On metric characterizations of some classes of Banach spaces

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Abstract. The first part of the paper is devoted to metric characterizations of Banach spaces with no cotype and no type > 1 in terms of graphs with uniformly bounded degrees. In the second part we prove that Banach spaces containing bilipschitz images of the infinite diamond do not have the Radon-Nikodým property and give a new proof of the Cheeger-Kleiner result on Banach spaces containing bilipschitz images of the Laakso space.

Keywords. Banach space, diamond graphs, expander graphs, Laakso graphs, Lipschitz embedding, Radon-Nikodým property

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1 Introduction

A mapping $F : X \to Y$ between metric spaces X and Y is called a *C-bilipschitz embedding* if there exists $r > 0$ such that $\forall u, v \in X \quad rd_X(u, v) \le d_Y(F(u), F(v)) \le rCd_X(u, v)$. A sequence $\{f_n\}$ of mappings $f_n : U_n \to Z_n$ between metric spaces is a sequence of *uniformly bilipschitz embeddings* if there is $C < \infty$ such that all of the embeddings are C-bilipschitz.

Many important classes of Banach spaces have been characterized in terms of uniformly bilipschitz embeddings of finite metric spaces. Bourgain [2] proved that a Banach space X is nonsuperreflexive if and only if there exist uniformly bilipschitz embeddings of finite binary trees $\{T_n\}_{n=1}^\infty$ of all depths into X. Similar characterization of spaces with no type > 1 was obtained by Bourgain, Milman, and Wolfson [3]: a Banach space X has no type > 1 if and only if there exist uniformly bilipschitz embeddings of Hamming cubes $\{H_n\}_{n=1}^\infty$ into X (see [16] for a simpler proof). Johnson and Schechtman [10] found a similar characterization of nonsuperreflexive spaces in terms of diamond graphs [8] and Laakso graphs [11]. Banach spaces without cotype were characterized by Mendel and Naor [15] in terms of lattice graphs $L_{m,n}$ whose vertex sets are $\{0, 1, \dots, m\}^n$, two vertices are joined by an edge if and only if their ℓ_∞-distance is equal to 1.

Characterizations in terms of bilipschitz embeddability of certain metric spaces are called *metric characterizations*. Observe that binary trees and Laakso graphs are graphs with uniformly bounded degrees. Degrees of Hamming cubes and the lattice graphs are unbounded. During the seminar "Nonlinear geometry of Banach spaces" (Workshop in Analysis and Probability at Texas A & M University, 2009) Johnson posed the following problem: Find metric characterizations of spaces with no type $p > 1$ and with no cotype in terms of graphs with uniformly bounded degrees. The first part of this paper is devoted to a solution of this problem.

In the second part of the paper we prove results related to another problem posed by Johnson during the mentioned seminar: Find metric characterizations of reflexivity and the Radon-Nikodým property (RNP). We prove that Banach spaces containing bilipschitz images of the infinite diamond do not have the RNP, but the converse is not true. We find a new proof of the Cheeger-Kleiner [6] result that Banach spaces containing bilipschitz images of the Laakso space do not have the RNP.

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2 Type and cotype in terms of graphs with uniformly bounded degrees

Theorem 2.1. *There exist metric characterizations of the classes of spaces with no type >1 and with no cotype in terms of graphs with maximum degree 3.*

We start with the characterization of cotype. This case is easier, because the well-known results ([13, Proposition 15.6.1] and [14]) imply that any family of finite metric spaces is uniformly bilipschitz embeddable into any Banach space with no cotype. Therefore to prove the cotype part of the theorem we need to show only that the graphs $L_{m,n}$ are uniformly bilipschitz embeddable into a family of graphs with uniformly bounded degrees. Thus it suffices to prove the following lemma.

Lemma 2.2. *Let $(V(G), d_G)$ be the vertex set of a graph G with its graph distance, and let $\varepsilon > 0$. Then there exist a graph M with maximum degree ≤ 3, $\ell \in \mathbb{N}$, and an embedding $F : V(G) \to V(M)$ such that*

$$\ell d_G(u,v) \le d_M(F(u),F(v)) \le (1+\varepsilon)\ell d_G(u,v) \quad \forall u,v \in V(G),$$

where d_M is the graph distance of M.

