, $\mathscr{L}(t;(1+\psi_n)z)$ converges, and the mapping F is sequentially continuous at zero.

Sufficiency. The mapping F constructed on the basis of the GF $\mathscr{L}(t;z)$, can be extended to the set $\mathscr{I}_{\Delta}=\{\psi:\psi=\chi_{\Delta}-1,\ \Delta\in\mathfrak{S}_0\}$ with preservation of continuity at those points where there is continuity. For this it suffices to choose a sequence $\psi_n\to\chi_{\Delta}-1$, and the sequence of functionals $\mathscr{L}(t;(1+\psi_n)z)$ corresponding to it converges, since $(1+\psi_n)z\to z_{\Delta}$, whereby the Bogolyubov equation has a unique (normalized) solution. This suffices to construct the extension [16] (we recall that the space $\mathscr{H}_p$ for simple lattice systems is metrizable). Suppose there exist two Gibbs distributions $\mathscr{L}_1(t; z)$, $\mathscr{L}_2(t; z)$ which both generate a mapping F continuous at zero. On the set $\mathscr{Y}_\Delta$ constructed on the basis of them the extensions coincide. We choose $\psi_{\Delta_n} \to 0$ $\psi_{\Delta_n} \in \mathscr{Y}_\Delta$; then

$$\mathscr{L}_1(t; z_{\Delta_n}) = \mathscr{L}_2(t; z_{\Delta_n}).$$

Because of the continuity of both mappings, the limiting procedure $n \to \infty$ gives $\mathscr{L}_1(t; z) = \mathscr{L}_2(t; z)$, i.e., the GF $\mathscr{L}_1$ and $\mathscr{L}_2$ coincide.

Gibbs distributions can be characterized by regularity conditions of various types [12]. Let P be a simple configuration measure with phase space Z^ν. The landing of particles in a region $V \in \mathfrak{S}_0$ ($\mathfrak{S}_0$ consists in this case of finite subsets of Z^ν) we call an event. Each such event A is identified with a system of subsets of V. If $c \subset V$, then {c} is an elementary event. In this terminology $\mathfrak{B}_V$ is the algebra of events. The measure P induces a probability distribution Pr on $[V, \mathfrak{B}_V]$. In correspondence with the notation of Sec. 1, $C_V{}^A = \mathscr{P}_V{}^{-1}(A)$ is a cylinder set. Thus,

$$\Pr(A) = P(C_V^A) = \sum_{c \in A} P_V(c), \tag{6.5}$$

where $P_V(c)$ is the probability of elementary events.

Let $\tilde{V} \in \mathfrak{S}_0$ and $V \cap \tilde{V} = \varnothing$. We denote by $d(V, \tilde{V})$ the distance between V and $\tilde{V}$. We set

$$\gamma(V, \tilde{V}) = \sup_{A \in \mathfrak{B}_V, B \in \mathfrak{B}_{\tilde{V}}, \Pr(B) > 0} |\Pr(A/B) - \Pr(A)|, \tag{6.6}$$

where the probability in (6.6) is computed by means of the measure P. $\gamma(V, \tilde{V})$ is a nonsymmetric function whose first argument is associated with the algebra of events $\mathfrak{B}_V$, and the second with the algebra of conditions $\mathfrak{B}_{\tilde{V}}$.

Definition 6.1. A distribution P possesses the property of external uniform regularity if $\forall \tilde{V} \in \mathfrak{S}_0$

$$\gamma(V, \tilde{V}) \leqslant \varphi_V(d(V, \tilde{V})), \tag{6.7}$$

where $\varphi_V(d) \to 0$ as $d \to \infty$ for fixed V.

Definition 6.2. The distribution P possesses the property of interior uniform regularity if $\forall V \in \mathfrak{S}_0$

$$\gamma(V, \tilde{V}) \leqslant \psi_{\tilde{V}}(d(V, \tilde{V})), \tag{6.8}$$

where $\psi_V(d) \to 0$ as $d \to \infty$ for fixed $\tilde{V}$.

Definition 6.3. A distribution P possesses the property of regularity if $\forall \tilde{V} \in \mathfrak{S}_0$

$$\delta(V, \tilde{V}) = \sup_{A \in \mathfrak{B}_V, B \in \mathfrak{B}_{\tilde{V}}} |\Pr(A \cap B) - \Pr(A)\Pr(B)| \leqslant \lambda_V(d(V, \tilde{V})), \tag{6.9}$$

where $\lambda_V(d) \to 0$ as $d \to \infty$ for fixed V. For simplicity we shall consider Gibbs distributions of systems without an external field and with localized repulsion, i.e., $f(x) = -1$ only for $x = 0$.