Proof. Let Δ_G be the maximum degree of G and let $r \in \mathbb{N}$ be such that $3 \cdot 2^{r-1} \ge \Delta_G$. Let $\ell \in \mathbb{N}$ be such that $\frac{\ell+2r}{\ell} < 1+\varepsilon$. We define the graph M in the following way. For each vertex v of G the graph M contains a 3-regular tree of depth r rooted at a vertex which we denote $m(v)$. For each edge uv of G we pick a leaf of the tree rooted at $m(v)$ and a leaf of the tree rooted at $m(u)$ and join them by a path of length ℓ. Leaves picked for different edges are different (this is possible because $3 \cdot 2^{r-1} \ge \Delta_G$), and there is no further interaction between the constructed trees and paths. It is easy to see that the maximum degree of M is 3.

We map $V(G)$ into $V(M)$ by mapping each v to the corresponding $m(v)$. It remains to show that

$$\ell d_G(u,v) \le d_M(m(u),m(v)) \le (\ell+2r)d_G(u,v) \tag{1}$$

The right-hand side inequality follows from the observation that if u and v are adjacent in G, then $d_M(m(u), m(v)) \le \ell + 2r$, the path of length $\ell + 2r$ can be constructed as the union of path in M corresponding to uv, and paths from $m(u)$ and $m(v)$ to the corresponding leaves.

To prove the left-hand side of (1) we consider a path joining $m(u)$ and $m(v)$. Let $m(u_1), \dots, m(u_k)$ be the set of roots of those trees which are visited by the path, listed in the order of visits. The description of M implies that $u, u_1, \dots, u_k, v$ is a uv-walk in G, hence its length is $\ge d_G(u, v)$. In order to get from one tree to another we need to traverse ℓ edges. Hence any path joining $m(u)$ and $m(v)$ has length $\ge \ell d_G(u, v)$. This completes the proof of the lemma and the cotype part of the theorem. $\square$

Proof of the type part of Theorem 2.1. By [14, Theorem 2.3], each space with no type > 1 contains subspaces whose Banach-Mazur distances to ℓ_1^d ($d \in \mathbb{N}$) are arbitrarily close to 1. Therefore it suffices to check that each of the graphs obtained in a similar way from Hamming cubes admits a uniformly bilipschitz embedding into ℓ_1^d for sufficiently large d. We denote by $\{S_n\}_{n=1}^\infty$ graphs obtained from $\{H_n\}_{n=1}^\infty$ using the procedure described in the proof of Lemma 2.2. We describe an embedding of the vertex set of S_n into ℓ_1^k. The images of vertices of S_n under this embedding are integer points of ℓ_1^k, edges of S_n correspond to line segments of length 1 parallel to unit vectors of ℓ_1^k. Having such a representation of S_n, it remains to show that the identity mappings of vertex sets of S_n endowed with their graph distances and their ℓ_1-distances are uniformly bilipschitz.

The graph H_n is n-regular, so we let $r \in \mathbb{N}$ be such that $n \le 3 \cdot 2^{r-1}$ and consider a rooted 3-regular tree of depth r. This tree can be isometrically embedded into ℓ_1^m, where $m = 3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{r-1}$. The embedding is the following: observing that m is the number of edges in the tree, we find a bijection between unit vectors in ℓ_1^m and edges of the tree. Now we map the root of the tree to $0 \in \ell_1^m$; if v is different from the root, we map v to the sum of unit vectors corresponding to the path from v to the root. We denote by T_r the image of the tree in ℓ_1^m.

We consider the natural isometric embedding of H_n into ℓ_1^n, with images of the vertices being all possible $0, 1$-sequences. We pick ℓ in the same way as in Lemma 2.2. We specify the position of the rooted tree corresponding to the vertex $v = \{\theta_i\}_{i=1}^n$ of H_n in $\ell_1^k = \ell_1^m \oplus_1 \ell_1^n$ as $T_r + \ell \cdot \{\theta_i\}$, where we mean that $T_r \subset \ell_1^m$ and $\ell \cdot \{\theta_i\}$ is a multiple of v considered as a vector in ℓ_1^n.