LEMMA 6.2. For simple Gibbs distributions of systems without an external field and with localized repulsion $\forall V \in \mathfrak{S}_0$ and $\forall A \in \mathfrak{B}_V$

$$\Pr(A) \geqslant \varepsilon_V > 0. \tag{6.10}$$

Proof. By definition (6.5)

$$\begin{aligned} \Pr(A) &\geqslant \min_{c \subset V} P_V(c) = \min_{c \subset V} \mathscr{D}(c) \mathscr{L}(-\chi_V) = \\ &= \min_{c \subset V} z^{|c|} \exp\{-\beta H(c)\} \mathscr{L}((1 + f[c])(1 - \chi_V) - 1) = \varepsilon_V > 0. \end{aligned} \tag{6.11}$$

Here $\mathscr{L}$ is the GF of the Gibbs distribution. At the last step we used system (3.16) and Theorem 4.4.

THEOREM 6.1 [20]. For the Gibbs distribution of simple lattice systems satisfying the conditions of Lemma 6.2 1) the property of external uniform regularity implies the property of interior uniform regularity; 2) the property of internal uniform regularity is equivalent to the property of regularity.

As shown in [12], the property of external uniform regularity is equivalent to uniqueness of the Gibbs distribution; it is therefore natural to compare it with the condition of stability used in Lemma 6.1. To prove the equivalence of these conditions we first carry out the following constructions. We set

$$P_{V,\tilde V}(c|\tilde c)\equiv \Pr(\{c\}/\{\tilde c\}),$$

where $c\subset V\in\mathfrak{S}_0$, $\tilde c\subset\tilde V\in\mathfrak{S}_0$ and $V\cap\tilde V=\varnothing$, and we define

$$\gamma_0(V,\tilde V)=\sup_{c\subset V,\tilde c\subset\tilde V}|P_{V,\tilde V}(c|\tilde c)-P_V(c)|. \tag{6.12}$$

Obviously,

$$\Pr(A\cap B)=\sum_{\substack{c\in A\\ \tilde c\in B}} P_{V,\tilde V}(c|\tilde c)P_{\tilde V}(\tilde c). \tag{6.13}$$

LEMMA 6.3. $\forall V\in\mathfrak{S}_0$, $\tilde V\in\mathfrak{S}_0$ such that $V\cap\tilde V=\varnothing$

$$\gamma_0(V,\tilde V)\leqslant\gamma(V,\tilde V)\leqslant 2^{|V|}\gamma_0(V,V_0). \tag{6.14}$$

The conditional probability $P_{V,\tilde V}(c|\tilde c)$ for Gibbs distributions can be expressed in terms of the GF [22]

$$P_{V,\tilde V}(c|\tilde c)=\Pr(\{c\}\cap\{\tilde c\})/\Pr(\{\tilde c\})=\mathcal{D}(c\cup\tilde c)\mathcal{L}(-\chi_{V\cup\tilde V})/\mathcal{D}(\tilde c)\mathcal{L}(-\chi_{\tilde V})=z^{|c|}\exp\{-\beta H(c,\tilde c)\}$$

$$\times\mathcal{L}((1-\chi_{V\cup\tilde V})(1+f[c\cup\tilde c])-1)/\mathcal{L}(1-\chi_{\tilde V})(1+f[\tilde c])-1)=\mathcal{D}(c)\mathcal{L}(-\chi_V;(1-\chi_{\tilde V})(1+f[\tilde c])z). \tag{6.15}$$

We have successively used definition (2.30), the system of equations (3.16), and the multiplicative condition (6.1). $H(c,\tilde c)$ is defined by (4.9) and $f[c]$ by (3.17), so that the equality $(1+f[c\cup\tilde c])=(1+f[c])(1+f[\tilde c])$ can be verified trivially. The dependence of the GF $\mathcal{L}$ on z is indicated in the notation only at the last step. Using (6.15), we write $\gamma_0(V,\tilde V)$,

$$\gamma_0(V,\tilde V)=\sup|\mathcal{D}(c)\mathcal{L}(-\chi_V;(1-\chi_{\tilde V})(1+f[\tilde c])z)-\mathcal{D}(c)\mathcal{L}(-\chi_{\tilde V};z)|. \tag{6.16}$$

This expression shows that external uniform regularity of a Gibbs distribution is connected with continuous dependence of the GF on the external field created by the boundary condition $\tilde c$. A rigorous proof of this requires two technical lemmas whose proofs we omit (see, for example, [20, 22]). The GF of simple lattice systems lie in the subspace $\mathcal{H}_c\subset\mathcal{H}$, defined by the conditions

$$\mathcal{D}(c)A(0)=0,$$

if c is not a simple finite configuration; $\mathcal{H}_c$ is weakly compact.

LEMMA 6.4. The weak topology on $\mathcal{H}_c$ can be defined by the system of seminorms: for $V\in\mathfrak{S}_0$

$$g_V(A)=\sup_{c\subset V}|\mathcal{D}(c)A(-\chi_V)|. \tag{6.17}$$

As should be the case, the weak topology on $\mathcal{H}_c$ is metrizable. We consider the mapping F constructed according to (6.1) on the set $f[\tilde c]$, where $\tilde c$ are all possible finite configurations. We denote this set by $\mathcal{I}_f$. Obviously, $\mathcal{I}_f\subset\mathcal{I}_1$.