We introduce and embed the paths of length ℓ (from the construction of Lemma 2.2) in the following way: Since n is $\le$ the number of leaves in T_r, there is a bijection between the unit vectors of ℓ_1^n and some subset of leaves of T_r. On the other hand, each edge of H_n is parallel to one of the unit vectors. We add ℓ-paths in the following way. The path corresponding to the edge between v and $v + e_t$ (e_t is a unit vector of ℓ_1^n) is the straight line path of length ℓ joining the leaves of $T_r + \ell v$ and $T_r + \ell(v + e_t)$; in each of the trees the leaf is chosen in such a way that its ℓ_1^m component is the leaf corresponding to e_t.

It is clear that the graph S_n obtained in this way fits the description of M in the proof of Lemma 2.2. Therefore the natural embedding of H_n into S_n is $(1 + \varepsilon)$-bilipschitz if both graphs are endowed with their graph distances. It remains to estimate the bilipschitz constants of natural embeddings of S_n into ℓ_1^k.

Observe that the graph distance between two vertices of S_n cannot be less than the distance between their images in $\ell_1^n \oplus_1 \ell_1^m$, because each edge corresponds to a line segment of length 1. It remains to show that the graph distance between two vertices of S_n cannot be much larger than the ℓ_1-distance. Let x, y be two vertices of S_n, we need to estimate $d_{S_n}(x, y)$ from above in terms of $||x - y||_1$.

For each set of vertices of the form $T_r + \ell v$ we consider its union with the set of all vertices of ℓ-paths going out of this set. It is easy to see that if both x and y belong to one of such sets, then $d_{S_n}(x, y) \leq ||x - y||_1$.

For $x \in V(S_n)$ denote the projection of x to ℓ_1^n by $\pi(x)$, and the i-th coordinate of this projection by $\pi(x)_i$. If the situation described in the previous paragraph does not occur then there exists $i \in \{1, \dots, n\}$ such that $|\pi(x)_i - \pi(y)_i| = \ell$. Let $k \leq n$ be the number of coordinates for which this equality holds.

We have $||x - y||_1 \geq ||\pi(x) - \pi(y)||_1 \geq k\ell$. To estimate $d_{S_n}(x, y)$ from above we construct the following xy-path in S_n. If one of the numbers $\pi(x)_i$ is strictly between 0 and ℓ we start by moving from x in the direction of $\pi(y)_i$ (which in this case should be 0 or ℓ) till we reach a set of the form $T_r + \ell w_x$ for some vertex w_x of H_n.

We do similar thing at the other end of the path (near y): If one of the numbers $\pi(y)_i$ is strictly between 0 and ℓ we end the path by moving from y in the direction of $\pi(x)_i$ (which in this case should be 0 or ℓ) till we reach a set of the form $T_r + \ell w_y$.

We find a shortest path between w_x and w_y in H_n. It is easy to see that it has length k. Now we continue construction of the xy-path in S_n. This path will contain all paths of length ℓ corresponding to the edges of the $w_x w_y$-path in H_n. Between these paths we add the pieces of the corresponding trees of the from $T_r + \ell u$, needed to make a path. As a result we get an xy-path of length $< 2\ell + k\ell + 2(k + 1)r$. If $4r \leq \ell$ (we can definitely assume this), we have $2\ell + k\ell + 2(k + 1)r \leq 4k\ell$. In such a case $d_{S_n}(x, y) \leq 4||x - y||_1$. This completes the proof of the type part of the theorem. □

Corollary 2.3. *There exists a family $\{K_n\}$ of constant degree expanders, such that a Banach space X for which there exist uniformly bilipschitz embeddings of $\{K_n\}$ into X, has no cotype.*

Proof. Let $\{M_n\}$ be graphs of maximum degree 3 from the metric characterization of Banach spaces with no cotype. It suffices to show that there exists a family $\{K_n\}$ of constant degree expanders containing subsets isometric to $\{M_n\}$. Consider any family $\{G_k\}$ of constant d-regular expanders with the growing number of vertices. Let D_n be the diameter of M_n and m_n be its number of vertices. It is clear that we may assume without loss of generality that G_n contains a D_n-separated set of cardinality m_n. We fix a bijection between this set and $V(M_n)$. We add to G_n edges between vertices corresponding to adjacent vertices of M_n. Since D_n is the diameter of M_n, the obtained graph contains an isometric copy of M_n. The maximum degree of the obtained graph is $\leq d + 3$. Its expanding properties are not worse than those of G_n. Adding as many self-loops to it as is needed we get a $(d+3)$-regular graph K_n. It is clear that $\{K_n\}$ is a desired family of $(d+3)$-regular expanders. □

3 Diamonds and Laakso graphs

The *diamond graph* of level 0 is denoted D_0. It has two vertices joined by an edge of length 1. D_i is obtained from D_{i-1} as follows. Given an edge $uv \in E(D_{i-1})$, it is replaced by a quadrilateral u, a, v, b with edge lengths 2^{-i}. We endow D_n with their shortest path metrics. We consider the vertex of D_n as a subset of the vertex set of D_{n+1}, it is easy to check that this defines an isometric embedding. We introduce D_ω as the union of the vertex sets of $\{D_n\}_{n=0}^{\infty}$. For $u, v \in D_\omega$ we introduce $d_{D_\omega}(u,v)$ as $d_{D_n}(u,v)$ where $n \in \mathbb{N}$ is any integer for which $u, v \in V(D_n)$. Since the natural embeddings $D_n \to D_{n+1}$ are isometric, it is easy to see that $d_{D_n}(u,v)$ does not depend on the choice of n for which $u, v \in V(D_n)$.

Definition 3.1 ([9] or [5, p. 34]). Let $\delta > 0$. A sequence $\{x_i\}_{i=1}^{\infty}$ is called a δ-*tree* if $x_i = \frac{1}{2}(x_{2i} + x_{2i+1})$ and $||x_{2i} - x_i|| = ||x_{2i+1} - x_i|| \geq \delta$.

Theorem 3.2. *If D_ω is bilipschitz embeddable into a Banach space X, then X contains a bounded δ-tree for some $\delta > 0$.*

It is well-known that Banach spaces with the RNP do not contain bounded δ-trees (see [5, p. 31]). On the other hand there exist Banach spaces without the RNP which do not contain bounded δ-trees, see [4, p. 54]. So Theorem 3.2 implies:

Corollary 3.3. *If D_ω is bilipschitz embeddable into a Banach space X, then X does not have the Radon-Nikodým property. The converse is not true.*

Proof of Theorem 3.2. Let $f : D_\omega \to X$ be a bilipschitz embedding. Without loss of generality we assume that

$$\delta d_{D_\omega}(x,y) \leq ||f(x) - f(y)|| \leq d_{D_\omega}(x,y) \tag{2}$$

for some $\delta > 0$.

Let us show that this implies that the unit ball of X contains a δ-tree. The first element of the tree will be $x_1 = f(u_0) - f(v_0)$, where $\{u_0, v_0\} = V(D_0)$.

Now we consider the quadrilateral u_0, a, v_0, b. Inequality (2) implies $||f(a) - f(b)|| \geq \delta$. Consider two pairs of vectors (corresponding to two different paths from u to v in D_1):

Pair 1: $f(v_0) - f(a)$, $f(a) - f(u_0)$. **Pair 2:** $f(v_0) - f(b)$, $f(b) - f(u_0)$.

The inequality $||f(a) - f(b)|| \geq \delta$ implies that at least one of the following is true

$$||(f(v_0) - f(a)) - (f(a) - f(u_0))|| \geq \delta \quad \text{or} \quad ||(f(v_0) - f(b)) - (f(b) - f(u_0))|| \geq \delta.$$

Suppose that the first inequality holds. We let

$$x_2 = 2(f(v_0) - f(a)) \quad \text{and} \quad x_3 = 2(f(a) - f(u_0)).$$

It is clear that both conditions of Definition 3.1 are satisfied. Also, the condition (2) implies that $||x_2||, ||x_3|| \leq 1$.