LEMMA 6.5. If $f(0)=-1$, then the sequence $f_n[c_n]\to 0$ in the topology τ_* if and only if

$$\min_{x\in c_n}|\tilde x|\to\infty,$$

i.e., the configurations c_n "go out" to infinity.

THEOREM 6.2. The Gibbs distribution of a simple lattice system with localized repulsion possesses the property of external uniform regularity if and only if the mapping $F:(\mathcal{I}_f,\tau_*)\to(\mathcal{H}_c,\tau)$ defined by (6.1) is continuous at zero.

Proof. By the estimates (6.14) $\gamma(V,\tilde V)$ in Definition 6.1 can be replaced by $\gamma_0(V,\tilde V)$. If $F:(\mathcal{I}_f,\tau_*)\to(\mathcal{H}_c,\tau)$, then by definition of the seminorms (6.17) and γ_0 it immediately follows that F is continuous at zero. Conversely, if F is continuous at zero, then $\gamma_0(V, \tilde{V})$ tends to zero as the configuration $\tilde{c}$ departs, since by Lemma 6.4 in this case $-\chi_{\tilde{V}} + f[c](1 - \chi_{\tilde{V}}) \to 0$.

Continuity at zero of the mapping F considered only on $\mathcal{Y}_f$, suffices for uniqueness of the Gibbs distribution, since points of the set $\mathcal{Y}_\Delta$ are limit points of the set $\mathcal{Y}_f$.

Quite different topologies on $\mathcal{Y}_f$ and $\mathcal{H}_c$ are required in order to express the property of regularity of a Gibbs distribution in terms of continuity of the mapping F. We define $\forall V \in \mathfrak{S}_0$, $\tilde{V} \in \mathfrak{S}_0$, $V \cap \tilde{V} = \varnothing$

$$\delta_0(V, \tilde{V}) = \sup_{c \subset V, B \in \mathfrak{B}_{\tilde{V}}} \left| \sum_{\tilde{c} \in B} P_{\tilde{V}, V}(\tilde{c} \mid c) - P_{\tilde{V}}(\tilde{c}) \right|. \tag{6.18}$$

LEMMA 6.6. $\forall V \in \mathfrak{S}_0$, $\tilde{V} \in \mathfrak{S}_0$, $V \cap \tilde{V} = \varnothing$,

$$2^{|V|} \varepsilon_V \delta_0(V, \tilde{V}) \leqslant \delta(V, \tilde{V}) \leqslant \delta_0(V, \tilde{V}) 2^{|V|}. \tag{6.19}$$

The proof of the lemma is similar to the proof of Lemma 6.3. The form of $\delta_0(V, \tilde{V})$ "suggests" the choice of topology on $\mathcal{H}_c$. We denote by L the linear space (subspace) of functionals in $\mathcal{H}_c$ satisfying the condition: for any expanding sequence $\Lambda_n \subset \mathbf{Z}^\nu$, $\bigcup_{n=1}^{\infty} \Lambda_n = \mathbf{Z}^\nu$ and $\tilde{V}_n \in \mathfrak{S}_0$, $\Lambda_n \cap \tilde{V}_n = \varnothing$,

$$\lim_{n \to \infty} \sup_{B \in \mathfrak{B}_{\tilde{V}_n}} \left| \sum_{\tilde{c} \in B} \mathcal{D}(\tilde{c}) A(-\chi_{\tilde{V}_n}) \right| = 0. \tag{6.20}$$

Sets of the form

$$U_{A_0} = A_0 + L \tag{6.21}$$

form a basis for a topology on $\mathcal{H}_c$, which we denote by τ_L. GF in L define measures on $[M, \mathfrak{M}]$, concentrated "mainly" on cylinder sets with bases containing zero, since by (6.20) the variation of the measure constructed from A and restricted to the measurable space $[C_{\overline{V}}, \mathfrak{M}_{\overline{V}}]$, where $\overline{V} = Z^\nu \setminus V$, tends to zero for expanding V. Measures constructed on the basis of two functionals whose difference belongs to L differ from one another only "locally."

Finally, in $\mathcal{Y}_f$ we introduce the trivial topology $\tau_a = \{\varnothing, \mathcal{Y}_f\}$. After these constructions the next result becomes obvious.

THEOREM 6.3. A Gibbs distribution possesses the property of regularity if and only if the mapping $F: (\mathcal{Y}_f, \tau_a) \to (\mathcal{H}_c, \tau_L)$ constructed according to (6.1) is continuous at zero.

The continuity of F defined in Theorems 6.2 and 6.3 expresses essentially different forms of stability of the physical system. Continuity defined by the pair (τ_*, τ) expresses, so to speak, the property of "elasticity" of the system: perturbations by a weak field leave no trace in the system after they are removed. Continuity defined by the pair (τ_a, τ_L) expresses the property of stability connected with localization of the effect on the system of a weak external field (the field created by a finite number of particles). By choosing other pairs of topologies, infinitely many versions of stability conditions can be obtained. Distinguishing among them the physically significant ones is a current problem of statistical physics.

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