We continue construction of the δ-tree in the unit ball of X in a similar manner. For example, to construct x_4 and x_5 we consider the corresponding quadrilateral a, a_1, v_0, b_1 in D_2. The inequality $||f(a_1) - f(b_1)|| \geq \delta/2$ implies that at least one of the following is true

$$||(f(v_0) - f(a_1)) - (f(a_1) - f(a))|| \geq \delta/2 \text{ or } ||(f(v_0) - f(b_1)) - (f(b_1) - f(a))|| \geq \delta/2.$$

Suppose that the second inequality holds. We let

$$x_4 = 4(f(v_0) - f(b_1)) \quad \text{and} \quad x_5 = 4(f(b_1) - f(a)).$$

It is clear that both conditions of Definition 3.1 are satisfied. Also (2) implies that $||x_4||, ||x_5|| \leq 1$. Proceeding in an obvious way we get a δ-tree in the unit ball of X. □

3.1 Finite version and the Johnson-Schechtman characterization of superreflexivity

Definition 3.4 ([9]). A Banach space X has the *finite tree property* if there exist $\delta > 0$ such that for each $k \in \mathbb{N}$ the unit ball of X contains a finite sequence $\{x_i : i = 1, \dots, 2^k - 1\}$ such that $x_i = \frac{1}{2}(x_{2i} + x_{2i+1})$ and $||x_{2i} - x_i|| = ||x_{2i+1} - x_i|| \geq \delta$ for each $i = 1, \dots, 2^{k-1} - 1$.

It is clear that the proof of Theorem 3.2 implies its finite version:

Corollary 3.5. *If there exist uniformly bilipschitz embeddings of $\{D_n\}_{n=1}^\infty$ into a Banach space X, then X has the finite tree property.*

Combining Corollary 3.5 with the well-known fact (see [9] and [7]) that the finite tree property is equivalent to nonsuperreflexivity, we get the second part of the result in [10, p. 181]: uniform bilipschitz embeddability of $\{D_n\}_{n=1}^\infty$ into X implies the nonsuperreflexivity of X.

3.2 Laakso space

Our version of the Laakso space (originally constructed in [11]) is similar to the version from [12, p. 290]. However, our version is a countable set (dense in the version of the space from [12]). The *Laakso graph* of level 0 is denoted L_0. It consists of two vertices joined by an edge of length 1. The *Laakso graph* L_i is obtained from L_{i-1} as follows. Each edge $uv \in E(L_{i-1})$ of length 4^{-i+1} is replaced by a graph with 6 vertices u, t_1, t_2, o_1, o_2, v where o_1, t_1, o_2, t_2 form a quadrilateral, and there are only two more edges ut_1 and vt_2, with all edge lengths 4^{-i}. We endow L_n with their shortest path metrics. We consider the vertex of L_n as a subset of the vertex set of L_{n+1}, it is easy to check that this defines an isometric embedding. We introduce the *Laakso space* L_ω as the union of the vertex sets of $\{L_n\}_{n=0}^{\infty}$. For $u, v \in L_\omega$ we introduce $d_{L_\omega}(u,v)$ as $d_{L_n}(u,v)$ where $n \in \mathbb{N}$ is any integer for which $u, v \in V(L_n)$. Since the natural embeddings $L_n \to L_{n+1}$ are isometric, it is easy to see that $d_{L_n}(u,v)$ does not depend on the choice of n for which $u, v \in V(L_n)$.

Our next purpose is to give a new proof of the following result of Cheeger and Kleiner [6, Corollary 1.7]:

Theorem 3.6. *If L_ω is bilipschitz embeddable into a Banach space X, then X does not have the Radon-Nikodým property.*

Proof. We do not know whether bilipschitz embeddability of L_ω into X implies the existence of a bounded δ-tree in X. To prove Theorem 3.6 we introduce the following definition.

Definition 3.7. Let $\delta > 0$. A sequence $\{x_i\}_{i=1}^{\infty}$ is called a δ-*semitree* if $x_i = \frac{1}{4}(x_{4i-2} + x_{4i-1} + x_{4i} + x_{4i+1})$ and $||(x_{4i-2} + x_{4i-1}) - (x_{4i} + x_{4i+1})|| \geq \delta$.

Our proof has two steps. First we show that bilipschitz embeddability of L_ω into X implies that X contains a bounded δ-semitree. The second step is to show that existence of a bounded δ-semitree in X implies that X does not have the RNP (this is almost standard, based on martingales).

Let $f : L_\omega \to X$ be a bilipschitz embedding. Without loss of generality we assume that

$$\delta d_{L_\omega}(x,y) \leq ||f(x) - f(y)|| \leq d_{L_\omega}(x,y) \tag{3}$$

for some $\delta > 0$.

We need to construct a δ-semitree in the unit ball of X. The first element of the semitree is $x_1 = f(u_0) - f(v_0)$, where $\{u_0, v_0\} = V(L_0)$.

Now we consider the 4-tuple u_0, o_1, v_0, o_2. Observe that (3) together with $d_{L_\omega}(o_1, o_2) \geq 1/2$ implies that $||o_1 - o_2|| \geq \delta/2$. Consider two pairs of vectors:

Pair 1: $f(v_0) - f(o_1),\ f(o_1) - f(u_0)$. **Pair 2:** $f(v_0) - f(o_2),\ f(o_2) - f(u_0)$.

The inequality $||f(o_1) - f(o_2)|| \geq \delta/2$ implies that at least one of the following is true

$$||(f(v_0) - f(o_1)) - (f(o_1) - f(u_0))|| \geq \delta/2 \text{ or } ||(f(v_0) - f(o_2)) - (f(o_2) - f(u_0))|| \geq \delta/2.$$

Suppose that the first inequality holds. We let

$$x_2 = 4(f(v_0) - f(t_2)), x_3 = 4(f(t_2) - f(o_1)), x_4 = 4(f(o_1) - f(t_1)), x_5 = 4(f(t_1) - f(u_0)).$$

It is easy to check that both conditions of Definition 3.7 are satisfied, we even get

$$||(x_2 + x_3) - (x_4 + x_5)|| = 4||(f(v_0) - f(o_1)) - (f(o_1) - f(u_0))|| \geq 2\delta.$$

Also, (3) applied to $d_{L_\omega}(u_0, t_1) = d_{L_\omega}(t_1, o_1) = d_{L_\omega}(o_1, t_2) = d_{L_\omega}(t_2, v_0) = 1/4$ implies that $||x_2||, ||x_3||, ||x_4||, ||x_5|| \leq 1$.

We continue our construction of the δ-semitree in the unit ball of X in a similar manner. For example, to construct x_6, x_7, x_8, and x_9, we consider the 6-tuple corresponding to the edge t_2v_0 of L_1 and repeat the same procedure as above for u_0v_0. Proceeding in an obvious way we get a δ-semitree in the unit ball of X.

To show that presence of a bounded δ-semitree implies absence of the RNP we use the same argument as for ε-bushes in [1, p. 111]. We construct an X-valued martingale $\{f_n\}_{n=0}^{\infty}$ on $[0,1]$. We let $f_0 = x_1$. The function f_2 is defined on four quarters of $[0,1]$ by x_2, x_3, x_4, x_5, respectively. To define the function f_3 we divide $[0,1]$ into 16 equal subintervals, and define f_3 as $x_6, \ldots, x_{21}$, on the respective subintervals, etc.

It is clear that we get a sequence of uniformly bounded functions. The first condition in the definition of a δ-semitree implies that this sequence is a martingale. The second condition implies that it is not convergent almost everywhere because it shows that on each interval of the form $\left[\frac{k}{4^n}, \frac{k+1}{4^n}\right]$ the average value of $||f_n - f_{n+1}||$ over the first half of the interval is $\geq \delta/4$, this implies that $||f_n(t) - f_{n+1}(t)|| \geq \delta/4$ on a subset in $[0,1]$ of measure $\geq \frac{1}{2}$. It remains to apply [1, Theorem 5.8]. □

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